NORMAL APPROXIMATION IN TOTAL VARIATION FOR STATISTICS IN GEOMETRIC PROBABILITY

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Abstract

We use Stein's method to establish the rates of normal approximation in terms of the total variation distance for a large class of sums of score functions of samples arising from random events driven by a marked Poisson point process on $\mathbb{R}^d$. As in the study under the weaker Kolmogorov distance, the score functions are assumed to satisfy stabilisation and moment conditions. At the cost of an additional non-singularity condition, we show that the rates are in line with those under the Kolmogorov distance. We demonstrate the use of the theorems in four applications: Voronoi tessellations, *k*-nearest-neighbours graphs, timber volume, and maximal layers.

Keywords: Total variation distance; non-singular distribution; Berry–Esseen bound; Stein's method

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1. Introduction

Limit theorems of functionals of Poisson point processes initiated by Avram and Bertsimas [1] have been of considerable interest in the literature; see, e.g., [30, 31, 37, 41, 46, 47] and references therein. To estimate the approximation errors, a number of tools have been developed, including Stein's method [3, 4], the Malliavin–Stein technique via the Wiener–Itô expansion [37] and the second-order Poincaré inequalities [31], and the stabilisation [38, 39]. The main feature of the stabilisation is that the insertion of a point into a Poisson point process induces only a local effect, in some sense; hence there is little change in the functionals. However, inserting an additional point into the Poisson point process results in the Palm process of the Poisson point process at the point [26, Chapter 10], and it is shown in [13, 14] that the magnitude of the difference between a point process and its Palm processes is directly linked to the accuracy of Poisson and normal approximations of the point process. This is also the fundamental reason why the limit theorems in the above-mentioned papers can be established.

The normal approximation theory is generally quantified in terms of the Kolmogorov distance d_K: for two random variables X_1 and X_2 with distributions F_1 and F_2,

$$d_K(X_1, X_2) := d_K(F_1, F_2) := \sup_{x \in \mathbb{R}} |F_1(x) - F_2(x)|.$$

The well-known Berry–Esseen theorem [7, 20] states the following. If X_i, $1 \le i \le n$, are independent and identically distributed (i.i.d.) random variables with mean 0, variance 1, and a finite third moment, define

$$Y_n = \frac{\sum_{i=1}^n X_i}{\sqrt{n}}, \qquad Z \sim N(0, 1),$$

where $\sim$ denotes 'is distributed as'. Then

$$d_K(Y_n, Z) \le \frac{C \mathbb{E}|X_1|^3}{\sqrt{n}}.$$

The Kolmogorov distance $d_K(F_1, F_2)$ measures the maximum difference between the distribution functions F_1 and F_2, but it does not say much about the difference between the probabilities $\mathbb{P}(X_1 \in A)$ and $\mathbb{P}(X_2 \in A)$ for a non-interval Borel subset A of the real space $\mathbb{R}$, e.g., $A = \cup_{i \in \mathbb{Z}} (2i, 2i + 0.5]$, where $\mathbb{Z}$ denotes the set of all integers. Such information is reflected in the total variation distance $d_{TV}(F_1, F_2)$, defined by

$$d_{TV}(X_1, X_2) := d_{TV}(F_1, F_2) := \sup_{A \in \mathscr{B}(\mathbb{R})} |F_1(A) - F_2(A)|,$$

where $\mathscr{B}(\mathbb{R})$ stands for the Borel σ-algebra on $\mathbb{R}$. Vanishing bounds for normal approximation under the total variation distance for statistics in geometric probability provide us with the theoretical foundation for constructing density estimators of the distributions in a wide range of applications documented in [15, Chapters 8–10] and bound the minimax probability of error for making statistical inference [49, pp. 80–81]. Moreover, since $d_K(X_1, X_2) \le d_{TV}(X_1, X_2)$, vanishing bounds on the total variation distance imply a stronger qualitative result than just a central limit theorem (CLT).

Although CLTs with errors measured in the total variation distance have been studied in some special circumstances (see, e.g., [19, 36, 2]), it is generally believed that the total variation distance is too strong for quantifying the errors in normal approximation (see, e.g., [9, 12, 21]). For example, the total variation distance between any discrete distribution and any normal distribution is always 1. To recover CLTs with errors measured in the total variation distance, a common approach is to discretise the distribution of interest and approximate it with a simple discrete distribution, e.g., a translated Poisson distribution [43, 44], a centred binomial distribution [45], a discretised normal distribution [12, 21], or a family of polynomial-type distributions [24]. The multivariate versions of these approximations are investigated by [6].

By discretising a distribution F of interest, we essentially group the probability of an area and put it at one point in the area. Hence the information of $F(A)$ for a general set $A \in \mathscr{B}(\mathbb{R})$ is completely lost. In this paper, we consider the normal approximation in terms of the total variation distance to the sum of random variables under various circumstances.

A motivating example: [22, p. 146]. Let $\{X_i : i \ge 1\}$ be a sequence of i.i.d. random variables taking values 0 and 1 with equal probability; then $X = \sum_{k=1}^{\infty} 2^{-k} X_k$ has the uniform distribution on $(0, 1)$. If we separate the even and odd terms into $U = \sum_{k=1}^{\infty} 2^{-2k} X_{2k}$

and $V = \sum_{k=1}^{\infty} 2^{-(2k-1)} X_{2k-1}$, then U and V are independent, $2U \overset{\mathrm{d}}{=} V$. Both U and V differ only by scale factors from the Cantor distribution, which is singular with respect to the Lebesgue measure [22, p. 146]. Now, we can construct mutually independent random variables $\{U_i, \ V_i : i \geq 1\}$ such that $U_i \overset{\mathrm{d}}{=} U - \mathbb{E}U$ and $V_i \overset{\mathrm{d}}{=} V - \mathbb{E}V$. Consider $\xi_1 = U_1 + V_1$, $\xi_2 = -V_1 - U_2$, $\xi_3 = U_2 + V_2, \ldots$; then $\{\xi_i\}$ is a sequence of 1-dependent and identically distributed random variables having the uniform distribution on $(-0.5, 0.5)$. One can easily verify that $\sum_{i=1}^{n} \xi_i$ does not converge to normal as $n \to \infty$; hence stronger conditions are needed to ensure normal approximation for the sum of dependent random variables.
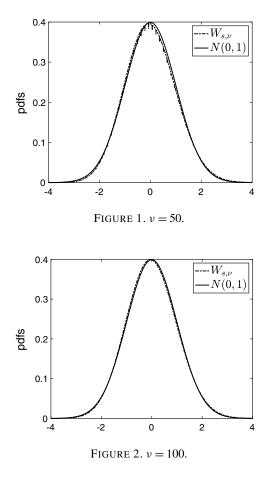
Under the Kolmogorov distance, user-friendly conditions are usually formulated to ensure that the variance of the sum becomes large as $n \to \infty$. In the motivating example, the variance of the partial sum $\sum_{i=1}^{n} \xi_i$ is bounded, which does not give sufficient offset in the ξ_i's to warrant a CLT. For the normal approximation of the sum of score functions, a typical condition for the variance to converge to infinity sufficiently fast is non-degeneracy—that is, the conditional variance of the sum given the information outside a local region is away from 0 (see [38, 51]).

The success of normal approximation with errors measured in terms of the total variation distance without discretisation hinges on non-singularity; that is, the distribution contains an absolutely continuous component with respect to the Lebesgue measure. More precisely, by the Lebesgue decomposition theorem [25, p. 134], any distribution function F on $\mathbb{R}$ can be represented as

$$F = (1 - \alpha_F)F_s + \alpha_F F_a, \tag{1.1}$$

where $\alpha_F \in [0, 1]$, and where F_s and F_a are two distribution functions such that, with respect to the Lebesgue measure on $\mathbb{R}$, F_a is absolutely continuous and F_s is singular [25, p. 126]; F is said to be *non-singular* if $\alpha_F > 0$. For convenience, we say that a random variable is non-singular if its distribution function is non-singular. It is clear that non-singularity implies non-degeneracy. Prohorov [40] proved that in one dimension, a necessary and sufficient condition for the convergence of the standardised partial sum of i.i.d. random variables X_i to the standard normal distribution in terms of the total variation distance is that there exists an $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{n_0} X_i$ is non-singular. The non-singularity is almost necessary because it is an essential ingredient in the special case of the sum of i.i.d. random variables; see [2] for a brief review of the development of the CLT in the total variation distance.

In the context of functionals of Poisson point processes, a prototypical example is the sum $W_\nu = \sum_{i=1}^{N} X_i$ of Poisson number $N \sim \mathrm{Poisson}(\nu)$ i.i.d. random variables $\{X_i, \ i \geq 1\}$ which are independent of N. If the distribution $\mathscr{L}(X_1)$ of X_1 is discrete, then the distribution of $W_{s,\nu} := (W_\nu - \mathbb{E}W_\nu)/\sqrt{\mathrm{Var}(W_\nu)}$ is also discrete, and hence $d_{TV}(W_{s,\nu}, Z) = 1$ for all ν, where $Z \sim N(0, 1)$. However, if $\mathscr{L}(X_1)$ is non-singular, the asymptotic behaviour of $d_{TV}(W_{s,\nu}, Z)$ in terms of large ν is very similar to that of $d_K(W_{s,\nu}, Z)$. For example, assume that $\mathbb{P}(Y_1 = 1) = 9/10$ and $\mathbb{P}(Y_1 \in dx) = 0.1dx$ for $x \in [0, 1)$. Then Figure 1 shows that when $\nu = 50$, the density function of the absolutely continuous part of $W_{s,\nu}$ is very close to that of $N(0, 1)$ in the total variation distance, giving a very small value of $d_{TV}(W_{s,\nu}, Z)$. Moreover, as shown in Figure 2, the distance $d_{TV}(W_{s,\nu}, Z)$ decreases very fast when ν becomes large. Briefly summarised, the main results of the paper state that, for the score functions of samples arising from random events driven by a marked Poisson point process on $\mathbb{R}^d$, if the conditional distribution of the sum given the information outside a neighbourhood is non-singular, then, with the cost of a logarithmic factor, the error of the normal approximation to the sum of such score functions in terms of the total variation distance is similar to the error with respect to the Kolmogorov distance established in, e.g., [38, 39, 30].

FIGURE 1. $\nu = 50$.

FIGURE 2. $\nu = 100$.

In Section 2, we give definitions of the concepts, state the conditions, and present the main theorems. In Section 3, these theorems are applied to establish error bounds for the normal approximation for statistics in Voronoi tessellations, *k*-nearest-neighbours graphs, timber volume, and maximal layers. The proofs of the main results in Section 2 rely on a number of preliminaries and lemmas which are given in Section 4. For ease of reading, all proofs are postponed to Section 5.

2. General results

We consider the functionals of a marked point process with a Poisson point process in $\mathbb{R}^d$ as its ground process, and each point carries a mark in a measurable space $(T, \mathscr{T})$ independently of other marks, where $\mathscr{T}$ is a σ-algebra on T. More precisely, let $S := \mathbb{R}^d \times T$ be equipped with the product σ-field $\mathscr{S} := \mathscr{B}(\mathbb{R}^d) \times \mathscr{T}$, where $\mathscr{B}(\mathbb{R}^d)$ is the Borel σ-algebra of $\mathbb{R}^d$. We use C_S to denote the space of all locally finite non-negative integer-valued measures ξ, often called *configurations*, on S such that $\xi(\{x\} \times T) \leq 1$ for all $x \in \mathbb{R}^d$. The space C_S is endowed with the σ-field $\mathscr{C}_S$ generated by the vague topology [26, p. 169]. A *marked point process* Ξ is a measurable mapping from $(\Omega, \mathscr{F}, \mathbb{P})$ to $(C_S, \mathscr{C}_S)$ [27, p. 49]. The induced simple point process $\bar{\Xi}(\cdot) := \Xi(\cdot \times T)$ is called the *ground process* [16, p. 3] or projection [27, p. 17] of the marked point process Ξ on $\mathbb{R}^d$. In this paper, we use $\mathscr{P}_{\lambda, \mathscr{L}_T}$ to denote the law of a marked Poisson

point process having a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity measure λdx as its ground process and i.i.d. marks on $(T, \mathscr{T})$ with the law $\mathscr{L}_T$.

For convenience, we write the restricted process of Ξ on a measurable set $A \in \mathscr{B}(\mathbb{R}^d)$ as Ξ_A, i.e., $\Xi_A(B \times D) := \Xi((A \cap B) \times D)$ for all $D \in \mathscr{T}$ and $B \in \mathscr{B}(\mathbb{R}^d)$. Statistics in geometric probability are often affected by the point configuration; hence both the restricted and unrestricted point patterns are of interest in applications. The functionals we study in the paper are defined on $\Gamma_\alpha := \left[-\frac{1}{2}\alpha^{\frac{1}{d}}, \frac{1}{2}\alpha^{\frac{1}{d}} \right]^d$ and have the forms

$$W_\alpha := \sum_{(x,m) \in \Xi_{\Gamma_\alpha}} \eta((x, m), \Xi)$$

and

$$\bar{W}_\alpha := \sum_{(x,m) \in \Xi_{\Gamma_\alpha}} \eta((x, m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha) = \sum_{(x,m) \in \Xi_{\Gamma_\alpha}} \eta((x, m), \Xi, \Gamma_\alpha),$$

where $\Xi \sim \mathscr{P}_{\lambda, \mathscr{L}_T}$. The function η is called a *score function* (resp. *restricted score function*), i.e., a measurable function on $(S \times C_S, \mathscr{S} \times \mathscr{C}_S)$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ (resp. a function mapping $S \times C_S \times \mathscr{B}(\mathbb{R}^d)$ to $\mathbb{R}$ which is measurable with respect to $(\mathscr{S} \cap (\Gamma_\alpha \times T)) \times \mathscr{C}_{S \cap (\Gamma_\alpha \times T)} \to \mathscr{B}(\mathbb{R})$ when the third coordinate is fixed), and it represents the interaction between a point and the configuration. Because our interest is in the values of the score function of the points in a configuration, for convenience, $\eta((x, m), \mathscr{X})$ (resp. $\eta((x, m), \mathscr{X}, \Gamma_\alpha)$) is understood as 0 for all $x \in \mathbb{R}^d$ and $\mathscr{X} \in C_S$ such that $(x, m) \notin \mathscr{X}$. We consider the score functions satisfying the following four conditions:

- stabilisation (Assumption 2.1)
- translation-invariance (Assumption 2.2)
- a moment condition (Assumption 2.3)
- non-singularity (Assumption 2.4)

Assumption 2.1. Stabilisation

Unrestricted case. For a locally finite configuration $\mathscr{X}$ and $z \in (\mathbb{R}^d \times T) \cup \{\emptyset\}$, write $\mathscr{X}^{\{z\}} = \mathscr{X}$ if $z = \emptyset$ and $\mathscr{X}^{\{z\}} = \mathscr{X} \cup \{z\}$ otherwise. We use δ_v to denote the Dirac measure at v, and $B(x, r)$ to denote the ball with centre x and radius r. The notion of stabilisation is introduced in [38], and we adapt it to our setup as follows.

Definition 2.1. *(Unrestricted case.)* A score function η on $\mathbb{R}^d \times T$ is range-bounded (resp. exponentially stabilising, polynomially stabilising of order $\beta > 0$) with respect to intensity λ and a probability measure $\mathscr{L}_T$ on T if for all $x \in \mathbb{R}^d$, $z \in (\mathbb{R}^d \times T) \cup \{\emptyset\}$, and almost all realisations $\mathscr{X}$ of the marked homogeneous Poisson point process $\Xi \sim \mathscr{P}_{\lambda, \mathscr{L}_T}$, there exists an $R := R(x) := R(x, m_x, \mathscr{X}^{\{z\}}) \in (0, \infty)$ (a radius of stabilisation) such that for all locally finite $\mathscr{Y} \subset (\mathbb{R}^d \backslash B(x, R)) \times T$, we have

$$R\left((x, m_x), \left[\mathscr{X}^{\{z\}} \cap (B(x, R) \times T)\right] \cup \mathscr{Y}\right) = R\left((x, m_x), \mathscr{X}^{\{z\}} \cap (B(x, R) \times T)\right),$$

$$\eta\left((x, m_x), \left[\mathscr{X}^{\{z\}} \cap (B(x, R) \times T)\right] \cup \mathscr{Y}\right) = \eta\left((x, m_x), \mathscr{X}^{\{z\}} \cap (B(x, R) \times T)\right),$$

and the tail probability

$$\tau(t) := \sup_{x \in \mathbb{R}^d, m_x \in \mathrm{supp}(\mathscr{L}_T)} \sup_{z \in (\mathbb{R}^d \times T) \cup \{\emptyset\}} \mathbb{P}\Big(R(x, m_x, \Xi^{\lfloor z \rfloor} + \delta_{(x,m_x)}) \geq t\Big)$$

satisfies

$$\tau(t) = 0 \text{ for some } t \in \mathbb{R}_+ \quad \Big(\text{resp. } \tau(t) \leq C_1 e^{-C_2 t}, \ \tau(t) \leq C_1 t^{-\beta} \text{ for all } t \in \mathbb{R}_+\Big)$$

for some positive constants C_1 and C_2.

Here and in the following, we write R or $R(x)$ (resp. $\bar{R}$ or $\bar{R}(x)$ in Definition 2.1) only if it will not cause any confusion. The definition ensures that $\{R((x, m_x), \mathscr{X}^{\lfloor z \rfloor}) \leq t\}$ is determined by $\mathscr{X}_{B(x,t)}$ for all $x \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$.

Restricted case. We have the following counterpart of stabilisation for the functionals with a restricted marked Poisson point process input. Note that the score function for the restricted input is not affected by points outside Γ_α.

Definition 2.2. *(Restricted case.)* A score function η is range-bounded (resp. exponentially stabilising, polynomially stabilising of order $\beta > 0$) with respect to intensity λ and a probability measure $\mathscr{L}_T$ on T if for all $\alpha \in \mathbb{R}_+$, $x \in \Gamma_\alpha$, and $z \in (\Gamma_\alpha \times T) \cup \{\emptyset\}$, and almost all realisations $\mathscr{X}$ of the marked homogeneous Poisson point process $\Xi \sim \mathscr{P}_{\lambda, \mathscr{L}_T}$, there exists an $\bar{R} := \bar{R}(x, \alpha) := \bar{R}(x, m_x, \alpha, \mathscr{X}^{\lfloor z \rfloor}) \in (0, \infty)$ (a radius of stabilisation) such that for all locally finite $\mathscr{Y} \subset (\Gamma_\alpha \backslash B(x, R)) \times T$, we have

$$\bar{R}\Big((x, m_x), \big[\mathscr{X}_{\Gamma_\alpha}^{\lfloor z \rfloor} \cap (B(x, \bar{R}) \times T)\big] \cup \mathscr{Y}, \Gamma_\alpha\Big) = \bar{R}\Big((x, m_x), \mathscr{X}_{\Gamma_\alpha}^{\lfloor z \rfloor} \cap (B(x, \bar{R}) \times T), \Gamma_\alpha\Big),$$

$$\eta\Big((x, m_x), \big[\mathscr{X}_{\Gamma_\alpha}^{\lfloor z \rfloor} \cap (B(x, \bar{R}) \times T)\big] \cup \mathscr{Y}, \Gamma_\alpha\Big) = \eta\Big((x, m_x), \mathscr{X}_{\Gamma_\alpha}^{\lfloor z \rfloor} \cap (B(x, \bar{R}) \times T), \Gamma_\alpha\Big),$$

$$(2.1)$$

and the tail probability

$$\bar{\tau}(t) := \sup_{x \in \mathbb{R}^d, m_x \in \mathrm{supp}(\mathscr{L}_T), \alpha \in \mathbb{R}_+} \sup_{z \in (\Gamma_\alpha \times T) \cup \{\emptyset\}} \mathbb{P}\Big(\bar{R}(x, m_x, \alpha, \Xi^{\lfloor z \rfloor} + \delta_{(x,m_x)}) \geq t\Big)$$

satisfies

$$\bar{\tau}(t) = 0 \text{ for some } t \in \mathbb{R}_+ \quad \Big(\text{resp. } \bar{\tau}(t) \leq C_1 e^{-C_2 t}, \ \bar{\tau}(t) \leq C_1 t^{-\beta} \text{ for all } t \in \mathbb{R}_+\Big)$$

for some positive constants C_1 and C_2.

As in the unrestricted case, the definition ensures that $\{\bar{R}((x, m_x), \alpha, \mathscr{X}^{\lfloor z \rfloor}) \leq t\}$ is a function of $\mathscr{X}_{B(x,t) \cap \Gamma_\alpha}$ for all $x \in \mathbb{R}^d$ and $\alpha, t \in \mathbb{R}_+$.

Assumption 2.2. Translation-invariance

We write $d(x, A) := \inf\{d(x, y); y \in A\}$, $A \pm B := \{x \pm y; x \in A, \ y \in B\}$ for $x \in \mathbb{R}^d$ and $A, B \in \mathscr{B}(\mathbb{R}^d)$, and define the shift operator as $\Xi^x(\cdot \times D) := \Xi((\cdot + x) \times D)$ for all $x \in \mathbb{R}^d$, $D \in \mathscr{T}$.

Unrestricted case. See also [38].

Definition 2.3. The score function η is *translation-invariant* if for all locally finite configurations $\mathscr{X}$, $x, y \in \mathbb{R}^d$, and $m \in T$, $\eta((x + y, m), \mathscr{X}) = \eta((x, m), \mathscr{X}^y)$.

Restricted case. A translation may send a configuration outside of Γ_α, resulting in a completely different configuration inside Γ_α. For example, for a configuration $\mathscr{X} \subset \Gamma_\alpha \times T$ and a point $x \neq \mathbf{0}$ in $\mathbb{R}^d$, the translation $\mathscr{X}^x$ may not be a configuration contained in $\Gamma_\alpha \times T$. To focus on the part that affects the score function, we expect the score function to take the same value for two configurations if the parts within their stabilising radii are completely inside Γ_α, and one is a translation of the other. More precisely, we have the following definition.

Definition 2.4. A stabilising score function is called *translation-invariant* if for any $\alpha > 0$, $x \in \Gamma_\alpha$, and $\mathscr{X} \in C_{\mathbb{R}^d \times T}$ such that $\bar{R}(x, m, \alpha, \mathscr{X}) \leq d(x, \partial\Gamma_\alpha)$, where ∂A stands for the boundary of A, we have $\eta\big((x, m), \mathscr{X}, \Gamma_\alpha\big) = \eta\big((x', m), \mathscr{X}', \Gamma_{\alpha'}\big)$ and $\bar{R}(x', m, \alpha', \mathscr{X}') = \bar{R}(x, m, \alpha, \mathscr{X})$ for all $\alpha' > 0$, $x' \in \Gamma_{\alpha'}$, and $\mathscr{X}' \in C_{\mathbb{R}^d \times T}$ such that $\bar{R}(x, m, \alpha, \mathscr{X}) \leq d(x', \partial\Gamma_{\alpha'})$ and

$$\left(\mathscr{X}'_{B\left(x', \bar{R}(x,m,\alpha,\mathscr{X})\right)} \right)^{x'} = \left(\mathscr{X}_{B\left(x, \bar{R}(x,m,\alpha,\mathscr{X})\right)} \right)^{x}.$$

Note that there is a tacit assumption of consistency in Definition 2.4, which implies that if η is translation-invariant under Definition 2.4, then there exists a $\bar{g} : C_{\mathbb{R}^d \times T} \to \mathbb{R}$ such that

$$\lim_{\alpha \to \infty} \eta((0, m), \mathscr{X}, \Gamma_\alpha) = \bar{g}\big(\mathscr{X} + \delta_{(0,m)}\big)$$

for $\mathscr{L}_T$-almost-sure $m \in T$ and almost all realisations $\mathscr{X}$ of the marked homogeneous Poisson point process $\Xi \sim \mathscr{P}_{\lambda, \mathscr{L}_T}$. Furthermore, we can see that for each score function η satisfying the translation-invariance in Definition 2.4, there exists a score function for the unrestricted case, obtained by setting

$$\bar{\eta}((x, m), \mathscr{X}) := \bar{g}(\mathscr{X}^x)\mathbf{1}_{(x,m) \in \mathscr{X}} \tag{2.2}$$

and using its corresponding radius of stabilisation in the sense of Definition 2.1 as R. From the construction, $\bar{\eta}$ is range-bounded (resp. exponentially stabilising, polynomially stabilising of order $\beta > 0$) in the sense of Definition 2.1 if η is range-bounded (resp. exponentially stabilising, polynomially stabilising of order $\beta > 0$) in the sense of Definition 2.2. Moreover, if $B(x, R(x)) \subset \Gamma_\alpha$, then $\bar{R}(x, \alpha) = R(x)$, and if $B(x, R(x)) \not\subset \Gamma_\alpha$, then $\bar{R}(x, \alpha) > d\big(x, \partial\Gamma_\alpha\big)$, but there is no definite relationship between $\bar{R}$ and R.

Assumption 2.3. Moment condition

Unrestricted case. A score function η is said to satisfy the *k*th moment condition if

$$\mathbb{E}\Big(\big| \eta\big((\mathbf{0}, M_3), \Xi + a_1 \delta_{(x_1, M_1)} + a_2 \delta_{(x_2, M_2)} + \delta_{(\mathbf{0}, M_3)}\big) \big|^k \Big) \leq C \tag{2.3}$$

for some positive constant C for all $a_i \in \{0, 1\}$, distinct $x_i \in \mathbb{R}^d$, $i \in \{1, 2\}$, and i.i.d. random elements $M_1, M_2, M_3 \sim \mathscr{L}_T$ that are independent of Ξ.

Restricted case. A score function η is said to satisfy the *k*th moment condition if there exists a positive constant C independent of α such that

$$\mathbb{E}\Big(\big| \eta\big((x_3, M_3), \Xi_{\Gamma_\alpha} + a_1 \delta_{(x_1, M_1)} + a_2 \delta_{(x_2, M_2)} + \delta_{(x_3, M_3)}\big) \big|^k \Big) \leq C \tag{2.4}$$

for all $a_1, a_2 \in \{0, 1\}$, distinct $x_1, x_2, x_3 \in \Gamma_\alpha$, and i.i.d. random elements $M_1, M_2, M_3 \sim \mathscr{L}_T$ that are independent of Ξ. From the construction, if η is stabilising, then $\bar{\eta}$ satisfies the moment condition of the same order in the sense of (2.3).

Assumption 2.4. Non-singularity

Unrestricted case. The score function is said to be *non-singular* if

$$\mathscr{L}\left(\sum_{(x,m)\in\Xi}\eta((x,m),\,\Xi)\,\mathbf{1}_{d(x,N_0)<R(x)}\,\middle|\,\sigma\left(\Xi_{N_0^c}\right)\right) \tag{2.5}$$

has a positive probability of being non-singular for some bounded set N_0. That is, the sum of the values of the score function affected by the points in N_0 is non-singular.

Restricted case. We define non-singularity when the score function is stabilising and translation-invariant. The score function η for restricted input is said to be *non-singular* if it is stabilising, and the corresponding $\bar{\eta}$ defined in (2.2) satisfies that

$$\mathscr{L}\left(\sum_{(x,m)\in\Xi}\bar{\eta}((x,m),\,\Xi)\,\mathbf{1}_{d(x,N_0)<R(x)}\,\middle|\,\sigma\left(\Xi_{N_0^c}\right)\right) \tag{2.6}$$

has a positive probability of being non-singular for some bounded set N_0.

The non-singularity assumptions above are stronger than the non-degeneracy conditions (1.10) and (1.12) in [51] for Poisson input. The latter is comparable to the non-degeneracy condition in [38, Theorem 2.1]; see the proof of Lemma 5.9.

The main result for W_α (the unrestricted case) is summarised below.

Theorem 2.1. *Let $Z_\alpha \sim N(\mathbb{E}W_\alpha, \mathrm{Var}(W_\alpha))$. Assume that the score function η is translation-invariant as in Definition 2.2 and satisfies the non-singularity in Assumption 2.4.*

(i) *If η is range-bounded as in Definition 2.1 and satisfies the third moment condition (2.3), then*

$$d_{TV}(W_\alpha, Z_\alpha) \le O\left(\alpha^{-\frac{1}{2}}\right).$$

(ii) *If η is exponentially stabilising as in Definition 2.1 and satisfies the third moment condition (2.3), then*

$$d_{TV}(W_\alpha, Z_\alpha) \le O\left(\alpha^{-\frac{1}{2}}\ln(\alpha)^{\frac{5d}{2}}\right).$$

(iii) *If η is polynomially stabilising as in Definition 2.1 with parameter $\beta > \frac{(15k-14)d}{k-2}$ and satisfies the k'th moment condition (2.3) with $k' > k \ge 3$, then*

$$d_{TV}(W_\alpha, Z_\alpha) \le O\left(\alpha^{-\frac{\beta(k-2)[\beta(k-2)-d(15k-14)]}{(k\beta-2\beta-dk)(5dk+2\beta k-4\beta)}}\right).$$

When approximation error is measured in terms of the Kolmogorov distance, the distributions of W_α and $\bar{W}_\alpha$ are often close for large α. However, in terms of the total variation distance, one cannot infer the accuracy of the normal approximation of W_α by taking the limit of that of $\bar{W}_\alpha$. For this reason, we need to adapt the conditions accordingly and tackle $\bar{W}_\alpha$ separately. We state the main result for $\bar{W}_\alpha$ (the restricted case) in the following theorem.

Theorem 2.2. *Let $\bar{Z}_\alpha \sim N(\mathbb{E}\bar{W}_\alpha, \mathrm{Var}(\bar{W}_\alpha))$. Assume that η is translation-invariant as in Definition 2.4 and satisfies the non-singularity in Assumption 2.4.*

(i) *If η is range-bounded as in Definition 2.2 and satisfies the third moment condition (2.4), then*

$$d_{TV}(\bar{W}_\alpha, \bar{Z}_\alpha) \le O\left(\alpha^{-\frac{1}{2}}\right).$$

(ii) *If η is exponentially stabilising as in Definition* 2.2 *and satisfies the third moment condition* (2.4), *then*

$$d_{TV}(\bar{W}_\alpha, \bar{Z}_\alpha) \le \mathrm{O}\left(\alpha^{-\frac{1}{2}} \ln(\alpha)^{\frac{5d}{2}} \right).$$

(iii) *If η is polynomially stabilising as in Definition* 2.2 *with parameter $\beta > \frac{(15k-14)d}{k-2}$ and satisfies the k'th moment condition* (2.4) *with $k' > k \ge 3$, then*

$$d_{TV}\left(\bar{W}_\alpha, \bar{Z}_\alpha \right) \le \mathrm{O}\left(\alpha^{-\frac{\beta(k-2)[\beta(k-2)-d(15k-14)]}{(k\beta-2\beta-dk)(5dk+2\beta k-4\beta)}} \right).$$

Remark 2.1. It is unclear whether the logarithmic factors in Theorem 2.1(ii) and Theorem 2.2(ii) are artefacts of the proofs.

3. Applications

Our main result can be applied to a wide range of geometric probability problems, including the normal approximation of functionals of k-nearest-neighbours graphs, Voronoi graphs, sphere-of-influence graphs, Delaunay triangulation graphs, Gabriel graphs, and relative neighbourhood graphs. To keep our article to a reasonable length, we give details only for the k-nearest-neighbours graph and the Voronoi graph. We can see that many functionals of the graphs, such as the total edge length, satisfy the conditions of the main theorems naturally, and the ideas for verifying these conditions are similar. For ease of reading, we briefly introduce these graphs below; more details can be found in [17, 48].

Let $\mathscr{X} \subset \mathbb{R}^d$ be a locally finite point set.

(i) *The k-nearest-neighbours graph.* The k-nearest-neighbours graph $NG(\mathscr{X})$ is the graph obtained by including $\{x, y\}$ as an edge whenever y is one of the k points nearest to x or x is one of the k points nearest to y. A variant of $NG(\mathscr{X})$ considered in the literature is the *directed* graph $NG'(\mathscr{X})$, which is constructed by inserting a directed edge (x, y) if y is one of the k nearest neighbours of x.

(ii) *Voronoi tessellations.* We enumerate the points in $\mathscr{X}$ as $\{x_1, x_2, \dots\}$, and for each $i \in \mathbb{N}$ we denote by $C(x_i) := C(x_i, \mathscr{X})$ the locus of points in $\mathbb{R}^d$ closer to x_i than to any other points in $\mathscr{X}$. We can see that $C(x_i)$ is the intersection of half-planes. In particular, when the point set $\mathscr{X}$ has $n < \infty$ points, $C(x_i)$ is a convex polygonal region with at most $n-1$ sides, $1 \le i \le n$. The cells $C(x_i)$ form a partition of $\mathbb{R}^d$. The partition is called a Voronoi tessellation, and the points in $\mathscr{X}$ are usually called Voronoi generators.

(iii) *The Delaunay triangulation graph.* The Delaunay triangulation graph puts an edge between two points in $\mathscr{X}$ if these points are centres of adjacent Voronoi cells; it is a dual of a Voronoi tessellation.

(iv) *The Gabriel graph.* The Gabriel graph puts an edge between two points x and y in $\mathscr{X}$ if the ball centred at the middle point $\frac{x+y}{2}$ with radius $\left\| \frac{x-y}{2} \right\|$ does not contain any other points in $\mathscr{X}$. We can see that the Gabriel graph is a subgraph of the Delaunay triangulation graph.

(v) *The relative neighbourhood graph.* The relative neighbourhood graph puts an edge between two points x and y in $\mathscr{X}$ if $B(x, \|x-y\|) \cap B(y, \|x-y\|) \cap \mathscr{X} = \emptyset$. When $d = 2$, this means that the vesica piscis between x and y does not contain any other points in $\mathscr{X}$. This graph is a subgraph of Gabriel graph, so it is also a subgraph of Delaunay triangulation graph.

(vi) *The sphere-of-influence graph.* The sphere-of-influence graph of a locally finite point set $\mathscr{X} \subset \mathbb{R}^d$ is the graph obtained by including $\{x, y\}$ as an edge whenever $x, y \in \mathscr{X}$ and $\|x - y\| \leq \|x - N(x, \mathscr{X})\| + \|y - N(y, \mathscr{X})\|$, where for $z \in \mathscr{X}$, $N(z, \mathscr{X})$ is the nearest point to z in $\mathscr{X}$. That is, for every point $z \in \mathscr{X}$, we draw a circle with centre z whose radius is the distance between z and the point nearest to it in $\mathscr{X}$; then two points x, y are connected if the circles centred at x and y intersect.

Remark 3.1. Using [30, Corollary 2.2] or [13, Theorem 3.1], it is possible to consider normal approximation to the statistics in Section 3.1 and Section 3.2 under the Kolmogorov distance with convergence rate $\mathrm{O}\big(\alpha^{-\frac{1}{2}}\big)$ instead of $\mathrm{O}\big(\alpha^{-\frac{1}{2}} \ln(\alpha)^{\frac{5d}{2}}\big)$.

3.1. The total edge length of the *k*-nearest-neighbours graph

Theorem 3.1. *If Ξ is a homogeneous Poisson point process, then the total edge length $\bar{W}_\alpha$ (resp. $\bar{W}'_\alpha$) of $NG(\Xi_{\Gamma_\alpha})$ (resp. $NG'(\Xi_{\Gamma_\alpha})$) satisfies*

$$d_{TV}\big(\bar{W}_\alpha, \bar{Z}_\alpha\big) \leq \mathrm{O}\Big(\alpha^{-\frac{1}{2}} \ln(\alpha)^{\frac{5d}{2}}\Big) \Big(\text{resp. } d_{TV}\big(\bar{W}'_\alpha, \bar{Z}'_\alpha\big) \leq \mathrm{O}\Big(\alpha^{-\frac{1}{2}} \ln(\alpha)^{\frac{5d}{2}}\Big)\Big),$$

where $\bar{Z}_\alpha$ (resp. $\bar{Z}'_\alpha$) is a normal random variable with the same mean and variance as $\bar{W}_\alpha$ (resp. $\bar{W}'_\alpha$).

Proof. We adapt the idea of the proof in [38] to our setup. We only show the claim for the total edge length of $NG(\Xi_{\Gamma_\alpha})$ since $NG'(\Xi_{\Gamma_\alpha})$ can be handled with the same idea. The score function in this case is

$$\eta(x, \mathscr{X}, \Gamma_\alpha) := \frac{1}{2} \sum_{y \in \mathscr{X}_{\Gamma_\alpha}} \|y - x\| \mathbf{1}_{\big\{(x,y) \in NG\big(\mathscr{X}_{\Gamma_\alpha}\big)\big\}},$$

which is clearly translation-invariant.

To apply Theorem 2.2, we need to check the moment condition (2.4), the non-singularity in Assumption 2.4, and the stabilisation condition in Definition 2.2. For simplicity, we show these conditions in two-dimensional cases; the argument can easily be extended to $\mathbb{R}^d$ with $d \geq 3$ (resp. $d = 1$) by using cones (resp. intervals) instead of triangles in the proof.

We start with the exponential stabilisation, and fix $\alpha > 0$ and $x \in \Gamma_\alpha$. Referring to Figure 3, for each $t > 0$, we construct six disjoint sectors of the same size $T_j(t)$, $1 \leq j \leq 6$, with x as the centre and angle $\frac{\pi}{3}$. In consideration of edge effects near the boundary of Γ_α, the sectors are rotated around x so that all straight edges of the sectors have angles at least $\pi/12$ with respect to the edges of Γ_α. It is clear that $T_j(t) \subset T_j(u)$ for all $0 < t < u$. Set $T_j(\infty) = \cup_{t>0} T_j(t)$ for $1 \leq j \leq 6$; then from the properties of the Poisson point process, there are infinitely many points in $\Xi \cap T_j(\infty)$ for all j, a.s. Let $|A|$ denote the cardinality of the set A; define

$$t_{x,\alpha} = \inf \big\{t : |T_j(t) \cap \Gamma_\alpha \cap \Xi| \geq k + 1 \text{ or } T_j(t) \cap \Gamma_\alpha = T_j(\infty) \cap \Gamma_\alpha, \ 1 \leq j \leq 6\big\} \qquad (3.1)$$

and $\bar{R}(x, \alpha) = 3t_{x,\alpha}$. We show that $\bar{R}$ is a radius of stabilisation, and its tail distribution can be bounded by an exponentially decaying function independent of α and x.

For the radius of stabilisation, there are two cases to consider. The first case is that none of $T_j(t_{x,\alpha}) \cap \Gamma_\alpha \cap \Xi$, $1 \leq j \leq 6$, contains at least $k + 1$ points; then $B\big(x, \bar{R}(x, \alpha)\big) \supset \Gamma_\alpha$, and (2.1) is obvious. The second case is that at least one of $T_j(t_{x,\alpha}) \cap \Gamma_\alpha \cap \Xi$, $1 \leq j \leq 6$, contains at least $k + 1$ points, which means that the k nearest neighbours $\{x_1, \ldots, x_k\}$ of x are
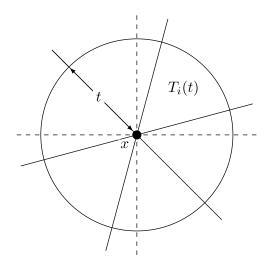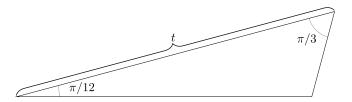
FIGURE 3. k-nearest: stabilisation.

FIGURE 4. k-nearest: A_t.

in $B(x, t_{x,\alpha})$. If a point $y \in \Gamma_\alpha \backslash B(x, t_{x,\alpha})$, then $y \in \Gamma_\alpha \cap (T_j(\infty) \backslash T_j(t_{x,\alpha}))$ for some j. Since $\Gamma_\alpha \cap (T_j(\infty) \backslash T_j(t_{x,\alpha}))$ is non-empty, $T_j(t_{x,\alpha})$ contains at least $k+1$ points $\{y_1, \ldots, y_{k+1}\}$ and $d(x, y) > d(y_i, y)$ for all $i \leq k+1$; hence y cannot have x as one of its k nearest neighbours. This ensures that all points having x as one of their k nearest neighbours are in $B(x, t_{x,\alpha})$. Noting that the diameter of $B(x, t_{x,\alpha})$ is $2t_{x,\alpha}$ and there are at least $k+1$ points in $B(x, t_{x,\alpha})$, we can see that whether a point y in $B(x, t_{x,\alpha})$ has x as one of its k nearest neighbours is entirely determined by $\Xi \cap B(y, 2t_{x,\alpha}) \subset \Xi \cap B(x, 3t_{x,\alpha})$. This guarantees that $\eta(x, \mathscr{X}, \Gamma_\alpha)$ is $\sigma(\Xi_{B(x, 3t_{x,\alpha})})$-measurable, and $\bar{R}(x, \alpha)$ is a radius of stabilisation. For the tail distribution of $\bar{R}$, referring to Figure 4, we consider the number of points of Ξ falling into a triangle A_t as a result of a sector being chopped off by the edge of Γ_α. This is the worst situation for capturing the number of points by one sector intersecting Γ_α in the sense of volume. A routine trigonometry calculation gives that the area of A_t is at least $0.116t^2$. Define $\tau := \inf\{t : |\Xi \cap A_t| \geq k+1\}$; then

$$\mathbb{P}(\bar{R}(x, \alpha) > t) \leq 6\mathbb{P}(\tau > t/3) \leq 6e^{-0.116\lambda(t/3)^2} \sum_{i=0}^{k} \frac{(0.116\lambda(t/3)^2)^i}{i!}, \ t > 0, \qquad (3.2)$$

which implies the exponential stabilisation.

The non-singularity in Assumption 2.4 can be proved through the corresponding unrestricted score function $\bar{\eta}(x, \mathscr{X}) = \frac{1}{2} \sum_{y \in \mathscr{X}} \|y - x\| \mathbf{1}_{\{(x,y) \in NG(\mathscr{X})\}}$. Referring to Figure 5, we take $N_0 := B(0, 0.5)$, observe that $\partial B(0, 6)$ can be covered by finitely many $B(x, 3)$ with $\|x\| = 5$, and write the centres of these balls as $x_1, \ldots, x_n$ (in the two-dimensional case, $n = 5$).
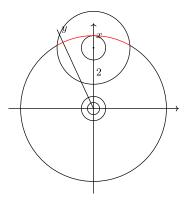
FIGURE 5. *k*-nearest: non-singularity.

Let E be the event that $|B(x_i, 1) \cap \Xi| \geq k+1$ for all $1 \leq i \leq n$, $|(B(0, 1) \backslash B(0, 0.5)) \cap \Xi| = k$, and $\Xi \cap \left(B(0, 6) \backslash \left(\cup_{i \leq n} B(x_i, 1) \cup B(0, 1)\right)\right)$ is empty; then E is $\sigma\left(\Xi_{N_0^c}\right)$-measurable and $\mathbb{P}(E) > 0$. Conditional on E, we can see that $E_1 := \{|\Xi \cap B(0, 0.5)| = 1\}$ satisfies $\mathbb{P}(E_1|E) > 0$, and on E_1, all of the summands in (2.6) that are random are those involving the point of $\Xi_{B(0,0.5)}$; thus we now establish that these random score functions are entirely determined by $\Xi_{B(0,1)}$. As a matter of fact, any point in $\Xi \cap \left(\cup_{1 \leq i \leq n} B(x_i, 1)\right)$ has k nearest points with distances no larger than 2, so points in $\Xi \cap B(0, 1)$ cannot be among the k nearest points to points in $\Xi \cap B(x_i, 1)$. For any point $y \in \Xi \cap B(0, 6)^c$, the line between 0 and y intersects $\partial B(0, 6)$ at y', which is in $B(x_{i_0}, 3)$ for some $1 \leq i_0 \leq n$, so the distances between y and points in $\Xi \cap B(x_{i_0}, 3)$ are at most $\|y - y'\| + 4 = \|y\| - 2$. The distances between y and points in $\Xi \cap B(0, 1)$ are at least $\|y\| - 1$, which ensures that points of $\Xi \cap B(0, 6)^c$ cannot have points in $\Xi \cap B(0, 1)$ as their k nearest neighbours. On the other hand, on $E \cap E_1$, there are $k+1$ points in $\Xi \cap B(0, 1)$. For any point $x \in B(0, 1)$, since the distances between x and other points in $B(0, 1)$ are less than 2, the points outside $B(x, 2) \subset B(0, 3)$ will not be among the k nearest points to x:

$$\bar{\eta}(x, \Xi) = \frac{1}{2} \sum_{y \in (\Xi \cap B(0,1)) \backslash \{x\}} \|x - y\|.$$

Hence, given E, all random score functions contributing to the sum of (2.6) are those completely determined by $\Xi_{B(0,2)}$, giving

$$\mathbf{1}_{E_1} \sum_{\substack{x \in \Xi \text{ such that} \\ \bar{\eta}(x, \Xi) \text{ is random given} \Xi_{N_0^c}}} \bar{\eta}(x, \Xi) \, \mathbf{1}_{d(x,N_0) < R(x)} = \mathbf{1}_{E_1} \left\{ \sum_{\substack{y \in \Xi \cap B(0,0.5), \\ x \in (B(0,1) \backslash B(0,0.5)) \cap \Xi}} \|x - y\| + X \right\},$$

where X is $\sigma\left(\Xi_{N_0^c}\right)$-measurable. Since this is a continuous function of $y \in \Xi \cap B(0, 0.5)$, the non-singularity in Assumption 2.4 follows.

For the moment condition (2.4), recalling the definition of $t_{x,\alpha}$ in (3.1), we replace x with x_3 to get $t_{x_3,\alpha}$. We now establish that

$$\eta\left(x_3, \Xi_{\Gamma_\alpha} + a_1 \delta_{x_1} + a_2 \delta_{x_2} + \delta_{x_3}\right) \leq 3.5 k t_{x_3,\alpha}. \tag{3.3}$$

In fact, the k nearest neighbours to x_3 have the contribution of the total edge length $\leq \frac{1}{2} k t_{x_3,\alpha}$. On the other hand, for $1 \leq j \leq 6$, each point in $\left(\Xi_{\Gamma_\alpha} + a_1 \delta_{x_1} + a_2 \delta_{x_2}\right) \cap T_j\left(t_{x_3,\alpha}\right)$ may take x_3
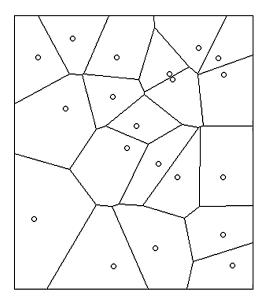
FIGURE 6. Voronoi tessellation.

as its k nearest neighbour, with the contribution to the total edge length $\leq \frac{1}{2} t_{x_3,\alpha}$. As there are six sectors $\left(\Xi_{\Gamma_\alpha} + a_1 \delta_{x_1} + a_2 \delta_{x_2}\right) \cap T_j\left(t_{x_3,\alpha}\right)$, $1 \leq j \leq 6$, and each sector has no more than k points, with x_3 as one of their k nearest neighbours, the contribution to the total edge length from this part is bounded by $3kt_{x_3,\alpha}$. By adding these two bounds, we obtain (3.3). Finally, we combine (3.3) and (3.2) to get

$$\mathbb{E}\left\{\eta\left(x_3, \Xi_{\Gamma_\alpha} + a_1 \delta_{x_1} + a_2 \delta_{x_2} + \delta_{x_3}\right)^3\right\} \leq 42.875 k^3 \mathbb{E}\left\{t_{x_3,\alpha}^3\right\} \leq C < \infty,$$

and the proof is completed by applying Theorem 2.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2. The total edge length of a Voronoi tessellation

Consider a finite point set $\mathscr{X} \subset \Gamma_\alpha$. The Voronoi tessellation in Γ_α generated by $\mathscr{X}$ is the partition formed by cells $C(x_i, \mathscr{X}) \cap \Gamma_\alpha$; see Figure 6. We write the graph of this tessellation as $V(\mathscr{X}, \alpha)$ and the total edge length of $V(\mathscr{X}, \alpha)$ as $\mathscr{W}(\mathscr{X}, \alpha)$.

Theorem 3.2. *If Ξ is a homogeneous Poisson point process, then*

$$d_{TV}\left(\mathscr{W}(\Xi_{\Gamma_\alpha}, \alpha), \bar{Z}_\alpha\right) \leq \mathrm{O}\left(\alpha^{-\frac{1}{2}} \ln(\alpha)^{\frac{5d}{2}}\right),$$

where $\bar{Z}_\alpha$ is a normal random variable with the same mean and variance as $\mathscr{W}(\mathscr{X}, \alpha)$.

Proof. The stabilising part of the proof is inspired by [38], and the idea can also be used to show the stabilisation property for Laguerre tessellations, which are a generalisation of Voronoi tessellations, as in [23].

Before going into details, we observe that

$$\mathscr{W}\left(\Xi_{\Gamma_\alpha}, \alpha\right) = \sum_{\{x,y\} \subset \Xi_{\Gamma_\alpha}, x \neq y} l\left(\partial C\left(x, \Xi_{\Gamma_\alpha}\right) \cap \partial C\left(y, \Xi_{\Gamma_\alpha}\right)\right)$$
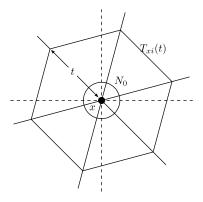$$+ l(\partial \Gamma_\alpha),$$

FIGURE 7. Voronoi: stabilisation.

where $l(\cdot)$ is the volume of a set in dimension $d - 1$. We restrict our attention to Voronoi tessellations of random point sets in $\mathbb{R}^2$; with greater notational complexity, the approach also works in $\mathbb{R}^d$ with $d \geq 3$, using cones instead of triangles. Because $l(\partial\Gamma_\alpha) = 4\alpha^{\frac{1}{2}}$ is a constant, by removing this constant, we have $\mathscr{V}'(\Xi_{\Gamma_\alpha}, \alpha) := \mathscr{V}(\Xi_{\Gamma_\alpha}, \alpha) - 4\alpha^{\frac{1}{2}}$ and $d_{TV}(\mathscr{V}(\Xi_{\Gamma_\alpha}, \alpha), \bar{Z}_\alpha) = d_{TV}(\mathscr{V}'(\Xi_{\Gamma_\alpha}, \alpha), \bar{Z}'_\alpha)$, where $\bar{Z}'_\alpha$ is a normal random variable having the same mean and variance as $\mathscr{V}'(\Xi_{\Gamma_\alpha}, \alpha)$. We can set the score function corresponding to $\mathscr{V}'$ as

$$\eta(x, \mathscr{X}, \Gamma_\alpha) = \frac{1}{2}\sum_{y \in \mathscr{X}, y \neq x} l(\partial C(x, \mathscr{X}) \cap \partial C(y, \mathscr{X})) = \frac{1}{2}l(\partial(C(x, \mathscr{X}) \cap \Gamma_\alpha)\backslash(\partial\Gamma_\alpha))$$

for all $x \in \mathscr{X} \subset \Gamma_\alpha$, i.e., $\eta(x, \mathscr{X}, \Gamma_\alpha)$ is half of the total length of edges of $C(x, \mathscr{X}) \cap \Gamma_\alpha$, excluding the boundary of Γ_α. The score function η is clearly translation-invariant. Thus, to apply Theorem 2.2, we need to verify the stabilisation in Definition 2.2, the moment condition (2.4), and the non-singularity in Assumption 2.4.

We start by showing that the score function is exponentially stabilising. Referring to Figure 7, similarly to Section 3.1, we construct six disjoint equilateral triangles $T_{xj}(t)$, $1 \leq j \leq 6$, such that x is a vertex of these triangles, and the triangles are rotated so that all edges with x as a vertex have angles at least $\pi/12$ relative to the edges of Γ_α. Let $T_{xj}(\infty) = \cup_{t \geq 0} T_{xj}(t)$, $1 \leq j \leq 6$; then $\cup_{1 \leq j \leq 6} T_{xj}(\infty) = \mathbb{R}^2$. Define

$$R_{xj} := R_{xj}(x, \alpha, \Xi_{\Gamma_\alpha}) := \inf\{t : T_{xj}(t) \cap \Xi_{\Gamma_\alpha} \neq \emptyset \text{ or } T_{xj}(t) \cap \Gamma_\alpha = T_{xj}(\infty) \cap \Gamma_\alpha\}$$

and

$$R_{x0} := R_{x0}(x, \alpha, \Xi_{\Gamma_\alpha}) := \max_{1 \leq j \leq 6} R_{xj}(x, \alpha, \Xi_{\Gamma_\alpha}).$$

We note that there is a minor issue with the counterpart of R_{x0} defined in [34] when x is close to the corners of Γ_α. We now show that $\bar{R}(x, \alpha) := 3R_{x0}(x, \alpha, \Xi_{\Gamma_\alpha})$ is a radius of stabilisation. In fact, for any point x' in $\Gamma_\alpha \backslash \overline{(\cup_{1 \leq j \leq 6} T_{xj}(R_{x0}))}$, x' is contained in a triangle $T_{xj_0}(\infty) \backslash \overline{T_{xj_0}(R_{x0})}$. This implies that $T_{xj_0}(R_{x0}) \cap \Xi_{\Gamma_\alpha} \neq \emptyset$, i.e., we can find a point $y \in T_{xj_0}(R_{x0}) \cap \Xi_{\Gamma_\alpha}$ and the point y satisfies $d(x', y) \leq d(x, x')$; hence $x' \notin C(x, \Xi_{\Gamma_\alpha}) \cap \Gamma_\alpha$, which ensures that $C(x, \Xi_{\Gamma_\alpha}) \subset \overline{(\cup_{1 \leq j \leq 6} T_{xj}(R_0))}$. Consequently, if a point y in Ξ_{Γ_α} generates an edge of $C(X) \cap \Gamma_\alpha$, then $d(x, y) \leq 2R_0$, and $\bar{R}(x, \alpha)$ satisfies Definition 2.2. As in Section 3.1, we use A_t in Figure 4 again to define $\tau := \inf\{t : |\Xi \cap A_t| \geq 1\}$; then

$$\mathbb{P}(\bar{R}(x) > t) \leq 6\mathbb{P}(\tau > t/3) \leq 6e^{-0.116\lambda(t/3)^2}, \ t > 0. \tag{3.4}$$

This completes the proof of the exponential stabilisation of η.

The non-singularity in Assumption 2.4 can be examined by using a non-restricted counterpart $\bar{\eta}$ of η, taking $N_0 = B(0, 1)$, and filling the moat $B(0, 4) \setminus B(0, 3)$ with sufficiently dense points of Ξ such that, when $\Xi_{N_0^c}$ is fixed, the random score functions contributing to the sum of (2.6) are purely determined by a point in $\Xi \cap N_0$. More precisely, we cover the circle $\partial B(0, 3)$ with disjoint squares having side length $\sqrt{2}/4$, and enumerate the squares as S_i, $1 \leq i \leq k$. Note that all the squares are contained in $B(0, 4) \setminus B(0, 2)$. Let $E = \cap_{1 \leq i \leq k} \{|\Xi \cap (S_i)| \geq 1\}$, $E_1 = \{|\Xi \cap N_0| = 1\}$; then E is $\sigma\left(\Xi_{N_0^c}\right)$-measurable, $\mathbb{P}(E) > 0$, and $\mathbb{P}(E_1 | E) > 0$. Since the points in $\Xi \cap \left(\cup_{i=1}^k S_i\right)$ have neighbours within distance 1, for any $x \in N_0$, $T_{xj}(6)$ contains at least one point from $\Xi \cap (B(0, 4) \setminus B(0, 2))$. As argued in the stabilisation, points in $\Xi \cap (B(0, 12)^c)$ do not affect the cell centred at $x \in N_0$, and by symmetry, $x \in N_0$ does not affect the Voronoi cells centred at points in $\Xi \cap (B(0, 12)^c)$. This ensures that, conditional on E, all random score functions contributing to the sum of (2.6) are those completely determined by Ξ_{N_0}, giving

$$\mathbf{1}_{E_1} \sum_{\substack{x \in \Xi \text{ such that} \\ \bar{\eta}(x, \Xi) \text{ is random given } \Xi_{N_0^c}}} \bar{\eta}(x, \Xi)\, \mathbf{1}_{d(x, N_0) < R(x)}$$

$$= \mathbf{1}_{E_1} \left\{ \sum_{x \in \Xi \cap B(0, 12)} \bar{\eta}(x, \Xi)\, \mathbf{1}_{d(x, N_0) < R(x)} + X \right\},$$

where X is $\sigma\left(\Xi_{N_0^c}\right)$-measurable. As $\mathbf{1}_{E_1} \bar{\eta}(x, \Xi)$ is an almost surely (a.s.) (in terms of the volume measure in $\mathbb{R}^2$) continuous function of $x \in \Xi \cap N_0$, the proof of the non-singularity in Assumption 2.4 is completed.

It remains to show the moment condition (2.4). In fact, as shown in the stabilisation property, we can see that $\bar{R}(x, \alpha)$ will not increase when points are added, so $C(x_3, \Xi_{\Gamma_\alpha} + a_1 \delta_{x_1} + a_2 \delta_{x_2} + \delta_{x_3}) \cap \Gamma_\alpha \subset B(x, \bar{R}(x, \alpha))$; then the number of edges of $C(x_3, \Xi_{\Gamma_\alpha} + a_1 \delta_{x_1} + a_2 \delta_{x_2} + \delta_{x_3}) \cap \Gamma_\alpha$ excluding those in the edge of Γ_α is less than or equal to $(\Xi_{\Gamma_\alpha} + a_1 \delta_{x_1} + a_2 \delta_{x_2})(B(x, \bar{R}(x, \alpha))) \leq \Xi_{\Gamma_\alpha}(B(x, \bar{R}(x, \alpha))) + 2$, and each of them has length less than $2\bar{R}(x, \alpha)$. We observe that Ξ restricted to outside of $\cup_{j=1}^6 T_{xj}(R_{xj})$ is independent of $\bar{R}(x, \alpha)$; hence

$$\Xi_{\Gamma_\alpha}\left(B(x, \bar{R}(x, \alpha))\right) \leq \Xi_{\Gamma_\alpha}\left(B(x, \bar{R}(x, \alpha)) \setminus \left(\cup_{j=1}^6 T_{xj}(R_{xj})\right)\right) + 6$$

$$\overset{\text{ST}}{\leq} \Xi'_{\Gamma_\alpha}\left(B(x, \bar{R}(x, \alpha))\right) + 6,$$

where $\overset{\text{ST}}{\leq}$ stands for 'stochastically less than or equal to', and Ξ' is an independent copy of Ξ. Therefore, using (3.4), we obtain

$$\mathbb{E}\left(\eta(x_3, \Xi_{\Gamma_\alpha} + a_1 \delta_{x_1} + a_2 \delta_{x_2} + \delta_{x_3})^3\right)$$

$$\leq \mathbb{E}\left(\left(\Xi'(B(x_3, \bar{R}(x, \alpha))) + 8\right)^3 (2\bar{R}(x_3, \alpha))^3\right)$$

$$\leq \int_0^\infty \sum_{i=0}^\infty (i+8)^3 (2r)^3 \frac{e^{-\lambda \pi r^2} (\lambda \pi r^2)^i}{i!} 6 e^{-0.116\lambda(t/3)^2} * (0.116\lambda/9)\, 2r\, dr$$

$$\leq C < \infty,$$

which ensures (2.4). The proof of Theorem 3.2 is completed by using Theorem 2.2. $\qquad \square$

3.3. Timber volume estimation

Timber volume estimation is an essential research topic in forest science and forest management [8, 32]. This example demonstrates that, with the marks, our theorem can be used to provide an error estimate for the normal approximation of the timber volume distribution. To this end, it is reasonable to assume that in a given range Γ_α of a natural forest, the locations of trees form a Poisson point process $\bar{\Xi}$. For $x \in \bar{\Xi}$, we can use a random mark $M_x \in T := \{1, \ldots, n\}$ to denote the species of the tree at position x; then $\Xi := \sum_{x \in \bar{\Xi}} \delta_{(x, M_x)}$ is a marked Poisson point process with independent marks. The timber volume of a tree at x is a combined result of the location, the species of the tree, the configuration of species of trees in a finite range around x, and some other random factors that cannot be explained by the configuration of trees in the range. We write $\eta((x, m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha)$ as the timber volume determined by the location x, the species m, and the configuration of trees, and ϵ_x as the adjusted timber volume at location x due to unexplained random factors.

Theorem 3.3. *Assume that η is a non-negative bounded score function such that*

$$\eta((x, m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha) = \eta\big((x, m), \Xi_{\Gamma_\alpha \cap B(x, r)}, \Gamma_\alpha\big)$$

for some positive constant r,

$$\eta((x, m), \Xi_{B(x, r)}, \Gamma_{\alpha_1}) = \eta\big((x, m), \Xi_{B(x, r)}, \Gamma_{\alpha_2}\big)$$

for all α_1 and α_2 with $B(x, r) \subset \Gamma_{\alpha_1 \wedge \alpha_2}$, η is translation-invariant as in Definition 2.4, the ϵ_x are i.i.d. random variables with a finite third moment, the positive part $\epsilon_x^+ = \max(\epsilon_x, 0)$ is non-singular, and the ϵ_x are independent of the configuration Ξ. Then the timber volume of the range Γ_α can be represented as

$$\bar{W}_\alpha := \sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} \left[\big(\eta((x, m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha) + \epsilon_x\big) \vee 0\right],$$

and it satisfies

$$d_{TV}\big(\bar{W}_\alpha, \bar{Z}_\alpha\big) \leq O\left(\alpha^{-\frac{1}{2}}\right),$$

where $\bar{Z}_\alpha$ is a normal random variable with the same mean and variance as $\bar{W}_\alpha$.

Proof. Before going into details, we first construct a new marked Poisson point process $\Xi' := \sum_{x \in \bar{\Xi}} \delta_{(x, (M_x, \epsilon_x))}$ with i.i.d. marks $(M_x, \epsilon_x) \in T \times \mathbb{R}$ independent of the ground process $\bar{\Xi}' = \bar{\Xi}$, and incorporate ϵ_x into a new score function on Ξ' as

$$\begin{aligned}
\eta'((x, (m, \epsilon)), \Xi', \Gamma_\alpha) &:= \eta'((x, (m, \epsilon)), \Xi'_{\Gamma_\alpha}, \Gamma_\alpha) \\
&:= \left[\big(\eta((x, m), \Xi_{\Gamma_\alpha}, \Gamma_\alpha) + \epsilon\big) \vee 0\right] \mathbf{1}_{(x, (m, \epsilon)) \in \Xi'_{\Gamma_\alpha}}.
\end{aligned}$$

We can see that

$$\bar{W}_\alpha = \sum_{x \in \bar{\Xi}'_{\Gamma_\alpha}} \eta'\Big((x, (m_x, \epsilon_x)), \Xi'_{\Gamma_\alpha}, \Gamma_\alpha\Big).$$

The score function η' is clearly translation-invariant. Thus, to apply Theorem 2.2, it is sufficient to verify that η' is range-bounded as in Definition 2.2 and satisfies the moment condition (2.4) and the non-singularity in Assumption 2.4.

The range-bounded property of the score function η' is inherited from the range-bounded property of η with the same radius of stabilisation $\bar{R}(x, \alpha) := r$. The moment condition (2.4) is a direct consequence of the boundedness of η, the finite third moment of ϵ_x, and the Minkowski inequality. Hence it remains to show the non-singularity. To this end, we observe that the corresponding unrestricted counterpart $\bar{\eta}$ of η' is defined by

$$\bar{\eta}\big((x, (m_x, \epsilon_x)), \mathscr{X}'\big) = \lim_{\alpha \to \infty} \eta'\big((x, (m_x, \epsilon_x)), \mathscr{X}', \Gamma_\alpha\big)$$
$$= \eta'\big((x, (m_x, \epsilon_x)), \mathscr{X}', \Gamma_{\alpha_x}\big) = \big(\eta\big((x, m_x), \mathscr{X}, \Gamma_{\alpha_x}\big) + \epsilon_x\big) \vee 0,$$

where $\alpha_x = 4(\|x\| + r)^2$, and $\mathscr{X}$ is the projection of $\mathscr{X}'$ on $\mathbb{R}^2 \times T$. Let $N_0 = B(0, 1)$, and let

$$E = \left\{ \left| \Xi'_{B(N_0, r) \setminus N_0} \right| = 0 \right\}, \qquad E_1 = \left\{ \left| \Xi'_{N_0} \right| = 1 \right\};$$

then E is $\sigma\left(\Xi'_{N_0^c}\right)$-measurable, $\mathbb{P}(E) > 0$, and $\mathbb{P}(E_1|E) > 0$. Writing $\epsilon_x^- = -\min(\epsilon_x, 0)$, $\bar{\Xi}'_{N_0} = \{x_0\}$ in E_1, given E we have

$$\mathbf{1}_{E_1} \sum_{x \in \bar{\Xi}'} \bar{\eta}\big((x, (m_x, \epsilon_x)), \Xi'\big) \mathbf{1}_{d(x, N_0) < r}$$

$$= \mathbf{1}_{E_1} \left(\eta\big((x_0, m_{x_0}), \delta_{(x_0, m_{x_0})}, \Gamma_{4(r+1)^2}\big) + \epsilon_{x_0} \right) \vee 0$$

$$= \mathbf{1}_{E_1} \left[\mathbf{1}_{\epsilon_{x_0} > 0} \left(\eta\big((x_0, m_{x_0}), \delta_{(x_0, m_{x_0})}, \Gamma_{4(r+1)^2}\big) + \epsilon_{x_0}^+ \right) \right.$$
$$\left. + \mathbf{1}_{\epsilon_{x_0} \leq 0} \left(\eta\big((x_0, m_{x_0}), \delta_{(x_0, m_{x_0})}, \Gamma_{4(r+1)^2}\big) - \epsilon_{x_0}^- \right) \right] \vee 0.$$

On $\{\epsilon_{x_0} > 0\}$, $\epsilon_{x_0}^+$ is independent of $\eta\big((x_0, m_{x_0}), \delta_{(x_0, m_{x_0})}, \Gamma_{4(r+1)^2}\big)$ and has a positive non-singular component; hence $\eta\big((x_0, m_{x_0}), \delta_{(x_0, m_{x_0})}, \Gamma_{4(r+1)^2}\big) + \epsilon_{x_0}^+$ is also non-singular. Together with the fact that $\{\epsilon_{x_0} > 0\}$ and $\{\epsilon_{x_0} \leq 0\}$ are disjoint, this implies the non-singularity. $\square$

Remark 3.2. If the timber volume of a tree is determined by its nearest neighbouring trees, then we can set the score function η as a function of weighted Voronoi cells. Using the idea of the proof of Theorem 3.2, we can establish the bound of the error of the normal approximation to the distribution of the timber volume $\bar{W}_\alpha$ as $d_{TV}\big(\bar{W}_\alpha, \bar{Z}_\alpha\big) \leq \mathrm{O}\big(\alpha^{-\frac{1}{2}} \ln(\alpha)^{\frac{5d}{2}}\big)$. Furthermore, by setting $\eta((x, m_x), \Xi, \Gamma_\alpha)$ as a function of the k_{m_x} nearest neighbourhoods of x instead of the configuration of species of trees in a bounded neighbourhood, we gain an example with stabilising radius R depending not only on x but also on m_x.

3.4. Maximal layers

Maximal layers of points have been of considerable interest since [42, 29], and have a wide range of applications; see [11] for a brief review of their applications. One of the applications is the *smallest colour-spanning interval* [28], which is a linear function of the distances between maximal points and the edge. In this subsection, we demonstrate that Theorem 2.2 with marks can easily be applied to estimate the error of the normal approximation to the distribution of the sum of distances between different maximal layers if the points are from a Poisson point process.
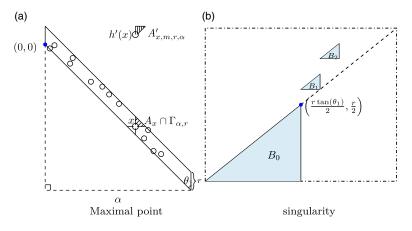
FIGURE 8. Maximal layers.

For $x \in \mathbb{R}^d$, we define $A_x = ([0, \infty)^d + x) \cap \Gamma_\alpha$. Given a locally finite point set $\mathscr{X} \subset \mathbb{R}^d$, a point x is called *maximal* in $\mathscr{X}$ if $x \in \mathscr{X}$ and there is no other point $(y_1, \ldots, y_d) \in \mathscr{X}$ satisfying $y_i \geq x_i$ for all $1 \leq i \leq d$ (see Figure 8(a)). Mathematically, x is maximal in $\mathscr{X}$ if $\mathscr{X} \cap A_x = \{x\}$. This enables us to write different maximal layers as follows: the kth maximal layer of points can be recursively defined as

$$\mathscr{X}_k := \sum_{x \in \mathscr{X}} \delta_x \mathbf{1}_{\left[A_x \cap \left(\mathscr{X} \setminus \left(\cup_{i=1}^{k-1} \mathscr{X}_i \right) \right) = \{x\} \right]}, \quad k \geq 1,$$

with the convention $\cup_{i=1}^{0} \mathscr{X}_i = \emptyset$.

For simplicity, we consider the restriction of the Poisson point process to a region in $\mathbb{R}^d$ between two parallel $(d-1)$-dimensional planes for $d \geq 2$. More precisely, the region of interest is

$$\Gamma_{\alpha,r} = \left\{ (x_1, x_2, \ldots, x_d); x_i \in \left[0, \alpha^{\frac{1}{d-1}} \right], i \leq d-1, x_d + \sum_{i=1}^{d-1} x_i \cot(\theta_i) \in [0, r] \right\}$$

for fixed $\theta_i \in \left(0, \frac{\pi}{2} \right)$, $1 \leq i \leq d-1$, and $\Xi_{\Gamma_{\alpha,r}}$ is a homogeneous Poisson point process with rate λ on $\Gamma_{\alpha,r}$. Define $\Xi_{k,r,\alpha}$ as the kth maximal layer of $\Xi_{\Gamma_{\alpha,r}}$; then the total distance between the points in $\Xi_{k,r,\alpha}$ and the upper plane

$$P := \left\{ (x_1, x_2, \ldots, x_d); x_i \in \left[0, \alpha^{\frac{1}{d-1}} \right], i \leq d-1, x_d = -\sum_{i=1}^{d-1} x_i \cot(\theta_i) + r \right\}$$

can be represented as $\bar{W}_{k,r,\alpha} := \sum_{x \in \Xi_{k,r,\alpha}} d(x, P)$.

Theorem 3.4. *With the above setup, when $r \in \mathbb{R}_+$ and $k \in \mathbb{N}$ is fixed,*

$$d_{TV}\left(\bar{W}_{k,r,\alpha}, \bar{Z}_{k,r,\alpha} \right) \leq \mathrm{O}\left(\alpha^{-\frac{1}{2}} \right),$$

where $\bar{Z}_{k,r,\alpha} \sim N\left(\mathbb{E}\left(\bar{W}_{k,r,\alpha} \right), \mathrm{Var}\left(\bar{W}_{k,r,\alpha} \right) \right)$.

Remark 3.3. It remains a challenge to consider maximal layers induced by a homogeneous Poisson point process on

$$\left\{ (v_1, v_2) : v_1 \in \left[0, \alpha^{1/(d-1)}\right]^{d-1}, 0 \le v_2 \le F(v_1) \right\},$$

where $F : [0, \alpha^{1/(d-1)}]^{d-1} \to [0, \infty)$ has continuous negative partial derivatives in all coordinates, the partial derivatives are bounded away from 0 and $-\infty$, and $|F| \le \mathrm{O}(\alpha^{1/(d-1)})$. We conjecture that the normal approximation in total variation for the total distance between the points in a maximal layer and the upper edge surface is still valid. The convergence rate of the total number of points in the kth maximal layer under the Kolmogorov distance can be found in [30, Theorem 3.3], but because the number of points is a discrete random variable, the convergence rate under the total variation distance is always 1.

Proof of Theorem 3.4. As the score function $d(\cdot, P)$ is not translation-invariant in the sense of Definition 2.4, we first turn the problem into that of a marked Poisson point process with independent marks. The idea is to project the points of $\Xi_{\Gamma_{\alpha,r}}$ on their first $d-1$ coordinates to obtain the ground Poisson point process, and send the last coordinate to marks with $T = [0, r]$. To this end, define a mapping $h' : \Gamma_{\infty,r} := \cup_{\alpha>0} \Gamma_{\alpha,r} \to [0, \infty)^{d-1} \times [0, r]$ such that

$$h'(x_1, \ldots, x_d) = (x_1, \ldots, x_d) + \left(0, \ldots, 0, \sum_{i=1}^{d-1} x_i \cot(\theta_i)\right)$$

and $h(\mathscr{X}) := \{h'(x) : x \in \mathscr{X}\}$. Then h' is a one-to-one mapping, and $\Xi' := h(\Xi_{\Gamma_{\infty,r}})$ can be regarded as a marked Poisson point process on $[0, \infty)^{d-1} \times [0, r]$ with rate $r\lambda$ and independent marks following the uniform distribution on $[0, r]$. Write the mark of $x \in \Xi'$ as m_x; then

$$\bar{W}_{k,r,\alpha} = C(\theta_1, \ldots, \theta_{d-1}) \sum_{x \in h(\Xi_{k,r,\alpha})} (r - m_x), \tag{3.5}$$

where $C(\theta_1, \ldots, \theta_{d-1})$ is a constant determined by $\theta_1, \ldots, \theta_{d-1}$. Let $\Gamma'_\alpha := \left[0, \alpha^{\frac{1}{d-1}}\right]^{d-1}$; then $h(\Xi_{\Gamma_{\alpha,r}}) = \Xi'_{\Gamma'_\alpha}$. For a point $(x, m) \in [0, \infty)^{d-1} \times [0, r]$, we write

$$A'_{x,m,r,\alpha} = h'\left(\left((h')^{-1}(x, m) + [0, \infty)^d\right) \cap \Gamma_{\alpha,r}\right)$$

(see Figure 8(b)) and

$$Xi'_{k,r,\alpha} := h(\Xi_{k,r,\alpha}) = \sum_{x \in \bar{\Xi}'_{\Gamma'_\alpha}} \delta_{(x,m_x)} \mathbf{1}_{A'_{x,m_x,r,\alpha} \cap \left(\Xi'_{\Gamma'_\alpha} \setminus \cup_{i=1}^{k-1} \Xi'_{i,r,\alpha}\right) = \{(x,m_x)\}}. \tag{3.6}$$

Combining (3.5) and (3.6), we can represent $\bar{W}_{k,r,\alpha}$ as the sum of values of the score function

$$\eta\left((x, m_x), \Xi', \Gamma'_\alpha\right) := C(\theta_1, \ldots, \theta_{d-1})(r - m_x)\mathbf{1}_{(x,m_x) \in \Xi'_{k,\alpha}}$$

over the range Γ'_α. To apply Theorem 2.2, we need to check that η is range-bounded as in Definition 2.2, and that it satisfies the moment condition (2.4) and the non-singularity in Assumption 2.4.

For simplicity, we only show the claim in the two-dimensional case; the argument for $d > 2$ is the same except for notational complexity. When $d = 2$, $\Gamma_{\alpha,r}$ is a parallelogram

with angle θ_1 as in Figure 8(a), and P and $C(\theta_1, \ldots, \theta_{d-1})$ reduce to an edge in $\mathbb{R}^2$ and $\sin(\theta_1)$, respectively. Since $\sin(\theta_1)(r - m_x)$ is given by the mark of x, to show η is range-bounded, it is sufficient to show that $\mathbf{1}_{\left\{(x,m_x)\in\Xi'_{k,r,\alpha}\right\}}$ is completely determined by $\Xi' \cap A'_{x,m_x,r,\alpha}$. In fact, we can accomplish this by observing that $(x, m_x) \in \Xi'_{k,r,\alpha}$ if and only if there is a sequence $\{(x_j, m_{x_j}), \ 1 \leq j \leq k\} \subset \Xi' \cap A'_{x,m_x,r,\alpha}$ (which ensures $A'_{x_j,m_{x_j},r,\alpha} \subset A'_{x,m_x,r,\alpha}$) such that $(x_k, m_{x_k}) = (x, m)$ and

$$A'_{x_j,m_{x_j},r,\alpha} \cap \left(\Xi' \setminus \cup_{i=1}^{j-1} \Xi'_{i,r,\alpha}\right) = \left\{(x_j, m_{x_j})\right\}$$

for $1 \leq j \leq k$. Since

$$\Xi' \cap A'_{x,m_x,\alpha} \subset \Xi'_{[x,x+r\tan(\theta_1)]},$$

we can see that η is range-bounded as in Definition 2.2 with $\bar{R}(x, \alpha) := r\tan(\theta_1)+1$. The moment condition follows from the fact that η is bounded above by r. For the non-singularity, we extend Ξ' to $\mathbb{R}^{d-1} \times [0, r]$, write

$$\Gamma^e_{\infty,r} := \{x \in \mathbb{R}^d : \text{there exists } y \in P \text{ such that } y - r(0, \ldots, 0, 1) \leq x \leq y\},$$

and let $(\Xi_{\Gamma^e_{\infty,r}})_j$ be the jth maximal layer of $\Xi_{\Gamma^e_{\infty,r}}$ and $\Xi'_j = h((\Xi_{\Gamma^e_{\infty,r}})_j)$. We can see that the corresponding unrestricted score function is $\bar{\eta}(x, \Xi') = \sin(\theta_1)(r - m_x)\mathbf{1}_{(x,m_x)\in\Xi'_j}$ with the stabilising radius $R(x) = r\tan(\theta_1)+1$. Referring to Figure 8(b), we set $N_0 := \left(0, \frac{r\tan(\theta_1)}{2}\right)$, $B_0 = \{(x, m); x \in N_0, 0 \leq m \leq x\cot(\theta_1)\}$, B_i as the triangle region with vertices

$$\left(r\tan(\theta_1)\left(\frac{1}{2} + \frac{i-1}{4(k-1)}\right), r\left(\frac{1}{2} + \frac{2i-1}{4(k-1)}\right)\right),$$
$$\left(r\tan(\theta_1)\left(\frac{1}{2} + \frac{i}{4(k-1)}\right), r\left(\frac{1}{2} + \frac{2i-1}{4(k-1)}\right)\right),$$
$$\left(r\tan(\theta_1)\left(\frac{1}{2} + \frac{i}{4(k-1)}\right), r\left(\frac{1}{2} + \frac{2i}{4(k-1)}\right)\right)$$

for $1 \leq i \leq k - 1$, and

$$D = \left(\left(\left[-r\tan(\theta_1)-1, \frac{3r\tan(\theta_1)}{2}+1\right] \setminus N_0\right) \times [0, r]\right) \setminus \left(\cup_{i=1}^{k-1}B_i\right).$$

Define $E := \left\{\Xi' \cap D = \emptyset, \left|\Xi' \cap B_i\right| = 1, 1 \leq i \leq k - 1\right\}$ and $E_0 := \left\{\left|\Xi'_{N_0}\right| = \left|\Xi' \cap B_0\right| = 1\right\}$. Then $E \in \sigma\left(\Xi'_{N_0^c}\right)$, $\mathbb{P}(E) > 0$, and $\mathbb{P}(E_0|E) > 0$. We can see that given E, the point in $\Xi' \cap B_i$ is in Ξ'_{k-i} for all $1 \leq i \leq k - 1$. Moreover, on $E \cap E_0$, the point (x_0, m_{x_0}) in Ξ'_{N_0} is in Ξ'_k. Hence, given E,

$$\mathbf{1}_{E_0} \sum_{x\in\bar{\Xi}'} \bar{\eta}(x, \Xi') \mathbf{1}_{d(x,N_0)<R(x)} = \mathbf{1}_{E_0} \sin(\theta_1)(r - m_{x_0})$$

is non-singular. $\qquad\square$

As a final remark of the section, we mention that unrestricted versions of all the examples considered here can be proved, because it is trivial to show that the unrestricted version of the score function $\bar{\eta}$ satisfies the stabilisation condition in Definition 2.1 and the moment condition (2.3) using the same method, and the non-singularity in Assumption 2.4 for the restricted case is the same as that in the unrestricted case with the score function $\bar{\eta}$.

4. Preliminaries and auxiliary results

We start with a few technical lemmas.

Lemma 4.1. *Assume $\xi_1, \ldots, \xi_n$ are i.i.d. random variables having the triangular density function*

$$\kappa_a(x) = \begin{cases} \frac{1}{a}\left(1 - \frac{|x|}{a}\right) & \text{for } |x| \leq a, \\ 0 & \text{for } |x| > a, \end{cases} \tag{4.1}$$

where $a > 0$. Let $T_n = \sum_{i=1}^n \xi_i$. Then for any $\gamma > 0$,

$$d_{TV}(T_n, T_n + \gamma) \leq \frac{\gamma}{a}\left\{\sqrt{\frac{3}{\pi n}} + \frac{2}{(2n-1)\pi^{2n}}\right\}. \tag{4.2}$$

The following lemma says that if the distribution of a random variable is non-singular, then the distributions of random variables which are not far away from it are also non-singular.

Lemma 4.2. *Let F be a non-singular distribution on $\mathbb{R}$ with $\alpha_F > 0$ in the decomposition (1.1). If G is a distribution satisfying $d_{TV}(F, G) < \alpha_F$, then the weight of the absolutely continuous component, α_G, in the Lebesgue decomposition of G,*

$$G = (1 - \alpha_G)G_s + \alpha_G G_a,$$

satisfies $\alpha_G \geq \alpha_F - d_{TV}(F, G)$.

We denote the convolution by $*$.

Lemma 4.3. *For any two non-singular distributions F_1 and F_2, there exist positive constants $a > 0$, $u \in \mathbb{R}$, $\theta \in (0, 1]$ and a distribution function H such that*

$$F_1 * F_2 = (1 - \theta)H + \theta K_a * \delta_u, \tag{4.3}$$

where K_a is the distribution of the triangle density κ_a in (4.1), and δ_u is the Dirac measure at u.

Lemma 4.3 says that $F_1 * F_2$ is the distribution function of $(X_1 + u)X_3 + X_2(1 - X_3)$, where $X_1 \sim K_a$, $X_2 \sim H$, $X_3 \sim \text{Bernoulli}(\theta)$ are independent random variables.

Remark 4.1. From the definition of the triangular density function, if a, u, θ satisfy (4.3) with a distribution H, then for arbitrary p, q such that $0 < q \leq p \leq 1$, we can find an H' satisfying the equation with $a' = pa$, $u' = u$, and $\theta' = q\theta$.

Using the properties of the triangular distributions, we can derive that the sum of the score functions restricted by their radii of stabilisation has a similar property in Lemma 4.1 when the score function is range-bounded, exponentially stabilising, or polynomially stabilising with suitable β.

Lemma 4.4. *Let Ξ be a marked homogeneous Poisson point process on $(\mathbb{R}^d \times T, \mathscr{B}(\mathbb{R}^d) \times \mathscr{T})$ with intensity λ and i.i.d. marks in $(T, \mathscr{T})$ following $\mathscr{L}_T$.*

(a) *(Unrestricted case.) Assume that the score function η satisfies the non-singularity in Assumption 2.4. If η is polynomially stabilising as in Definition 2.1 with order $\beta > d + 1$, then*

$$d_{TV}(W_{\alpha,r}, W_{\alpha,r} + \gamma) \leq C(|\gamma| \vee 1)\left(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}}\right) \tag{4.4}$$

for any $\gamma \in \mathbb{R}$ and $r > R_0$, where C and R_0 are positive constants independent of γ. If η is range-bounded as in Definition 2.1, then

$$d_{TV}\left(W_\alpha, W_\alpha + \gamma\right) \leq C(|\gamma| \vee 1)\alpha^{-\frac{1}{2}} \tag{4.5}$$

for some positive constant C independent of γ.

(b) *(Restricted case.) Assume that the score function η satisfies the non-singularity in Assumption 2.4. If η is polynomially stabilising as in Definition 2.2 with order $\beta > d + 1$, then*

$$d_{TV}(\bar{W}_{\alpha,r}, \bar{W}_{\alpha,r} + \gamma) \leq C(|\gamma| \vee 1)\left(\alpha^{-\frac{1}{2}}r^{\frac{d}{2}}\right) \tag{4.6}$$

for any $\gamma \in \mathbb{R}$ and $r > R_0$, where C and R_0 are positive constants independent of γ. If η is range-bounded as in Definition 2.2, then

$$d_{TV}(\bar{W}_\alpha, \bar{W}_\alpha + \gamma) \leq C(|\gamma| \vee 1)\alpha^{-\frac{1}{2}} \tag{4.7}$$

for some positive constant C independent of γ.

Remark 4.2. Since exponential stabilisation implies polynomial stabilisation, the statements (4.4) and (4.6) also hold under the corresponding exponential stabilisation conditions.

We can generalise Lemma 4.4 by replacing γ with a function of Ξ_N for some Borel set N and the expectation with a conditional expectation.

Corollary 4.1. *For $\alpha, r > 0$, let $\left\{N_{\alpha,r}^{(k)}\right\}_{k \in \{1,2,3\}} \subset \mathscr{B}(\mathbb{R}^d)$ be such that*

$$\left(N_{\alpha,r}^{(1)} \cup N_{\alpha,r}^{(2)} \cup N_{\alpha,r}^{(3)}\right) \cap \Gamma_\alpha \subset B\left(x, q\alpha^{\frac{1}{d}}\right)$$

for a point $x \in \mathbb{R}^d$ and a positive constant $q \in \left(0, \frac{1}{2}\right)$, let $\mathscr{F}_{0,\alpha,r}$ be a sub-σ-algebra of $\sigma\left(\Xi_{N_{\alpha,r}^{(1)}}\right)$, and let $h_{\alpha,r}$ be a measurable function mapping configurations on $N_{\alpha,r}^{(2)} \times T$ to the real space.

(a) *(Unrestricted case.) Define*

$$W'_{\alpha,r} := \sum_{(x,m) \in \Xi_{\Gamma_\alpha \setminus N_{\alpha,r}^{(3)}}} \eta(\,(x,m),\,\Xi)\mathbf{1}_{R(x) \leq r}.$$

If the conditions of Lemma 4.4(a) hold, then

$$d_{TV}\left(W'_{\alpha,r}, W'_{\alpha,r} + h_{\alpha,r}\left(\Xi_{N_{\alpha,r}^{(2)}}\right)\Big|\mathscr{F}_{0,\alpha,r}\right)$$
$$\leq \mathbb{E}\left(\left|h_{\alpha,r}\left(\Xi_{N_{\alpha,r}^{(2)}}\right)\right| \vee 1\Big|\mathscr{F}_{0,\alpha,r}\right) \mathrm{O}\left(\alpha^{-\frac{1}{2}}r^{\frac{d}{2}}\right) \ a.s. \tag{4.8}$$

for $r > R_0$, where $R_0 > 0$ is a constant, and $\mathrm{O}\left(\alpha^{-\frac{1}{2}}r^{\frac{d}{2}}\right)$ is independent of the sets $\left\{N_{\alpha,r}^{(k)}\right\}_{\alpha,r \in \mathbb{R}_+, k \in \{1,2,3\}}$, functions $\{h_{\alpha,r}\}_{\alpha,r \in \mathbb{R}_+}$ and σ-algebras $\{\mathscr{F}_{0,\alpha,r}\}_{\alpha,r \in \mathbb{R}_+}$.

(b) (Restricted case.) Define

$$\bar{W}'_{\alpha,r} := \sum_{(x,m) \in \Xi_{\Gamma_\alpha \setminus N^{(3)}_{\alpha,r}}} \eta(\,(x,m),\, \Xi,\, \Gamma_\alpha) \mathbf{1}_{\bar{R}(x,\alpha)<r}.$$

If the conditions of Lemma 4.4(b) hold, then

$$d_{TV}\Big(\bar{W}'_{\alpha,r},\, \bar{W}'_{\alpha,r} + h_{\alpha,r}\Big(\Xi_{N^{(2)}_{\alpha,r}}\Big)\Big|\mathscr{F}_{0,\alpha,r}\Big)$$
$$\leq \mathbb{E}\Big(\Big|h_{\alpha,r}\Big(\Xi_{N^{(2)}_{\alpha,r}}\Big)\Big| \vee 1\Big|\mathscr{F}_{0,\alpha,r}\Big)\, O\Big(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}}\Big)\ a.s. \qquad (4.9)$$

for $r > R_0$, where $R_0 > 0$ is a constant, and $O\Big(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}}\Big)$ is independent of the sets $\{N^{(k)}_{\alpha,r}\}_{\alpha,r\in\mathbb{R}_+, k\in\{1,2,3\}}$, functions $\{h_{\alpha,r}\}_{\alpha,r\in\mathbb{R}_+}$ and σ-algebras $\{\mathscr{F}_{0,\alpha,r}\}_{\alpha,r\in\mathbb{R}_+}$.

As discussed in the motivating example, the orders of $\mathrm{Var}(W_\alpha)$ and $\mathrm{Var}(\bar{W}_\alpha)$ play a pivotal role in the accuracy of the normal approximation. The next lemma says that the optimal order of the variance can be achieved under exponential stabilisation.

Lemma 4.5.

(a) (Unrestricted case.) If the score function η satisfies the third moment condition (2.3), the non-singularity in Assumption 2.4, and the exponential stabilisation in Definition 2.1, then $\mathrm{Var}(W_\alpha) = \Omega(\alpha)$.

(b) (Restricted case.) If the score function η satisfies the third moment condition (2.4), the non-singularity in Assumption 2.4, and the exponential stabilisation in Definition 2.2, then $\mathrm{Var}(\bar{W}_\alpha) = \Omega(\alpha)$.

We cannot obtain the optimal order of the variances in the polynomially stabilising case, but the following lower bound can be established.

Lemma 4.6.

(a) (Unrestricted case.) If the score function η satisfies the k'th moment condition (2.3) with $k' > k \geq 3$ and the non-singularity in Assumption 2.4, and is polynomially stabilising as in Definition 2.1 with parameter $\beta > (3k-2)d/(k-2)$, then

$$\mathrm{Var}(W_\alpha) \geq O\Big(\alpha^{\frac{k\beta-2\beta-3dk+2d}{k\beta-2\beta-dk}}\Big).$$

(b) (Restricted case.) If the score function η satisfies the k'th moment condition (2.4) with $k' > k \geq 3$ and the non-singularity in Assumption 2.4, and is polynomially stabilising as in Definition 2.2 with parameter $\beta > (3k-2)d/(k-2)$, then

$$\mathrm{Var}(\bar{W}_\alpha) \geq O\Big(\alpha^{\frac{k\beta-2\beta-3dk+2d}{k\beta-2\beta-dk}}\Big).$$

5. The proofs of the auxiliary and main results

We need Palm processes and reduced Palm processes in our proofs, and for ease of reading, we briefly recall their definitions. Let E be a Polish space with Borel σ-algebra $\mathscr{E}$ and configuration space $(\boldsymbol{C}_E, \mathscr{C}_E)$, let Ψ be a point process on $(E, \mathscr{E})$, and write the mean measure of Ψ as

$\psi(dx) := \mathbb{E}\Psi(dx)$. The family of point processes $\{\Psi_x : x \in E\}$ are said to be the *Palm processes* associated with Ψ if for any measurable function $f : E \times C_E \to [0, \infty)$,

$$\mathbb{E}\left[\int_E f(x, \Psi)\Psi(dx)\right] = \int_E \mathbb{E}f(x, \Psi_x)\psi(dx); \tag{5.1}$$

see [26, Section 10.1]. A Palm process Ψ_x contains a point at x, and it is often more convenient to consider the *reduced Palm process* $\Psi_x - \delta_x$ at x by removing the point x from Ψ_x. Furthermore, suppose that the *factorial moments*

$$\psi^{[2]}(dx, dy) := \mathbb{E}[\Psi(dx)(\Psi - \delta_x)(dy)],$$

$$\psi^{[3]}(dx, dy, dz) := \mathbb{E}[\Psi(dx)(\Psi - \delta_x)(dy)(\Psi - \delta_x - \delta_y)(dz)]$$

are locally finite; then we can respectively define the *second-order Palm processes* $\{\Psi_{xy} : x, y \in E\}$ and *third-order Palm processes* $\{\Psi_{xyz} : x, y, z \in E\}$ associated with Ψ by

$$\mathbb{E}\left[\iint_{E^2} f(x, y; \Psi)\Psi(dx)(\Psi - \delta_x)(dy)\right] = \iint_{E^2} \mathbb{E}f(x, y; \Psi_{xy})\psi^{[2]}(dx, dy), \tag{5.2}$$

$$\mathbb{E}\left[\iiint_{E^3} f(x, y, z; \Psi)\Psi(dx)(\Psi - \delta_x)(dy)(\Psi - \delta_x - \delta_y)(dz)\right]$$
$$= \iiint_{E^3} \mathbb{E}f(x, y, z; \Psi_{xyz})\psi^{[3]}(dx, dy, dz), \tag{5.3}$$

for all measurable functions $f : E^2 \times C_E \to [0, \infty)$ in (5.2) and $f : E^3 \times C_E \to [0, \infty)$ in (5.3) [26, Section 12.3]. Using reduced Palm processes, the Slivnyak–Mecke theorem [35] states that the distributions of the reduced Palm processes of a point process are the same as that of the point process if and only if the point process is a Poisson point process. Then we can see that for a homogeneous Poisson point process with rate λ, its mean measure can be written as $\Lambda(dx) = \lambda dx$; its Palm processes satisfy $\Psi_x \stackrel{d}{=} \Psi + \delta_x$, $\Psi_{xy} \stackrel{d}{=} \Psi + \delta_x + \delta_y$, and $\Psi_{xyz} \stackrel{d}{=} \Psi + \delta_x + \delta_y + \delta_z$; and the factorial moments $\psi^{[2]}(dx, dy) = \lambda^2 dxdy$ and $\psi^{[3]}(dx, dy, dz) = \lambda^3 dxdydz$ for all distinct $x, y, z \in E = \mathbb{R}^d$.

We can adapt (5.1), (5.2), and (5.3) to the marked point process Ξ. To this end, we assume that the rate of $\bar{\Xi}$ is λ, and that $\{M_i\}_{1 \le i \le 3}$ are i.i.d. random elements on $(T, \mathscr{T})$ following the distribution $\mathscr{L}_T$, which are independent of Ξ. We use M_x to denote the mark of x if $x \in \bar{\Xi}$. Because of the independence of the marks, we can obtain the following corollaries of the *Campbell–Mecke theorem* (also known as the *Mecke equation*) and the *multivariate Mecke equation* [15, p. 130]:

$$\mathbb{E}\left[\int_{\mathbb{R}^d} f((x, M_x), \Xi)\bar{\Xi}(dx)\right] = \int_{\mathbb{R}^d} \mathbb{E}f((x, M_1), \Xi + \delta_{(x, M_1)})\lambda dx, \tag{5.4}$$

$$\mathbb{E}\left[\iint_{(\mathbb{R}^d)^2} f((x, M_x), (y, M_y); \Xi)\bar{\Xi}(dx)(\bar{\Xi} - \delta_x)(dy)\right]$$
$$= \iint_{(\mathbb{R}^d)^2} \mathbb{E}f((x, M_1), (y, M_2); \Xi + \delta_{(x, M_1)} + \delta_{(y, M_2)})\lambda^2 dxdy, \tag{5.5}$$

$$
\mathbb{E}\left[\iiint\limits_{(\mathbb{R}^d)^3} f((x, M_x), (y, M_y), (z, M_z); \Xi)\bar{\Xi}(dx)(\bar{\Xi} - \delta_x)(dy)(\bar{\Xi} - \delta_x - \delta_y)(dz)\right]
$$

$$
= \iiint\limits_{(\mathbb{R}^d)^3} \mathbb{E}f((x, M_1), (y, M_2), (z, M_3); \Xi + \delta_{(x,M_1)} + \delta_{(y,M_2)} + \delta_{(z,M_3)})\lambda^3 dxdydz, \qquad (5.6)
$$

for all measurable functions $f : S \times C_S \to [0, \infty)$ in (5.4), $f : S^2 \times C_S \to [0, \infty)$ in (5.5), and $f : S^3 \times C_S \to [0, \infty)$ in (5.6).

Recalling the shift operator defined in Section 2, we can write $g(\mathscr{X}^x) := \eta((x, m), \mathscr{X})$ (resp. $g_\alpha(x, \mathscr{X}) := \eta((x, m), \Xi, \mathscr{X}, \Gamma_\alpha))$ for every configuration $\mathscr{X}$, $(x, m) \in \mathscr{X}$, and $\alpha > 0$, so that the notation can be simplified significantly; e.g.,

$$
W_\alpha = \sum_{(x,m) \in \Xi_{\Gamma_\alpha}} \eta((x, m), \Xi) = \int_{\Gamma_\alpha} g(\Xi^x)\bar{\Xi}(dx) = \sum_{x \in \bar{\Xi}} g(\Xi^x),
$$

$$
\bar{W}_\alpha = \sum_{(x,m) \in \Xi_{\Gamma_\alpha}} \eta((x, m), \Xi, \Gamma_\alpha) = \int_{\Gamma_\alpha} g_\alpha(x, \Xi)\bar{\Xi}(dx),
$$

where $\bar{\Xi}$ is the projection of Ξ on $\mathbb{R}^d$, and R and $\bar{R}(x, \alpha)$ are the corresponding radii of stabilisation. Here, $g(\mathscr{X})$ (resp. $g_\alpha(x, \mathscr{X})$) is not affected by m since $\eta((x, m), \mathscr{X})$ (resp. $\eta((x, m), \Xi, \mathscr{X}, \Gamma_\alpha)$) is understood as 0 if $(x, m) \notin \mathscr{X}$.

Since the mean and variance of $\bar{W}_\alpha$ are generally different from those of $\bar{W}_{\alpha,r}$, in the proof of Theorem 2.2, we will need the bound of the total variation distance between two normal distributions.

Lemma 5.1. ([18, Theorem 1.3].) *Let $F_{\mu,\sigma}$ be the distribution of $N(\mu, \sigma^2)$; then*

$$
d_{TV}\left(F_{\mu_1,\sigma_1}, F_{\mu_2,\sigma_2}\right) \leq \frac{3\left|\sigma_1^2 - \sigma_2^2\right|}{2 \max\left\{\sigma_1^2, \sigma_2^2\right\}} + \frac{|\mu_1 - \mu_2|}{2 \max(\sigma_1, \sigma_2)}.
$$

To find suitable normal approximations for W_α and $W_{\alpha,r}$ (resp. $\bar{W}_\alpha$ and $\bar{W}_{\alpha,r}$), we need to analyse the first two moments of W_α and $W_{\alpha,r}$ (resp. $\bar{W}_\alpha$ and $\bar{W}_{\alpha,r}$). Before doing this, we establish a few lemmas needed in the proofs.

Lemma 5.2. (Conditional total variance formula.) *Let X be a random variable on the probability space $(\Omega, \mathscr{G}, \mathbb{P})$ with a finite second moment, and let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two sub-σ-algebras of $\mathscr{G}$ such that $\mathscr{G}_1 \subset \mathscr{G}_2$. Then*

$$
\mathrm{Var}(X|\mathscr{G}_1) = \mathbb{E}(\mathrm{Var}(X|\mathscr{G}_2)|\mathscr{G}_1) + \mathrm{Var}(\mathbb{E}(X|\mathscr{G}_2)|\mathscr{G}_1).
$$

Proof. From the definition of the conditional variance, we can see that

$$
\begin{aligned}
\mathrm{Var}(X|\mathscr{G}_1) &= \mathbb{E}\left(X^2\Big|\mathscr{G}_1\right) - \mathbb{E}(X|\mathscr{G}_1)^2 \\
&= \mathbb{E}\left(\mathbb{E}\left(X^2\Big|\mathscr{G}_2\right)\Big|\mathscr{G}_1\right) - \mathbb{E}(\mathbb{E}(X|\mathscr{G}_2)|\mathscr{G}_1)^2 \\
&= \mathbb{E}\left(\mathbb{E}\left(X^2\Big|\mathscr{G}_2\right)\Big|\mathscr{G}_1\right) - \mathbb{E}\left(\mathbb{E}(X|\mathscr{G}_2)^2\Big|\mathscr{G}_1\right) + \mathbb{E}\left(\mathbb{E}(X|\mathscr{G}_2)^2\Big|\mathscr{G}_1\right) \\
&\quad - \mathbb{E}(\mathbb{E}(X|\mathscr{G}_2)|\mathscr{G}_1)^2 \\
&= \mathbb{E}(\mathrm{Var}(X|\mathscr{G}_2)|\mathscr{G}_1) + \mathrm{Var}(\mathbb{E}(X|\mathscr{G}_2)|\mathscr{G}_1),
\end{aligned}
$$

so the statement holds.   □

Also, given the value of a random variable in a certain event, we can find a lower bound for the conditional variance.

Lemma 5.3. *Let X be a random variable on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with a finite second moment. For any event A and σ-algebra such that $\mathcal{F} \subset \mathcal{G}$,*

$$\operatorname{Var}(X|\mathcal{F}) \geq \operatorname{Var}\left(X\mathbf{1}_A + \frac{\mathbb{E}(X\mathbf{1}_A|\mathcal{F})}{\mathbb{P}(A|\mathcal{F})}\mathbf{1}_{A^c}\,\middle|\,\mathcal{F}\right), \tag{5.7}$$

where $\frac{0}{0} = 0$ by convention.

Proof of Lemma 5.3. The statement is trivially true if $\mathbb{P}(A) = 0$, so we focus on the case that $\mathbb{P}(A) > 0$. Let $A \cap \mathcal{F} = \{B \cap A; B \in \mathcal{F}\}$, which is a σ-algebra on A, and let $\mathbb{P}_A$ be a probability measure on $(A, A \cap \mathcal{F})$ such that $\mathbb{P}_A(B \cap A) = \mathbb{P}(B|A)$ for all $B \in \mathcal{F}$. Then we have the corresponding conditional expectation

$$\mathbb{E}_A(X|A \cap \mathcal{F}) = \mathbb{E}_A(X\mathbf{1}_A|A \cap \mathcal{F}) = \mathbf{1}_A\mathbb{E}_A(X|A \cap \mathcal{F}),$$

which equals 0 on $\mathbf{1}_{A^c}$. The proof relies on the following observations.

Lemma 5.4. *For any random variable Y with $\mathbb{E}|Y| < \infty$ and $A \in \mathcal{G}$ such that $\mathbb{P}(A) > 0$,*

$$\mathbb{E}(\mathbb{E}_A(Y|A \cap \mathcal{F})|\mathcal{F}) = \mathbb{E}(\mathbf{1}_A Y|\mathcal{F}).$$

Proof. Both sides are $\mathcal{F}$-measurable, and for any $B \in \mathcal{F}$,

$$\begin{aligned}
\mathbb{E}(\mathbb{E}(\mathbb{E}_A(Y|A \cap \mathcal{F})|\mathcal{F})\,\mathbf{1}_B) &= \mathbb{E}(\mathbb{E}_A(Y|A \cap \mathcal{F})\,\mathbf{1}_B) \\
&= \mathbb{E}(\mathbb{E}_A(Y|A \cap \mathcal{F})\,\mathbf{1}_{B \cap A}) \\
&= \mathbb{E}(\mathbb{E}_A(\mathbf{1}_{B \cap A}Y|A \cap \mathcal{F})) \\
&= \mathbb{P}(A)\mathbb{E}_A(\mathbb{E}_A(\mathbf{1}_{B \cap A}Y|A \cap \mathcal{F})) \\
&= \mathbb{P}(A)\mathbb{E}_A(\mathbf{1}_{B \cap A}Y) = \mathbb{E}(\mathbf{1}_{B \cap A}Y) = \mathbb{E}(\mathbb{E}(\mathbf{1}_A Y|\mathcal{F})\,\mathbf{1}_B),
\end{aligned}$$

as claimed. $\qquad\square$

Lemma 5.5. *For any random variable Y with $\mathbb{E}|Y| < \infty$ and $A \in \mathcal{G}$ such that $\mathbb{P}(A) > 0$,*

$$\frac{\mathbb{E}(\mathbf{1}_A Y|\mathcal{F})}{\mathbb{P}(A|\mathcal{F})}\mathbf{1}_A = \mathbb{E}_A(Y|\mathcal{F}). \tag{5.8}$$

Proof. Both sides equal 0 on A^c and are measurable on $A \cap \mathcal{F}$ when restricted to A. From the construction of $A \cap \mathcal{F}$, any set $B' \in A \cap \mathcal{F}$ is of the form $B \cap A$ for some $B \in \mathcal{F}$. Hence (5.8) is equivalent to

$$\mathbb{E}_A\left(\frac{\mathbb{E}(\mathbf{1}_A Y|\mathcal{F})}{\mathbb{P}(A|\mathcal{F})}\mathbf{1}_A\mathbf{1}_{A \cap B}\right) = \mathbb{E}_A(\mathbb{E}_A(Y|\mathcal{F})\,\mathbf{1}_{A \cap B})$$

for all $B \in \mathscr{F}$. Now we have

$$
\begin{aligned}
\mathbb{E}_A\left(\frac{\mathbb{E}(\mathbf{1}_A Y|\mathscr{F})}{\mathbb{E}(\mathbf{1}_A|\mathscr{F})}\mathbf{1}_{A\cap B}\right) &= \frac{1}{\mathbb{P}(A)}\mathbb{E}\left(\frac{\mathbb{E}(\mathbf{1}_A Y|\mathscr{F})}{\mathbb{E}(\mathbf{1}_A|\mathscr{F})}\mathbf{1}_A\mathbf{1}_B\right) \\
&= \frac{1}{\mathbb{P}(A)}\mathbb{E}\left(\mathbb{E}\left(\frac{\mathbb{E}(\mathbf{1}_A Y|\mathscr{F})}{\mathbb{E}(\mathbf{1}_A|\mathscr{F})}\mathbf{1}_A\mathbf{1}_B\bigg|\mathscr{F}\right)\right) \\
&= \frac{1}{\mathbb{P}(A)}\mathbb{E}(\mathbb{E}(\mathbf{1}_A Y|\mathscr{F})\,\mathbf{1}_B) = \frac{1}{\mathbb{P}(A)}\mathbb{E}(\mathbb{E}(\mathbf{1}_A\mathbf{1}_B Y|\mathscr{F})) \\
&= \frac{1}{\mathbb{P}(A)}\mathbb{E}(\mathbf{1}_A\mathbf{1}_B Y) = \mathbb{E}_A(\mathbb{E}_A(Y\mathbf{1}_A\mathbf{1}_B|\mathscr{F})) = \mathbb{E}_A(\mathbb{E}_A(Y|\mathscr{F})\mathbf{1}_{A\cap B}),
\end{aligned}
$$

completing the proof. $\qquad\square$

Proof of Lemma 5.3 (*continued*). We start from the left-hand side of (5.7):

$$
\begin{aligned}
&\operatorname{Var}(X|\mathscr{F}) \\
&= \mathbb{E}\left(X^2\big|\mathscr{F}\right) - \mathbb{E}(X|\mathscr{F})^2 \\
&= \mathbb{E}\left(X^2\big|\mathscr{F}\right) - (\mathbb{E}(X\mathbf{1}_A|\mathscr{F}) + \mathbb{E}(X\mathbf{1}_{A^c}|\mathscr{F}))^2 \\
&= \mathbb{E}\left(X^2\big|\mathscr{F}\right) - \mathbb{E}\big(\mathbb{E}_A(X\mathbf{1}_A|A\cap\mathscr{F}) + \mathbb{E}_{A^c}\left(X\big|A^c\cap\mathscr{F}\right)\big|\mathscr{F}\big)^2 \\
&\geq \mathbb{E}\left(X^2\big|\mathscr{F}\right) - \mathbb{E}\big(\big(\mathbb{E}_A(X\mathbf{1}_A|A\cap\mathscr{F}) + \mathbb{E}_{A^c}\left(X\big|A^c\cap\mathscr{F}\right)\big)^2\big|\mathscr{F}\big) \\
&= \mathbb{E}\big(X^2\left(\mathbf{1}_A + \mathbf{1}_{A^c}\right)\big|\mathscr{F}\big) - \mathbb{E}\big(\mathbb{E}_A(X\mathbf{1}_A|A\cap\mathscr{F})^2 + \mathbb{E}_{A^c}\left(X\big|A^c\cap\mathscr{F}\right)^2\big|\mathscr{F}\big) \\
&= \mathbb{E}\big(\mathbb{E}_A\left(X^2\big|A\cap\mathscr{F}\right) + \mathbb{E}_{A^c}\left(X^2\big|A^c\cap\mathscr{F}\right)\big|\mathscr{F}\big) \\
&\quad - \mathbb{E}\big(\mathbb{E}_A(X\mathbf{1}_A|A\cap\mathscr{F})^2 + \mathbb{E}_{A^c}\left(X\big|A^c\cap\mathscr{F}\right)^2\big|\mathscr{F}\big) \\
&= \mathbb{E}\big(\operatorname{Var}_A\left(X|A\cap\mathscr{F}\right) + \operatorname{Var}_{A^c}\left(X\big|A^c\cap\mathscr{F}\right)\big|\mathscr{F}\big), \qquad\qquad\qquad (5.9)
\end{aligned}
$$

where the inequality follows from Jensen's inequality, and the third equality and the second-to-last equality are from Lemma 5.4. On the other hand, the right-hand side of (5.7) can be written as

$$
\begin{aligned}
&\mathbb{E}\left(X^2\mathbf{1}_A + \left(\frac{\mathbb{E}(X\mathbf{1}_A|\mathscr{F})}{\mathbb{P}(A|\mathscr{F})}\right)^2\mathbf{1}_{A^c}\bigg|\mathscr{F}\right) - \left(\mathbb{E}(X\mathbf{1}_A|\mathscr{F}) + \mathbb{E}\left(\frac{\mathbb{E}(X\mathbf{1}_A|\mathscr{F})}{\mathbb{P}(A|\mathscr{F})}\mathbf{1}_{A^c}\bigg|\mathscr{F}\right)\right)^2 \\
&= \mathbb{E}\left(X^2\mathbf{1}_A\big|\mathscr{F}\right) + \left(\frac{\mathbb{E}(X\mathbf{1}_A|\mathscr{F})}{\mathbb{P}(A|\mathscr{F})}\right)^2\mathbb{P}(A^c|\mathscr{F}) - \mathbb{E}(X\mathbf{1}_A|\mathscr{F})^2\frac{1}{\mathbb{P}(A|\mathscr{F})^2} \\
&= \mathbb{E}\left(X^2\mathbf{1}_A\big|\mathscr{F}\right) - \left(\frac{\mathbb{E}(X\mathbf{1}_A|\mathscr{F})}{\mathbb{P}(A|\mathscr{F})}\right)^2\mathbb{P}(A|\mathscr{F}) \\
&= \mathbb{E}\left(\mathbb{E}_A\left(X^2\big|A\cap\mathscr{F}\right) - \left(\frac{\mathbb{E}(X\mathbf{1}_A|\mathscr{F})}{\mathbb{P}(A|\mathscr{F})}\right)^2\mathbf{1}_A\bigg|\mathscr{F}\right) \\
&= \mathbb{E}\left(\mathbb{E}_A\left(X^2\big|A\cap\mathscr{F}\right) - \mathbb{E}_A(X|A\cap\mathscr{F})^2\big|\mathscr{F}\right) \\
&= \mathbb{E}(\operatorname{Var}_A\left(X|A\cap\mathscr{F}\right)|\mathscr{F}), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.10)
\end{aligned}
$$

where the third equality follows from Lemma 5.4, and the second-to-last equality is from Lemma 5.5. Combining (5.9), (5.10), and Lemma 5.2 completes the proof. $\qquad\square$

Before going to the proof of Lemma 4.4, we need a lemma to show that under the stabilisation conditions, the cost of throwing away the terms with a large radius of stabilisation is negligible.

Lemma 5.6.

(a) *(Unrestricted case.) If the score function is exponentially stabilising as in Definition 2.1, then we have*

$$d_{TV}(W_\alpha, W_{\alpha,r}) \le C_1 \alpha e^{-C_2 r}$$

for some positive constants C_1, C_2. If the score function is polynomially stabilising with parameter β in Definition 2.1, then we have

$$d_{TV}(W_\alpha, W_{\alpha,r}) \le C\alpha r^{-\beta}$$

for some positive constant C.

(b) *(Restricted case.) If the score function is exponentially stabilising as in Definition 2.2, then we have*

$$d_{TV}(\bar{W}_\alpha, \bar{W}_{\alpha,r}) \le C_1 \alpha e^{-C_2 r}$$

for some positive constants C_1, C_2. If the score function is polynomially stabilising with parameter β in Definition 2.2, then we have

$$d_{TV}(\bar{W}_\alpha, \bar{W}_{\alpha,r}) \le C\alpha r^{-\beta}$$

for some positive constant C.

Proof. We first show that the statement is true for $\bar{W}_\alpha$ and $\bar{W}_{\alpha,r}$. Recall that we defined $M_1 \sim \mathscr{L}_T$ as a random element independent of Ξ. From the construction of $\bar{W}_\alpha$ and $\bar{W}_{\alpha,r}$, we can see that the event $\{\bar{W}_\alpha \ne \bar{W}_{\alpha,r}\} \subset \{$at least one $x \in \bar{\Xi} \cap \Gamma_\alpha$ with $\bar{R}(x,\alpha) > r\}$. From (5.4), we have

$$
\begin{aligned}
d_{TV}(\bar{W}_\alpha, \bar{W}_{\alpha,r}) &\le \mathbb{P}(\{\bar{W}_\alpha \ne \bar{W}_{\alpha,r}\}) \\
&\le \mathbb{P}(\{\text{at least one } x \in \bar{\Xi} \cap \Gamma_\alpha \text{ such that } \bar{R}(x,\alpha) > r\}) \\
&\le \mathbb{E} \int_{\Gamma_\alpha} \mathbf{1}_{\bar{R}(x,\alpha)>r} \bar{\Xi}(dx) \\
&= \int_{\Gamma_\alpha} \mathbb{E}\left(\mathbf{1}_{\bar{R}(x,M_1,\alpha,\Xi+\delta_{(x,M_1)})>r}\right) \lambda dx \\
&= \int_{\Gamma_\alpha} \mathbb{P}(\bar{R}(x,M_1,\alpha,\Xi+\delta_{(x,M_1)}) > r) \lambda dx \\
&\le \alpha\lambda\bar{\tau}(r),
\end{aligned}
$$

which, together with the stabilisation conditions, gives the claim for $\bar{W}_\alpha$.

The statement is also true for W_α, which can be proved by replacing $\bar{W}_\alpha$ with the corresponding W_α, $\bar{W}_{\alpha,r}$ with $W_{\alpha,r}$, $\bar{R}(x,\alpha)$ with $R(x)$, $\bar{R}(x,M_1,\alpha,\Xi+\delta_{(x,M_1)})$ with $R(x,M_1,\Xi+\delta_{(x,M_1)})$, and $\bar{\tau}$ with τ. $\qquad\square$

Proof of Lemma 4.1. For convenience, we write G_n, g_n, and ψ_n for the distribution, density, and characteristic functions, respectively, of T_n. It is well known that the triangular density κ_a has the characteristic function

$$\psi_1(s) = \frac{2(1 - \cos(as))}{(as)^2},$$

which gives

$$\psi_n(s) = \left(\frac{2(1 - \cos(as))}{(as)^2} \right)^n.$$

Using the fact that the convolution of two symmetric unimodal distributions on $\mathbb{R}$ is unimodal [50], we can conclude that the distribution of T_n is unimodal and symmetric. This ensures that

$$d_{TV}(T_n, T_n + \gamma) = \sup_{x \in \mathbb{R}} |G_n(x) - G_n(x - \gamma)| = \int_{-\gamma/2}^{\gamma/2} g_n(x)dx. \tag{5.11}$$

Applying the inversion formula, we have

$$\begin{aligned} g_n(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-isx} \psi_n(s)ds = \frac{1}{2\pi} \int_{\mathbb{R}} \cos(sx) \psi_n(s)ds \\ &= \frac{1}{a\pi} \int_0^\infty \cos(sx/a) \left(\frac{2(1 - \cos s)}{s^2} \right)^n ds, \end{aligned}$$

where $i = \sqrt{-1}$, and the second equality is due to the fact that $\sin(sx)\psi_n(s)$ is an odd function. Obviously, $g_n(x) \leq g_n(0)$, so we need to establish an upper bound for $g_n(0)$. A direct verification gives

$$0 \leq \frac{2(1 - \cos s)}{s^2} \leq e^{-\frac{s^2}{12}} \quad \text{for } 0 < s \leq 2\pi,$$

which implies

$$\begin{aligned} g_n(0) &\leq \frac{1}{a\pi} \left\{ \int_0^{2\pi} e^{-\frac{ns^2}{12}} ds + \int_{2\pi}^\infty \left(\frac{4}{s^2} \right)^n ds \right\} \\ &\leq \frac{1}{a\pi \sqrt{n}} \int_0^\infty e^{-\frac{s^2}{12}} ds + \frac{2}{a(2n-1)\pi^{2n}} \\ &= \frac{1}{a} \sqrt{\frac{3}{\pi n}} + \frac{2}{a(2n-1)\pi^{2n}}. \end{aligned} \tag{5.12}$$

Now, combining (5.11) with (5.12) gives (4.2). □

Proof of Lemma 4.2. We construct a maximal coupling [5, p. 254] (X, Y) such that $X \sim F$, $Y \sim G$, and $d_{TV}(F, G) = \mathbb{P}(X \neq Y)$. The Lebesgue decomposition (1.1) ensures that there exists an $A \in \mathscr{B}(\mathbb{R})$ such that $F_a(A) = 1$ and $F_s(A) = 0$. Define $\mu_G(B) = \mathbb{P}(X \in B \cap A, X = Y) \leq \alpha_F F_a(B)$ for $B \in \mathscr{B}(\mathbb{R})$, so μ_G is absolutely continuous with respect to the Lebesgue measure. On the other hand,

$$G(B) \geq G(B \cap A) \geq \mathbb{P}(Y \in B \cap A, X = Y) = \mu_G(B), \text{ for } B \in \mathscr{B}(\mathbb{R}) ;$$

hence $\alpha_G \geq \mu_G(\mathbb{R}) = \alpha_F - \mathbb{P}(X \neq Y) = \alpha_F - d_{TV}(F, G) > 0$. □
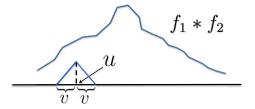
FIGURE 9. Existence of u and v.

Proof of Lemma 4.3. Since F_i is non-singular, there exists a non-zero sub-probability measure μ_i with a density f_i such that $\mu_i(dx) = f_i(x)dx \leq dF_i(x)$ for $x \in \mathbb{R}$. Without loss of generality, we can assume that both f_1 and f_2 are bounded with bounded support, which ensures that $f_1 * f_2$ is continuous (for the case of $f_1 = f_2$, see [33, p. 79]). In fact, as f_1 is a density, one can find a sequence of continuous functions $\{f_{1n} : n \geq 1\}$ satisfying $|f_{1n} - f_1|_1 \to 0$ as $n \to \infty$, where $|\cdot|_1$ is the L_1 norm. Now, with $|\cdot|_\infty$ denoting the supremum norm, $|f_{1n} * f_2 - f_1 * f_2|_\infty \leq |f_2|_\infty |f_{1n} - f_1|_1 \to 0$ as $n \to \infty$. However, since the continuity is preserved under the supremum norm, the continuity of $f_1 * f_2$ follows.

Referring to Figure 9, since $f_1 * f_2 \not\equiv 0$, we can find $u \in \mathbb{R}$ and $v > 0$ such that $f_1 * f_2(u) > 0$ and

$$\min_{x \in [u-v, u+v]} f_1 * f_2(x) \geq \frac{1}{2} f_1 * f_2(u) =: b.$$

Let $\theta = vb$ and $a = v$; then $H = \frac{1}{1-\theta}(F_1 * F_2 - \theta K_a * \delta_u)$ is a distribution function, and the claim follows. □

Proof of Lemma 4.4. The idea of the proof is to use the radius of stabilisation to limit the effect of dependence, pass the non-singularity property to the truncated score function $\eta((x, m), \Xi)\mathbf{1}_{R(x) \leq r}$ (resp. $\eta((x, m), \Xi, \Gamma_\alpha)\mathbf{1}_{\bar{R}(x, \alpha) \leq r}$), and dig out the maximum number of cubes in the carrier space Γ_α such that the truncated score functions on these cubes are independent. Then we can apply Lemmas 4.2 and 4.3 to find the component with the form of a convolution of triangular distributions, which, together with the property of the triangular distributions in Lemma 4.1, implies the conclusion. The order of the bound is then determined by the reciprocal of the number of the cubes, as in the Berry–Esseen bound. Except for notational complexity, the proof of the restricted case is the same, so we first focus on the unrestricted case.

From Assumption 2.2, for the restricted case, we can find $\bar{g}$, $\bar{\eta}$, and R corresponding to η such that the stabilisation radius R of $\bar{\eta}$ satisfies the same stabilisation property as η in the sense of Definition 2.1. Because N_0 is a bounded set, there exists an $r_1 \in \mathbb{R}_+$ such that $N_0 \subset B(0, r_1)$. For convenience, we write the random variables

$$Y := \sum_{x \in \bar{\Xi}} \bar{g}(\Xi^x)\mathbf{1}_{d(x, N_0) < R(x)}, \qquad Y_r := \sum_{x \in \bar{\Xi}} \bar{g}(\Xi^x)\mathbf{1}_{d(x, N_0) < R(x) < r},$$

and write the event $\{Y \neq Y_r\}$ as E_r for $r \in \mathbb{R}_+$. We can see that

$$\mathbb{P}(E_r) \leq \mathbb{P}(\{\text{there is at least one point } x \in \bar{\Xi} \text{ such that } d(x, N_0) \vee r < R(x)\}) =: \mathbb{P}(E_r'),$$

and the right-hand side is a decreasing function of r. Under any stabilisation condition in Lemma 4.4, we can show that $\mathbb{P}(E_r) \to 0$ as $r \to \infty$, that is, Y_r converges to Y a.s. In fact,

$$\mathbb{P}(E_r')$$
$$\leq \mathbb{P}\big(\{\text{there is at least one point } x \in \bar{\Xi} \cap B(0, r_1 + r) \text{ such that } r \leq R(x)\}\big) \tag{5.13}$$
$$+ \mathbb{P}\big(\{\text{there is at least one point } x \in \bar{\Xi} \cap B(0, r_1 + r)^c \text{ such that } |x| - r_1 \leq R(x)\}\big).$$

Using the property of the Palm process (5.4), we can see that the first term of (5.13) satisfies

$$\mathbb{P}\big(\{\text{there is at least one point } x \in \bar{\Xi} \cap B(0, r_1 + r) \text{ such that } r \leq R(x)\}\big)$$

$$\leq \mathbb{E} \int_{B(0,r_1+r)} \mathbf{1}_{R(x) \geq r} \bar{\Xi}(dx) = \int_{B(0,r_1+r)} \mathbb{E}\mathbf{1}_{R\left(x, M_1, \Xi + \delta_{(x,M_1)}\right) \geq r} \lambda dx$$

$$= \int_{B(0,r_1+r)} \mathbb{P}\left(R\left(x, M_1, \Xi + \delta_{(x,M_1)}\right) \geq r\right) \lambda dx$$

$$\leq \int_{B(0,r_1+r)} \tau(r) \lambda dx = \frac{\lambda(r_1 + r)^d \pi^{d/2} \tau(r)}{\Gamma\left(\frac{d}{2} + 1\right)}, \tag{5.14}$$

and the second term is bounded by

$$\mathbb{P}\big(\{\text{there is at least one point } x \in \bar{\Xi} \cap B(0, r_1 + r)^c \text{ such that } |x| - r_1 \leq R(x)\}\big)$$

$$\leq \mathbb{E} \int_{B(0,r_1+r)^c} \mathbf{1}_{R(x) \geq |x| - r_1} \bar{\Xi}(dx) = \int_{B(0,r_1+r)^c} \mathbb{E}\mathbf{1}_{R\left(x, M_1, \Xi + \delta_{(x,M_1)}\right) \geq |x| - r_1} \lambda dx$$

$$= \int_{B(0,r_1+r)^c} \mathbb{P}\left(R\left(x, M_1, \Xi + \delta_{(x,M_1)}\right) \geq |x| - r_1\right) \lambda dx$$

$$\leq \int_{B(0,r_1+r)^c} \tau(|x| - r_1) \lambda dx = \int_{r_1+r}^{\infty} \frac{d\lambda t^{d-1} \pi^{d/2} \tau(t - r_1)}{\Gamma\left(\frac{d}{2} + 1\right)} dt. \tag{5.15}$$

When the score function satisfies one of the stabilisation conditions, both bounds in (5.14) and (5.15) converge to 0 as $r \to \infty$, so $\mathbb{P}(E_r) \leq \mathbb{P}(E_r') \to 0$ as $r \to \infty$.

For two measures μ_1, μ_2, we write $\mu_1 \preceq \mu_2$ if $\mu_1(A) \leq \mu_2(A)$ for all measurable sets A. The non-singularity in Assumption 2.4 ensures that, with positive probability, the conditional distribution $\mathscr{L}\big(Y \big| \sigma\big(\Xi_{N_0^c}\big)\big)$ is non-singular. So we can find a $\sigma\big(\Xi_{N_0^c}\big)$-measurable random measure ξ on $\mathbb{R}$ such that $\xi \preceq \mathscr{L}\big(Y \big| \sigma\big(\Xi_{N_0^c}\big)\big)$ a.s., $\mathbb{P}(\xi(\mathbb{R}) > 0) > 0$, and ξ is absolutely continuous [26, Lemma 2.1]. Since

$$\lim_{u \downarrow 0} \mathbb{P}(\xi(\mathbb{R}) > u) = \mathbb{P}(\xi(\mathbb{R}) > 0) > 0,$$

we can find a $p > 0$ such that $\mathbb{P}(\xi(\mathbb{R}) > p) > 4p$. Because E_r' is decreasing in the sense of inclusion in r, and $\mathbb{P}(E_r') \to 0$ as $r \to \infty$, we can find an $R_0 \in \mathbb{R}_+$ such that $\mathbb{P}(E_{R_0}') \leq p^2$, which ensures

$$\mathbb{P}\left(\mathbb{P}\left(E_{R_0}' \big| \sigma\big(\Xi_{N_0^c}\big)\right) > \frac{p}{2}\right) \leq 2p. \tag{5.16}$$

If we write $\tilde{Y} := Y\mathbf{1}_{E_{R_0}'^c}$, $A_1 := \left\{d_{TV}\left(Y, \tilde{Y}\big| \sigma\big(\Xi_{N_0^c}\big)\right) > p/2\right\}$, $A_2 := \{\xi(\mathbb{R}) > p\}$, then A_1 and A_2 are both $\sigma\big(\Xi_{N_0^c}\big)$-measurable, and

$$\mathbb{P}\left(\mathbb{P}\left(d_{TV}\left(Y, \tilde{Y}\big| \sigma\big(\Xi_{N_0^c}\big)\right)\right) > \frac{p}{2}\right) \leq \mathbb{P}\left(\mathbb{P}\left(E_{R_0}'\big| \sigma\big(\Xi_{N_0^c}\big)\right) > \frac{p}{2}\right) \leq 2p,$$

giving $\mathbb{P}\big(A_2 \cap A_1^c\big) > 2p$. For $\omega \in A_1^c \cap A_2$, $d_{TV}\big(Y, \tilde{Y} \big| \sigma\big(\Xi_{N_0^c}\big)\big)(\omega) < p/2$ and $\xi(\omega)(\mathbb{R}) > p$. By Lemma 4.2 and (5.16), there exists an absolutely continuous $\sigma\big(\Xi_{N_0^c}\big)$-measurable random measure $\tilde{\xi}$ such that $\tilde{\xi} \preceq \mathscr{L}\big(\tilde{Y}\big|\sigma\big(\Xi_{N_0^c}\big)\big)$ a.s. and $\mathbb{P}\big(\tilde{\xi}(\mathbb{R}) > \frac{p}{2}\big) > 2p$. We write Ξ' as an independent copy of Ξ, and the corresponding Y_r and $\tilde{\xi}$ as Y_r' and $\tilde{\xi}'$, respectively. Using Lemma 4.3, we can find $\sigma\big(\Xi_{N_0^c}, \Xi'_{N_0^c}\big)$-measurable random variables $\Theta_1 \geq 0$, $\Theta_2 \geq 0$ and $U \in \mathbb{R}$ such that $\mathbb{P}(\Theta_1 > 0, \Theta_2 > 0) = 4p^2$, and

$$\tilde{\xi} \star \tilde{\xi}' \succeq \Theta_1 K_{\Theta_2} \star \delta_U.$$

However,

$$\lim_{\epsilon \downarrow 0} \mathbb{P}\{\Theta_1/\Theta_2 \geq \epsilon, \Theta_2 \geq \epsilon\} = \mathbb{P}\{\Theta_1 > 0, \Theta_2 > 0\},$$

and from Remark 4.1, we can find an $\epsilon > 0$ such that

$$\mathbb{P}\Big\{\tilde{\xi} \star \tilde{\xi}' \succeq \epsilon^2 K_\epsilon \star \delta_U\Big\} \geq 2p^2$$

for a $\sigma\big(\Xi_{N_0^c}, \Xi'_{N_0^c}\big)$-measurable U. From the fact that we can write $Y_r = \tilde{Y} + Y_r' \mathbf{1}_{E'_{R_0}}$, we have, for any $B \in \mathscr{B}(\mathbb{R}\backslash\{0\})$,

$$\mathbb{P}\Big(Y_r \in B \big| \sigma\big(\Xi_{N_0^c}\big)\Big) = \mathbb{P}\Big(\tilde{Y} \in B, E'_{R_0}{}^c \big| \sigma\big(\Xi_{N_0^c}\big)\Big) + \mathbb{P}\Big(Y_r \in B, E'^{R_0} \big| \sigma\big(\Xi_{N_0^c}\big)\Big)$$

$$\geq \mathbb{P}\Big(\tilde{Y} \in B, E'^{R_0}{}^c \big| \sigma\big(\Xi_{N_0^c}\big)\Big)$$

$$= \mathbb{P}\Big(\tilde{Y} \in B \big| \sigma\big(\Xi_{N_0^c}\big)\Big).$$

Hence

$$\mathscr{L}\Big(Y_r \big| \sigma\big(\Xi_{N_0^c}\big)\Big)(\cdot) \geq \mathscr{L}\Big(\tilde{Y}\big|\sigma\big(\Xi_{N_0^c}\big)\Big)(\cdot\backslash\{0\})$$

$$\geq \tilde{\xi}(\cdot\backslash\{0\}) = \tilde{\xi}(\cdot) \text{ a.s. for all } r \geq R_0. \tag{5.17}$$

Therefore, using $U \in \mathscr{A}$ to stand for U being $\mathscr{A}$-measurable, we have

$$\sup_{U \in \sigma\big(\Xi_{B(N_0, 2r)\backslash N_0}, \Xi'_{B(N_0, 2r)\backslash N_0}\big)} \mathbb{P}\left\{\mathscr{L}\big(Y_r\big|\sigma\big(\Xi_{B(N_0, 2r)\backslash N_0}\big)\big) \star \mathscr{L}\big(Y_r'\big|\sigma\big(\Xi'_{B(N_0, 2r)\backslash N_0}\big)\big) \succeq \epsilon^2 K_\epsilon \star \delta_U\right\}$$

$$= \sup_{U \in \sigma\big(\Xi_{N_0^c}, \Xi'_{N_0^c}\big)} \mathbb{P}\left\{\mathscr{L}\big(Y_r\big|\sigma\big(\Xi_{B(N_0, 2r)\backslash N_0}\big)\big) \star \mathscr{L}\big(Y_r'\big|\sigma\big(\Xi'_{B(N_0, 2r)\backslash N_0}\big)\big) \succeq \epsilon^2 K_\epsilon \star \delta_U\right\}$$

$$= \sup_{U \in \sigma\big(\Xi_{N_0^c}, \Xi'_{N_0^c}\big)} \mathbb{P}\left\{\mathscr{L}\big(Y_r\big|\sigma\big(\Xi_{N_0^c}\big)\big) \star \mathscr{L}\big(Y_r'\big|\sigma\big(\Xi'_{N_0^c}\big)\big) \succeq \epsilon^2 K_\epsilon \star \delta_U\right\}$$

$$\geq \sup_{U \in \sigma\big(\Xi_{N_0^c}, \Xi'_{N_0^c}\big)} \mathbb{P}\left\{\tilde{\xi} \star \tilde{\xi}' \succeq \epsilon^2 K_\epsilon \star \delta_U\right\}$$

$$\geq 2p^2,$$

which ensures that, for any $r > R_0$, we can find a $\sigma\left(\Xi_{B(N_0,2r)\setminus N_0}, \Xi'_{B(N_0,2r)\setminus N_0}\right)$-measurable U such that

$$\mathbb{P}\left\{\mathscr{L}\left(Y_r\big|\sigma\left(\Xi_{B(N_0,2r)\setminus N_0}\right)\right) \star \mathscr{L}\left(Y'_r\big|\sigma\left(\Xi'_{B(N_0,2r)\setminus N_0}\right)\right) \succeq \epsilon^2 K_\epsilon \star \delta_U\right\} \geq p^2. \tag{5.18}$$

If $\alpha \leq (2(4r + 2r_1))^d$, (4.4) is trivial with

$$C = \left\{2\left(4 + \frac{2r_1}{R_0}\right)\right\}^{d/2},$$

so we now assume $\alpha > \{2(4r + 2r_1)\}^d$. From the structure of Ξ, we can see that $\Xi(A, D) \stackrel{d}{=} \Xi(x+A, D)$, and $\Xi(A, D)$ is independent of $\Xi(B, D)$ for all disjoint $A, B \in \mathscr{B}(\mathbb{R}^d)$, $D \in \mathscr{T}$, and $x \in \mathbb{R}^d$. For a fixed $r > R_0$, we can divide Γ_α into disjoint cubes $\mathbb{C}_1, \cdots, \mathbb{C}_{m_{\alpha,r}}$ with edge length $4r + 2r_1$ and centres $c_1, \cdots, c_{m_{\alpha,r}}$, aiming to maximise the number of cubes, so $m_{\alpha,r} \sim \alpha(4r + 2r_1)^{-d}$, which has order $O(\alpha r^{-d})$. Without loss of generality, we can assume that $m_{\alpha,r}$ is even or simply delete one of them, and the above properties still hold. For $i \leq m_{\alpha,r}$, we define

$$A_i = c_i + N_0, \qquad B_i = B(A_i, r), \qquad C_i = B(B_i, r), \qquad D_i = C_i\setminus A_i,$$
$$\mathcal{N}_{0,\alpha,r} := \cup_{1 \leq i \leq m_{\alpha,r}} A_i, \qquad \mathcal{N}_{1,\alpha,r} := \cup_{1 \leq i \leq m_{\alpha,r}} B_i, \qquad \mathcal{N}_{2,\alpha,r} := \cup_{1 \leq i \leq m_{\alpha,r}} D_i,$$
$$\mathscr{F}_{1,\alpha,r} := \sigma\left(\Xi_{\mathbb{R}^d\setminus\mathcal{N}_{0,\alpha,r}}\right), \qquad \mathscr{F}_{2,\alpha,r} := \sigma\left(\Xi_{\mathcal{N}_{2,\alpha,r}}\right),$$
$$W^0_{\alpha,r} = \int_{\mathcal{N}_{1,\alpha,r}} \bar{g}(\Xi^x)\mathbf{1}_{R(x)<r}\bar{\Xi}(dx), \qquad W^1_{\alpha,r} = W_{\alpha,r} - W^0_{\alpha,r}.$$

Note that for all x such that $d(x, \partial\Gamma_\alpha) \geq r$,

$$\eta((x, m), \Xi, \Gamma_\alpha)\mathbf{1}_{\bar{R}(x,\alpha)<r} = \bar{\eta}((x, m), \Xi)\mathbf{1}_{R(x)<r}$$

for all $(x, m) \in \Xi$ a.s. From the definition of total variation distance,

$$d_{TV}\left(W_{\alpha,r}, W_{\alpha,r} + \gamma\right) = \sup_{A\in\mathscr{B}(R)} \left(\mathbb{P}\left(W_{\alpha,r} \in A\right) - \mathbb{P}\left(W_{\alpha,r} \in A - \gamma\right)\right);$$

hence the tower property ensures that

$$\begin{aligned}
&d_{TV}\left(W_{\alpha,r}, W_{\alpha,r} + \gamma\right) \\
&= \sup_{A\in\mathscr{B}(\mathbb{R})} \mathbb{E}\left(\mathbf{1}_{W_{\alpha,r}\in A} - \mathbf{1}_{W_{\alpha,r}\in A-\gamma}\right) \\
&= \sup_{A\in\mathscr{B}(\mathbb{R})} \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{W_{\alpha,r}\in A} - \mathbf{1}_{W_{\alpha,r}\in A-\gamma}|\mathscr{F}_{1,\alpha,r}\right)\right) \\
&= \sup_{A\in\mathscr{B}(\mathbb{R})} \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{W^0_{\alpha,r}\in A-W^1_{\alpha,r}} - \mathbf{1}_{W^0_{\alpha,r}\in A-\gamma-W^1_{\alpha,r}}|\mathscr{F}_{1,\alpha,r}\right)\right) \\
&\leq \mathbb{E}\left(\sup_{A\in\mathscr{B}(\mathbb{R})}\left[\mathbb{E}\left(\mathbf{1}_{W^0_{\alpha,r}\in A} - \mathbf{1}_{W^0_{\alpha,r}\in A-\gamma}|\mathscr{F}_{1,\alpha,r}\right)\right]\right) \\
&= \mathbb{E}\left(\sup_{A\in\mathscr{B}(\mathbb{R})}\left[\mathbb{E}\left(\mathbf{1}_{W^0_{\alpha,r}\in A} - \mathbf{1}_{W^0_{\alpha,r}\in A-\gamma}|\mathscr{F}_{2,\alpha,r}\right)\right]\right), \tag{5.19}
\end{aligned}$$

where the last equality follows from the fact that $W^0_{\alpha,r}$ depends on $\mathscr{F}_{2,\alpha,r}$ in $\mathscr{F}_{1,\alpha,r}$.

From (5.19), to show (4.4), it is sufficient to show that

$$\mathbb{E}\left(\sup_{A\in\mathscr{B}(\mathbb{R})}\left[\mathbb{E}\left(\mathbf{1}_{W^0_{\alpha,r}\in A}-\mathbf{1}_{W^0_{\alpha,r}\in A-\gamma}|\mathscr{F}_{2,\alpha,r}\right)\right]\right)\leq(|\gamma|\vee 1)\,\mathrm{O}\left(\alpha^{-\frac{1}{2}}r^{\frac{d}{2}}\right).$$

Using the fact that $\int_{B_i}\bar{g}(\Xi^x)\mathbf{1}_{R(x)<r}\overline{\Xi}(dx)$ depends only on $\sigma(\Xi_{D_i})$ in $\mathscr{F}_{2,\alpha,r}$ for $i\leq m_{\alpha,r}$, and from the independence of $\sigma(\Xi_{D_i})$ for different i, we can see that

$$\mathscr{L}\left(W^0_{\alpha,r}|\mathscr{F}_{2,\alpha,r}\right)=\mathscr{L}\left(\sum_{i=1}^{m_{\alpha,r}}\int_{B_i}\bar{g}(\Xi^x)\mathbf{1}_{R(x)<r}\overline{\Xi}(dx)\bigg|\mathscr{F}_{2,\alpha,r}\right)$$

$$=\mathscr{L}\left(\sum_{i=1}^{m_{\alpha,r}}\int_{B_i}\bar{g}(\Xi^x)\mathbf{1}_{R(x)<r}\overline{\Xi}(dx)\bigg|\sigma\left(\Xi_{D_i},\,i\leq m_{\alpha,r}\right)\right). \qquad (5.20)$$

Using (5.18), we obtain

$$\mathscr{L}\left(\sum_{i=2j-1}^{2j}\int_{B_i}\bar{g}(\Xi^x)\mathbf{1}_{R(x)<r}\overline{\Xi}(dx)\bigg|\sigma\left(\Xi_{D_i},\,i\leq m_{\alpha,r}\right)\right)$$
$$=\mathscr{L}\left(X_{1,j}\left(1-J_{1,j}\right)+X_{2,j}J_{1,j}\left(1-J_{2,j}\right)+(X_{3,j}+U_j)J_{1,j}J_{2,j}\big|\sigma\left(\Xi_{D_{2j-1}},\,\Xi_{D_{2j}}\right)\right),$$

where $J_{1,j}$, $J_{2,j}$ and U_j are $\sigma\left(\Xi_{D_{2j-1}},\,\Xi_{D_{2j}}\right)$-measurable with $\mathbb{P}(J_{1,j}=1)=1-\mathbb{P}(J_{1,j}=0)=p^2$, $\mathbb{P}(J_{2,j}=1)=1-\mathbb{P}(J_{2,j}=0)=\epsilon^2$, $J_{1,j}\perp\!\!\!\perp J_{2,j}$, $X_{1,j}$ and $X_{2,j}$ are $\sigma\left(\Xi_{B_{2j-1}},\,\Xi_{B_{2j}}\right)$-measurable, and $X_{3,j}\sim K_\epsilon$, $1\leq j\leq m_{\alpha,r}/2$, are i.i.d. and independent of $\sigma\left(\Xi_{D_i},\,i\leq m_{\alpha,r}\right)$. Define

$$\Sigma_1:=\sum_{j=1}^{m_{\alpha,r}/2}\left(X_{1,j}\left(1-J_{1,j}\right)+X_{2,j}J_{1,j}\left(1-J_{2,j}\right)+(X_{3,j}+U_j)J_{1,j}J_{2,j}\right),$$

$$\Sigma_2:=\sum_{j=1}^{m_{\alpha,r}/2}X_{3,j}J_{1,j}J_{2,j},$$

$$\Sigma_{3,l}:=\sum_{j=1}^{l}X_{3,j},$$

and let $I\sim\mathrm{Binomial}\left(m_{\alpha,r}/2,\,\epsilon^2p^2\right)$ be independent of $\{X_{3,j}:j\leq m_{\alpha,r}/2\}$. It follows from (5.19) and (5.20) that

$$d_{TV}\left(W_{\alpha,r},\,W_{\alpha,r}+\gamma\right)$$

$$\leq\mathbb{E}\left(\sup_{A\in\mathscr{B}(\mathbb{R})}\left[\mathbb{E}\left(\mathbf{1}_{\Sigma_1\in A}-\mathbf{1}_{\Sigma_1\in A-\gamma}|\sigma\left(\Xi_{D_i},\,i\leq m_{\alpha,r}\right)\right)\right]\right)$$

$$\leq\mathbb{E}\left(\sup_{A\in\mathscr{B}(\mathbb{R})}\left[\mathbb{E}\left(\mathbf{1}_{\Sigma_2\in A}-\mathbf{1}_{\Sigma_2\in A-\gamma}|\sigma\left(\Xi_{D_i},\,i\leq m_{\alpha,r}\right)\right)\right]\right)$$

$$=\mathbb{E}\left(\sup_{A\in\mathscr{B}(\mathbb{R})}\left[\mathbb{E}\left(\mathbf{1}_{\Sigma_{3,I}\in A}-\mathbf{1}_{\Sigma_{3,I}\in A-\gamma}|I\right)\right]\right)$$

$$\leq \mathbb{P}(I \leq (\mathbb{E}I)/2) + \sum_{(\mathbb{E}I)/2 < j \leq m_{\alpha,r}/2} \sup_{A \in \mathscr{B}(\mathbb{R})} \left[\mathbb{E}\left(\mathbf{1}_{\Sigma_{3,j} \in A} - \mathbf{1}_{\Sigma_{3,j} \in A - \gamma}\right) \right] \mathbb{P}(I = j)$$

$$\leq O\left(\alpha^{-1} r^d\right) + O\left(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}}\right) |\gamma| = (|\gamma| \vee 1) \, O\left(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}}\right), \tag{5.21}$$

where the first term of (5.21) is from Chebyshev's inequality, and the second term is due to Lemma 4.1. This completes the proof of (4.4).

In terms of (4.5), since range-boundedness implies polynomial stabilisation with arbitrary order β, (4.4) still holds for all $r > R_0$. On the other hand, $\bar{W}_\alpha = \bar{W}_{\alpha,r}$ a.s. when $r > t$ for some positive constant t; (4.5) follows by taking $r = R_0 \vee t + 1$.

The claim (4.6) can be proved by replacing W_α with $\bar{W}_\alpha$, $W_{\alpha,r}$ with $\bar{W}_{\alpha,r}$, $\bar{g}$ with g, $W_{\alpha,r}^0$ with $\bar{W}_{\alpha,r}^0$, $W_{\alpha,r}^1$ with $\bar{W}_{\alpha,r}^1$, Ξ^x by $\Xi^{\Gamma_\alpha,x}$, $R(x)$ with $\bar{R}(x,\alpha)$, and $R(x, M_1, \Xi + \delta_{(x,M_1)})$ with $\bar{R}(x, M_1, \alpha, \Xi + \delta_{(x,M_1)})$, and redefining $\mathscr{F}_{1,\alpha,r} := \sigma\left(\Xi_{\Gamma_\alpha \setminus \mathcal{N}_{0,\alpha,r}}\right)$. The bound (4.7) can be argued in the same way as that for (4.5). $\qquad\square$

Proof of Corollary 4.1. The proof can easily be adapted from the second half of the proof of Lemma 4.4, and we start with (4.9). If $\alpha^{-\frac{1}{d}}(1 - 2q)^{-1}(4r + 2r_1) > \frac{1}{3}$, (4.9) is obvious because the total variation distance is bounded above by 1. Now we assume $\alpha^{-\frac{1}{d}}(1 - 2q)^{-1}(4r + 2r_1) \leq \frac{1}{3}$. Similarly to the proof of Lemma 4.4, we embed disjoint cubes with edge length $4r + 2r_1$ into $\Gamma_\alpha \setminus \left(N_{\alpha,r}^{(1)} \cup N_{\alpha,r}^{(2)} \cup N_{\alpha,r}^{(3)} \right)$, aiming to maximise the number $m_{\alpha,r}$ of cubes. Without loss of generality, we assume that $m_{\alpha,r}$ is even. Then we have

$$\alpha(1 - 2q)^d(12r + 6r_1)^{-d} - 1 \leq m_{\alpha,r} \leq \alpha(1 - 2q)^d(4r + 2r_1)^{-d},$$

giving $m_{\alpha,r} = O(\alpha r^{-d})$.

We use the same notation as in the proof of Lemma 4.4 but with Γ_α replaced by $\Gamma_\alpha \setminus \left(N_{\alpha,r}^{(1)} \cup N_{\alpha,r}^{(2)} \cup N_{\alpha,r}^{(3)} \right)$, and define

$$\mathscr{F}'_{2,\alpha,r} := \sigma\left(\Xi_{\mathcal{N}_{2,\alpha,r} \cup N_{\alpha,r}^{(2)}} \right).$$

Bearing in mind that $N_{\alpha,r}^{(1)} \cup N_{\alpha,r}^{(2)} \cup N_{\alpha,r}^{(3)}$ is excluded from the $m_{\alpha,r}$ cubes, we have $\mathscr{F}_{0,\alpha,r} \subset \mathscr{F}_{1,\alpha,r}$, giving the following result analogous to (5.19):

$$d_{TV}\left(\bar{W}'_{\alpha,r}, \bar{W}'_{\alpha,r} + h_{\alpha,r}\left(\Xi_{N_{\alpha,r}^{(2)}} \right) \Big| \mathscr{F}_{0,\alpha,r} \right)$$

$$= \sup_{A \in \mathscr{B}(\mathbb{R})} \mathbb{E}\left(\mathbf{1}_{\bar{W}'_{\alpha,r} \in A} - \mathbf{1}_{\bar{W}_{\alpha,r} \in A - h_{\alpha,r}\left(\Xi_{N_{\alpha,r}^{(2)}} \right)} \Big| \mathscr{F}_{0,\alpha,r} \right)$$

$$= \sup_{A \in \mathscr{B}(\mathbb{R})} \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\bar{W}_{\alpha,r} \in A} - \mathbf{1}_{\bar{W}_{\alpha,r} \in A - h_{\alpha,r}\left(\Xi_{N_{\alpha,r}^{(2)}} \right)} \Big| \mathscr{F}_{1,\alpha,r} \right) \Big| \mathscr{F}_{0,\alpha,r} \right)$$

$$= \sup_{A \in \mathscr{B}(\mathbb{R})} \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\bar{W}_{\alpha,r}^0 \in A} - \mathbf{1}_{\bar{W}_{\alpha,r}^0 \in A - h_{\alpha,r}\left(\Xi_{N_{\alpha,r}^{(2)}} \right)} \Big| \mathscr{F}_{1,\alpha,r} \right) \Big| \mathscr{F}_{0,\alpha,r} \right)$$

$$\leq \mathbb{E}\left(\sup_{A \in \mathscr{B}(\mathbb{R})} \left[\mathbb{E}\left(\mathbf{1}_{\bar{W}^0_{\alpha,r} \in A} - \mathbf{1}_{\bar{W}^0_{\alpha,r} \in A - h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right)} \Bigg| \mathscr{F}_{1,\alpha,r} \right) \right] \Bigg| \mathscr{F}_{0,\alpha,r} \right)$$

$$= \mathbb{E}\left(\sup_{A \in \mathscr{B}(\mathbb{R})} \left[\mathbb{E}\left(\mathbf{1}_{\bar{W}^0_{\alpha,r} \in A} - \mathbf{1}_{\bar{W}^0_{\alpha,r} \in A - h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right)} \Bigg| \mathscr{F}'_{2,\alpha,r} \right) \right] \Bigg| \mathscr{F}_{0,\alpha,r} \right).$$

The rest is a line-by-line repetition of the proof of Lemma 4.4 with γ replaced by $h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right)$, and the expectations replaced by the conditional expectations given $\mathscr{F}_{0,\alpha,r}$, leading to

$$d_{TV}\left(\bar{W}'_{\alpha,r}, \bar{W}'_{\alpha,r} + h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right) \Big| \mathscr{F}_{0,\alpha,r} \right)$$

$$\leq \mathbb{E}\left(\sup_{A \in \mathscr{B}(\mathbb{R})} \left[\mathbb{E}\left(\mathbf{1}_{\bar{W}^0_{\alpha,r} \in A} - \mathbf{1}_{\bar{W}^0_{\alpha,r} \in A - h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right)} \Bigg| \mathscr{F}'_{2,\alpha,r} \right) \right] \Bigg| \mathscr{F}_{0,\alpha,r} \right)$$

$$\leq \mathbb{E}\left(\mathbb{E}\left[\sup_{A \in \mathscr{B}(\mathbb{R})} \left(\mathbf{1}_{\Sigma_{3,I} \in A} - \mathbf{1}_{\Sigma_{3,I} \in A - h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right)} \Bigg| I, \sigma\left(\Xi_{N^{(2)}_{\alpha,r}}\right) \right) \right] \Bigg| \mathscr{F}_{0,\alpha,r} \right) \quad (5.22)$$

$$\leq \sum_{\mathbb{E}I < 2j \leq m_{\alpha,r}} \mathbb{P}(I = j)\mathbb{E}\left(\mathbb{E}\left[\sup_{A \in \mathscr{B}(\mathbb{R})} \left(\mathbf{1}_{\Sigma_{3,j} \in A} - \mathbf{1}_{\Sigma_{3,j} \in A - h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right)} \Bigg| \sigma\left(\Xi_{N^{(2)}_{\alpha,r}}\right) \right) \right] \Bigg| \mathscr{F}_{0,\alpha,r} \right)$$

$$+ \mathbb{P}(I \leq (\mathbb{E}I)/2)$$

$$\leq O\left(\alpha^{-1}r^d \right) + O\left(\alpha^{-\frac{1}{2}}r^{\frac{d}{2}} \right) \mathbb{E}\left(\left| h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right) \right| \Big| \mathscr{F}_{0,\alpha,r} \right) \quad (5.23)$$

$$= \mathbb{E}\left(\left| h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right) \right| \vee 1 \Big| \mathscr{F}_{0,\alpha,r} \right) O\left(\alpha^{-\frac{1}{2}}r^{\frac{d}{2}} \right),$$

where (5.22) follows from the fact that

$$\sup_{A \in \mathscr{B}(\mathbb{R})} \left[\mathbf{1}_{\Sigma_{3,I} \in A} - \mathbf{1}_{\Sigma_{3,I} \in A - h_{\alpha,r}\left(\Xi_{N^{(2)}_{\alpha,r}}\right)} \right]$$

is a function of I and $\Xi_{N^{(2)}_{\alpha,r}}$, the first term of (5.23) is from Chebyshev's inequality, and the second term is due to Lemma 4.1. This completes the proof for the statement of $\bar{W}_{\alpha,r}$.

The claim (4.8) can be proved by replacing $\bar{W}_{\alpha,r}$ with $W_{\alpha,r}$, $\bar{W}'_{\alpha,r}$ with $W'_{\alpha,r}$, and $\bar{W}^0_{\alpha,r}$ with $W^0_{\alpha,r}$. $\square$

The moments of $W_{\alpha,r}$ and W_α (resp. $\bar{W}_{\alpha,r}$ and $\bar{W}_\alpha$) are established in the following lemmas using the ideas in [51, Section 4]. Let $\|X\|_p := \mathbb{E}(|X|^p)^{\frac{1}{p}}$ be the L_p norm of X, provided it is finite.

Lemma 5.7.

(a) (*Unrestricted case.*) *If the score function η satisfies the k'th moment condition 2.3 with $k' > k \geq 1$, then* $\max_{0 < l \leq k}\left\{ \|W_\alpha\|_l, \|W_{\alpha,r}\|_l \right\} \leq C\alpha.$

(b) (*Restricted case.*) *If the score function η satisfies the k'th moment condition (2.4) with $k' > k \geq 1$, then* $\max_{0 < l \leq k}\left\{ \|\bar{W}_\alpha\|_l, \|\bar{W}_{\alpha,r}\|_l \right\} \leq C\alpha.$

Proof. The proof is adapted from that of [51, Lemma 4.1]. We use the same notation as in the proof of Lemma 4.4, and start with the restricted case. To this end, it suffices to show $\|\bar{W}_\alpha\|_k \vee \|W_{\alpha,r}\|_k \leq C\alpha$, and the claim follows from Hölder's inequality. Let $N_\alpha := \left|\bar{\Xi}_{\Gamma_\alpha}\right|$; then N_α follows the Poisson distribution with parameter $\alpha\lambda$. Using Minkowski's inequality, we obtain

$$
\begin{aligned}
\|\bar{W}_\alpha\|_k &\leq \left\|\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\right\|_k \\
&= \left\|\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)| \left(\mathbf{1}_{N_\alpha\leq\alpha\lambda} + \sum_{j=0}^{\infty} \mathbf{1}_{\alpha\lambda 2^j < N_\alpha \leq \alpha\lambda 2^{j+1}}\right)\right\|_k \\
&\leq \left\|\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\, \mathbf{1}_{N_\alpha\leq\alpha\lambda}\right\|_k + \sum_{j=0}^{\infty} \left\|\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\, \mathbf{1}_{\alpha\lambda 2^j < N_\alpha \leq \alpha\lambda 2^{j+1}}\right\|_k. \quad (5.24)
\end{aligned}
$$

Let $s = \frac{k'}{k} > 1$ and let t be its conjugate, i.e., $\frac{1}{s} + \frac{1}{t} = 1$. Using Hölder's inequality and Minkowski's inequality, for any $j \in \mathbb{N}$ we have

$$
\begin{aligned}
&\left\|\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\, \mathbf{1}_{\alpha\lambda 2^j < N_\alpha \leq \alpha\lambda 2^{j+1}}\right\|_k \\
&= \left\|\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\, \mathbf{1}_{N_\alpha\leq\alpha\lambda 2^{j+1}} \mathbf{1}_{\alpha\lambda 2^j < N_\alpha}\right\|_k \\
&\leq \left\|\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\, \mathbf{1}_{N_\alpha\leq\alpha\lambda 2^{j+1}}\right\|_{k'} \left(\mathbb{P}\big(N_\alpha > \alpha\lambda 2^j\big)\right)^{\frac{1}{kt}} \\
&= \left\|\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\, \mathbf{1}_{N_\alpha\leq\alpha\lambda 2^{j+1}}\right\|_{k'} \mathbb{P}\big(N_\alpha - \alpha\lambda > \alpha\lambda\big(2^j - 1\big)\big)^{\frac{1}{kt}}. \quad (5.25)
\end{aligned}
$$

For the term $\|\cdot\|_{k'}$ in (5.25), we have

$$
\begin{aligned}
&\left\|\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\, \mathbf{1}_{N_\alpha\leq n}\right\|_{k'} \\
&= \left\{\mathbb{E}\left[\left(\sum_{j=1}^{n}\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\, \mathbf{1}_{N_\alpha=j}\right)^{k'}\right]\right\}^{\frac{1}{k'}} \\
&= \left\{\sum_{j=1}^{n}\mathbb{E}\left[\left(\sum_{x\in\bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|\, \mathbf{1}_{N_\alpha=j}\right)^{k'}\right]\right\}^{\frac{1}{k'}}, \quad (5.26)
\end{aligned}
$$

where the first equality holds because

$$\sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)|^{k'} \mathbf{1}_{N_\alpha=0} = 0,$$

and the last equality follows from the fact that $\{N_\alpha = j\}$, $1 \le j \le n$, are disjoint events. On $\{N_\alpha = j\}$ for some fixed $j \in \mathbb{N}$, if we write j points in $\Xi \cap \Gamma_\alpha$ as $\{(x_1, m_1), \dots, (x_j, m_j)\}$, and let $\{(U_{\alpha,i}, M_i)\}_{i \in \mathbb{N}}$ be a sequence of i.i.d. random elements that have distribution $U(\Gamma_\alpha) \times \mathscr{L}_T$ and are independent of Ξ, where $U(\Gamma_\alpha)$ is the uniform distribution on Γ_α, then

$$\mathbb{E}\left[\left(\sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)| \, \mathbf{1}_{N_\alpha=j} \right)^{k'} \right]$$

$$\le \left\{ \sum_{i=1}^{j} \mathbb{E}\left[|g_\alpha((x_i, m_i), \Xi)|^{k'} \, \mathbf{1}_{N_\alpha=j} \right]^{\frac{1}{k'}} \right\}^{k'}$$

$$= j^{k'} \mathbb{E}\left[\left| g_\alpha\left((U_{\alpha,i}, M_i), \left(\sum_{i=1}^{j} \delta_{(U_{\alpha,i}, M_i)} \right) \right) \right|^{k'} \mathbf{1}_{N_\alpha=j} \right], \tag{5.27}$$

where the inequality follows from Minkowski's inequality, and the equality follows from the fact that when $|\bar{\Xi} \cap \Gamma_\alpha|$ is fixed, points in $\bar{\Xi} \cap \Gamma_\alpha$ are independent and follow the uniform distribution on Γ_α. Combining (5.26) and (5.27), we have

$$\left\| \sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)| \, \mathbf{1}_{N_\alpha \le n} \right\|_{k'}$$

$$\le \left\{ \sum_{j=1}^{n} j^{k'} \mathbb{E}\left[\left| g_\alpha\left((U_{\alpha,1}, M_1), \left(\sum_{i=1}^{j} \delta_{(U_{\alpha,i}, M_i)} \right) \right) \right|^{k'} \mathbf{1}_{N_\alpha=j} \right] \right\}^{\frac{1}{k'}}$$

$$= \left\{ \sum_{j=1}^{n} \lambda\alpha j^{k'-1} \mathbb{E}\left[\left| g_\alpha\left((U_{\alpha,1}, M_1), \left(\sum_{i=1}^{j} \delta_{(U_{\alpha,i}, M_i)} \right) \right) \right|^{k'} \mathbf{1}_{N_\alpha=j-1} \right] \right\}^{\frac{1}{k'}}$$

$$\le (\lambda\alpha)^{\frac{1}{k'}} n^{\frac{k'-1}{k'}} \left\{ \mathbb{E}\left[\sum_{j=0}^{n-1} \left| g_\alpha\left((U_{\alpha,1}, M_1), \left(\sum_{i=1}^{j+1} \delta_{(U_{\alpha,i}, M_i)} \right) \right) \right|^{k'} \mathbf{1}_{N_\alpha=j} \right] \right\}^{\frac{1}{k'}}$$

$$\le (\lambda\alpha)^{\frac{1}{k'}} n^{\frac{k'-1}{k'}} \left\{ \mathbb{E}\left[\sum_{j=0}^{\infty} \left| g_\alpha\left((U_{\alpha,1}, M_1), \left(\sum_{i=1}^{j+1} \delta_{(U_{\alpha,i}, M_i)} \right) \right) \right|^{k'} \mathbf{1}_{N_\alpha=j} \right] \right\}^{\frac{1}{k'}}$$

$$= (\lambda\alpha)^{\frac{1}{k'}} n^{\frac{k'-1}{k'}} \left\{ \int_{\Gamma_\alpha} \mathbb{E}\left[|g_\alpha((x, M), \Xi_{\Gamma_\alpha} + \delta_{(x,M)})|^{k'} \frac{1}{\alpha} dx \right] \right\}^{\frac{1}{k'}}$$

$$\le (\lambda\alpha)^{\frac{1}{k'}} n^{\frac{k'-1}{k'}} C_0^{\frac{1}{k'}}, \tag{5.28}$$

where the first equality follows from the fact that N_α is independent of $\{(U_{\alpha,i}, M_i)\}_{i \in \mathbb{N}}$, $\mathbb{P}(N_\alpha = j) = \frac{\lambda\alpha}{j}\mathbb{P}(N_\alpha = j - 1)$, and the last equality follows from the construction of the marked Poisson point process.

Combining (5.25) and (5.28), we have

$$\left\| \sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)| \mathbf{1}_{\alpha\lambda 2^j < N_\alpha \le \alpha\lambda 2^{j+1}} \right\|_k \le \alpha\lambda 2^{\frac{(k'-1)(j+1)}{k'}} C_0^{\frac{1}{k'}} \mathbb{P}(N_\alpha - \alpha\lambda > \alpha\lambda(2^j - 1))^{\frac{1}{kt}}. \quad (5.29)$$

Using (5.28) and Hölder's inequality, we have

$$\left\| \sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)| \mathbf{1}_{N_\alpha \le \alpha\lambda} \right\|_k \le \left\| \sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)| \mathbf{1}_{N_\alpha \le \alpha\lambda} \right\|_{k'} \le \alpha\lambda C_0^{\frac{1}{k'}}. \quad (5.30)$$

Combining (5.29) and (5.30), together with the fact that $\mathbb{P}(N_\alpha - \alpha\lambda > \alpha\lambda k)$ decreases exponentially fast with respect to k, we have from (5.24) that

$$\|\bar{W}_\alpha\|_k \le \left\| \sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)| \right\|_k \le C\alpha.$$

The proof of (b) is completed by observing that, for arbitrary $r \in \mathbb{R}_+$,

$$\|\bar{W}_{\alpha,r}\|_k = \left\| \sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} g_\alpha(x, \Xi) \mathbf{1}_{R(x) \le r} \right\|_k \le \left\| \sum_{x \in \bar{\Xi}_{\Gamma_\alpha}} |g_\alpha(x, \Xi)| \right\|_k \le C\alpha.$$

The claim in (a) can be established by replacing $\bar{W}_\alpha$ with W_α, $\bar{W}_{\alpha,r}$ with $W_{\alpha,r}$, $g_\alpha(x, \mathscr{X})$ with $g(\mathscr{X}^x)$, and $\sum_{i=1}^j \delta_{(U_{\alpha,i}, M_i)}$ with $\sum_{i=1}^j \delta_{(U_{\alpha,i}, M_i)} + \Xi_{\Gamma_\alpha^c}$. $\qquad\square$

Remark 5.1. The proof of Lemma 5.7 does not depend on the shape of Γ_α, so the claims still hold if we replace Γ_α with a set $A \in \mathscr{B}(\mathbb{R}^d)$ and α in the upper bound with the volume of A.

With these preparations, we are ready to bound the differences $|\text{Var}(W_\alpha) - \text{Var}(W_{\alpha,r})|$ and $|\text{Var}(\bar{W}_\alpha) - \text{Var}(\bar{W}_{\alpha,r})|$.

Lemma 5.8.

(a) *(Unrestricted case.) Assume that the score function η satisfies the k'th moment condition (2.3) for some $k' > 2$. If η is exponentially stabilising as in Definition 2.1, then there exist positive constants α_0 and C such that*

$$\left| \text{Var}(W_\alpha) - \text{Var}(W_{\alpha,r}) \right| \le \frac{1}{\alpha} \quad (5.31)$$

for all $\alpha \ge \alpha_0$ and $r \ge C \ln(\alpha)$. If η is polynomially stabilising as in Definition 2.1 with parameter β, then for any $k \in (2, k')$, there exists a positive constant C such that

$$\left| \text{Var}(W_\alpha) - \text{Var}(W_{\alpha,r}) \right| \le C\left(\alpha^{\frac{3k-2}{k}} r^{-\beta\frac{k-2}{k}} \right) \vee \left(\alpha^{\frac{3k-1}{k}} r^{-\beta\frac{k-1}{k}} \right) \quad (5.32)$$

for all $r \le \alpha^{\frac{1}{d}}$.

(b) *(Restricted case.) Assume that the score function η satisfies the k'th moment condition* (2.4) *for some $k' > 2$. If η is exponentially stabilising as in Definition* 2.2, *then there exist positive constants α_0 and C such that*

$$\left| \mathrm{Var}\big(\bar{W}_\alpha\big) - \mathrm{Var}\big(\bar{W}_{\alpha,r}\big) \right| \leq \frac{1}{\alpha} \tag{5.33}$$

for all $\alpha \geq \alpha_0$ and $r \geq C \ln(\alpha)$. If η is polynomially stabilising as in Definition 2.2, *with parameter β, then for any $k \in (2, k')$, there exists a positive constant C such that*

$$\left| \mathrm{Var}\big(\bar{W}_\alpha\big) - \mathrm{Var}\big(\bar{W}_{\alpha,r}\big) \right| \leq C\left(\alpha^{\frac{3k-2}{k}} r^{-\beta \frac{k-2}{k}} \right) \vee \left(\alpha^{\frac{3k-1}{k}} r^{-\beta \frac{k-1}{k}} \right) \tag{5.34}$$

for all $r \leq \alpha^{\frac{1}{d}}$.

Proof. We start with (5.33). From Lemma 5.7(b), for fixed $k \in (2, k')$, we have

$$\max_{0 < l \leq k} \left\{ \|\bar{W}_\alpha\|_l, \|\bar{W}_{\alpha,r}\|_l \right\} \leq C_0 \alpha \tag{5.35}$$

for some positive constant C_0. Without loss of generality, we assume $\alpha_0 > 1$. Since

$$\left| \mathrm{Var}\big(\bar{W}_\alpha\big) - \mathrm{Var}\big(\bar{W}_{\alpha,r}\big) \right| \leq \left| \mathbb{E}\big(\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2 \big) \right| + \left| \big(\mathbb{E}\bar{W}_\alpha\big)^2 - \big(\mathbb{E}\bar{W}_{\alpha,r}\big)^2 \right|, \tag{5.36}$$

assuming that the score function is exponentially stabilising as in Definition 2.2, we show that each of the terms in the right-hand side of (5.36) is bounded by $\frac{1}{2\alpha}$ for α and r sufficiently large. Clearly, the definition of $\bar{W}_{\alpha,r}$ implies that $\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2 = 0$ if $\bar{R}(x, \alpha) \leq r$ for all $x \in \bar{\Xi}_{\Gamma_\alpha}$; hence it remains to tackle $E_{r,\alpha} := \{\bar{R}(x, \alpha) \leq r$ for all $x \in \bar{\Xi}_{\Gamma_\alpha}\}^c$. As shown in the proof of Lemma 5.6, $\mathbb{P}\big(E_{r,\alpha}\big) \leq \alpha C_1 e^{-C_2 r}$, which, together with Hölder's inequality, ensures that

$$\begin{aligned}
\left| \mathbb{E}\big(\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2 \big) \right| &= \left| \mathbb{E}\Big[\big(\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2 \big) \mathbf{1}_{E_{r,\alpha}} \Big] \right| \\
&\leq \|\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2\|_{\frac{k}{2}} \|\mathbf{1}_{E_{r,\alpha}}\|_{\frac{k}{k-2}} \\
&\leq \left(\|\bar{W}_\alpha^2\|_{\frac{k}{2}} + \|\bar{W}_{\alpha,r}^2\|_{\frac{k}{2}} \right) \mathbb{P}(E_{r,\alpha})^{\frac{k-2}{k}} \\
&= \left(\|\bar{W}_\alpha\|_k^2 + \|\bar{W}_{\alpha,r}\|_k^2 \right) \mathbb{P}(E_{r,\alpha})^{\frac{k-2}{k}} \leq 2(C_0\alpha)^2 \left(\alpha C_1 e^{-C_2 r} \right)^{\frac{k-2}{k}}. \tag{5.37}
\end{aligned}$$

For the remaining term of (5.36), we have

$$\left| \big(\mathbb{E}\bar{W}_\alpha\big)^2 - \big(\mathbb{E}\bar{W}_{\alpha,r}\big)^2 \right| = \left| \mathbb{E}\bar{W}_\alpha - \mathbb{E}\bar{W}_{\alpha,r} \right| \left| \mathbb{E}\bar{W}_\alpha + \mathbb{E}\bar{W}_{\alpha,r} \right|.$$

The bound (5.35) implies $\left| \mathbb{E}\bar{W}_\alpha + \mathbb{E}\bar{W}_{\alpha,r} \right| \leq 2C_0\alpha$. However, using Hölder's inequality, Minkowski's inequality, and (5.35) again, we have

$$\begin{aligned}
\left| \mathbb{E}\bar{W}_\alpha - \mathbb{E}\bar{W}_{\alpha,r} \right| &= \left| \mathbb{E}\big[\big(\bar{W}_\alpha - \bar{W}_{\alpha,r} \big) \mathbf{1}_{E_{r,\alpha}} \big] \right| \\
&\leq \|\bar{W}_\alpha - \bar{W}_{\alpha,r}\|_k \|\mathbf{1}_{E_{r,\alpha}}\|_{\frac{k}{k-1}} \\
&\leq \left(\|\bar{W}_\alpha\|_k + \|\bar{W}_{\alpha,r}\|_k \right) \mathbb{P}(E_{r,\alpha})^{\frac{k-1}{k}} \\
&\leq 2C_0\alpha \left(\alpha C_1 e^{-C_2 r} \right)^{\frac{k-1}{k}}, \tag{5.38}
\end{aligned}$$

giving

$$\left| \left(\mathbb{E}\bar{W}_\alpha\right)^2 - \left(\mathbb{E}\bar{W}_{\alpha,r}\right)^2 \right| \leq 4\left(C_0\alpha\right)^2 \left(\alpha C_1 e^{-C_2 r}\right)^{\frac{k-1}{k}}. \tag{5.39}$$

We set $r = C\ln(\alpha)$ in the upper bounds of (5.37) and (5.39), and find C such that both bounds are bounded by $1/(2\alpha)$, completing the proof of (5.33).

The same proof can be adapted for (5.34). With (5.36) in mind, recalling the fact established in the proof of Lemma 5.6 that $\mathbb{P}(E_{r,\alpha}) \leq C_1\alpha r^{-\beta}$, we replace the last inequalities of (5.37), (5.38), and (5.39) with the corresponding bound of $\mathbb{P}(E_{r,\alpha})$ to obtain

$$\left| \mathbb{E}\left(\bar{W}_\alpha^2 - \bar{W}_{\alpha,r}^2\right) \right| \leq 2(C_0\alpha)^2 \left(C_1\alpha r^{-\beta}\right)^{\frac{k-2}{k}}, \tag{5.40}$$

$$\left| \mathbb{E}\bar{W}_\alpha - \mathbb{E}\bar{W}_{\alpha,r} \right| \leq 2C_0\alpha \left(C_1\alpha r^{-\beta}\right)^{\frac{k-1}{k}}, \tag{5.41}$$

$$\left| \left(\mathbb{E}\bar{W}_\alpha\right)^2 - \left(\mathbb{E}\bar{W}_{\alpha,r}\right)^2 \right| \leq 4(C_0\alpha)^2 \left(C_1\alpha r^{-\beta}\right)^{\frac{k-1}{k}}. \tag{5.42}$$

The claim (5.34) follows from combining (5.40) and (5.42), extracting α and r, and then taking C as the sum of the remaining constants.

A line-by-line repetition of the above proof with $\bar{W}_\alpha$ and $\bar{W}_{\alpha,r}$ replaced by W_α and $W_{\alpha,r}$ gives (5.31) and (5.32) respectively. $\qquad\square$

Next, we apply Lemma 5.2 and Lemma 5.3 to establish lower bounds for $\mathrm{Var}\left(W_{\alpha,r}\right)$ and $\mathrm{Var}\left(\bar{W}_{\alpha,r}\right)$.

Lemma 5.9.

(a) *(Unrestricted case.) If the score function η satisfies the polynomial stabilisation in Definition 2.1 with order $\beta > d+1$ and the non-singularity in Assumption 2.4, then $\mathrm{Var}\left(W_{\alpha,r}\right) \geq C\alpha r^{-d}$ for $R_0 \leq r \leq \alpha^{1/d}/6$, where $C, R_0 > 0$ are independent of α.*

(b) *(Restricted case.) If the score function η satisfies the polynomial stabilisation in Definition 2.2 with order $\beta > d+1$ and the non-singularity in Assumption 2.4, then $\mathrm{Var}\left(\bar{W}_{\alpha,r}\right) \geq C\alpha r^{-d}$ for $R_0 \leq r \leq \alpha^{1/d}/6$, where $C, R_0 > 0$ are independent of α.*

Proof. For (b), recalling the notation in the paragraph after (5.19), we obtain from the total variance formula that

$$\begin{aligned}
\mathrm{Var}\left(\bar{W}_{\alpha,r}\right) &= \mathbb{E}\left(\mathrm{Var}\left(\bar{W}_{\alpha,r}\big|\mathscr{F}_{2,\alpha,r}\right)\right) + \mathrm{Var}\left(\mathbb{E}\left(\bar{W}_{\alpha,r}\big|\mathscr{F}_{2,\alpha,r}\right)\right) \\
&\geq \mathbb{E}\left(\mathrm{Var}\left(\bar{W}_{\alpha,r}\big|\mathscr{F}_{2,\alpha,r}\right)\right) \\
&= \sum_{i=1}^{m_{\alpha,r}} \mathbb{E}\left(\mathrm{Var}\left(\sum_{x\in\overline{\Xi}\cap B_i} \bar{g}(\Xi^{\Gamma_\alpha,x})\mathbf{1}_{\bar{R}(x,\alpha)\leq r}\bigg| \Xi_{D_i}\right)\right) \\
&= m_{\alpha,r}\mathbb{E}\left(\mathrm{Var}\left(Y_r\big|\Xi_{N_0^c}\right)\right), \tag{5.43}
\end{aligned}$$

where $m_{\alpha,r}$ is the number of disjoint cubes with length $4r + 2r_1$ embedded into Γ_α.

Using (5.17), there exists an $R_0 \geq r_1 > 0$ such that for all $r > R_0$,

$$\mathscr{L}\left(Y_r\mathbf{1}_{\left(E'_{R_0}\right)^c}\bigg|\Xi_{N_0^c}\right) = \mathscr{L}\left(\tilde{Y}\big|\Xi_{N_0^c}\right) \geq \tilde{\xi} \text{ a.s.,}$$

where $\tilde{\xi}$ is an absolutely continuous $\sigma\big(\Xi_{N_0^c}\big)$-measurable random measure satisfying $\mathbb{P}\big(\tilde{\xi}(\mathbb{R}) > \frac{p}{2}\big) > 2p$. Hence, for $r > R_0$, we apply Lemma 5.3 with $X := Y_r$, $A := \big(E'_{R_0}\big)^c$, and use the fact that $Y_r \mathbf{1}_{\big(E'_{R_0}\big)^c} = \tilde{Y}$ for all $r \geq R_0$ to obtain

$$\mathbb{E}\mathrm{Var}\Big(Y_r \,\Big|\, \Xi_{N_0^c}\Big) \geq \mathbb{E}\mathrm{Var}\left(\tilde{Y} + \frac{\mathbb{E}\big(\tilde{Y}\big|\Xi_{N_0^c}\big)}{\mathbb{P}\big(\big(E'_{R_0}\big)^c\big|\Xi_{N_0^c}\big)}\mathbf{1}_{E'_{R_0}} \,\Bigg|\, \Xi_{N_0^c}\right) =: b > 0. \qquad (5.44)$$

The proof of the claim (b) is completed by combining (5.43) and (5.44), and observing that $R_0 \leq r \leq \alpha^{1/d}/6$ ensures that $m_{\alpha,r} \geq 12^{-d}\alpha r^{-d}$.

The claim (a) can be proved by replacing $\bar{W}_{\alpha,r}$ with $W_{\alpha,r}$ and $\bar{g}$ with g throughout the above argument. $\qquad\square$

Finally, we make use of [51, Lemma 4.6], Lemma 5.8, and Lemma 5.9 to establish Lemma 4.5.

Proof of Lemma 4.5. To begin with, we combine (5.31), (5.33), and Lemma 5.9(a) to find an $r := C_1 \ln(\alpha)$ such that

$$\big|\mathrm{Var}(W_\alpha) - \mathrm{Var}\big(W_{\alpha,r}\big)\big| \leq \frac{1}{\alpha}, \qquad (5.45)$$

$$\big|\mathrm{Var}(\bar{W}_\alpha) - \mathrm{Var}\big(\bar{W}_{\alpha,r}\big)\big| \leq \frac{1}{\alpha}, \qquad (5.46)$$

$$\mathrm{Var}\big(W_{\alpha,r}\big) \geq C_2\alpha \ln(\alpha)^{-d}, \qquad (5.47)$$

for positive constants C_1, C_2. The inequalities (5.45) and (5.47) imply $\mathrm{Var}(W_\alpha) \geq \mathrm{O}\big(\alpha \ln(\alpha)^{-d}\big)$; hence the claim (a) follows from the dichotomy established in [51, Lemma 4.6] saying either $\mathrm{Var}(W_\alpha) = \Omega(\alpha)$ or $\mathrm{Var}(W_\alpha) = \mathrm{O}\big(\alpha^{\frac{d-1}{d}}\big)$.

For (b), it suffices to show that $\mathrm{Var}\big(\bar{W}_\alpha\big) - \mathrm{Var}(W_\alpha) = \mathrm{o}(\alpha)$ if we take $\bar{\eta}$ as the score function in the unrestricted case. To this end, noting that (5.45) and (5.46), it remains to show that $\mathrm{Var}\big(W_{\alpha,r}\big) - \mathrm{Var}\big(\bar{W}_{\alpha,r}\big) = \mathrm{o}(\alpha)$. However, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\big|\mathrm{Var}\big(W_{\alpha,r}\big) - \mathrm{Var}\big(\bar{W}_{\alpha,r}\big)\big| \\
&= \big|\mathrm{Var}\big(W_{\alpha,r} - \bar{W}_{\alpha,r}\big) - 2\mathrm{cov}\big(W_{\alpha,r} - \bar{W}_{\alpha,r}, W_{\alpha,r}\big)\big| \\
&\leq \mathrm{Var}\big(W_{\alpha,r} - \bar{W}_{\alpha,r}\big) + 2\sqrt{\mathrm{Var}\big(W_{\alpha,r} - \bar{W}_{\alpha,r}\big)\mathrm{Var}\big(W_{\alpha,r}\big)},
\end{aligned}$$

and it follows from $\mathrm{Var}(W_\alpha) = \Omega(\alpha)$, $\mathrm{Var}\big(W_{\alpha,r}\big) = \mathrm{O}(\alpha)$, and (5.45) that $\mathrm{Var}\big(W_{\alpha,r}\big) = \Omega(\alpha)$; hence the proof is reduced to showing $\mathrm{Var}\big(W_{\alpha,r} - \bar{W}_{\alpha,r}\big) = \mathrm{o}(\alpha)$.

Since $g_\alpha(x, \Xi)\mathbf{1}_{\bar{R}(x,\alpha) < r} = \bar{g}\left(\Xi^x\right)\mathbf{1}_{R(x) < r}$ if $d(x, \partial\Gamma_\alpha) > r$, we have $W_{\alpha,r} - \bar{W}_{\alpha,r} = W_{1,\alpha,r} - W_{2,\alpha,r}$ where

$$W_{1,\alpha,r} := \sum_{x\in\bar{\Xi}_{B(\partial\Gamma_\alpha,r)\cap\Gamma_\alpha}} \bar{g}\left(\Xi^x\right)\mathbf{1}_{R(x)<r}, \; W_{2,\alpha,r} := \sum_{x\in\bar{\Xi}_{B(\partial\Gamma_\alpha,r)\cap\Gamma_\alpha}} g_\alpha\left(x, \Xi\right)\mathbf{1}_{\bar{R}(x,\alpha)<r}.$$

As the summands of $W_{1,\alpha,r}$ and $W_{2,\alpha,r}$ are in the moat within distance r from the boundary of Γ_α, both $\text{Var}\big(W_{1,\alpha,r'_\alpha}\big)$ and $\text{Var}\big(W_{2,\alpha,r'_\alpha}\big)$ are of order $\text{o}(\alpha)$, as detailed below. In fact, it follows from (5.4) that

$$\mathbb{E}\big(g_\alpha(x,\,\Xi)\,\mathbf{1}_{\bar{R}(x,\alpha)<r}\bar{\Xi}(dx)\big)$$
$$= \mathbb{E}\left(g_\alpha\left(x,\,\Xi+\delta_{(x,M_1)}\right)\mathbf{1}_{\bar{R}(x,M_1,\alpha,\Xi+\delta_{(x,M_1)})<r}\right)\lambda dx$$
$$=: P_{x,\alpha,r}dx; \qquad (5.48)$$

if we set

$$\bar{\Xi}^*_\alpha(dx) := g_\alpha(x,\,\Xi)\mathbf{1}_{\bar{R}(x,\alpha)<r}\bar{\Xi}(dx) - P_{x,\alpha,r}dx, \qquad (5.49)$$

then $\mathbb{E}\big(\bar{\Xi}^*_\alpha(dx)\bar{\Xi}^*_\alpha(dy)\big) = \mathbb{E}\big(\bar{\Xi}^*_\alpha(dx)\big)\mathbb{E}\big(\bar{\Xi}^*_\alpha(dy)\big) = 0$ if $d(x,\,y) > 2r$. Therefore,

$$\text{Var}\big(W_{2,\alpha,r}\big)$$
$$= \int_{x,y\in B(\partial\Gamma_\alpha,r)\cap\Gamma_\alpha}\mathbb{E}\big(\bar{\Xi}^*_\alpha(dx)\bar{\Xi}^*_\alpha(dy)\big)$$
$$= \int_{x,y\in B(\partial\Gamma_\alpha,r)\cap\Gamma_\alpha,d(x,y)\leq 2r}\mathbb{E}\big(\bar{\Xi}^*_\alpha(dx)\bar{\Xi}^*_\alpha(dy)\big)$$
$$= \int_{x,y\in B(\partial\Gamma_\alpha,r)\cap\Gamma_\alpha,d(x,y)\leq 2r}\Big\{\mathbb{E}\Big[g_\alpha(x,\,\Xi)\mathbf{1}_{\bar{R}(x,\alpha)<r}g_\alpha(y,\,\Xi)\mathbf{1}_{\bar{R}(y,\alpha)<r}\bar{\Xi}(dx)\bar{\Xi}(dy)\Big]$$
$$-P_{x,\alpha,r}P_{y,\alpha,r}dxdy\Big\}. \qquad (5.50)$$

Recalling the second-order Palm distribution in (5.5), we can use the moment condition (2.4) together with Hölder's inequality to obtain

$$\mathbb{E}\Big[|g_\alpha(x,\,\Xi)|\,\mathbf{1}_{\bar{R}(x,\alpha)<r}\,|g_\alpha(y,\,\Xi)|\,\mathbf{1}_{\bar{R}(y,\alpha)<r}\bar{\Xi}(dx)\bar{\Xi}(dy)\Big] \leq C^2(\lambda^2dxdy + \lambda dx), \qquad (5.51)$$

$$\big|P_{x,\alpha,r}\big|\,\big|P_{y,\alpha,r}\big|\,dxdy \leq C^2\lambda^2dxdy, \qquad (5.52)$$

where $C \geq 1$. Combining these estimates with (5.50) gives

$$\text{Var}\big(W_{2,\alpha,r}\big) = \text{O}\Big(\alpha^{\frac{d-1}{d}}r^{d+1}\Big) = \text{o}(\alpha).$$

The proof of $\text{Var}\big(W_{1,\alpha,r}\big) = \text{o}(\alpha)$ is similar, except we replace (2.4) with (2.3). Consequently,

$$\text{Var}\big(W_{\alpha,r} - \bar{W}_{\alpha,r}\big) = \text{Var}\big(W_{1,\alpha,r} - W_{2,\alpha,r}\big) \leq 2\big(\text{Var}\big(W_{1,\alpha,r}\big) + \text{Var}\big(W_{2,\alpha,r}\big)\big) = \text{o}(\alpha),$$

and the statement follows. $\qquad\qquad\square$

As the lower bounds in Lemma 4.6 are very conservative, their proofs are less demanding, as demonstrated below.

Proof of Lemma 4.5. We start with (b). The bound (5.34) ensures that

$$\big|\text{Var}\big(\bar{W}_\alpha\big) - \text{Var}\big(\bar{W}_{\alpha,r}\big)\big| \leq C_1\left(\alpha^{\frac{3k_0-2}{k_0}}r^{-\beta\frac{k_0-2}{k_0}}\right)\vee\left(\alpha^{\frac{3k_0-1}{k_0}}r^{-\beta\frac{k_0-1}{k_0}}\right) \qquad (5.53)$$

for all $r \leq \alpha^{\frac{1}{d}}$ and $k' > k_0 > k \geq 3$. On the other hand, Lemma 5.9(b) says

$$\text{Var}\big(\bar{W}_{\alpha,r}\big) \geq C_1\alpha r^{-d}$$

for $0 < R_0 \le r \le \alpha^{1/d}/6$. Let $r_\alpha := \alpha^{\frac{2k-2}{k\beta - 2\beta - dk}}$. The assumption $\beta > (3k-2)d/(k-2)$ ensures that $r_\alpha < \alpha^{1/d}/6$ for large α, and $k_0 > k$ guarantees $\left|\mathrm{Var}(\bar{W}_\alpha) - \mathrm{Var}(\bar{W}_{\alpha,r_\alpha})\right| \ll \mathrm{Var}(\bar{W}_{\alpha,r_\alpha})$ for large α. Hence

$$\mathrm{Var}(\bar{W}_\alpha) \ge C_1 \alpha r_\alpha^{-d} = \mathrm{O}\left(\alpha^{\frac{k\beta - 2\beta - 3dk + 2d}{k\beta - 2\beta - dk}} \right),$$

completing the proof.

For the proof of (a), we can proceed as in the proof of (b), replacing $\bar{W}_\alpha$ with W_α and $\bar{W}_{\alpha,r}$ with $W_{\alpha,r}$. $\qquad\square$

The proof of Lemma 4.6 enables us to get slightly better bounds for $\mathrm{Var}(W_{\alpha,r})$ and $\mathrm{Var}(\bar{W}_{\alpha,r})$.

Lemma 5.10.

(a) *(Unrestricted case.) If the score function η satisfies the conditions of Lemma 4.6(a), then*

$$\mathrm{Var}(W_{\alpha,r}) \ge C(\alpha r^{-d}) \vee \left(\alpha^{\frac{k\beta - 2\beta - 3dk + 2d}{k\beta - 2\beta - dk}} \right)$$

for $R_0 \le r \le \alpha^{1/d}/6$, where $C, R_0 > 0$ are independent of α.

(b) *(Restricted case.) If the score function η satisfies the conditions of Lemma 4.6(b), then*

$$\mathrm{Var}(\bar{W}_{\alpha,r}) \ge C(\alpha r^{-d}) \vee \left(\alpha^{\frac{k\beta - 2\beta - 3dk + 2d}{k\beta - 2\beta - dk}} \right)$$

for $R_0 \le r \le \alpha^{1/d}/6$, where $C, R_0 > 0$ are independent of α.

Proof of Theorem 2.2. Let $\sigma_\alpha^2 := \mathrm{Var}(\bar{W}_\alpha)$, $\sigma_{\alpha,r}^2 := \mathrm{Var}(\bar{W}_{\alpha,r})$, and $\bar{Z}_{\alpha,r} \sim N(\mathbb{E}\bar{W}_{\alpha,r}, \sigma_{\alpha,r}^2)$; then it follows from the triangle inequality that

$$d_{TV}(\bar{W}_\alpha, \bar{Z}_\alpha) \le d_{TV}\left(\bar{W}_\alpha, \bar{W}_{\alpha,r} \right) + d_{TV}\left(\bar{Z}_\alpha, \bar{Z}_{\alpha,r} \right) + d_{TV}\left(\bar{W}_{\alpha,r}, \bar{Z}_{\alpha,r} \right). \tag{5.54}$$

We take R_0 as the maximum of the quantities R_0 in Lemma 4.4(b), Corollary 4.1(b), and Lemma 5.9(b). We start with the exponentially stabilising case (ii).

(ii) The first term of (5.54) can be bounded using Lemma 5.6(b), giving

$$d_{TV}\left(\bar{W}_\alpha, \bar{W}_{\alpha,r} \right) \le C_1 \alpha e^{-C_1 r} \le \frac{1}{\alpha}, \tag{5.55}$$

for $r > C_3 \ln(\alpha)$.

We can establish an upper bound for the second term $d_{TV}\left(\bar{Z}_\alpha, \bar{Z}_{\alpha,r} \right)$ of (5.54) using Lemma 5.1. To this end, (5.33) gives

$$\left| \sigma_\alpha^2 - \sigma_{\alpha,r}^2 \right| \le \frac{1}{\alpha}, \tag{5.56}$$

which, together with Lemma 4.5(b), implies

$$\sigma_{\alpha,r}^2 = \Omega(\alpha), \qquad \sigma_\alpha^2 = \Omega(\alpha), \tag{5.57}$$

for $r > C_4 \ln(\alpha)$. We combine (5.57) and (5.38) to obtain

$$\frac{\left| \mathbb{E}(\bar{Z}_\alpha) - \mathbb{E}(\bar{Z}_{\alpha,r}) \right|}{\max(\sigma_\alpha, \sigma_{\alpha,r})} = \frac{\left| \mathbb{E}(\bar{W}_\alpha) - \mathbb{E}(\bar{W}_{\alpha,r}) \right|}{\max(\sigma_\alpha, \sigma_{\alpha,r})} \le \mathrm{O}\left(\alpha^{-2} \right) \tag{5.58}$$

for $r > C_5 \ln(\alpha)$. Therefore, it follows from (5.56), (5.57), (5.58), and Lemma 5.1 that

$$d_{TV}(\bar{Z}_\alpha, \bar{Z}_{\alpha,r}) \leq \frac{|\mathbb{E}(\bar{Z}_\alpha) - \mathbb{E}(\bar{Z}_{\alpha,r})|}{2 \max(\sigma_\alpha, \sigma_{\alpha,r})} + \frac{3 |\text{Var}(\bar{W}_\alpha) - \text{Var}(\bar{W}_{\alpha,r})|}{2 \max(\text{Var}(\bar{W}_\alpha), \text{Var}(\bar{W}_{\alpha,r}))} \tag{5.59}$$

$$\leq \text{O}(\alpha^{-2}) \tag{5.60}$$

for $r > C_6 \ln(\alpha)$.

For the last term of (5.54), as a linear transformation does not change the total variation distance, we can rewrite it as

$$d_{TV}(\bar{W}_{\alpha,r}, \bar{Z}_{\alpha,r}) = d_{TV}(V_{\alpha,r}, Z),$$

where $V_{\alpha,r} := (\bar{W}_{\alpha,r} - \mathbb{E}\bar{W}_{\alpha,r})/\sigma_{\alpha,r}$ and $Z \sim N(0, 1)$. We now appeal to Stein's method to tackle the problem. Briefly speaking, Stein's method for normal approximation hinges on a Stein equation (see [10, p. 15])

$$f'(w) - wf(w) = h(w) - Nh, \tag{5.61}$$

where $Nh := \mathbb{E}h(Z)$. The solution of (5.61) satisfies (see [10, p. 16])

$$\|f_h'\| := \sup_w |f_h'(w)| \leq 2\|h(\cdot) - Nh\|.$$

Hence, for $h = \mathbf{1}_A$ with $A \in \mathscr{B}(\mathbb{R})$, the solution $f_h =: f_A$ satisfies

$$\|f_h'\| \leq 2. \tag{5.62}$$

The Stein equation (5.61) enables us to bound $d_{TV}(V_{\alpha,r}, Z)$ through a functional form of $V_{\alpha,r}$ only, giving

$$d_{TV}(V_{\alpha,r}, Z) \leq \sup_{\{f: \|f'\| \leq 2\}} \mathbb{E}[f'(V_{\alpha,r}) - V_{\alpha,r}f(V_{\alpha,r})]. \tag{5.63}$$

Recalling (5.48) and (5.49), we can represent $V_{\alpha,r}$ through $V(dx) := \frac{1}{\sigma_{\alpha,r}} \bar{\Xi}_\alpha^*(dx)$, giving $V_{\alpha,r} = \int_{\Gamma_\alpha} V(dx)$. Let $N'_{x,\alpha,r} = B(x, 2r) \cap \Gamma_\alpha$ and $N''_{x,\alpha,r} = B(x, 4r) \cap \Gamma_\alpha$. Define

$$S'_{x,\alpha,r} = \int_{N'_{x,\alpha,r}} V(dy), \qquad S''_{x,\alpha,r} = \int_{N''_{x,\alpha,r}} V(dy).$$

Since $V(dx)$ is independent of $V(dy)$ if $|x - y| > 2r$, $V(dx)$ is independent of $V_{\alpha,r} - S'_{x,\alpha,r}$, $S'_{x,\alpha,r}V(dx)$ is independent of $V_{\alpha,r} - S''_{x,\alpha,r}$, $1 = \text{Var}(V_{\alpha,r}) = \mathbb{E}\int_{\Gamma_\alpha} S'_{x,\alpha,r}V(dx)$, and

$$\mathbb{E}[f'(V_{\alpha,r}) - V_{\alpha,r}f(V_{\alpha,r})]$$

$$= \mathbb{E}f'(V_{\alpha,r}) - \mathbb{E}\int_{\Gamma_\alpha}(f(V_{\alpha,r}) - f(V_{\alpha,r} - S'_{x,\alpha,r}))V(dx)$$

$$= \mathbb{E}f'(V_{\alpha,r}) - \mathbb{E}\int_{\Gamma_\alpha}\int_0^1 f'(V_{\alpha,r} - uS'_{x,\alpha,r})S'_{x,\alpha,r}duV(dx)$$

$$= \mathbb{E}\int_{\Gamma_\alpha}\mathbb{E}[f'(V_{\alpha,r})]S'_{x,\alpha,r}V(dx) - \mathbb{E}\int_{\Gamma_\alpha}\int_0^1 f'(V_{\alpha,r} - uS'_{x,\alpha,r})S'_{x,\alpha,r}duV(dx)$$

$$= \mathbb{E}\int_{\Gamma_\alpha}\mathbb{E}[f'(V_{\alpha,r}) - f'(V_{\alpha,r} - S''_{x,\alpha,r})]S'_{x,\alpha,r}V(dx)$$

$$\quad - \mathbb{E}\int_{\Gamma_\alpha}\int_0^1(f'(V_{\alpha,r} - uS'_{x,\alpha,r}) - f'(V_{\alpha,r} - S''_{x,\alpha,r}))S'_{x,\alpha,r}duV(dx). \tag{5.64}$$

By the definition of the total variation distance, we have

$$d_{TV}\left(V_{\alpha,r}, V_{\alpha,r} + \gamma\right) = d_{TV}\left(\bar{W}_{\alpha,r}, \bar{W}_{\alpha,r} + \sigma_{\alpha,r}\gamma\right) \tag{5.65}$$

for any $\gamma \in \mathbb{R}$. Using Corollary 4.1(b) with $N_{\alpha,r}^{(1)} = N_{\alpha,r}^{(3)} := \emptyset$ and $N_{\alpha,r}^{(2)} := B\left(N_{x,\alpha,r}'', r\right)$, for $r \leq C_7 \alpha^{\frac{1}{d}}$, the non-singularity in Assumption 2.4 ensures that

$$d_{TV}\left(\bar{W}_{\alpha,r}, \bar{W}_{\alpha,r} + h_{\alpha,r}\left(\Xi_{N_{\alpha,r}^{(2)}}\right)\right) \leq \mathbb{E}\left(\left|h_{\alpha,r}\left(\Xi_{N_{\alpha,r}^{(2)}}\right)\right| \vee 1\right) \mathrm{O}\left(\alpha^{-\frac{1}{2}}r^{\frac{d}{2}}\right),$$

which, together with (5.65), implies

$$\left|\mathbb{E}\left[f'\left(V_{\alpha,r}\right) - f'\left(V_{\alpha,r} - S_{x,\alpha,r}''\right)\right]\right| \leq 2\|f'\| \mathrm{O}\left(\alpha^{-\frac{1}{2}}r^{\frac{d}{2}}\right) \mathbb{E}\left[\left|\sigma_{\alpha,r}S_{x,\alpha,r}''\right| \vee 1\right]. \tag{5.66}$$

Recalling (5.49), we have

$$\sigma_{\alpha,r}S_{x,\alpha,r}'' = \int_{N_{x,\alpha,r}''} \bar{\Xi}_\alpha^*(dy).$$

Using the first-order Palm distribution (5.4), the third-order Palm distribution (5.6), and the moment condition (2.4), we obtain

$$\mathbb{E}\left[|g_\alpha(z, \Xi)| \, \mathbf{1}_{\bar{R}(z,\alpha)<r}\bar{\Xi}(dz) \, |g_\alpha(y, \Xi)| \, \mathbf{1}_{\bar{R}(y,\alpha)<r}\bar{\Xi}(dy) \, |g_\alpha(x, \Xi)| \, \mathbf{1}_{\bar{R}(x,\alpha)<r}\bar{\Xi}(dx)\right]$$
$$\leq C_8 \left(\lambda^3 dzdydx + \lambda^2 dzdx + \lambda^2 dydx + \lambda dx\right),$$
$$\mathbb{E}\left[|g_\alpha(y, \Xi)| \, \mathbf{1}_{\bar{R}(y,\alpha)<r}\bar{\Xi}(dy)\right] \leq C_1 \lambda dy,$$
$$\left|P_{y,\alpha,r}\right| \leq C_1 \lambda,$$

which, together with (5.51) and (5.52), yields

$$\mathbb{E}\left|\bar{\Xi}_\alpha^*(dy)\right| \leq C_1 1\lambda dy, \tag{5.67}$$

$$\mathbb{E}\left|\bar{\Xi}_\alpha^*(dy)\bar{\Xi}_\alpha^*(dx)\right| \leq C_1 2\left(\lambda^2 dydx + \lambda dx\right), \tag{5.68}$$

$$\mathbb{E}\left|\bar{\Xi}_\alpha^*(dz)\bar{\Xi}_\alpha^*(dy)\bar{\Xi}_\alpha^*(dx)\right| \leq C_1 3\left(\lambda^3 dzdydx + \lambda^2 dzdx + \lambda^2 dydx + \lambda dx\right). \tag{5.69}$$

Since $N_{x,\alpha,r}'' \subset B(x, 4r)$, the volume of $N_{x,\alpha,r}''$ is of order $\mathrm{O}(r^d)$. Then

$$\mathbb{E}\int_{N_{x,\alpha,r}'} \left|\bar{\Xi}_\alpha^*(dy)\right| \leq \mathbb{E}\int_{N_{x,\alpha,r}''} \left|\bar{\Xi}_\alpha^*(dy)\right| \leq \mathrm{O}(r^d),$$
$$\mathbb{E}\left[\left|\sigma_{\alpha,r}S_{x,\alpha,r}''\right| \vee 1\right] \leq 1 + \mathbb{E}\int_{N_{x,\alpha,r}''} \left|\bar{\Xi}_\alpha^*(dy)\right| \leq \mathrm{O}(r^d). \tag{5.70}$$

Combining (5.66), (5.70), and (5.62), we have

$$\left|\mathbb{E}\left[f'\left(V_{\alpha,r}\right) - f'\left(V_{\alpha,r} - S_{x,\alpha,r}''\right)\right]\right| \leq \mathrm{O}\left(\alpha^{-\frac{1}{2}}r^{\frac{3d}{2}}\right).$$

Hence the first term of (5.64) can be bounded as

$$
\left| \mathbb{E} \int_{\Gamma_\alpha} \mathbb{E}\big[f'\left(V_{\alpha,r} \right) - f'\left(V_{\alpha,r} - S''_{x,\alpha,r} \right) \big] S'_{x,\alpha,r} V(dx) \right|
$$

$$
\leq \mathrm{O}\!\left(\alpha^{-\frac{1}{2}} r^{\frac{3d}{2}} \right) \sigma_{\alpha,r}^{-2} \mathbb{E} \int_{\Gamma_\alpha} \int_{N'_{x,\alpha,r}} \left| \bar{\Xi}_\alpha^*(dy) \right| \left| \bar{\Xi}_\alpha^*(dx) \right|
$$

$$
\leq \mathrm{O}\!\left(\alpha^{-\frac{1}{2}} r^{\frac{3d}{2}} \right) \sigma_{\alpha,r}^{-2} \int_{\Gamma_\alpha} \left(\int_{N'_{x,\alpha,r}} \lambda\, dy + 1 \right) \lambda\, dx = \mathrm{O}\!\left(\sigma_{\alpha,r}^{-2} \alpha^{\frac{1}{2}} r^{\frac{5d}{2}} \right), \qquad (5.71)
$$

where the last inequality is from (5.68), and the last equality follows from the fact that $\mathrm{Vol}(N'_{x,\alpha,r}) \leq \mathrm{Vol}\left(B(x, 2r) \right) = \mathrm{O}(r^d)$.

For the second term of (5.64), we have from Corollary 4.1 with $N_{\alpha,r}^{(1)} := B\big(N'_{x,\alpha,r}, r \big)$, $N_{\alpha,r}^{(2)} := B\big(N''_{x,\alpha,r}, r \big)$, $N_{\alpha,r}^{(3)} := N''_{x,\alpha,r}$, for $r \leq C_1 4\alpha^{\frac{1}{d}}$, and the non-singularity in Assumption 2.4 that

$$
\left| \mathbb{E}\left[\left. \int_0^1 \left(f'\left(V_{\alpha,r} - u S'_{x,\alpha,r} \right) - f'\left(V_{\alpha,r} - S''_{x,\alpha,r} \right) \right) du \right| \Xi_{B(N'_{x,\alpha,r}, r)} \right] \right|
$$

$$
\leq 2 \int_0^1 \mathbb{E} d_{TV}\left(V_{\alpha,r} - u S'_{x,\alpha,r},\; V_{\alpha,r} - S''_{x,\alpha,r} \,\middle|\, \Xi_{B(N'_{x,\alpha,r}, r)} \right) du
$$

$$
\leq \mathrm{O}\!\left(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}} \right) \mathbb{E}\left(\left. \int_{N''_{x,\alpha,r}} \left| \bar{\Xi}_\alpha^*(dz) \right| + 1 \,\middle|\, \Xi_{B(N'_{x,\alpha,r}, r)} \right) ;
$$

hence

$$
\left| \mathbb{E} \int_{\Gamma_\alpha} \int_0^1 \left(f'\left(V_{\alpha,r} - u S'_{x,\alpha,r} \right) - f'\left(V_{\alpha,r} - S''_{x,\alpha,r} \right) \right) S'_{x,\alpha,r}\, du\, V(dx) \right|
$$

$$
= \left| \mathbb{E} \int_{\Gamma_\alpha} \mathbb{E}\left[\left. \int_0^1 \left(f'\left(V_{\alpha,r} - u S'_{x,\alpha,r} \right) - f'\left(V_{\alpha,r} - S''_{x,\alpha,r} \right) \right) du \right| \Xi_{B(N'_{x,\alpha,r}, r)} \right] S'_{x,\alpha,r} V(dx) \right|
$$

$$
\leq \mathrm{O}\!\left(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}} \right) \mathbb{E} \int_{\Gamma_\alpha} \mathbb{E}\left(\left. \int_{N''_{x,\alpha,r}} \left| \bar{\Xi}_\alpha^*(dz) \right| + 1 \,\middle|\, \Xi_{B(N'_{x,\alpha,r}, r)} \right) \left| S'_{x,\alpha,r} \right| \left| V(dx) \right|
$$

$$
\leq \mathrm{O}\!\left(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}} \right) \sigma_{\alpha,r}^{-2} \mathbb{E} \int_{\Gamma_\alpha} \left[\int_{N''_{x,\alpha,r}} \int_{N'_{x,\alpha,r}} \left| \bar{\Xi}_\alpha^*(dz) \bar{\Xi}_\alpha^*(dy) \right| + \int_{N'_{x,\alpha,r}} \left| \bar{\Xi}_\alpha^*(dy) \right| \right] \left| \bar{\Xi}_\alpha^*(dx) \right|
$$

$$
\leq \mathrm{O}\!\left(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}} \right) \sigma_{\alpha,r}^{-2} \int_{\Gamma_\alpha} \left(\int_{N''_{x,\alpha,r}} \int_{N'_{x,\alpha,r}} \lambda^2 dz\, dy + \int_{N''_{x,\alpha,r}} \lambda\, dz + \int_{N'_{x,\alpha,r}} \lambda\, dy + 1 \right) \lambda\, dx
$$

$$
\leq \mathrm{O}\!\left(\alpha^{-\frac{1}{2}} r^{\frac{d}{2}} \right) \sigma_{\alpha,r}^{-2}\, \mathrm{O}(\alpha r^{2d})
$$

$$
= \mathrm{O}\!\left(\sigma_{\alpha,r}^{-2} \alpha^{\frac{1}{2}} r^{\frac{5d}{2}} \right), \qquad (5.72)
$$

where the second-to-last inequality follows from (5.67), (5.68), and (5.69), and the last inequality is due to the fact that the volumes of $N'_{x,\alpha,r}$ and $N''_{x,\alpha,r}$ are of order $\mathrm{O}(r^d)$. Recalling (5.63) and (5.64), we add up the bounds of (5.71) and (5.72) to obtain

$$
d_{TV}\left(\bar{W}_{\alpha,r}, \bar{Z}_{\alpha,r} \right) = d_{TV}\left(V_{\alpha,r}, Z \right) \leq \mathrm{O}\!\left(\sigma_{\alpha,r}^{-2} \alpha^{\frac{1}{2}} r^{\frac{5d}{2}} \right). \qquad (5.73)
$$

The proof of (ii) is completed by using (5.54), taking $r = \max(C_3, C_4, C_5)\ln(\alpha)$ for large α, and collecting the bounds in (5.55), (5.60), and (5.73) and substituting $\sigma_{\alpha,r}^2 = \Omega(\alpha)$, as shown in (5.57).

(i) There exists an $r_1 > 0$ such that $\bar{W}_{\alpha,r_1} = \bar{W}_\alpha$ a.s. for all α, which implies $\mathbb{E}\bar{W}_{\alpha,r_1} = \mathbb{E}\bar{W}_\alpha$, $\mathrm{Var}(\bar{W}_{\alpha,r_1}) = \mathrm{Var}(\bar{W}_\alpha)$; hence $d_{TV}(\bar{W}_\alpha, \bar{Z}_\alpha) = d_{TV}(\bar{W}_{\alpha,r_1}, \bar{Z}_{\alpha,r_1})$. On the other hand, range-boundedness implies exponential stabilisation; with r_1 in place of r, (5.57) and (5.73) still hold. However, r_1 is a constant independent of α; the conclusion follows.

(iii) We take $r = r_\alpha := R_0 \vee \alpha^{\frac{5k-4}{5dk+2\beta k-4\beta}}$. Lemma 5.6(b) gives

$$d_{TV}(\bar{W}_\alpha, \bar{W}_{\alpha,r}) \leq \mathrm{O}(\alpha r^{-\beta}) < \mathrm{O}\left(\alpha^{-\frac{\beta(k-2)[\beta(k-2)-d(15k-14)]}{(k\beta-2\beta-dk)(5dk+2\beta k-4\beta)}}\right). \tag{5.74}$$

Next, applying Lemma 4.6(b) and Lemma 5.10(b), we have

$$\mathrm{Var}(\bar{W}_{\alpha,r}) \wedge \mathrm{Var}(\bar{W}_\alpha) \geq \mathrm{O}\left(\alpha^{\frac{k\beta-2\beta-3dk+2d}{k\beta-2\beta-dk}}\right), \tag{5.75}$$

which, together with (5.34), (5.41), and (5.59), yields

$$d_{TV}(\bar{Z}_\alpha, \bar{Z}_{\alpha,r}) \leq \mathrm{O}\left(\frac{\alpha^{\frac{2k-1}{k}}r^{-\beta\frac{k-1}{k}}}{\alpha^{\frac{1}{2}\frac{k\beta-2\beta-3dk+2d}{k\beta-2\beta-dk}}} \vee \frac{\alpha^{\frac{3k-2}{k}}r^{-\beta\frac{k-2}{k}}}{\alpha^{\frac{k\beta-2\beta-3dk+2d}{k\beta-2\beta-dk}}} \vee \frac{\alpha^{\frac{3k-1}{k}}r^{-\beta\frac{k-1}{k}}}{\alpha^{\frac{k\beta-2\beta-3dk+2d}{k\beta-2\beta-dk}}}\right) \tag{5.76}$$

for $R_0 < r < C_1 5\alpha^{\frac{1}{d}}$. Recalling that $\beta > \frac{(15k-14)d}{k-2}$, the dominating term of (5.76) is

$$\frac{\alpha^{\frac{3k-2}{k}}r^{-\beta\frac{k-2}{k}}}{\alpha^{\frac{k\beta-2\beta-3dk+2d}{k\beta-2\beta-dk}}},$$

giving

$$d_{TV}(\bar{Z}_\alpha, \bar{Z}_{\alpha,r}) \leq \mathrm{O}\left(\frac{\alpha^{\frac{3k-2}{k}}r^{-\beta\frac{k-2}{k}}}{\alpha^{\frac{k\beta-2\beta-3dk+2d}{k\beta-2\beta-dk}}}\right) = \mathrm{O}\left(\alpha^{-\frac{\beta(k-2)[\beta(k-2)-d(15k-14)]}{(k\beta-2\beta-dk)(5dk+2\beta k-4\beta)}}\right). \tag{5.77}$$

For $d_{TV}(\bar{W}_{\alpha,r}, \bar{Z}_{\alpha,r})$, we make use of (5.73) and (5.75), and replace r with r_α to obtain

$$d_{TV}(\bar{W}_{\alpha,r}, \bar{Z}_{\alpha,r}) \leq \mathrm{O}\left(\alpha^{\frac{1}{2}}r^{\frac{5d}{2}}\right)\mathrm{O}\left(\alpha^{-\frac{k\beta-2\beta-3dk+2d}{k\beta-2\beta-dk}}\right) = \mathrm{O}\left(\alpha^{-\frac{\beta(k-2)[\beta(k-2)-d(15k-14)]}{(k\beta-2\beta-dk)(5dk+2\beta k-4\beta)}}\right). \tag{5.78}$$

Finally, the proof is completed by combining (5.54), (5.74), (5.77), and (5.78). $\square$

Proof of Theorem 2.1. One can repeat the proof of Theorem 2.2 by replacing $\bar{W}_\alpha$, $\bar{W}_{\alpha,r}$, $\bar{Z}_\alpha$, $\bar{Z}_{\alpha,r}$, $g_\alpha(x, \Xi)$, and $\bar{R}(x, \alpha)$ with W_α, $W_{\alpha,r}$, Z_α, $Z_{\alpha,r}$, $g(\Xi^x)$, and $R(x)$. $\square$

Remark 5.2. If we aim to find the order of the total variation distance between $\bar{W}_\alpha$ and a normal distribution instead of a normal distribution with the same mean and variance in the polynomially stabilising case, we can get a better upper-bound approximation error with a weaker condition. When

$$\beta > \frac{5dk - 7d + \sqrt{20d^2k^2 - 60d^2k + 49d^2}}{k-2},$$

combining (5.73) and the fact that $d_{TV}(\bar{W}_\alpha, \bar{W}_{\alpha,r}) \leq C\alpha\lambda r^{-\beta}$, and taking

$$r_\alpha := \alpha^{\frac{3\beta k - 7dk + 4d - 6\beta}{(\beta k - dk - 2\beta)(5d+2\beta)}},$$

we have

$$d_{TV}(\bar{W}_\alpha, \bar{Z}_{\alpha, r_\alpha}) \le d_{TV}(\bar{W}_\alpha, \bar{W}_{\alpha, r_\alpha}) + d_{TV}(\bar{W}_{\alpha, r_\alpha}, \bar{Z}_{\alpha, r_\alpha})$$
$$\le \mathrm{O}\left(\alpha^{\frac{-\beta^2(k-2)+10\beta dk-14\beta d-5d^2 k}{(\beta k - dk - 2\beta)(5d + 2\beta)}} \right).$$

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