SOME INTEGRALS INVOLVING LEGENDRE FUNCTIONS

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1. Introduction

In this note generalisations of certain integrals involving Legendre functions including the Mehler-Dirichlet integral for Legendre functions of the first kind are given, these new results expressing associated Legendre functions of the first or second kinds as integrals involving corresponding functions of the same degree but different order. These integrals appear to be analogous to Sonine's integral in the theory of Bessel functions.

2. An Integral Expression for $T_\nu^{-(\mu+\sigma)}(x)$, $|x|<1$

We consider the integral

$$I_1=\int_x^1 T_\nu^{-\mu}(t)(t-x)^{\sigma-1}(1-t^2)^{\frac{1}{2}\mu}\,dt, \quad \text{.....................(1.1)}$$

where $Re\mu>-1$, $Re\sigma>0$, $|x|<1$ and $T_\nu^{-\mu}(t)$ is Ferrers' associated Legendre function. The function $T_\nu^{-\mu}(t)$ is given as a hypergeometric function (**1**, p. 143) as

$$T_\nu^{-\mu}(t)=\frac{1}{\Gamma(1+\mu)}\left(\frac{1-t}{1+t}\right)^{\frac{1}{2}\mu}F\left(-\nu,\,\nu+1\,;\,1+\mu\,;\,\tfrac{1}{2}-\tfrac{1}{2}t\right)$$

$$=\frac{1}{\Gamma(1+\mu)}\left(\frac{1-t}{1+t}\right)^{\frac{1}{2}\mu}\sum_{r=0}^{\infty}\frac{(-\nu)_r(\nu+1)_r(1-t)^r}{2^r(1+\mu)_r r!}, \quad \text{...............(1.2)}$$

where $(a)_r=\Gamma(a+r)/\Gamma(a)$. On substituting this expansion in (1.1) and interchanging the order of integration and summation, this operation being justified, we obtain

$$I_1=\frac{1}{\Gamma(1+\mu)}\sum_{r=0}^{\infty}\frac{(-\nu)_r(\nu+1)_r}{2^r(1+\mu)_r r!}\int_x^1(t-x)^{\sigma-1}(1-t)^{r+\mu}\,dt \quad \text{.........(1.3)}$$

The integrals occurring in this expression are evaluated by a substitution given by Bateman (**1**, p. 159). On writing

$$t=1-v(1-x),$$

we obtain

$$\int_x^1(t-x)^{\sigma-1}(1-t)^{r+\mu}\,dt=(1-x)^{\mu+r+\sigma}\int_0^1 v^{r+\mu}(1-v)^{\sigma-1}\,dv$$

$$=(1-x)^{\mu+r+\sigma}B(r+\mu+1,\,\sigma),$$

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this being valid for $Re\mu > -1$, $Re\sigma > 0$ (1, p. 9). Equation (1.3) now gives

$$I_1 = \frac{\Gamma(\sigma)}{\Gamma(1+\mu+\sigma)}(1-x)^{\mu+\sigma}\sum_{r=0}^{\infty}\frac{(-\nu)_r(\nu+1)_r}{(\mu+\sigma+1)_r r!}(\tfrac{1}{2}-\tfrac{1}{2}x)^r$$

$$= \frac{\Gamma(\sigma)}{\Gamma(1+\mu+\sigma)}(1-x)^{\mu+\sigma}F(-\nu,\nu+1\,;\,\mu+\sigma+1\,;\,\tfrac{1}{2}-\tfrac{1}{2}x)$$

$$= \Gamma(\sigma)(1-x^2)^{\frac{1}{2}(\mu+\sigma)}T_\nu^{-(\mu+\sigma)}(x).$$

Thus we have

$$\Gamma(\sigma)T_\nu^{-(\mu+\sigma)}(x) = (1-x^2)^{-\frac{1}{2}(\mu+\sigma)}\int_x^1 T_\nu^{-\mu}(t)(t-x)^{\sigma-1}(1-t^2)^{\frac{1}{2}\mu}\,dt, \quad \text{......(1.4)}$$

provided $Re\mu > -1$, $Re\sigma > 0$. When $\mu = 0$, (1.4) reduces to an integral given by Bateman (1, p. 159).

We next note that

$$\sqrt{\pi}T_\nu^{\frac{1}{2}}(\cos\theta) = (\tfrac{1}{2}\sin\theta)^{-\frac{1}{2}}\cos(\nu+\tfrac{1}{2})\theta\,;$$

so, if in (1.4) we put $\mu = -\frac{1}{2}$, $\sigma = \lambda + \frac{1}{2}$, and make the substitutions $x = \cos\theta$, $t = \cos v$, we obtain

$$T_\nu^{-\lambda}(\cos\theta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{(\sin\theta)^{-\lambda}}{\Gamma(\lambda+\frac{1}{2})}\int_0^\theta(\cos v - \cos\theta)^{\lambda-\frac{1}{2}}\cos(\nu+\tfrac{1}{2})v\,dv,$$

provided $Re\lambda > -\frac{1}{2}$ and $0 < \theta < \pi$. This is the well-known Mehler-Dirichlet integral (1, p. 159).

3. An Integral Expression for $P_\nu^{-(\mu+\sigma)}(x)$, $|1-x| < 2$

We next consider the integral

$$I_2 = \int_1^x P_\nu^{-\mu}(t)(x-t)^{\sigma-1}(t^2-1)^{\frac{1}{2}\mu}\,dt, \quad \text{....................(2.1)}$$

where $Re\mu > -1$, $Re\sigma > 0$, $|1-x| < 2$, and $P_\nu^{-\mu}(t)$ is the associated Legendre function of the first kind. As before, we express the Legendre function as a hypergeometric function for $|1-t| < 2$ in the form

$$P_\nu^{-\mu}(t) = \frac{1}{\Gamma(1+\mu)}\left(\frac{t-1}{t+1}\right)^{\frac{1}{2}\mu}F(-\nu,\nu+1\,;\,1+\mu\,;\,\tfrac{1}{2}-\tfrac{1}{2}t)$$

$$= \frac{1}{\Gamma(1+\mu)}\left(\frac{t-1}{t+1}\right)^{\frac{1}{2}\mu}\sum_{r=0}^{\infty}\frac{(-\nu)_r(\nu+1)_r(-1)^r}{(1+\mu)_r r!}\left(\frac{t}{2}-\frac{1}{2}\right)^r. \quad \text{............(2.2)}$$

On substituting in (2.1) and interchanging the order of integration and summation, this being justified, we obtain

$$I_2 = \frac{1}{\Gamma(1+\mu)}\sum_{r=0}^{\infty}\frac{(-\nu)_r(\nu+1)_r(-1)^r}{(1+\mu)_r r!\,2^r}\int_1^x(x-t)^{\sigma-1}(t-1)^{r+\mu}\,dt. \quad \text{......(2.3)}$$

Following Bateman (1, p. 159) we evaluate the integrals in (2.3) by means of the substitution

$$t = v(x-1)+1$$

to obtain

$$\int_1^x (x-t)^{\sigma-1}(t-1)^{r+\mu}\,dt=(x-1)^{\sigma+r+\mu}\int_0^1 v^{r+\mu}(1-v)^{\sigma-1}\,dv$$
$$=(x-1)^{\sigma+r+\mu}B(r+\mu+1,\sigma),$$

provided $Re\,\mu>-1$, $Re\,\sigma>0$. Hence we have

$$I_2=\frac{\Gamma(\sigma)(x-1)^{\sigma+\mu}}{\Gamma(\mu+\sigma+1)}F(-\nu,\nu+1\,;\,\mu+\sigma+1\,;\,\tfrac{1}{2}-\tfrac{1}{2}x)$$
$$=\Gamma(\sigma)(x^2-1)^{\frac{1}{2}(\mu+\sigma)}P_\nu^{-(\mu+\sigma)}(x),$$

and thus obtain

$$\Gamma(\sigma)P_\nu^{-(\mu+\sigma)}(x)=(x^2-1)^{-\frac{1}{2}(\mu+\sigma)}\int_1^x P_\nu^{-\mu}(t)(x-t)^{\sigma-2}(t^2-1)^{\frac{1}{2}\mu}\,dt,\quad\ldots(2.4)$$

provided $Re\,\mu>-1$, $Re\,\sigma>0$ and $|1-x|<2$. We note that, when $\mu=0$, (2.4) reduces to an integral given by Bateman (1, p. 159).

Further, since

$$P_\nu^{\frac{1}{2}}(\cosh t)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(\sinh t)^{-\frac{1}{2}}\cosh(\nu+\tfrac{1}{2})t,$$

we write $\mu=-\frac{1}{2}$, $\sigma=\lambda+\frac{1}{2}$, $x=\cosh\alpha$ and $t=\cosh v$ in (2.4) to obtain

$$\Gamma(\lambda+\tfrac{1}{2})P_\nu^{-\lambda}(\cosh\alpha)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(\sinh\alpha)^{-\lambda}\int_0^\alpha(\cosh\alpha-\cosh v)^{\lambda-\frac{1}{2}}\cosh(\nu+\tfrac{1}{2})v\,dv,$$

where $Re\,\lambda>-\frac{1}{2}$ and $\alpha>0$. This integral is given by Bateman (1, p. 156).

4. An Integral Expression for $Q_\nu^{-(\mu+\sigma)}(x)$, $x>1$

The integral

$$I_3=\int_x^\infty Q_\nu^{-\mu}(t)(t-x)^{\sigma-1}(t^2-1)^{\frac{1}{2}\mu}\,dt,\quad\ldots\ldots\ldots\ldots(3.1)$$

where $Re\sigma>0$, $Re(\nu-\mu-\sigma+1)>0$ and $x>1$, $Q_\nu^{-\mu}(t)$ being the associated Legendre function of the second kind, can be evaluated by means of the expansion (1, p. 122)

$$\Gamma(\nu+\tfrac{3}{2})Q_\nu^{-\mu}(t)=e^{-\mu i\pi}2^{-(\nu+1)}\pi^{\frac{1}{2}}\Gamma(\nu-\mu+1)t^{-\nu+\mu-1}(t^2-1)^{-\frac{1}{2}\mu}$$
$$\times F(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+1,\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+\tfrac{1}{2}\,;\,\nu+\tfrac{3}{2}\,;\,t^{-2}),$$

where $t>1$. On substituting in (3.1) and interchanging the order of integration and summation, this being justified, we obtain

$$I_3=e^{-\mu i\pi}2^{-(\nu+1)}\pi^{\frac{1}{2}}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\frac{3}{2})}$$
$$\times\sum_{r=0}^{\infty}\frac{(\frac{1}{2}-\frac{1}{2}\mu+1)_r(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2})_r}{(\nu+\frac{3}{2})_r r!}\int_x^\infty t^{-2r-\nu+\mu-1}(t-x)^{\sigma-1}\,dt.\quad\ldots\ldots(3.2)$$

Again, following Bateman (1, p. 160), we evaluate the integrals in this expression by means of the substitution

$$t=vx+x$$

to obtain

$$\int_x^\infty t^{-2r-\nu+\mu-1}(t-x)^{\sigma-1}\,dt = x^{-\nu+\mu-2r+\sigma-1}\int_0^\infty v^{\sigma-1}(1+v)^{-\nu+\mu-1-2r}\,d\sigma$$
$$= x^{-\nu+\mu-2r+\sigma-1}B(\sigma,\ \nu-\mu-\sigma+1+2r), \qquad (3.3)$$

provided $Re\,\sigma>0$, $Re(\nu-\mu-\sigma+1)>0$ (**1**, p. 9).

On applying Legendre's duplication formula (**1**, p. 5) to the gamma functions occurring in (3.3), we obtain

$$\Gamma(\nu-\mu-\sigma+2r+1)$$
$$=2^{\nu-\mu-\sigma+2r}\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma+r+\tfrac{1}{2})\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma+r+1)\,;$$

so, by using (3.3) and this result, we find that

$$I_3 = e^{-\mu i\pi}\,2^{-\nu-1}\pi^{\frac{1}{2}}\,\frac{\Gamma(\sigma)\Gamma(\nu-\mu-\sigma+1)}{\Gamma(\nu+\frac{3}{2})}\,x^{-\nu+\mu+\sigma-1}$$
$$\times F(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma+1,\ \tfrac{1}{2}\nu-\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma+\tfrac{1}{2}\,;\ \nu+\tfrac{3}{2}\,;\ x^{-2})$$
$$= e^{-i\pi\sigma}\Gamma(\sigma)(x^2-1)^{\frac{1}{2}(\mu+\sigma)}Q_\nu^{-(\mu+\sigma)}(x).$$

We finally have

$$\Gamma(\sigma)Q_\nu^{-(\mu+\sigma)}(x) = e^{-i\pi\sigma}(x^2-1)^{-\frac{1}{2}(\mu+\sigma)}\int_x^\infty Q_\nu^{-\mu}(t)(t-x)^{\sigma-1}(t^2-1)^{\frac{1}{2}\mu}\,dt, \qquad (3.4)$$

provided $Re\,\sigma>0$, $Re(\nu-\mu-\sigma+1)>0$ and $x>1$. When $\mu=0$, this integral reduces to one given by Bateman (**1**, p. 160).

Further, since

$$Q_\nu^{\frac{1}{2}}(\cosh\alpha) = i\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\sinh\alpha)^{-\frac{1}{2}}e^{-(\nu+\frac{1}{2})\alpha},$$

we write $\mu=-\frac{1}{2}$, $\sigma=\lambda+\frac{1}{2}$, $x=\cosh\alpha$, $t=\cosh v$, in (3.4) to obtain

$$\Gamma(\lambda+\tfrac{1}{2})Q_\nu^{-\lambda}(\cosh\alpha) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}e^{-i\pi\lambda}(\sinh\alpha)^{-\lambda}\int_\alpha^\infty(\cosh v-\cosh\alpha)^{\lambda-\frac{1}{2}}e^{-(\nu+\frac{1}{2})v}\,dv,$$

provided $Re\,\lambda>-\frac{1}{2}$, $Re(\nu-\lambda+1)>0$ and $\alpha>0$ (**1**, p. 155).

5. An Integral Representation for $P_\nu^{(\sigma+\mu)}(x)$, $x>1$

Since (**1**, p. 140)

$$e^{-i(\mu+\sigma)\pi}\cos(\nu\pi)\Gamma(\nu+1-\mu-\sigma)\Gamma(-\mu-\sigma-\nu)P_\nu^{(\mu+\sigma)}(x)$$
$$= Q_{-(\nu+1)}^{-(\mu+\sigma)}(x) - Q_\nu^{-(\mu+\sigma)}(x), \qquad (4.1)$$

we use (3.4) to obtain

$$\Gamma(\sigma)e^{-i\pi(\mu+\sigma)}\cos(\nu\pi)\Gamma(\nu-\mu-\sigma+1)\Gamma(-\mu-\sigma-\nu)P_\nu^{(\mu+\sigma)}(x)$$
$$= e^{-i\pi\sigma}(x^2-1)^{-\frac{1}{2}(\mu+\sigma)}\int_x^\infty(Q_{-(\nu+1)}^{-\mu}(t)-Q_\nu^{-\mu}(t))(t-x)^{\sigma-1}(t^2-1)^{\frac{1}{2}\mu}\,dt,$$

provided $Re\,\sigma>0$, $Re(\nu-\mu-\sigma+1)>0$ and $Re(\nu+\mu+\sigma)<0$.

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On using (4.1) with $\sigma=0$, we now obtain

$$B(\sigma,\ \nu-\mu-\sigma+1)B(\sigma,\ -\nu-\mu-\sigma)P_\nu^{(\mu+\sigma)}(x)$$
$$=\Gamma(\sigma)(x^2-1)^{-\frac{1}{2}(\mu+\sigma)}\int_x^\infty P_\nu^\mu(t)(t-x)^{\sigma-1}(t^2-1)^{\frac{1}{2}\mu}\,dt, \qquad \text{.........(4.2)}$$

provided $Re\,\sigma>0$, $Re(\nu+1)>Re(\mu+\sigma)$, $Re(\nu+\mu+\sigma)<0$ and $x>1$.

If we now write $\mu=0$, $\nu=-\frac{1}{2}+i\lambda$ in (4.2), we obtain

$$\Gamma(\sigma)\Gamma(\tfrac{1}{2}-\sigma+i\lambda)\Gamma(\tfrac{1}{2}-\sigma-i\lambda)P^\sigma_{-\frac{1}{2}+i\lambda}(x)$$
$$=(x^2-1)^{-\frac{1}{2}\sigma}\pi\ \mathrm{sech}\ \pi\lambda\int_x^\infty P_{-\frac{1}{2}+i\lambda}(t)\,(t-x)^{\sigma-1}\,dt,$$

provided $0<Re\,\sigma<\frac{1}{2}$.

Finally, writing $\sigma=-\mu=\frac{1}{2}$, $\nu=-\frac{1}{2}+i\lambda$ in (4.2) and noting that

$$P^{-\frac{1}{2}}_{-\frac{1}{2}+i\lambda}(\cosh t)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{(\sinh t)^{-\frac{1}{2}}\sin\lambda t}{\lambda},$$

we obtain

$$P_{-\frac{1}{2}+i\lambda}(\cosh\alpha)=\frac{2\coth\pi\lambda}{\pi}\int_\alpha^\infty\frac{\sin\lambda t\,dt}{(2(\cosh t-\cosh\alpha))^{\frac{1}{2}}},$$

an integral given by Fock (2).

REFERENCES

(1) Bateman Manuscript Project, *Higher Transcendental Functions*, vol. I (McGraw-Hill, 1953).

(2) V. A. Fock, *C.R.* (*Doklady*) *Acad. Sci. U.R.S.S.* (N.S.) (1943), pp. 253-256.

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