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ON THE SEMIPRIMITIVITY OF SKEW POLYNOMIAL RINGS

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Let R be a left Noetherian ring with the ascending chain condition on right annihilators, let α be a ring monomorphism of R and δ an α-derivation of R. We prove that, if R is semiprime or α-prime, then $R[X;\alpha,\delta]$ is semiprimitive (and left Goldie), and that $J(R[X;\alpha])$ equals $N(R)[X;\alpha]$.

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Let R be a ring. A well known result of Amitsur [**1**] states that if R has no nil ideals then the polynomial ring $R[x]$ is semiprimitive. Various authors, for example Bedi and Ram [**2**], Bell [**3**], and C. R. Jordan and D. A. Jordan [**13**], have extended this result to skew polynomial rings of the form $R[x;\ \alpha,\delta]$, where α is an automorphism of R, and δ is an α-derivation of R. Most of these have worked either with the case $\delta=0$ and α an automorphism or the case where α is the identity. El Ahmar [**8**] has shown that, if R is right and left Noetherian, α: $R\to R$ a monomorphism, then $R[x;\ \alpha]$ is semiprimitive. An example [**13**, §5] shows that some conditions on R and α are necessary if results of this kind are to be valid. Bell [**3**] has shown that if R is semiprime left Goldie with α: $R\to R$ an automorphism and δ an α-derivation then $R[x;\ \alpha,\delta]$ is semiprimitive left Goldie, and has commented that it is not known whether this generalizes to the case where α is not surjective.

The situation we shall be concerned with is that of a ring R and a monomorphism α: $R\to R$ which is not assumed to be surjective. Let R be a left Noetherian ring with ascending chain condition on right annihilators, $N(R)$ its nilpotent radical. Dean [**7**] has shown that $\alpha(N(R))\subseteq N(R)$. We use methods adapted from those of some of the above authors together with Dean's result and the construction of the ring $A(R,\alpha)$ of Jordan [**14**], to show that $R[x;\alpha,\delta]$ is semiprimitive left Goldie if R is semiprime or α-prime. We also show that if R is semiprime left or right Goldie and α: $R\to R$ is a monomorphism then $R[x;\ \alpha]$ is semiprimitive. We use this result to see that if R is left Noetherian with ascending chain condition on right annihilators then

$$J(R[x;\ \alpha])=N(R[x;\ \alpha])=N(R)[x;\ \alpha].$$

1. Preliminaries

All rings in this paper have 1 and all endomorphisms are assumed to preserve 1.

Let R be a ring and $\alpha: R \to R$ a monomorphism. An α-derivation is an additive map δ from R to R such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for all $a, b \in R$. Given such an α and δ, the Ore extension $R[x; \alpha, \delta]$ is the ring of all formal linear combinations $\sum_{i=0}^{n} r_i x^i$, $r_i \in R$, $n \geqq 0$, with multiplication subject to the relation $xr = \alpha(r)x + \delta(r)$. If $\delta = 0$ we have the skew polynomial ring $R[x; \alpha]$. Then the set $\{x^i\}_{i \geqq 1}$ is easily seen to be a left Ore set so that one can localize and form the skew Laurent polynomial ring, $R[x, x^{-1}; \alpha]$. Elements of $R[x, x^{-1}; \alpha]$ are finite sums of elements of the form $x^{-i} r x^j$ where $r \in R$, $i \geqq 0$, $j \geqq 0$. Multiplication is subject to $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$.

In **[14]**, D. A. Jordan has constructed an overring $A(R, \alpha)$ of R, which is, in a sense, the minimal overring of R to which α extends as an automorphism. Consider an element of $R[x, x^{-1}; \alpha]$ of the form $x^{-i} r x^i$, $r \in R$, $i \geqq 0$. Then, for $j \geqq 0$, $x^{-j} r x^j = x^{-(i+j)} \alpha^i(r) x^{(i+j)}$. It follows that the set of all such elements forms a subring of $R[x, x^{-1}; \alpha]$, with

$$x^{-i} r x^i + x^{-j} s x^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s)) x^{(i+j)} \text{ and}$$

$$x^{-i} r x^i . \ x^{-j} s x^j = x^{-(i+j)}(\alpha^j(r) . \ \alpha^i(s)) x^{(i+j)}, \text{ with } r, s \in R.$$

This subring is denoted $A(R, \alpha)$. We extend α to $A(R, \alpha)$, by setting $\alpha(x^{-i} r x^i) = x^{-i} \alpha(r) x^i$. Since $\alpha(x^{-(i+1)} r x^{(i+1)}) = x^{-i} r x^i$, α is an automorphism of $A(R, \alpha)$.

Definition 1.1. Let I be an ideal of a ring R, $\alpha: R \to R$ be a monomorphism. Then I is said to be an *α-ideal* if $\alpha(I) \subseteq I$; I is said to be *α-invariant* if $\alpha^{-1}(I) = I$; I is said to be *α-prime* if it is α-invariant and for any α-ideals A and B of R, $AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$. The ring R is *α-prime* if 0 is an α-prime ideal of R.

Lemma 1.2. *The ring R is α-prime if and only if, for $a, b \in R$, $\alpha^t(a) R \alpha^s(b) = 0$ for all $t \geqq 0$, $s \geqq 0$, implies that $a = 0$ or $b = 0$.*

Proof. The proof is straightforward.

Definition 1.3. We mean by a *right annihilator ideal* (*left annihilator ideal*), an ideal of the form rann I, (respectively lann I), where I is an ideal of R. If an ideal of R is of the form rann I = lann I, where I is an α-ideal of R, then it is said to be an *annihilator α-ideal.*

By an *annihilator α-prime ideal*, we mean an annihilator α-ideal which is also α-prime.

The nilpotent radical of a ring R will be taken to be the sum of the nilpotent ideals of R and will be denoted by $N(R)$. The Jacobson radical of R will be denoted by $J(R)$.

Lemma 1.4. *Let R be a ring satisfying the ascending chain condition on right annihilators and let I be an α-ideal of R. Then* rann I *is an α-invariant ideal.*

Proof. Since rann $I \subseteq \text{rann}(\alpha(I)) \subseteq \text{rann}(\alpha^2(I)) \subseteq \ldots$, we have that, for some $k \geqq 0$, $\text{rann}(\alpha^k(I)) = \text{rann}(\alpha^{k+1}(I))$. Let $a \in \text{rann } I$. Then $\alpha^{k+1}(a) \in \text{rann}(\alpha^k(I))$, so $\alpha(a) \in \text{rann } I$. If $\alpha(a) \in \text{rann } I$, then $\alpha^{k+1}(a) \in \text{rann}(\alpha^{k+1}(I))$, so $a \in \text{rann } I$.

Lemma 1.5. *Let R be a ring satisfying the ascending chain condition on right annihilators and P be an α-invariant ideal of R. Then, (i) for each $t \geqq 1$, $P \subseteq$* $\mathrm{lann}(\alpha^t(\mathrm{rann}(P)))$, *and, (ii) if $P = \mathrm{lann}\ L$ for a subset L of R, then* $\mathrm{lann}\ L \subseteq \mathrm{lann}(\alpha^t(L))$, $t \geqq 1$.

Proof. (i) The chain $\mathrm{rann}\ P \subseteq \mathrm{rann}(\alpha(P)) \subseteq \dots$ terminates. So for some integer $k \geqq 0$ and for all $t \geqq 0$, $\mathrm{rann}(\alpha^k(P)) = \mathrm{rann}(\alpha^{k+t}(P))$. Let $t \geqq 1$, and $b \in \mathrm{rann}\ P$. Then $Pb = 0$, so $\alpha^{k+t}(b) \in \mathrm{rann}(\alpha^k(P))$. Since α^k is injective $P\alpha^t(b) = 0$.

(ii) We have for each $t \geqq 1$, $P = \mathrm{lann}\ L \subseteq \mathrm{lann}(\alpha^t(\mathrm{rann}(P))) = \mathrm{lann}(\alpha^t(\mathrm{rann}(\mathrm{lann}(L))))$. But $\alpha^t(L) \subseteq \alpha^t(\mathrm{rann}(\mathrm{lann}(L)))$. Thus

$$\mathrm{lann}(\alpha^t(\mathrm{rann}(\mathrm{lann}(L)))) \subseteq \mathrm{lann}(\alpha^t(L)), t \geqq 1.$$

Proposition 1.6. *Let R be a semiprime ring satisfying the ascending chain condition on right annihilators. Then R has only a finite number of minimal α-prime ideals, and their intersection is zero. An α-prime ideal of R is minimal if and only if it is an annihilator α-ideal.*

Proof. First we show that each annihilator α-ideal of R contains a product of annihilator α-prime ideals. Suppose not and let L be an annihilator α-ideal of R which is maximal with respect to not containing a product of annihilator α-prime ideals. So $L = \mathrm{rann}\ I = \mathrm{lann}\ I$, for some α-ideal I of R. Now, L cannot be an α-prime ideal, otherwise it is, itself, an annihilator α-prime ideal. Hence there are α-ideals T and K of R which strictly contain L such that $TK \subseteq L$. Take $C = \mathrm{rann}(IT)$ and $B = \mathrm{lann}(CT)$. By [**16**, Proposition 2.2.14], $C = \mathrm{rann}(IT) = \mathrm{lann}(IT)$, and $B = \mathrm{lann}(CI) = \mathrm{rann}(CI)$. Since $IL = 0$ and $TK \subseteq L$, $ITK = 0$. Since $IT \subseteq I$, $L = \mathrm{rann}\ I \subseteq \mathrm{rann}(IT)$, whence $C \supseteq L$. Also $B = \mathrm{lann}(CI) \supseteq \mathrm{lann}\ I$, so $B \supseteq L$. But $B = \mathrm{lann}(CI)$ thus $BC \subseteq \mathrm{lann}\ I = L$. Since I and T are α-ideals, C is an annihilator α-ideal. Also B is an annihilator α-ideal. We have $ITK = 0$, and $CIT = 0$, so $K \subseteq C$ and $T \subseteq B$. Since T and K strictly contain L, the annihilator α-ideals B and C strictly contain L with $BC \subseteq L$. By the choice of L, B and C each contain a product of annihilator α-prime ideals so also does L. Therefore we can deduce that every annihilator α-ideal of R contains a product of annihilator α-prime ideals. Since the zero ideal of R is an annihilator α-ideal, there are annihilator α-prime ideals $P_1, P_2, \dots, P_n$ of R say, such that $P_1 P_2 \dots P_n = 0$. We have $(P_1 \cap P_2 \cap \dots \cap P_n)^n = 0$. Since R is semiprime, $P_1 \cap P_2 \cap \dots \cap P_n = 0$.

Now let P be a minimal α-prime ideal of R. Then $P_1 P_2 \dots P_n \subseteq P$, where $P_1, P_2, \dots, P_n$ are annihilator α-prime ideals of R. Thus $P_i \subseteq P$ for some $1 \leqq i \leqq n$. Hence $P = P_i$ for some $1 \leqq i \leqq n$.

Conversely, suppose that P is an annihilator α-prime ideal of R. Let P' be an α-prime ideal of R with $P' \subseteq P$. Suppose that $\mathrm{rann}\ P \subseteq P'$, then $\mathrm{rann}\ P \subseteq P$ so that $(\mathrm{rann}\ P)^2 = 0$. Hence $\mathrm{rann}\ P = 0$. But P is an annihilator α-ideal and this is a contradiction. Therefore $\mathrm{rann}\ P \not\subseteq P'$. Now $P.\mathrm{rann}\ P \subseteq P'$, P and $\mathrm{rann}\ P$ are α-ideals, thus $P \subseteq P'$. Hence $P = P'$.

Proposition 1.7. *Let R be a semiprime left (or right) Goldie ring, α: $R \to R$ be a monomorphism. For an α-invariant ideal I of R, as in* [14], *let $I_i = \{a \in R\colon x^{-i}ax^i \in I\}$.*

(i) *Let P be a minimal α-prime ideal of R. Then $\Delta(P) = \bigcup_{i \geqq 0} x^{-i}Px^i$ is a minimal α-prime ideal of $A(R,\alpha)$.*

(ii) *Let P be an α-invariant ideal of $A(R,\alpha)$. Then for each $i \geqq 0$, P_i is an α-invariant ideal of R. Also $P_i = P_j$ for each $j \geqq 0$.*

(iii) *Let P be a minimal α-prime ideal of $A(R,\alpha)$. Then P_i is a minimal α-prime ideal of R. So that there is a $1-1$ correspondence between the minimal α-prime ideals of R and of $A(R,\alpha)$ via $P \mapsto \Delta(P)$.*

Proof. (i) Let P be an α-prime ideal of R. We show that $\Delta(P)$ is an α-prime ideal of $A(R,\alpha)$. We have $\alpha(\Delta(P)) = \Delta(P)$. Let I and J be α-ideals of $A(R,\alpha)$ with $IJ \subseteq \Delta(P)$ and $J \not\subseteq \Delta(P)$. We have $(\Delta(P))_i = P$, and $J_i \not\subseteq P$ for some $i \geqq 0$. Since $I_iJ_i \subseteq (IJ)_i$, $I_iJ_i \subseteq P$, so $I_i \subseteq P$. We show that for each $t \geqq 0$, $I_t \subseteq P$. Let $a \in I_t$. Then $x^{-t}ax^t \in I$. If $t \geqq i$ then $x^{-i}ax^i = x^{-t}\alpha^{t-i}(a)x^t = \alpha^{t-i}(x^{-t}ax^t) \in I$, whence $a \in I_i \subseteq P$. If $t < i$, then $\alpha^{i-t}(x^{-i}ax^i) = x^{-i}\alpha^{i-t}(a)x^i = x^{-t}ax^t \in I$, so $\alpha^{i-t}(a) \in I_i \subseteq P$. Thus $I \subseteq \Delta(P)$.

Now, suppose that P is a minimal α-prime ideal of R. By Proposition 1.6, $P = \text{lann } I = \text{rann } I$, for an α-ideal I of R. We show that $\Delta(P) = \text{lann}(\Delta(I))$. To see this, let $x^{-i}ax^i \in \Delta(P)$, with $a \in P$. Let $x^{-j}bx^j \in \Delta(I)$, with $b \in I$. Then $x^{-i}ax^ix^{-j}bx^j = x^{-(i+j)}\alpha^j(a)\alpha^i(b)x^{(i+j)}$, with $\alpha^j(a) \in P$ and $\alpha^i(b) \in I$. So $\Delta(P) \subseteq \text{lann } \Delta(I)$. Conversely, let $x^{-i}ax^i \in \text{lann } \Delta(I)$. Then $x^{-i}ax^i$. $\Delta(I) = 0$. Let $b \in I$. Then $x^{-i}ax^i.x^{-i}bx^i = 0$, so $a \in \text{lann } I = P$. Thus lann $\Delta(I) \subseteq \Delta(P)$. By Proposition 1.6, $\Delta(P)$ is a minimal α-prime ideal of $A(R,\alpha)$.

(ii) The proof is straightfoward.

(iii) Let I and J be α-ideals of R with $IJ \subseteq P_i$, then $\Delta(I)$ and $\Delta(J)$ are α-ideals of $A(R,\alpha)$, with $\Delta(I)\Delta(J) \subseteq P$; so $I \subseteq P_i$ or $J \subseteq P_i$. Thus P_i is an α-prime ideal of R. By Proposition 1.6, $P = \text{lann } M$ for an α-ideal M of $A(R,\alpha)$. Then M_i is an α-ideal of R. A similar argument shows that $P_i = \text{lann } M_i$ and P_i is therefore a minimal α-prime ideal of R.

Corollary 1.8. *A ring R is α-prime if and only if $A(R,\alpha)$ is α-prime.*

2. Semiprimitivity of $R[x; \alpha, \delta]$

Throughout the remainder of the paper, let R be a ring with α: $R \to R$ a monomorphism and $\delta: R \to R$ an α-derivation. Let I be a non-zero ideal of $R[x; \alpha, \delta]$. For each $n \geqq 0$, let $\tau_n(I) = \{a \in R$: there exists a non-zero polynomial in I with degree n and leading coefficient $a\} \cup \{0\}$.

Then $\tau_n(I)$ is a non-zero left ideal of R, with $\alpha(\tau_n(I)) \subseteq \tau_{n+1}(I)$ and $\tau_n(I) \supseteq \tau_n(I)\alpha^n(R)$, $n \geqq 0$.

Theorem 2.1. *Let R be an α-prime left Noetherian ring. Then $R[x; \alpha, \delta]$ is prime.*

Proof. The chain $\tau_0(I) \subseteq \tau_1(I) \subseteq \cdots \subseteq \tau_n(I) \subseteq \ldots$, will terminate for some integer

$p \geqq 0$. We then have $\tau_p(I) = \tau_{p+1}(I)$. Then $\alpha(\tau_p(I)) \subseteq \tau_p(I)$ and $\tau_p(I) \supseteq \tau_p(I)\alpha^p(R)$. Put $\tau_p(I) \cap \alpha^p(R) = \alpha^p(M)$. Then $\tau_p(I) \subseteq M$ and M is a non-zero α-ideal of R. By Lemma 1.5, rann M is an α-ideal of R. We have $M.\text{rann } M = 0$, and R is α-prime, so rann $M = 0$. We show that rann $I = 0$. To see this, let $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \text{rann } I$, with $a_n, a_{n-1}, \ldots, a_1, a_0 \in R$. Let $b \in M$. Then $\alpha^p(b) \in \tau_p(I)$, and there exists a polynomial

$$\alpha^p(b)x^p + b_{p-1}x^{p-1} + \cdots + b_1 x + b_0 \in I,$$

with $b, b_{p-1}, \ldots, b_1, b_0 \in R$, such that $\alpha^p(b)\alpha^p(a_n) = 0$. Thus $Ma_n = 0$ and $a_n = 0$. Therefore the result follows.

Theorem 2.2. *Let R be α-prime left Noetherian with ascending chain condition on right annihilators, $\alpha: R \to R$ a monomorphism and $\delta: R \to R$ an α-derivation. Then $R[x; \alpha, \delta]$ is semiprimitive.*

Proof. Let J be the Jacobson radical of $R[x; \alpha, \delta]$ and suppose that $J \neq 0$. As in the proof of Theorem 2.1, there exists an integer $p \geqq 0$ such that $\tau_p(J)$ is a non-zero left ideal of R with $\alpha(\tau_p(J)) \subseteq \tau_p(J)$ and $\tau_p(J) \supseteq \tau_p(J)\alpha^p(R)$. Now, the subset $\Delta = \bigcup_{i \geqq p} x^i \tau_p(J) x^i$ of $A(R, \alpha)$ is an ideal of $A(R, \alpha)$. To see this, let $x^{-i}ax^i \in \Delta$, $x^{-j}rx^j \in A(R, \alpha)$, with $i \geqq P$, $j \geqq 0$, $a \in \tau_p(J)$ and $r \in R$. Then $x^{-i}ax^i . x^{-j}rx^j = x^{-(i+j)}\alpha^j(a)\alpha^i(r)x^{(i+j)}$. We have $\alpha^j(a) \in \tau_p(J)$ and $\alpha^j(a)\alpha^i(r) \in \tau_p(J)\alpha^i(R) \subseteq \tau_p(J)$.

We show that every element of Δ is a zero divisor. Let $x^{-i}ax^i \in \Delta$, $i \geqq p$ and $a \in \tau_p(J)$. Then for some polynomial $f(x) = ax^p + a_{p-1}x^{p-1} + \cdots + a_1 x + a_0 \in J$ and a polynomial $g(x) = c_q x^q + c_{q-1}x^{q-1} + \cdots + c_1 x + c_0 \in R[x; \alpha, \delta]$, with $c_q \neq 0$, we have $(1 + f(x)x)g(x) = 1$. By comparing the leading term in this equation we have $a\alpha^{p+1}(c_q) = 0$, so $x^{-i}ax^i . x^{-i}\alpha^{p+1}(c_q)x^i = 0$. Thus $x^{-i}ax^i$ is a zero divisor. Since R is left Noetherian with ascending chain condition on right annihilators, by [7], we have $\alpha(N(R)) \subseteq N(R)$. But R is α-prime, and $\{N(R)\}^n = 0$ for some $n \geqq 0$. Then R is semiprime. By [**14**, Corollary 7.5], $A(R, \alpha)$ is semiprime left Goldie. By [**16**, Proposition 2.3.5.], Δ cannot be essential as a left ideal. Hence there exists a left ideal L of $A(R, \alpha)$ which is non zero and $\Delta . L \subseteq \Delta \cap L = 0$. Since the intersection of minimal prime ideals of $A(R, \alpha)$ is zero, [**16**, Theorem 2.2.15], some minimal prime ideal of $A(R, \alpha)$ must contain Δ. Since by Lemma 1.3, $A(R, \alpha)$ is α-prime, minimal prime ideals of $A(R, \alpha)$ form a single orbit under α and yet, as above $\alpha(\Delta) \subseteq \Delta$. Therefore $\Delta = 0$ and $J = 0$.

Theorem 2.3. *Let R be a semiprime left Noetherian ring with ascending chain condition on right annihilators, $\alpha: R \to R$ a monomoprhism and $\delta: R \to R$ an α-derivation. Then $R[x; \alpha, \delta]$ is semiprimitive left Goldie.*

Proof. By Proposition 1.6, R contains finitely many minimal α-prime ideals $P_1, P_2, \ldots, P_n$, say, with $P_1 \cap P_2 \cap \cdots \cap P_n = 0$. Let $\mathscr{C} = \mathscr{C}_R(0)$ denote the set of regular elements of R. By Goldie's Theorem, [**16**, Theorem 2.3.6.], R has a semisimple Artinian quotient ring $Q = \mathscr{C}^{-1}R$. By [**12**, Proposition 2.4]. we have $\alpha^{-1}(\mathscr{C}) = \mathscr{C}$. We extend α and δ to Q, with

$$\tilde{\alpha}(c^{-1}r) = \alpha(c)^{-1}\alpha(r) \text{ and } \tilde{\delta}(c^{-1}r) = \alpha(c)^{-1}\delta(r) - \alpha(c)^{-1}\delta(r)c^{-1}r,$$

for $c \in \mathscr{C}$, $r \in R$. We adapt the proof of Proposition 2.1 of Bell [**3**]. Here α is not assumed to be surjective. Let $1 \leqq i \leqq n$. By [**16**, Proposition 2.1.16 (vi)], QP_i is an ideal of Q. By Corollary 1 of [**10**, Theorem 1.4.2], $QP_i = Qe_i$ for a central idempotent $e_i \in Q$. Thus e_i is the identity of Qe_i. Let $c^{-1}a \in QP_i$, with $c \in \mathscr{C}$ and $a \in P_i$. Then $\tilde{\alpha}(c^{-1}a) = \alpha(c)^{-1}\alpha(a) \in QP_i$, because $\alpha^{-1}(P_i) = P_i$ and $\alpha(c) \in \mathscr{C}$. Thus $\tilde{\alpha}(QP_i) \subseteq QP_i$. We have $\tilde{\alpha}(e_i)e_i = e_i$ and $(\tilde{\alpha}(e_i) - e_i)e_i = 0$, so $\tilde{\alpha}(e_i) = e_i$. We have also,

$$\tilde{\delta}(e_i) = \tilde{\delta}(e_i^2) = \tilde{\delta}(e_i)e_i + \tilde{\alpha}(e_i)\tilde{\delta}(e_i) = 2e_i\tilde{\delta}(e_i).$$

Hence $e_i\tilde{\delta}(e_i) = 2e_i^2\tilde{\delta}(e_i) = 2e_i\tilde{\delta}(e_i)$, so $e_i\tilde{\delta}(e_i) = 0$ and $\tilde{\delta}(e_i) = 2e_i\tilde{\delta}(e_i) = 0$. Thus $\tilde{\delta}(QP_i) = \tilde{\delta}(\mathscr{C}^{-1}P_i) = \tilde{\delta}(Qe_i) \subseteq Qe_i = \mathscr{C}^{-1}P_i$ and $\delta(P_i) \subseteq \mathscr{C}^{-1}P_i \cap R = P_i$. Therefore for each $1 \leqq i \leqq n$, $\alpha^{-1}(P_i) = P_i$ and $\delta(P_i) \subseteq P_i$. This implies that $P_iR[x; \alpha, \delta]$ is an ideal of $R[x; \alpha, \delta]$.

There are induced monomorphisms and derivations $\bar{\alpha}_i$ and $\bar{\delta}_i$ on R/P_i, and we have

$$R[x; \alpha, \delta]/P_iR[x; \alpha, \delta] \cong (R/P_i)[x; \bar{\alpha}_i, \bar{\delta}_i].$$

For each $1 \leqq i \leqq n$, R/P_i is $\bar{\alpha}_i$-prime left Noetherian. We show that the ascending chain condition on annihilator right ideals passes to R/P_i, for each $1 \leqq i \leqq n$. By Proposition 1.6, $P_i = \operatorname{rann}(K_i)$, with K_i an α-ideal of R, for $1 \leqq i \leqq n$. We have P_i is an annihilator α-ideal. Let $\bar{0} \neq \bar{M} \subseteq R/P_i$ be a right annihilator in R/P_i. We show that $M = \{a \in R \colon \bar{a} \in \bar{M}\}$ is a right annihilator in R. Suppose that $\bar{M} = \operatorname{rann} \bar{I}$. Let I be the inverse image of $\bar{I}$ in R. We have $\bar{I} \neq 0$, otherwise $\bar{M} = R/P_i$.

Thus $I \not\subseteq P_i$, so $K_iI \neq 0$. We show that $M = \operatorname{rann}(K_iI)$. Since $\bar{I}\bar{M} = 0$, $K_iIM = 0$. Hence $IM \subseteq P_i$. Also, if $K_iIa = 0$, then $Ia \subseteq P_i$, so $\bar{I}\bar{a} = 0$ and $\bar{a} \in \bar{M}$. Now, let $\bar{M}_1 \subseteq \bar{M}_2 \subseteq \cdots \subseteq \bar{M}_n \subseteq \cdots$, be a chain of right annihilators in R/P_i. By above, $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \ldots$, is a chain of right annihilators in R, which terminates. Therefore for each $1 \leqq i \leqq n$, R/P_i is a left Noetherian ring satisfying the ascending chain condition on right annihilators. By Theorem 2.2, $(R/P_i)[x; \bar{\alpha}_i, \bar{\delta}_i]$ is semiprimitive. Thus $P_iR[x; \alpha, \delta]$ is a semiprimitive ideal of $R[x; \alpha, \delta]$. Since these ideals have zero intersection, $R[x; \alpha, \delta]$ is semiprimitive.

By Goldie's theorem [**16**, Theorem 2.3.6], $\mathscr{C} = \mathscr{C}_R(0)$ is a left Ore set in R and $Q = \mathscr{C}^{-1}R$ is a semisimple Artinian ring. By [**12**, Proposition 2.4], $\alpha^{-1}(\mathscr{C}) = \mathscr{C}$. One can show in a manner similar to that in [**9**, Lemma 1.4] that $\mathscr{C}$ is a left Ore set of regular elements in $R[x; \alpha, \delta]$. We then have

$$(\mathscr{C}^{-1}R)[x; \alpha, \delta] \cong \mathscr{C}^{-1}(R[x; \alpha, \delta]).$$

By [**5**, Theorem 3.2], $Q[x; \alpha, \delta])$ is semiprime left Noetherian. We have that Q is α-prime, so by Theorem 2.3, $Q[x; \alpha, \delta]$ is prime left Noetherian. By [**4**, Proposition 2.3], 0 is a left Goldie ideal of $R[x; \alpha, \delta]$, which means that $R[x; \alpha, \delta]$ is left Goldie.

Theorem 2.5. *Let R be a semiprime Goldie ring, $\alpha \colon R \to R$ a monomorphism such that α extends to an automorphism $\bar{\alpha}$ of the quotient ring $Q(R)$ of R. Let $\delta \colon R \to R$ be an α-derivation, then $R[x; \alpha, \delta]$ is semiprimitive left Goldie.*

Proof. By Proposition 1.6, R contains finitely many minimal α-prime ideals

$P_1, P_2, \ldots, P_n$, say, with $P_1 \cap P_2 \cap \cdots \cap P_n = 0$. For each $1 \leqq i \leqq n$, P_i is an annihilator α-prime ideal. By Theorem 2.3, we have

$$R[x;\alpha,\delta]/P_iR[x;\alpha,\delta] \cong (R/P_i)[x;\bar{\alpha},\bar{\delta}], \text{ and } \bigcap_{i=0} P_iR[x;\alpha,\delta]=0.$$

Let $\bar{S}=\{\bar{c}\in\bar{R}=R/P_i$: c is regular in $R\}$. We show that $\bar{S}$ consists of regular elements of $\bar{R}=R/P_i$ and satisfies the left Ore condition. To see this, let $\bar{c}\bar{r}=0$, with $c\in\mathscr{C}$, $r\in R$. Then $cr\in P_i$. But P_i is an annihilator α-ideal, so for an ideal A of R/P_i, we have $P_i=\text{lann } A=\text{rann } A$. Then $\text{Arc}=0$ implies that $r\in P_i$. Also, since $P_i=\text{lann } A$, $\bar{r}\bar{c}=0$, gives $\bar{r}=\bar{0}$. Now, if $\bar{c}\in\bar{S}$ and $\bar{a}\in\bar{R}$, then c is regular in R. So there is $b\in R$ and $d\in\mathscr{C}_R(0)$ such that $da=bc$. Hence $\bar{d}\bar{a}=\bar{b}c$, and by [**16**, Theorem 2.1.12], $\bar{R}_{\bar{S}}$ exists.

Now, let $a\in R$. Let $\tilde{\alpha}$ be as in the proof of Theorem 2.3. Since $\tilde{\alpha}^n$ is an automorphism of $Q(R)$, there is for each $n\geqq 0$, $b\in R$ and $c\in\mathscr{C}_R(0)$ such that $a=\tilde{\alpha}^n(bc^{-1})$. Thus, there exists $c\in\mathscr{C}$ such that, for each $n\geqq 0$, $a\alpha^n(c)\in\alpha(R)$. Therefore for each $\bar{a}\in\bar{R}$, and $n\geqq 0$, there exists $\bar{c}\neq 0$ such that $\bar{a}\bar{\alpha}^n(\bar{c})=\bar{\alpha}^n(\bar{b})$ with $\bar{b}\in\bar{R}$. Since $\alpha^{-1}(\mathscr{C})=\mathscr{C}$, $\bar{\alpha}^n(\bar{c})$ is regular in R/P_i. So $\bar{b}\neq\bar{0}$.

Let $\bar{J}=J(\bar{R}[x;\bar{\alpha},\bar{\delta}])$, with $\bar{R}=R/P_i$. Let $\bar{J}\neq 0$. By the above argument

$$\tau=\{a\in\bar{R}:\bar{\alpha}^n(a)x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\in\bar{J}, \text{ for some } n>0\}$$

is non-zero. Also τ is an $\bar{\alpha}$-ideal of $\bar{R}$. Then $\Delta(\tau)=\bigcup_{i\geqq 0}x^{-i}\tau x^i$ is an $\bar{\alpha}$-ideal of $A(\bar{R},\bar{\alpha})$.

Now, one can show that

$$A(R/P_i,\bar{\alpha})\cong A(R,\alpha)/\Delta(P_i)$$

with $a^{-i}\bar{a}x^i\mapsto x^{-i}ax^i+\Delta(P_i)$, where $\Delta(P_i)=\bigcup_{i\geqq 0}x^{-i}P_ix^i$.

By [**14**, Corollary 7.5], $A(R,\alpha)$ is semiprime left Goldie. By Proposition 1.7, $\Delta(P_i)$ is a minimal α-prime ideal of $A(R,\alpha)$. So $\Delta(P_i)$ is a finite intersection of minimal prime ideals of $A(R,\alpha)$. Using [**16**, Proposition 3.2.5], one can show that $A(R,\alpha)/\Delta(P_i)$ is α-prime left Goldie. So $A(R/P_i,\alpha)$ is semiprime left Goldie. Hence $\Delta(\tau)$ cannot be essential as a left ideal of $A(R/P_i,\bar{\alpha})$. Therefore as in the proof of Theorem 2.3, the result follows.

3. Semiprimitivity of $R[x;\alpha]$

Let R be a ring, $\alpha: R\to R$ a monomorphism. Let $f(x)=\sum_{i=m}^n a_ix^i\in R[x;\alpha]$, with $a_n\neq 0$ and $a_m\neq 0$, $n\geqq 0$, $m\geqq 0$. The *length* of $f(x)$ is the non-negative integer $n-m$. For an ideal I of $R[x;\alpha]$ we denote by $\mu(I)$ the set of non-zero elements of I of minimal length. We note that the set $\mu(I)\cup\{0\}$ is closed under multiplication on either side by elements of $R[x;\alpha]$ of length 0, i.e. elements of the form rx^i, $r\in R$, $i\geqq 0$.

The following lemma is proved in [**13**], where α is assumed to be an automorphism. However the proof in [**13**] remains valid in our more general situation.

Lemma 3.2. *Let J be the Jacobson radical of $R[x;\alpha]$. Let $f(x)=\sum_{i=m}^n a_ix^i\in\mu(J)$ be such that $m>0$. Then there exists an integer $s>0$ such that $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\ldots\alpha^{sn}(a_n)=0$.*

Definition 3.3. Let R be a ring, $\alpha: R \to R$ a monomorphism. Then the element $r \in R$ is said to be *α-nilpotent* if for all integers $k > 0$, there exists a positive integer $s = s(r,k)$ such that

$$r\alpha^k(r)\alpha^{2k}(r)\dots\alpha^{sk}(r) = 0.$$

An ideal (left ideal) of R is said to be *α-nil*, if every element of I is *α-nilpotent.*

Theorem 3.4. *Let R be a semiprime right Goldie ring. Let α: $R \to R$ be a monomorphism. Then $R[x; \alpha]$ is semiprimitive.*

Proof. Let J be the Jacobson radical of $R[x;\alpha]$ and assume that $J \neq 0$. Let $\tau = \{a \in R$: there exists a polynomial of minimal length in J with a its leading coefficient$\} \cup \{0\}$, which is non-zero left ideal of R, with $\alpha(\tau) \subseteq \tau$. By Lemma 3.2, τ is α-nil. By [**14**, Corollary 7.5], $A(R,\alpha)$ is semiprime right Goldie. Let $\Delta = \bigcup_{i \geqq 0} x^{-i}\tau x^i$. Then Δ is an α-nil left ideal of $A(R,\alpha)$. Suppose that Δ is not a nil left ideal of $A(R,\alpha)$. We adapt the proof of Theorem 2.1 of Ram [**18**], to show that for any $a \in \Delta$ which is a non-nilpotent element, there exists a positive integer n such that $a\alpha^n(a) \neq 0$. To see this let us suppose that $a\alpha^n(a) = 0$ for every positive integer n. Define

$$I_m = aA(R,\alpha) + \alpha^{-1}(a)A(R,\alpha) + \cdots + \alpha^{-m}(a)A(R,\alpha).$$

Note that α is an automorphism of $A(R,\alpha)$. Then we have

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m \dots,$$

and

$$\text{lann } I_0 \supseteq \text{lann } I_1 \supseteq \cdots \supseteq \text{lann } I_m \supseteq \cdots.$$

But the ascending chain condition on right annihilators is equivalent to the descending chain condition on left annihilators. Therefore for some positive integer t, lann I_t = lann I_{t+1}. Now, $a\alpha^n(a) = 0$ for each $n \geqq 1$ if and only if $\alpha^{-n}(a)a = 0$ for each $n \geqq 1$. Since $\alpha^{-(t+1)}(a) \in \text{lann } I_t$, $\alpha^{-(t+1)}(a) \in \text{lann } I_{t+1}$. Thus $\alpha^{-(t+1)}(a)\alpha^{-(t+1)}(a) = 0$ and $a^2 = 0$. This contradicts the assumption that a is not nilpotent. Thus, there exists $n \geqq 1$ such that $a\alpha^n(a) \neq 0$.

Since Δ is α-nil, for some positive integer $s \geqq 1$, $a\alpha^n(a)\dots\alpha^{sn}(a) = 0$. Let s be the least such integer. We have $\alpha^{-sn}(a)\alpha^{(1-s)n}(a)\dots\alpha^{-n}(a)a = 0$. Put $u = \alpha^{(1-s)n}(a)\dots\alpha^{-n}(a)a$. Then $u \neq 0$. If the left ideal $A(R,\alpha)u$ is not nil, then bu is not nilpotent for some $b \in A(R,\alpha)$. Put $a_1 = a$ and $a_2 = bu$. We show that $\text{rann}(a_1) \subsetneqq \text{rann}(a_2)$. Since $\alpha^{-sn}(a)\alpha^{(1-s)n}(a)\dots\alpha^{-n}(a)a = 0$, $\alpha^{(1-s)n}(a)\dots\alpha^{-n}(a)a\alpha^n(a) = 0$. Hence $a_2\alpha_n(a) = 0$. But $a\alpha^n(a) \neq 0$, thus $\text{rann}(a_1) \subsetneqq \text{rann}(a_2)$. Replace $a = a_1$ by a_2 and $a_2\alpha^{n_2}(a_2) \neq 0$ for some n_2. Repeat the argument to get a_3 such that $\text{rann}(a_1) \subsetneqq \text{rann}(a_2) \subsetneqq \text{rann}(a_3)$. Continuing in this way we get a strictly ascending chain of right annihilators in R. This is a contradiction. Therefore there is a non-zero nil left ideal in $A(R,\alpha)$. Since $A(R,\alpha)$ is semiprime right Goldie, then by [**15**, Theorem 1] every nil left ideal is nilpotent. Therefore $R[x;\alpha]$ is semiprimitive.

Remark. Since α is not assumed to be surjective on R, the assymmetry of the construction of $R[x;\alpha]$ means that symmetry cannot be cited to give the next result as a corollary of Theorem 3.4.

Theorem 3.5. *Let R be a semiprime left Goldie ring and $\alpha: R \to R$ a monomorphism of R. Then $R[x;\alpha]$ is semiprimitive.*

Proof. Since R is semiprime left Goldie, so it has a semisimple Artinian quotient ring, by [**16**, Theorem 2.3.6]. Hence one can show that R satisfies the ascending chain condition on right annihilators. Also $A(R,\alpha)$ is semiprime left Goldie. The rest of the proof is similar to the proof of Theorem 3.4.

Corollary 3.6. *Let R be a left Noetherian ring satisfying the ascending chain condition on right annihilators, $\alpha: R \to R$ a monomorphism, then*

$$J(R[x;\alpha]) = N(R[x;\alpha]) = N(R)[x;\alpha].$$

Proof. By [7] we have $\alpha(N(R)) \subseteq N(R)$. So $N(R)[x;\alpha]$ is an ideal of $R[x;\alpha]$. We show that $\alpha^{-1}(N(R)) = N(R)$. To see this let $\alpha(a) \in N(R)$, $a \in R$. Then $(R\alpha(a)R)^n = 0$ for some $n \geqq 0$. It follows that RaR is a nil ideal of R, so we have $\alpha^{-1}(N) = N$. Let $\bar{\alpha}$ be the homomorphism induced on $R/N(R)$, by α, given by $\bar{\alpha}(a+N(R)) = \alpha(a) + N(R)$. Then $\bar{\alpha}$ is injective. We have $R/N(R)$ is semiprime left Noetherian and $\bar{\alpha}$ is a monomorphism of $R/N(R)$. By Theorem 3.4, $(R/N(R))[x;\bar{\alpha}]$ is semiprimitive. We have

$$(R/N(R))[x;\bar{\alpha}] \cong R[x;\alpha]/N(R)[x;\alpha].$$

Thus $N(R)[x;\alpha]$ is a semiprimitive ideal of $R[x;\alpha]$ and $J(R[x;\alpha]) \subseteq N(R)[x;\alpha]$.

Since $\alpha(N(R)) \subseteq N(R)$ and $\{N(R)\}^k = 0$ for some $k > 0$, $(N(R)[x;\alpha])^k = 0$. So the result follows.

Corollary 3.7. *Let R be a right Noetherian ring satisfying the ascending chain condition on left annihilators. Then we have the same result as Corollary 3.6.*

Example 3.8. There are examples which show that some conditions on R and α are necessary if results of the nature of Theorems 2.2, 2.3 and 3.4 are to be valid. One is the example constructed by Jordan [**13**, §5]. In that example 0 is a semiprime ideal of R which is α-prime but it is not strongly α-prime. But $R[x;\alpha]$ has a non zero nil ideal. Another example is from Pearson and Stephenson in [**17**, §2]. They have constructed an α-prime commutative ring, and automorphism α of R such that $R[x;\alpha]$ is prime with a non zero nil ideal.

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