THE TRANSLATIONAL HULL OF AN INVERSE SEMIGROUP

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An ideal extension of one semigroup by another is determined by a partial homomorphism into the translational hull of the first semigroup [3, §2, Theorem 5]. In most instances, the development of the theory of ideal extensions has been hindered by inadequate knowledge of the translational hull; it is our purpose here to characterize certain basic structures in the translational hull of an arbitrary inverse semigroup.

For an inverse semigroup S, the translational hull of S, $\Omega(S)$, is again an inverse semigroup, and thus the idempotents of $\Omega(S)$ form a semilattice. How the structure of this semilattice, $E_{\Omega(S)}$, is influenced by the structure of the semilattice of idempotents of S, E_S, is seen in one of our main results: $E_{\Omega(S)} \simeq \Omega(E_S)$.

Since $\Omega(S)$ always possesses an identity, the group of units of $\Omega(S)$ is another structure which is of interest. We give here a characterization of this group in terms of automorphisms of the semilattice of S.

There are two sections dealing with applications of the characterizations given for $E_{\Omega(S)}$ and the group of units of $\Omega(S)$.

1. Notation and preliminary results. For a semigroup S, E_S denotes the set of idempotents of S. Except when otherwise stated, the notation and definitions used can be found in [**1**].

For a semigroup S, define

$$\Lambda(S) = \{\lambda : S \to S \mid \lambda(xy) = (\lambda x)y, \quad \text{for all} \quad x, y \in S\},$$
$$P(S) = \{\rho : S \to S \mid (xy)\rho = x(y\rho), \quad \text{for all} \quad x, y \in S\}.$$

These sets are semigroups under composition of mappings. Further, let

$$\Omega(S) = \{(\lambda, \rho) \in \Lambda(S) \times P(S) \mid x(\lambda y) = (x\rho)y, \quad \text{for all} \quad x, y \in S\}.$$

With multiplication defined by $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$, $\Omega(S)$ is also a semigroup. Let

$$\tilde{\Lambda}(S) = \{\lambda \in \Lambda(S) \mid (\lambda, \rho) \in \Omega(S) \quad \text{for some} \quad \rho\};$$
$$\tilde{P}(S) = \{\rho \in P(S) \mid (\lambda, \rho) \in \Omega(S) \quad \text{for some} \quad \lambda\};$$
$$\Pi(S) = \{(\lambda_a, \rho_a) \mid a \in S, \lambda_a x = ax, x\rho_a = xa, \quad \text{for all} \quad x \in S\};$$
$$\Sigma(S) = \{(\lambda, \rho) \in \Omega(S) \mid (\lambda, \rho)(\mu, \tau) = (\mu, \tau)(\lambda, \rho) = i \quad \text{for some } (\mu, \tau)\},$$

where $i = (i_S, i_S)$ and i_S is the identity map on S. The semigroup $\Omega(S)$ is called the translational hull of S, and $\Sigma(S)$ is the group of units of $\Omega(S)$; if (λ, ρ) is in $\Omega(S)$, we say that λ and ρ are linked. In case S is a weakly reductive semigroup (which is certainly true if S is an inverse semigroup), then S is isomorphic to $\Pi(S)$ under the mapping $x \to (\lambda_x, \rho_x)$; since $\Pi(S)$ is an ideal of $\Omega(S)$, we may consider S contained in $\Omega(S)$. A full treatment of the above semigroups is given in [3]. In particular, the following proposition appears there (see §2 of [3]).

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PROPOSITION 1.1. *Let S be a semigroup.*

(i) *If S is reductive, then $\Omega(S) \simeq \tilde{\Lambda}(S) \simeq \tilde{P}(S)$.*

(ii) *If S is commutative and reductive, then $\Omega(S) \simeq \Lambda(S)$.*

A semigroup $S = S^0$ is the *orthogonal sum* (called 0-*direct union* in **[1]**) of semigroups $S_\alpha = S_\alpha^0(\alpha \in A)$, if $S = \bigcup_{\alpha \in A} S_\alpha$ and $S_\alpha S_\beta = S_\alpha \cap S_\beta = \{0\}$ whenever $\alpha \neq \beta$. The usual direct product of semigroups T_i $(i \in I)$, is denoted by $\mathop{\times}\limits_{i \in I} T_i$.

PROPOSITION 1.2. *Let S be a semigroup which is the orthogonal sum of subsemigroups S_α, with $S_\alpha^2 = S_\alpha(\alpha \in A)$. Then $\Omega(S) \simeq \mathop{\times}\limits_{\alpha \in A} \Omega(S_\alpha)$.*

Proof. This result follows easily on using the fact that, if $(\lambda, \rho) \in \Omega(S)$, then $(\lambda|_{S_\alpha}, \rho|_{S_\alpha})$ is in $\Omega(S_\alpha)$ for all α in A.

The next proposition, which is due to Ponizovski **[5]**, will be used repeatedly without special reference.

PROPOSITION 1.3. *For an inverse semigroup S, $\Omega(S)$ is again an inverse semigroup. In particular, if (λ, ρ) is in $\Omega(S)$, then its inverse, $(\lambda^{-1}, \rho^{-1})$, is defined by*

$$\lambda^{-1}x = (x^{-1}\rho)^{-1}, \quad x\rho^{-1} = (\lambda x^{-1})^{-1}, \quad \textit{for all } x \textit{ in } S.$$

Since every element of an inverse semigroup S has a left and right identity, many of the properties of $\Lambda(S)$, $P(S)$, and $\Omega(S)$ can be simplified. For λ a mapping, it is understood that (λx) means $\lambda(x)$.

LEMMA 1.4. *Let S be an inverse semigroup. The following statements hold.*

(i) *For λ, λ' in $\Lambda(S)$, $\lambda = \lambda'$ if and only if $\lambda|_{E_S} = \lambda'|_{E_S}$.*

(ii) *Let λ be a mapping of S into itself. Then λ is in $\Lambda(S)$ if and only if $(\lambda e)f = \lambda f$ for e, f in E_S with $f \leqq e$, and $\lambda x = (\lambda x x^{-1})x$ for all x in S.*

(iii) *Let ρ be a mapping of S into itself. Then ρ is in $P(S)$ if and only if $f(e\rho) = f\rho$, for e, f in E_S with $f \leqq e$, and $x\rho = x(x^{-1}x\rho)$ for all x in S.*

(iv) *For $\lambda \in \Lambda(S)$, $\rho \in P(S)$, (λ, ρ) is in $\Omega(S)$ if and only if $e(\lambda f) = (e\rho)f$ for all e, f in E_S.*

Proof. (i) Let $\lambda, \lambda' \in \Lambda(S)$ with $\lambda|_{E_S} = \lambda'|_{E_S}$. For x in S,

$$\lambda x = \lambda x x^{-1} x = (\lambda x x^{-1})x = (\lambda' x x^{-1})x = \lambda' x x^{-1} x = \lambda' x;$$

that is, $\lambda = \lambda'$.

(ii) Let λ be a mapping of S into S satisfying $\lambda x = (\lambda x x^{-1})x$ for all x in S and $(\lambda e)f = \lambda f$ for $f \leqq e$. Then, for all x, y in S,

$$\begin{aligned} \lambda(xy) &= [\lambda(xy)(xy)^{-1}](xy) = [\lambda(xx^{-1})(xy)(xy)^{-1}](xy) \\ &= [\lambda(xx^{-1})](xy)(xy)^{-1}(xy) = [\lambda(xx^{-1})]xy = (\lambda x)y. \end{aligned}$$

Therefore λ is in $\Lambda(S)$.

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The proof of (iii) is symmetric to that of (ii); statement (iv) follows from the fact that every element has a left and right identity.

For the remainder of the paper, we shall be concerned exclusively with inverse semigroups. Consequently, we shall not necessarily include in the hypothesis of every proposition the fact that S is an inverse semigroup; this will be assumed without express mention. We note that Proposition 1.1(i) holds since every inverse semigroup is reductive.

2. The idempotents of $\Omega(S)$. In this section we give characterizations of the semilattice of idempotents of $\Omega(S)$ in terms of $\Omega(E_S)$ and in terms of a certain subsemigroup of $\Lambda(S)$. The next lemma is crucial for both of these characterizations.

LEMMA 2.1. *For $(\lambda, \rho) \in \Omega(S)$, (λ, ρ) is idempotent if and only if $\lambda(E_S) \subseteq E_S$, $(E_S)\rho \subseteq E_S$.*

Proof. Let (λ, ρ) be an idempotent in $\Omega(S)$. Then $\lambda^{-1} = \lambda$, $\rho^{-1} = \rho$, and, by Proposition 1.3, $e\rho = e\rho^{-1} = (\lambda e)^{-1}$ for all e in E_S. Therefore

$$\lambda e = (\lambda e)(e\rho)(\lambda e) = (\lambda e)e\lambda(\lambda e) = (\lambda e)(\lambda e),$$

since $\lambda^2 = \lambda$. Thus λe is in E_S. It can be shown similarly that $e\rho \in E_S$ for all e in E_S.

Conversely, let $\lambda(E_S) \subseteq E_S$, $(E_S)\rho \subseteq E_S$. Then, for e in E_S, $\lambda e = (\lambda e)e = e(\lambda e)$, and thus

$$\lambda^2 e = \lambda(\lambda e) = \lambda(e(\lambda e)) = (\lambda e)(\lambda e) = \lambda e.$$

Hence, by Lemma 1.4(i), $\lambda^2 = \lambda$. By a symmetric argument we have $\rho^2 = \rho$, and (λ, ρ) is idempotent.

THEOREM 2.2. *For an inverse semigroup S,*

$$E_{\Omega(S)} \simeq \{\lambda \in \Lambda(S) \mid \lambda(E_S) \subseteq E_S\}.$$

Proof. Define $\pi : E_{\Omega(S)} \to \Lambda(S)$ by $(\lambda, \rho)\pi = \lambda$. Then, by Lemma 2.1, π maps $E_{\Omega(S)}$ into $\{\lambda \in \Lambda(S) \mid \lambda(E_S) \subseteq E_S\}$. Further, if $\lambda \in \Lambda(S)$ with $\lambda(E_S) \subseteq E_S$, then define ρ on S by

$$x\rho = x\lambda(x^{-1}x), \quad \text{for all } x \text{ in } S.$$

For e in E_S, $e\rho = e(\lambda e) = (\lambda e)e = \lambda e$, since λe is in E_S. Therefore, if $e, f \in E_S$, then

$$(ef)\rho = \lambda(ef) = (\lambda f)e = e(\lambda f) = e(f\rho);$$

also, for x in S,

$$x(x^{-1}x\rho) = x(x^{-1}x(\lambda x^{-1}x)) = x(\lambda x^{-1}x) = x\rho.$$

Consequently, by Lemma 1.4(iii), ρ is in $P(S)$. In addition, for e, f in E_S,

$$e(\lambda f) = e(f\rho) = (ef)\rho = (fe)\rho = f(e\rho) = (e\rho)f,$$

so that, by Lemma 1.4(iv), (λ, ρ) is in $\Omega(S)$. The element (λ, ρ) is idempotent by Lemma 2.1, and thus π maps $E_{\Omega(S)}$ onto the subsemigroup $\{\lambda \in \Lambda(S) \mid \lambda(E_S) \subseteq E_S\}$. Finally, π is one-to-one due to the fact that S is reductive; π is a homomorphism since it is just the projection into $\Lambda(S)$.

PROPOSITION 2.3. *Let $\lambda \in \Lambda(S)$, $\rho \in P(S)$, with $\lambda^2 = \lambda$, $\rho^2 = \rho$. Then $(\lambda, \rho) \in \Omega(S)$ if and only if $\lambda e = e\rho$ for all e in E_S.*

Proof. Let (λ, ρ) be in $\Omega(S)$ with $\lambda^2 = \lambda$ and $\rho^2 = \rho$, and let $e \in E_S$. Then, by Lemma 2.1, $e\rho$ and λe are idempotents. Thus

$$e\rho = e\rho^{-1} = (\lambda e)^{-1} = \lambda e.$$

Conversely, suppose that $\lambda \in \Lambda(S)$, $\rho \in P(S)$ with $\lambda^2 = \lambda$, $\rho^2 = \rho$, and let $e\rho = \lambda e$ for all e in E_S. Then, for all $e, f \in E_S$,

$$(e\rho)^2 = (\lambda e)(e\rho) = ((\lambda e)e)\rho = (\lambda e)\rho = (e\rho)\rho = e\rho^2 = e\rho,$$

and so

$$(e\rho)f = f(e\rho) = (fe)\rho = (ef)\rho = e(f\rho) = e(\lambda f).$$

Therefore, by Lemma 1.4(iv), (λ, ρ) is in $\Omega(S)$.

THEOREM 2.4. *For S an inverse semigroup, the semilattices $\Omega(E_S)$ and $E_{\Omega(S)}$ are isomorphic.*

Proof. It is an immediate consequence of Lemma 2.1 that $\Omega(E_S)$ is indeed a semilattice. Consider the mapping $\theta : E_{\Omega(S)} \to \Omega(E_S)$ defined by $(\lambda, \rho)\theta = (\lambda|_{E_S}, \rho|_{E_S})$. By Lemma 2.1, $\lambda|_{E_S}$ is in $\Omega(E_S)$ and $\rho|_{E_S}$ is in $P(E_S)$, and certainly $\lambda|_{E_S}$ is linked to $\rho|_{E_S}$. Hence θ maps into $\Omega(E_S)$. That θ is one-to-one follows from Lemma 1.4(i) and its dual; θ is a homomorphism since $\lambda\lambda'|_{E_S} = \lambda|_{E_S}\lambda'|_{E_S}$ and $\rho\rho'|_{E_S} = \rho|_{E_S}\rho'|_{E_S}$. To see that θ maps onto $\Omega(E_S)$, let (λ_0, ρ_0) be in $\Omega(E_S)$. Define λ and ρ on S by

$$\lambda x = (\lambda_0 xx^{-1})x, \quad x\rho = x(x^{-1}x\rho_0).$$

By Lemma 1 4(ii), λ is in $\Lambda(S)$, and by Lemma 1.4(iii), ρ is in $P(S)$; (λ, ρ) is in $\Omega(S)$ by Lemma 1.4(iv). Finally, $\lambda|_{E_S} = \lambda_0$, $\rho|_{E_S} = \rho_0$; so θ is a mapping onto $\Omega(E_S)$. Therefore $E_{\Omega(S)}$ is isomorphic to $\Omega(E_S)$.

3. Applications. Using Theorem 2.4, we now give several examples of when conditions on S dictate properties of $\Omega(S)$. To do this we need a characterization of $\Omega(E_S)$ given by Petrich in [3]. For a semilattice E and x in E, the principal ideal of E generated by x is the set $I_x = \{y \mid y \leqq x\}$.

PROPOSITION 3.1. *Let E be a semilattice. Then $\Omega(E)$ is isomorphic to the semilattice $\mathscr{P}$, where $\mathscr{P} = \{I \mid I$ an ideal of E, $I \cap I_x$ principal for all $x \in E \backslash I\}$, with multiplication defined as intersection.*

We shall now assume that $\Omega(E) = \mathscr{P}$, with E embedded in $\mathscr{P}$ under the mapping $e \to I_e$.

PROPOSITION 3.2. *Let S be an inverse semigroup with zero element 0. Then 0 is a prime ideal of S if and only if 0 is prime in $\Omega(S)$.*

Proof. Since S is embedded as an ideal in $\Omega(S)$, if 0 is prime in $\Omega(S)$, it is certainly prime in S.

Conversely, let 0 be prime in S. Since $\Omega(S)$ is an inverse semigroup, it suffices to show that 0 is prime in $E_{\Omega(S)}$. Since $\Omega(E_S) \simeq E_{\Omega(S)}$, we need only prove that 0 is prime in $\Omega(E_S)$.

Let I and J be in $\Omega(E_S)$ with $IJ = 0$. Since product in $\Omega(E_S)$ is defined as intersection, we have $I \cap J = 0$. If $e \in I, f \in J$, then $ef \in I \cap J = 0$. But 0 is prime in E_S; so $e = 0$ or $f = 0$. That is, $I = 0$ or $J = 0$.

PROPOSITION 3.3. *The semilattice E_S is finite if and only if $E_{\Omega(S)}$ is a finite lattice.*

Proof. Assume that E_S is finite. Then the power set of E is finite and, by Proposition 3.1, it contains $\Omega(E_S)$. Further, using [3, §2, Proposition 8] we see that $\Omega(E_S)$ is a lattice. Since $E_{\Omega(S)} \simeq \Omega(E_S)$ by Theorem 2.4, $E_{\Omega(S)}$ is a finite lattice.

A semilattice E is a *tree* if, for e, f, g in E, $e < g, f < g$ implies that $e \leqq f$ or $f \leqq e$.

PROPOSITION 3.4. *If E_S is a tree, then $E_{\Omega(S)}$ is a lattice.*

Proof. Since $E_{\Omega(S)}$ is isomorphic to $\Omega(E_S)$, we need only show that, if I and J are in $\Omega(E_S)$, then $I \cup J$ is again in $\Omega(E_S)$. That is, for every x in $E \backslash (I \cup J)$, $(I \cup J) \cap I_x$ must be principal. This follows since

$$(I \cup J) \cap I_x = (I \cap I_x) \cup (J \cap I_x) = I_y \cup I_z$$

for some $y < x$, $z < x$. But E_S is a tree; so y and z are comparable and $I \cup J$ is in $\Omega(E_S)$.

For T a weakly reductive semigroup, T^1 can be embedded in $\Omega(T)$ under the mapping $x \to (\lambda_x, \rho_x)$ $(x \in T^1)$, where $\rho_1 = \lambda_1 = i_T$. We shall call this mapping the *extended embedding.*

THEOREM 3.5. *For an inverse semigroup S, $\Omega(S) = \Pi(S) \cup \Sigma(S)$ if and only if $\Omega(E_S)$ is isomorphic to E_S^1 under the extended embedding.*

Proof. Let $\Omega(S) = \Pi(S) \cup \Sigma(S)$. If $\Omega(S) = \Pi(S)$, then S has an identity and $\Omega(E_S) \simeq E_{\Omega(S)} = E_{\Pi(S)} \simeq E_S = E_S^1$. If $\Omega(S) \neq \Pi(S)$, then $\Pi(S) \cap \Sigma(S) = \phi$, and $\Omega(E_S) \simeq E_{\Omega(S)} = E_{\Pi(S)}^1 \simeq E_S^1$. In either case, using the isomorphism defined in Theorem 2.4 and the natural embedding of S onto $\Pi(S)$, we can see that $\Omega(E_S)$ is isomorphic to E_S^1 under the extended embedding.

Conversely, let $\Omega(E_S) \simeq E_S^1$ under the extended embedding. We shall first show that $E_{\Omega(S)} = E_{\Pi(S)}^1$. For, if $(\lambda, \rho) \in E_{\Omega(S)}$, then $(\lambda|_{E_S}, \rho|_{E_S})$ is in $\Omega(E_S)$ (see Theorem 2.4). But $\Omega(E_S)$ is isomorphic to E_S^1 under the extended embedding, so that there exists e in E_S^1 satisfying $\lambda|_{E_S} = \lambda'_e$, $\rho|_{E_S} = \rho'_e$, where $(\lambda'_e, \rho'_e) \in \Pi(E_S)$ or $(\lambda'_e, \rho'_e) = (i_{E_S}, i_{E_S})$. Hence, by Lemma 1.4(i), $\lambda = \lambda_e$, $\rho = \rho_e$, and $(\lambda, \rho) \in E_{\Pi(S)}^1$. Thus $E_{\Omega(S)} = E_{\Pi(S)}^1$.

Now every (λ, ρ) in $\Omega(S)$ has an inverse $(\lambda^{-1}, \rho^{-1})$ in $\Omega(S)$ and

$$(\lambda\lambda^{-1}, \rho\rho^{-1}) \in E_{\Pi(S)} \Leftrightarrow (\lambda, \rho) \in \Pi(S) \Leftrightarrow (\lambda^{-1}\lambda, \rho^{-1}\rho) \in E_{\Pi(S)},$$

since $\Pi(S)$ is an ideal of $\Omega(S)$. Assume that (λ, ρ) is not in $\Pi(S)$. Then $(\lambda\lambda^{-1}, \rho\rho^{-1})$ is in $E_{\Omega(S)} \backslash E_{\Pi(S)}$. But $E_{\Omega(S)} = E_{\Pi(S)}^1$. Therefore $(\lambda\lambda^{-1}, \rho\rho^{-1}) = (i_S, i_S)$. Similarly, $(\lambda^{-1}\lambda, \rho^{-1}\rho) = (i_S, i_S)$ and thus $(\lambda, \rho) \in \Sigma(S)$. Consequently $\Omega(S) = \Pi(S) \cup \Sigma(S)$.

To see which semilattices have the property mentioned in Theorem 3.5, we state a special case of Proposition 3.1.

PROPOSITION 3.6. *Let E be a semilattice. If, for every proper ideal I of E, I is principal whenever $I \cap I_x$ is principal for all $x \in E \backslash I$, then $\Omega(E) \simeq E^1$.*

COROLLARY 3.7. *Let E be a chain. Then* $\Omega(E) \simeq E^1$.

Proof. This follows easily from Proposition 3.6 since, if I is a proper ideal such that $I \cap I_x$ is principal for all $x \notin I$, then $I \cap I_x = I$ and thus I is principal.

PROPOSITION 3.8. *Let* $S = S^0$ *be an inverse semigroup with* E_S *the orthogonal sum of chains* C_α $(\alpha \in A)$. *Then* $E_{\Omega(S)} \simeq \bigtimes_{\alpha \in A} C_\alpha^1$.

Proof. By Proposition 1.2, we have $\Omega(E_S) \simeq \bigtimes_{\alpha \in A} \Omega(C_\alpha)$. By the preceding corollary, it follows that

$$E_{\Omega(S)} \simeq \Omega(E_S) \simeq \bigtimes_{\alpha \in A} \Omega(C_\alpha) \simeq \bigtimes_{\alpha \in A} C_\alpha^1.$$

4. The group of units of $\Omega(S)$. We are motivated by Theorem 3.5 to characterize the group of units, $\Sigma(S)$, of $\Omega(S)$.

THEOREM 4.1. *Let S be an inverse semigroup. Let* $\lambda \in \Lambda(S)$ *satisfy the following conditions:*

(i) $\theta : E_S \to E_S$ *defined by* $e\theta = (\lambda e)(\lambda e)^{-1}$ *is an automorphism;*
(ii) $\psi : E_S \to E_S$ *defined by* $e\psi = (\lambda e)^{-1}(\lambda e)$ *is the identity map.*

Then there exists a unique $\rho \in P(S)$ *such that* $(\lambda, \rho) \in \Sigma(S)$.
Conversely, for every $(\lambda, \rho) \in \Sigma(S)$, λ *satisfies* (i) *and* (ii).

Proof. Let $\lambda \in \Lambda(S)$ satisfy (i) and (ii). Define $\rho : S \to S$ by

$$x\rho = x(\lambda e), \quad \text{where} \quad e\theta = x^{-1}x.$$

By Lemma 1.4(iii), to see that ρ is in $P(S)$, it is sufficient to show that $x\rho = x(x^{-1}x\rho)$ for all x in S and $e(f\rho) = e\rho$ for $e, f \in E_S$ with $e \leqq f$. First, if $x^{-1}x = e\theta$, then

$$x(x^{-1}x\rho) = x(x^{-1}x(\lambda e)) = x(\lambda e) = x\rho.$$

Let e, f be idempotents with $e \leqq f$. By definition of ρ, $e(f\rho) = e(f(\lambda g))$, where $g\theta = f$. That is, $(\lambda g)(\lambda g)^{-1} = f$. Since θ maps onto E_S, $e = (\lambda h)(\lambda h)^{-1}$ for some h in E_S. From the fact that θ is an automorphism, we have $h \leqq g$. Thus

$$(\lambda h)^{-1}(\lambda g) = [\lambda(gh)]^{-1}(\lambda g) = [(\lambda g)h]^{-1}(\lambda g) = h^{-1}(\lambda g)^{-1}(\lambda g) = hg = h,$$

and consequently

$$e(f\rho) = e(\lambda g) = e(\lambda h)(\lambda h)^{-1}(\lambda g) = e(\lambda h)h = e(\lambda h) = e\rho.$$

Therefore ρ is in $P(S)$.
If e, f are in E_S, then $e = g\theta$ for some g in E_S and

$$(e\rho)f = e(\lambda g)f = e(\lambda gf) = e((gf)\theta\rho) = e((g\theta)((f\theta)\rho)) = e((f\theta)\rho) = e(\lambda f).$$

By Lemma 1.4(iv), (λ, ρ) is in $\Omega(S)$.

Now $(\lambda, \rho)(\lambda^{-1}, \rho^{-1}) = (\lambda\lambda^{-1}, \rho\rho^{-1})$. If e is in E_S, then there exists an f in E_S such that $e = (\lambda f)(\lambda f)^{-1}$. Recalling the definitions of λ^{-1} and ρ^{-1}, we have

$$\lambda\lambda^{-1}e = \lambda(e\rho)^{-1} = \lambda(\lambda f)^{-1} = \lambda(f\rho^{-1}) = \lambda(f(f\rho^{-1})) = (\lambda f)(f\rho^{-1}) = (\lambda f)(\lambda f)^{-1} = e.$$

Hence $\lambda\lambda^{-1} = i_S$. Since S is reductive, i_S is linked to just one element of $P(S)$ and therefore $\rho\rho^{-1} = i_S$. Using (ii) and the definition of ρ, a similar argument yields $\lambda^{-1}\lambda = i_S$, $\rho^{-1}\rho = i_S$. Hence (λ, ρ) is in $\Sigma(S)$. By the reductivity of S, ρ is unique.

Conversely, let $(\lambda, \rho) \in \Sigma(S)$. Then $\lambda\lambda^{-1} = \lambda^{-1}\lambda = i_S$, $\rho\rho^{-1} = \rho^{-1}\rho = i_S$. Define θ as in (i). Then, for e, f in E_S, we have

$$\begin{aligned}(e\theta)(f\theta) &= (\lambda e)(\lambda e)^{-1}(\lambda f)(\lambda f)^{-1} = (\lambda e)(e\rho^{-1})(\lambda f)(\lambda f)^{-1}\\ &= (\lambda e)(\lambda^{-1}\lambda f)(\lambda f)^{-1} = (\lambda e)f(\lambda f)^{-1} = (\lambda ef)e(\lambda f)^{-1}\\ &= (\lambda ef)e(f\rho^{-1}) = (\lambda ef)(ef\rho^{-1}) = (\lambda ef)(\lambda ef)^{-1} = (ef)\theta;\end{aligned}$$

therefore θ is a homomorphism. Since $\lambda\lambda^{-1}$, $\lambda^{-1}\lambda$, $\rho\rho^{-1}$, $\rho^{-1}\rho$ are all equal to the identity on S, λ, λ^{-1}, ρ, ρ^{-1} are all one-to-one mappings of S onto itself.

Since λ maps onto S, if e is in E_S, then there exists x in S such that $\lambda x = e$. Therefore

$$\begin{aligned}(xx^{-1})\theta &= \lambda(xx^{-1})[\lambda(xx^{-1})]^{-1} = \lambda(xx^{-1})[(\lambda x)x^{-1}]^{-1}\\ &= \lambda(xx^{-1})x(\lambda x)^{-1} = (\lambda x)(\lambda x)^{-1} = ee = e,\end{aligned}$$

and thus θ maps E_S onto E_S.

If $f\theta = g\theta$, then $(\lambda f)(\lambda f)^{-1} = (\lambda g)(\lambda g)^{-1}$. That is, $\lambda[f(\lambda f)^{-1}] = \lambda[g(\lambda g)^{-1}]$ and thus $f(\lambda f)^{-1} = g(\lambda g)^{-1}$. By definition of ρ^{-1}, this equation can be written as $f(f\rho^{-1}) = g(g\rho^{-1})$. Therefore $f\rho^{-1} = g\rho^{-1}$ and, since ρ^{-1} is one-to-one, $f = g$. Consequently θ is one-to-one and thus an automorphism.

Since $(\lambda e)^{-1}(\lambda e) = (e\rho^{-1})(\lambda e) = e\lambda^{-1}\lambda e = ee = e$, ψ as defined in (ii) is the identity map.

Corollary 4.2. *Let S be an inverse semigroup. Then*

$$\Sigma(S) \simeq \{\lambda \in \Lambda(S) \mid \lambda \textit{ satisfies } \text{(i), (ii)} \textit{ of Theorem } 4.1\}.$$

Proof. This follows directly from Theorem 4.1.

We shall henceforth take $\Sigma(S)$ equal to the subsemigroup of $\Lambda(S)$ whose elements satisfy (i) and (ii) of Theorem 4.1.

5. Example. Let $S = S^0$ be an inverse semigroup with E_S the orthogonal sum of chains C_α each having an identity e_α $(\alpha \in A)$. Define $\sim$ on A by

$$\alpha \sim \beta \quad \text{if } e_\alpha \text{ and } e_\beta \text{ are in the same } \mathscr{D}\text{-class of } S.$$

Then $\sim$ is an equivalence relation on A. Let $\{A_i\}_{i \in I}$ be the set of distinct equivalence classes of $\sim$, and, for $i \in I$, fix α_i in A_i. For any α in A, let R_α be the $\mathscr{R}$-class containing e_α and L_α be the $\mathscr{L}$-class containing e_α.

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The symmetric group on a set A will be denoted by $\mathscr{S}(A)$. The wreath product of a group G with $\mathscr{S}(A)$, denoted by $G \operatorname{wr} \mathscr{S}(A)$, is defined on the set

$$\{(\theta, \gamma) \mid \gamma \in \mathscr{S}(A), \theta : A \to G\},$$

with multiplication $(\theta, \gamma)(\theta', \gamma') = (\theta'', \gamma\gamma')$, where $i\theta'' = (i\theta)(i\gamma\theta')$ for all i in A. Under this multiplication, $G \operatorname{wr} \mathscr{S}(A)$ is again a group.

THEOREM 5.1. *For S as described above,*

$$E_{\Omega(S)} \simeq \bigtimes_{\alpha \in A} C_\alpha, \quad \textit{and} \quad \Sigma(S) \simeq \bigtimes_{i \in I} (G_i \operatorname{wr} \mathscr{S}(A_i)),$$

where $G_i = R_{\alpha_i} \cap L_{\alpha_i}$.

Proof. The first part follows directly from Proposition 3.8. To prove the second, we shall use Corollary 4.2 to find all elements of $\Sigma(S)$ and set up the isomorphism.

Let $\eta_i : A_i \to A_i$ be a permutation for all $i \in I$, and let $\alpha\eta = \alpha\eta_i$ if $\alpha \in A_i$.

Define λ on S as follows. For $\alpha \in A$, pick λe_α in $R_{\alpha\eta} \cap L_\alpha$. Then, for x in S, define $\lambda x = (\lambda e_\alpha)x$, where $xx^{-1} \leqq e_\alpha$: by Lemma 1.4(ii), it is clear that λ is in $\Lambda(S)$.

By Lemma 1.2 of [2] $\theta_\alpha : C_\alpha \to C_{\alpha\eta}$, defined by $e\theta_\alpha = (\lambda e_\alpha)e(\lambda e_\alpha)^{-1}$ is an isomorphism. But $(\lambda e_\alpha)e(\lambda e_\alpha)^{-1} = (\lambda e)(\lambda e)^{-1}$, so that $\theta : E_S \to E_S$ defined by $e\theta = (\lambda e)(\lambda e)^{-1}$ is an automorphism. In addition, for $e \leqq e_\alpha$, $\lambda e_\alpha \in L_\alpha$ implies that

$$(\lambda e)^{-1}(\lambda e) = e(\lambda e_\alpha)^{-1}(\lambda e_\alpha)e = ee_\alpha e = e.$$

Therefore, by Corollary 4.2, λ is in $\Sigma(S)$.

Conversely, if $\lambda \in \Sigma(S)$, then $e = (\lambda e)^{-1}(\lambda e)$ for all e in E_S and $\theta : e \to (\lambda e)(\lambda e)^{-1}$ is an automorphism of E_S which maps every element into its own $\mathscr{D}$-class. Thus, for $\alpha \in A$, $e_\alpha \theta$ is a maximal idempotent, and $\eta : A \to A$ defined by $\alpha\eta = \beta$ if $e_\alpha\theta = e_\beta$, is a permutation which maps A_i onto A_i for all i in I. Furthermore, by definition of θ and ψ, λe_α is in $R_{\alpha\eta} \cap L_\alpha$.

We define

$$\mathscr{M}(S) = \{x \in S \mid x \in R_\alpha \cap L_\beta \quad \text{for some} \quad \alpha, \beta \in A\} \cup \{0\},$$

with multiplication defined by

$$x \cdot y = \begin{cases} xy & \text{if} \quad xy \in \mathscr{M}(S), \\ 0 & \text{otherwise.} \end{cases}$$

It can easily be seen that $\mathscr{M}(S)$ is a semigroup. In fact, $\mathscr{M}(S)$ is a primitive inverse semigroup which is the orthogonal sum of Brandt semigroups B_i $(i \in I)$, where $e_\alpha \in B_i$ if and only if $\alpha \in A_i$.

Now, for $\lambda \in \Sigma(S)$, the permutation η maps A_i onto itself, with $\lambda e_\alpha \in R_{\alpha\eta} \cap L_\alpha$. Hence λe_α is in B_i whenever $e_\alpha \in B_i$ and, since θ is an automorphism, λ maps $B_i \backslash 0$ onto itself for all $i \in I$. On B_i, define λ_i by

$$\lambda_i(x) = \begin{cases} \lambda x & \text{if} \quad x \in B_i \backslash 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

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Then λ_i is in $\Lambda(B_i)$ and, in particular, λ_i is in $\Sigma(B_i)$. Define χ on $\Sigma(S)$ by $\lambda\chi = (\lambda_i)_{i\in I}$. Then it can easily be seen that χ is a homomorphism into $\bigtimes_{i\in I} \Sigma(B_i)$. The mapping is one-to-one by definition of λ_i; it is onto since, if λ_i is in $\Sigma(B_i)$, for all i in I, we can define λ on S by $\lambda x = (\lambda_i e_\alpha)x$, where $xx^{-1} \leqq e_\alpha$, $e_\alpha \in B_i$. Then λ is in $\Sigma(S)$ and $\lambda|_{B_i\backslash 0} = \lambda_i$ for all $i \in I$.

Since B_i is a Brandt semigroup, $\Sigma(B_i)$ is isomorphic to $G_i \,\mathrm{wr}\, \mathscr{S}(A_i)$, where $G_i = R_{\alpha_i} \cap L_{\alpha_i}$ [**4**, Theorem 1]. Consequently

$$\Sigma(S) \simeq \bigtimes_{i\in I} \Sigma(B_i) \simeq \bigtimes_{i\in I} (G_i \,\mathrm{wr}\, \mathscr{S}(A_i)).$$

COROLLARY 5.2. *Let S be a 0-bisimple semigroup with E_S the orthogonal sum of chains C_α each with identity e_α $(\alpha \in A)$. Then $\Sigma(S) \simeq G \,\mathrm{wr}\, \mathscr{S}(A)$, where $G = R_\alpha \cap L_\alpha$, for any $\alpha \in A$.*

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