BILINEAR FORMS ON VECTOR HARDY SPACES

by GORDON BLOWER

(Received 27 May, 1996)

Abstract. Let $\Phi:\tilde{H}^2\mathcal{K}\times\tilde{H}^2\mathcal{K}\to\mathbb{C}$ be a bilinear form on vector Hardy space. Introduce the symbol φ of Φ by $\langle\varphi(z_1,z_2),a\otimes b\rangle=\Phi(k_{z_1}\otimes a,k_{z_2}\otimes b)$, where k_w is the reproducing kernel for $w\in D$. We show that Φ extends to a bounded bilinear form on $\tilde{H}^1\mathcal{K}\times\tilde{H}^1\mathcal{K}$ provided that the gradient $\|\bar{\partial}_1\bar{\partial}_2\varphi\|_{\mathrm{Bi}(\mathcal{K},\mathcal{K})}A(dz_1)A(dz_2)$ defines a Carleson measure in the bidisc D^2. We obtain a sufficient condition for Φ to extend to a Hilbert space. For vectorial bilinear Hankel forms we obtain an analogue of Nehari's Theorem.

§1. Introduction. For any complex Banach spaces X and Y we denote by $\mathrm{Bi}(X,Y)$ the space of bounded bilinear forms $\Phi:X\times Y\to\mathbb{C}$ with the norm $\|\Phi\|_{\mathrm{Bi}(X,Y)}=\sup\{\mathcal{R}\Phi(x,y):\|x\|_X=\|y\|_Y=1\}$. Here we consider bilinear forms on the Hardy spaces $H^p\mathcal{K}$. These are spaces of analytic functions $f:D\to\mathcal{K}$, with values in the compact operators $\mathcal{K}$ on separable Hilbert space H, for which $\|f\|_{H^p\mathcal{K}}=\sup_{0<r<1}\|f(re^{i\theta})\|_{L^p(d\theta;\mathcal{K})}<\infty$. The matrix disc algebra $A\mathcal{K}$ is the closure in $H^\infty\mathcal{K}$ of the analytic trigonometric polynomials with coefficients from $\mathcal{K}$. The closure of $A\mathcal{K}$ in $H^2\mathcal{K}$ will be denoted $\tilde{H}^2\mathcal{K}$, and $L^p(d\theta;\mathcal{K})$ is the Bochner-Lebesgue space.

We are concerned with a particular question [**9**, Conjecture 8.3].

Given a bounded bilinear form $\Phi:A\mathcal{K}\times A\mathcal{K}\to\mathbb{C}$, when can one find a Hilbert space G and a bounded linear map $V:A\mathcal{K}\to G$ such that for all $f,g\in A\mathcal{K}$ we have

$$\Re\Phi(f,g)\leq\|Vf\|_G\,\|Vg\|_G?$$

The results of [**2**] for the disc algebra A suggest that this may *always* be possible. An application of such a factorization property for bilinear forms is suggested by [**7**, IV(a)]. In this paper I continue the approach to factorization initiated in [**1**], emphasizing the role of measures on the disc. The classical Nehari theorem [**10**], [**6**, p. 322] suggests which conditions to impose upon bilinear Hankel forms.

A positive Radon measure μ on the unit disc D is said to be a *Carleson measure* if there is a constant C_* such that $\mu(S(I))\leq C_*|I|$ for each subinterval I of $[0,2\pi]$, where $S(I)$ is the sector $S(I)=\{re^{i\theta}\in D:r\geq 1-|I|,\ \theta\in I\}$ based upon I. See [**6**, p. 258].

THEOREM 1.1. (Nehari, C. Fefferman-Stein). *Let $\Phi:H^2\times H^2\to\mathbb{C}$ be the bilinear Hankel form with analytic symbol φ that satisfies*

$$\Phi(g,h)=\int_{\mathbb{T}}\varphi(e^{-i\theta})g(e^{i\theta})h(e^{i\theta})\frac{d\theta}{2\pi}\quad(g,h\in H^2).\tag{1.1}$$

Then Φ is bounded if and only if Q_φ defines a Carleson measure on D, where

$$Q_\varphi(dr\,d\theta)=(1-r)\,|\varphi'(re^{i\theta})|^2r\,dr\,d\theta.\tag{1.2}$$

In Section 4 we obtain an analogous sufficient condition for bilinear Hankel forms on $\tilde{H}^2\mathcal{K}\times\tilde{H}^2\mathcal{K}$ to be bounded and extend to bounded bilinear forms on $\mathcal{G}\times\mathcal{G}$, where $\mathcal{G}$ is some Hilbert space.

For general bilinear forms it is useful to introduce another scale of Banach spaces.

Glasgow Math. J. **39** (1997) 371–378.

https://doi.org/10.1017/S0017089500032286 Published online by Cambridge University Press

For any Banach space X we let $G^p(X)$ be the Banach space of analytic functions $g:D\to X$ for which the norm

$$\|g\|_{G^p(X)}=\|g(0)\|_X+\left\{\int_{\mathbb{T}}\left(\int_0^1(1-r)\,\|g'(re^{i\theta})\|_X^2\,r\,dr\right)^{p/2}\frac{d\theta}{2\pi}\right\}^{1/p} \tag{1.3}$$

is finite. When $X=H$ is a Hilbert space, $G^2(H)$ has a norm equivalent to that of $H^2(H)$, by (3.17) below. However, when $X=\mathcal{K}$, the space $G^2(\mathcal{K})$ does not contain $A\mathcal{K}$. See [**1**, 6.5(i)]. Nevertheless, the Poisson semigroup $P_rg(z)=g(rz)$ satisfies $\|P_rg-g\|_{G^2(\mathcal{K})}\to 0$ as $r\to 1-$. Hence the algebraic tensor product $A\otimes\mathcal{K}$ is a dense subspace of $G^2(\mathcal{K})$. The spaces of functions with $f(0)=0$ are denoted by G_0^p, H_0^p and so forth.

In Section 2 we introduce the notion of the *symbol* of a bounded bilinear form Φ on $H^2\mathcal{K}$ and obtain a sufficient condition for Φ to extend to a Hilbert space containing $G^2(\mathcal{K})$. In the next section we achieve a Carleson measure condition involving the symbol for such a Φ to be bounded on $\tilde{H}^1\mathcal{K}\times\tilde{H}^1\mathcal{K}$.

NOTATION. For $a\in\mathcal{K}$ we write $|a|_S=(2^{-1}(a^*a+aa^*))^{1/2}$ for the symmetric modulus of a. The dual space of $\mathcal{K}$ is the space c^1 of trace class operators under the pairing $\langle a,b\rangle=\operatorname{trace}(ab)$. We shall use the same notation for the pairing of a bilinear form φ with an elementary tensor $a\otimes b$, so that $\langle\varphi,a\otimes b\rangle=\varphi(a,b)$. The space of Hilbert-Schmidt operators will be denoted by c^2.

By a *dyadic sector* of the disc we mean a set such as

$$R_{jk}=\{re^{i\theta}\in D:1-2^{-j}\le r<1-2^{-j-1},\ k2^{-j}\le\theta/(2\pi)<(k+1)2^{-j}\}, \tag{1.4}$$

where $k=0,1,2,\dots,2^j-1$ and $j\ge 0$. We write $A(dz)=r\,dr\,d\theta$ for area measure on the disc. For partial derivatives on the bidisc D^2 we write $\partial_j=\dfrac{\partial}{\partial z_j}$ and $\bar\partial_j=\dfrac{\partial}{\partial\bar z_j}$. By C we mean a constant, not necessarily the same at each occurrence. Also $\mathbf{1}_R$ is the indicator function of R.

§2. The symbol of a bilinear form. Let Φ be a bounded bilinear form on $\tilde{H}^2\mathcal{K}\times\tilde{H}^2\mathcal{K}$. Let $k_w(z)=(1-z\bar w)^{-1}$ be the reproducing kernel function for $w\in D$ that satisfies

$$f(w)=\langle f,k_w\rangle_{H^2}\quad(f\in H^2). \tag{2.1}$$

By the Riesz–Fréchet Theorem, k_w is uniquely determined as the vector in H^2 satisfying (2.1). Note that $w\mapsto k_w$ is anti-analytic, so that $\dfrac{\partial}{\partial w}k_w(z)=0$.

There is for each $(z_1,z_2)\in D^2$ a bounded bilinear form $\varphi(z_1,z_2)$ on $\mathcal{K}\times\mathcal{K}$ satisfying

$$\langle\varphi(z_1,z_2),a\otimes b\rangle=\Phi(k_{z_1}\otimes a,k_{z_2}\otimes b)\ (a,b\in\mathcal{K}). \tag{2.2}$$

By Morera's Theorem $\partial_1\varphi=\partial_2\varphi=0$ and so we call φ the *anti-analytic symbol of* Φ.

THEOREM 2.1. *Let Φ be a bilinear form on $\tilde{H}_0^2\mathcal{K}\times\tilde{H}_0^2\mathcal{K}$ whose symbol φ satisfies*

$$\sup_{z_2}\int_D\|\bar\partial_1\bar\partial_2\varphi(z_1,z_2)\|_{\mathrm{Bi}(\mathcal{K},\mathcal{K})}\log\frac{1}{|z_1|}A(dz_1) \tag{2.3}$$

$$+\sup_{z_1}\int_D\|\bar\partial_1\bar\partial_2\,\varphi(z_1,z_2)\|_{\mathrm{Bi}(\mathcal{K},\mathcal{K})}\log\frac{1}{|z_2|}A(dz_2)<\infty. \tag{2.4}$$

Then there is a Hilbert space G_μ containing $G_0^2(\mathscr{K})$ such that Φ extends to a bounded bilinear form on $G_\mu \times G_\mu$.

Proof. Let $f, g \in \tilde{H}_0^2\mathscr{K}$. Then, in the sense of Abel summation,

$$\Phi(f,g) = \frac{4}{\pi^2}\iint_{D\times D} \langle \bar{\partial}_1\,\bar{\partial}_2\varphi(z_1,z_2), \partial_1 f(z_1)\otimes\partial_2 g(z_2)\rangle \log\frac{1}{|z_1|}\log\frac{1}{|z_2|}A(dz_1)A(dz_2). \quad (2.5)$$

This identity may readily be established for monomials $f = z_1^n\otimes a$ and $g = z_2^m \otimes b$ by comparing coefficients in the power series development of $\varphi(z_1, z_2)$. One then uses linearity and density to obtain the general case. Compare [**6**, p. 304].

Now $\bar{\partial}_1\,\bar{\partial}_2\varphi(z_1,z_2)$ is a bounded bilinear form on the C^*-algebra $\mathscr{K}$, and by the Grothendieck–Pisier Theorem [**8**, Theorem 9.1] there is a universal constant K with the following property. For each $(z_1,z_2)\in D^2$, there is a positive $\nu(z_1,z_2)\in c^1$ with

$$\|\nu(z_1,z_1)\|_{c^1} \le K\,\|\bar{\partial}_1\,\bar{\partial}_2\varphi(z_1,z_2)\|_{\mathrm{Bi}(\mathscr{K},\mathscr{K})} \quad (2.6)$$

that satisfies

$$|\langle\bar{\partial}_1\,\bar{\partial}_2\varphi(z_1,z_2), a\otimes b\rangle|^2 \le \langle |a|_S^2, \nu(z_1,z_2)\rangle\langle |b|_S^2, \nu(z_1,z_2)\rangle \quad (a, b\in\mathscr{K}). \quad (2.7)$$

The norm of our Hilbert space is obtained from $\nu(z_1,z_2)$ as follows. We apply the Cauchy-Schwarz inequality to (2.5) and use (2.7) and Fubini's Theorem to obtain

$$|\Phi(f,g)|^2 \le \frac{2}{\pi}\int_D \langle |\partial_1 f(z_1)|_S^2, \mu_1(z_1)\rangle \log\frac{1}{|z_1|}A(dz_1)\,\frac{2}{\pi}\int_D \langle |\partial_2 g(z_2)|_S^2, \mu_2(z_2)\rangle\log\frac{1}{|z_2|}A(dz_2), \quad (2.8)$$

where we have introduced the positive c^1-valued functions

$$\mu_1(z_1) = \int_D \nu(z_1,z_2)\log\frac{1}{|z_2|}A(dz_2) \quad (z_1\in D), \quad (2.9)$$

$$\mu_2(z_2) = \int_D \nu(z_1,z_2)\log\frac{1}{|z_1|}A(dz_1) \quad (z_2\in D). \quad (2.10)$$

The required Hilbert space G_μ is the completion of $A_0\otimes\mathscr{K}$ for the norm given by

$$\|f\|_{G_\mu}^2 = \frac{2}{\pi}\int_D \langle |\partial f(z)|_S^2, \mu(z)\rangle\log\frac{1}{|z|}A(dz), \quad (2.11)$$

where $\mu(z) = \mu_1(z) + \mu_2(z)$.

Using (2.6) we see that under the hypothesis of the Theorem $\|\mu(z)\|_{c^1}\le C$, for $z\in D$, and consequently the formal inclusion map $G_0^2(\mathscr{K})\to G_\mu$ is bounded.

§3. Carleson measures on the bidisc. Let $E\subseteq\mathbb{T}\times\mathbb{T}$ be an open subset of the bi-torus. We define

$$S(E) = \bigcup\{S(I)\times S(J)\}, \quad (3.1)$$

where the union of products of sectors is taken over all possible products of open intervals $I \times J$ contained in E. Then a positive Radon measure μ on the bidisc D^2 is said to be a *Carleson measure* if there is $C_* < \infty$ satisfying $\mu(S(E)) \le C_* |E|$, for all connected open sets E, where $|E|$ is the area of E. See [**4, 5**]. (It is not enough for μ to satisfy the inequality merely for open rectangles E.)

THEOREM 3.1. *Let* $\Phi: \tilde{H}_0^2\mathcal{H} \times \tilde{H}_0^2\mathcal{H} \to \mathbb{C}$ *be a bounded bilinear form whose symbol* φ *has the property that*

$$\mu(dz_1\, dz_2) = \|\bar{\partial}_1\, \bar{\partial}_2 \varphi(z_1, z_2)\|_{\mathrm{Bi}(\mathcal{H},\mathcal{H})} A(dz_1)A(dz_2) \tag{3.2}$$

defines a Carleson measure on D^2. *Then* Φ *extends to define a bounded bilinear form on* $\tilde{H}_0^1\mathcal{H} \times \tilde{H}_0^1\mathcal{H}$.

Proof. We have, by the Littlewood–Paley identity (2.5), for $f, g \in \tilde{H}_0^2\mathcal{H}$

$$\Re\Phi(f,g)$$

$$\le \frac{4}{\pi^2} \iint_{D\times D} \|\bar{\partial}_1\, \bar{\partial}_2 \varphi(z_1, z_2)\|_{\mathrm{Bi}(\mathcal{H},\mathcal{H})} \|\partial_1 f(z_1)\|_{\mathcal{H}} \|\partial_2 g(z_2)\|_{\mathcal{H}} \log\frac{1}{|z_1|} \log\frac{1}{|z_2|} A(dz_1)A(dz_2). \tag{3.3}$$

Let R be a typical dyadic sector in D, as in (1.4), and let $\tilde{R}$ be its dilate about the centre of mass with scale factor 3/2. Then, by the Cauchy Integral Formula,

$$\log\frac{1}{|z|} \|\partial f(z)\|_{\mathcal{H}} \le \frac{C}{|R|} \int_{\tilde{R}} \|f(\zeta)\|_{\mathcal{H}} A(d\zeta) \quad (z \in R), \tag{3.4}$$

$$\log\frac{1}{|z|} \|\partial g(z)\|_{\mathcal{H}} \le \frac{C}{|R|} \int_{\tilde{R}} \|g(\zeta)\|_{\mathcal{H}} A(d\zeta) \quad (z \in R). \tag{3.5}$$

Hence we can estimate (3.3) by an integral involving μ

$$\Re\Phi(f,g) \le C \iint_{D\times D} F(\zeta)G(\eta)\mu(d\zeta\, d\eta), \tag{3.6}$$

where we have introduced

$$F(\zeta) = \sum_R \mathbf{1}_R(\zeta) \frac{1}{|R|} \int_{\tilde{R}} \|f(z)\|_{\mathcal{H}} A(dz) \quad (\zeta \in D), \tag{3.7}$$

$$G(\eta) = \sum_R \mathbf{1}_R(\eta) \frac{1}{|R|} \int_{\tilde{R}} \|g(z)\|_{\mathcal{H}} A(dz) \quad (\eta \in D). \tag{3.8}$$

These resemble the conditional expectations of $\|f(z)\|_{\mathcal{H}}$ and $\|g(z)\|_{\mathcal{H}}$ with respect to the σ-algebra generated by the dyadic sectors. By R. Fefferman's Theorem [**5**, p. 403] on Carleson measures

$$\Re\Phi(f,g) \le CC_*(\mu) \int_{\mathbb{T}} \sup_{\zeta\in\Gamma(\theta)} F(\zeta) \frac{d\theta}{2\pi} \times \int_{\mathbb{T}} \sup_{\eta\in\Gamma(\phi)} G(\eta) \frac{d\phi}{2\pi}, \tag{3.9}$$

where $C_*(\mu)$ is the Carleson constant of μ and the maximal functions are taken over the nontangential approach regions $\Gamma(\theta)$ based at $e^{i\theta}$. Enlarging the region $\Gamma(\theta)$ to $\tilde{\Gamma}(\theta)$, we see that

$$\sup_{\zeta\in\Gamma(\theta)} F(\zeta) \le C \sup_{\zeta\in\tilde{\Gamma}(\theta)} \|f(\zeta)\|_{\mathcal{H}}, \tag{3.10}$$

since only boundedly many $\tilde{R}$ can overlap at any point in the disc. Hence we can conclude, by applying Bourgain's maximal inequality [3, p. 13] to (3.9), that

$$\Re\Phi(f,g) \le CC_*(\mu) \int_{\mathbb{T}} \|f(e^{i\theta})\|_{\mathcal{H}} \frac{d\theta}{2\pi} \times \int_{\mathbb{T}} \|g(e^{i\phi})\|_{\mathcal{H}} \frac{d\phi}{2\pi}. \tag{3.11}$$

For bilinear forms on H^1c^1 we can use a factorization technique to obtain a statement involving a *quadratic* expression in the symbol. Let us recall that, since c^1 is a separable dual space, $H^1c^1 = \tilde{H}^1c^1$.

THEOREM 3.2. *Let $\Phi: H_0^2c^1 \times H_0^2c^1 \to \mathbb{C}$ be a bilinear form whose symbol φ has the property that*

$$Q_\varphi(dz_1\, dz_2) = (1-|z_1|)(1-|z_2|)\, \|\bar{\partial}_1 \bar{\partial}_2 \varphi(z_1,z_2)\|^2_{\mathrm{Bi}(c^1,c^1)} A(dz_1)A(dz_2) \tag{3.12}$$

defines a Carleson measure on D^2. Then Φ extends to a bounded bilinear form $H_0^1c^1 \times H_0^1c^1 \to \mathbb{C}$.

Proof. Let $f_j(z_j) \in H^1c^1$ for $j=1,2$. Then we can use the Sarason Factorization Theorem [9, p. 62] to write $f_j(z_j) = g_j(z_j)h_j(z_j)$ for $z_j \in D$, where $g_j \in H^2c^2$, $h_j \in H^2c^2$ with

$$\|g_j\|^2_{H^2c^2} = \|h_j\|^2_{H^2c^2} = \|f_j\|_{H^1c^1} \quad (j=1,2). \tag{3.13}$$

Let us note that by Leibniz's formula the integrand of (2.5) may be bounded using

$$\begin{aligned}
&\Re\langle \bar{\partial}_1 \bar{\partial}_2 \varphi(z_1,z_2), \partial_1 f_1(z_1) \otimes \partial_2 f_2(z_2)\rangle \\
&\quad = \Re\langle \bar{\partial}_1 \bar{\partial}_2 \varphi(z_1,z_2), \partial_1 g_1(z_1) h_1(z_1) \otimes \partial_2 g_2(z_2) h_2(z_2)\rangle + \text{similar terms} \qquad (3.14)\\
&\quad \le \|\bar{\partial}_1 \bar{\partial}_2 \varphi(z_1,z_2)\|_{\mathrm{Bi}(c^1,c^1)} \|\partial_1 g_1(z_1)\|_{c^2} \|h_1(z_1)\|_{c^2} \|\partial_2 g_2(z_2)\|_{c^2} \|h_2(z_2)\|_{c^2} \\
&\qquad + \text{similar terms.} \qquad (3.15)
\end{aligned}$$

Hence by (2.5) and the Cauchy–Schwarz inequality

$$\begin{aligned}
\Re\Phi(f_1,f_2) \le C&\left\{\int_D \log\frac{1}{|z_1|} \|\partial_1 g_1(z_1)\|^2_{c^2} A(dz_1)\right\}^{1/2} \left\{\int_D \log\frac{1}{|z_2|} \|\partial_2 g_2(z_2)\|^2_{c^2} A(dz_2)\right\}^{1/2} \\
&\times \left\{\iint_{D^2} \|h_1(z_1)\|^2_{c^2} \|h_2(z_2)\|^2_{c^2} Q_\varphi(dz_1\, dz_2)\right\}^{1/2} + \text{similar terms.} \qquad (3.16)
\end{aligned}$$

By the Littlewood–Paley identity for c^2-valued functions [6, p. 304], we have

$$\left\{\frac{2}{\pi}\int_D \log\frac{1}{|z_j|} \|\partial_j g_j(z_j)\|^2_{c^2} A(dz_j)\right\}^{1/2} \le \left\{\int_{\mathbb{T}} \|g_j(e^{i\theta})\|^2_{c^2} \frac{d\theta}{2\pi}\right\}^{1/2} \quad (j=1,2) \tag{3.17}$$

https://doi.org/10.1017/S0017089500032286 Published online by Cambridge University Press

and hence we can bound the first two factors in (3.16) by Hardy norms. Using the hypothesis on Q_φ and Theorem 1 of [**4**] we can bound the third factor in (3.16) by

$$CC_*(Q_\varphi)^{1/2}\left\{\int_{\mathbb{T}} \sup_{0<r<1} \|h_1(re^{i\theta})\|_{c^2}^2 \frac{d\theta}{2\pi}\right\}^{1/2}\left\{\int_{\mathbb{T}} \sup_{0<r<1} \|h_2(re^{i\theta})\|_{c^2}^2 \frac{d\theta}{2\pi}\right\}^{1/2}. \tag{3.18}$$

By the Hardy Littlewood Maximal Theorem [**6**, p. 237] this is bounded by

$$CC_*(Q_\varphi)^{1/2}\, \|h_1\|_{H^2c^2}\, \|h_2\|_{H^2c^2}. \tag{3.19}$$

Combining the estimates (3.19) and (3.17) arising from each summand in (3.16) we have the required estimate

$$\Re\Phi(f_1,f_2) \le CC_*(Q_\varphi)^{1/2}\, \|g_1\|_{H^2c^2}\, \|g_2\|_{H^2c^2}\, \|h_1\|_{H^2c^2}\, \|h_2\|_{H^2c^2} \tag{3.20}$$

$$\le CC_*(Q_\varphi)^{1/2}\, \|f_1\|_{H^1c^1}\, \|f_2\|_{H^1c^1}. \tag{3.21}$$

§4. Hankel forms. A bilinear form Φ is said to be a *Hankel form* if

$$\Phi(fg,h) = \Phi(g,fh) \quad (g,h \in H^2\mathcal{H}), \tag{4.1}$$

for all $f \in A$. For each such bilinear form we can introduce a *symbol* $\varphi(z)$ which is a function of a single variable. There is a unique analytic power series $\varphi(z)$ with coefficients in $\mathrm{Bi}(\mathcal{H},\mathcal{H})$ that satisfies the identity

$$\Phi(g,h) = \int_{\mathbb{T}} \langle \varphi(e^{-i\theta}), (g\otimes h)(e^{i\theta})\rangle \frac{d\theta}{2\pi}, \tag{4.2}$$

where g,h are analytic trigonometric polynomials with coefficients in $\mathcal{H}$. When Φ is bounded on some Hardy space we obtain an analytic function $\varphi: D \to \mathrm{Bi}(\mathcal{H},\mathcal{H})$.

THEOREM 4.1. *Let Φ be a bilinear Hankel form on $\tilde{H}_0^2\mathcal{H}\times\tilde{H}_0^2\mathcal{H}$ with symbol φ. Suppose that $\sigma(dz) = \|\varphi'(z)\|_{\mathrm{Bi}(\mathcal{H},\mathcal{H})}A(dz)$ defines a Carleson measure on D. Then there is a Hilbert space $\mathcal{G}$ for which*

(i) *the inclusion map $\tilde{H}_0^2\mathcal{H} \to \mathcal{G}$ is bounded,*
(ii) *Φ extends to define a bounded bilinear form on $\mathcal{G}\times\mathcal{G}$.*

Proof. (ii) By the Littlewood–Paley identity [**6**, p. 304] we have that

$$\Phi(g,h) = \frac{2}{\pi}\iint_D \langle \varphi'(\bar z), g'(z)\otimes h(z)\rangle \log\frac{1}{|z|}A(dz)$$

$$+\frac{2}{\pi}\iint_D \langle \varphi'(\bar z), g(z)\otimes h'(z)\rangle \log\frac{1}{|z|}A(dz) \quad (g,h \in \tilde{H}_0^2\mathcal{H}). \tag{4.3}$$

Now for each $z \in D$, the map $a\otimes b \mapsto \langle \varphi'(z), a\otimes b\rangle$ defines a bounded bilinear form on $\mathcal{H}\times\mathcal{H}$. Hence, by the Grothendieck–Pisier Theorem [**8**, Theorem 9.1], there is an absolute constant K and a positive $\nu(z) \in c^1$ with

$$\|\nu(z)\|_{c^1} \le K\, \|\varphi'(z)\|_{\mathrm{Bi}(\mathcal{H},\mathcal{H})} \quad (z \in D) \tag{4.4}$$

for which

$$|\langle \varphi'(z), a\otimes b\rangle|^2 \le \langle |a|_S^2, \nu(z)\rangle\langle |b|_S^2, \nu(z)\rangle \quad (a,b \in \mathcal{H}, z \in D). \tag{4.5}$$

By the Cauchy–Schwarz inequality applied to (4.3) we have

$$\|\Phi(f,g)\|^2 \le \iint_D \langle |g'(z)|_S^2, v(\bar z)\rangle \left(\log\frac{1}{|z|}\right)^2 A(dz) \times \iint_D \langle |h(z)|_S^2, v(\bar z)\rangle A(dz) + \text{similar term}. \tag{4.6}$$

Hence Φ defines a bounded bilinear form on the Hilbert space $\mathscr{G}$ formed by completing $A_0 \otimes \mathscr{K}$ for the norm

$$\|f\|_{\mathscr{G}}^2 = \iint_D \langle |f'(z)|_S^2, v(\bar z)\rangle \left(\log\frac{1}{|z|}\right)^2 A(dz) + \iint_D \langle |f(z)|_S^2, v(\bar z)\rangle A(dz). \tag{4.7}$$

(i) To verify that the inclusion $\tilde H_0^2 \mathscr{K} \to \mathscr{G}$ is bounded, we consider the first summand in (4.7); our proof also deals with the second summand. We note that, by the Cauchy integral formula, we have

$$(1-|z|)^2 |f'(z)|_S^2 \le \frac{C}{|R|} \iint_{\tilde R} |f(\zeta)|_S^2 A(d\zeta) \quad (z \in R) \tag{4.8}$$

(as positive operators on Hilbert space), where $\tilde R$ is the dilatation of the dyadic sector R about its centre of mass by scale factor 3/2. See (1.4). Hence, by (4.4), we have the inequality

$$\|f\|_{\mathscr{G}}^2 \le C \iint_D F(z)\, \|\varphi'(\bar z)\|_{\mathrm{Bi}(\mathscr{K},\mathscr{K})} A(dz), \tag{4.9}$$

where we have introduced

$$F(z) = \sum_R \mathbf{1}_R(z) \frac{1}{|R|} \iint_{\tilde R} \|f(\zeta)\|_{\mathscr{K}}^2 A(d\zeta) \quad (z \in D). \tag{4.10}$$

By Carleson's Theorem [**6**, p. 258] we can estimate (4.9) by

$$\iint_D F(z)\sigma(dz) \le CC_*(\sigma) \int_{\mathbb{T}} \sup_{z \in \Gamma(\theta)} F(z) \frac{d\theta}{2\pi} \tag{4.11}$$

$$\le CC_*(\sigma) \int_{\mathbb{T}} \sup_{z \in \tilde\Gamma'(\theta)} \|f(z)\|_{\mathscr{K}}^2 \frac{d\theta}{2\pi} \tag{4.12}$$

$$\le CC_*(\sigma) \int_{\mathbb{T}} \|f(e^{i\theta})\|_{\mathscr{K}}^2 \frac{d\theta}{2\pi}, \tag{4.13}$$

where the last step follows from the Hardy–Littlewood maximal theorem [**6**, p. 237].

REFERENCES

1. G. Blower, Quadratic integrals and factorization of linear operators, *J. London Math. Soc.* (2), to appear.

2. J. Bourgain, New Banach space properties of the disc algebra and H^∞, *Acta Math*. **152** (1984), 1–48.

3. J. Bourgain, Vector-valued singular integrals and the H^1-BMO duality, in *Probability*

https://doi.org/10.1017/S0017089500032286 Published online by Cambridge University Press

Theory and Harmonic Analysis, Editors J. A. Chao and W. A. Woyczyński (Marcel Dekker, New York, 1986), 1–19.

4. S-Y. A. Chang, Carleson measure on the bi-disc, *Annals of Math.* **109** (1979), 613–620.

5. R. Fefferman, Bounded mean oscillation on the polydisk, *Annals of Math.* **110** (1979), 395–406.

6. P. Koosis, *Introduction to H_p spaces*, London Mathematical Society Lecture Note Series 40 (Cambridge University Press, Cambridge 1980).

7. C. Le Merdy, The Adamjan-Arov-Krein theorem on H^p-spaces ($2 \leq p \leq \infty$) and on the disc algebra, *Bull. London Math. Soc.* **25** (1993), 275–281.

8. G. Pisier, *Factorization of linear operators and geometry of Banach spaces* (AMS Regional Conference Series 60. AMS, Providence Rhode Island, 1986).

9. G. Pisier, Factorization of operator valued analytic functions, *Adv. Math.* **93** (1992), 61–125.

10. S. C. Power, Hankel operators on Hilbert space, *Bull. London Math. Soc.* **12** (1980), 422–442.

DEPARTMENT OF MATHEMATICS AND STATISTICS,
LANCASTER UNIVERSITY,
LANCASTER LA1 4YF,
ENGLAND, U.K.