Sums of distinct fifth roots of unity and the regular dodecahedron

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Introduction

Integer linear combinations of cube, fourth, or sixth roots of unity form lattices in the complex plane $\mathbb{C}$. In contrast, integer linear combinations of fifth roots of unity do not form a lattice; in fact, they are dense in $\mathbb{C}$. Nevertheless, the geometry for fifth roots of unity has considerable structure. Here we consider only sums of distinct fifth roots of unity, and show that 20 of these sums are orthogonal projections of the vertices of a regular dodecahedron. Pentagonal symmetry here is only to be expected, but it is a little surprising to encounter a plane projection of a polyhedron with much richer dodecahedral symmetry.

More precisely, let ζ_n be a primitive nth root of unity. Then the algebraic number field $\mathbb{Q}(\zeta_n)$, consisting of all polynomials in ζ_n over the rational numbers $\mathbb{Q}$, is a *cyclotomic field*. Its *ring of integers* is $\mathbb{Z}(\zeta_n)$, consisting of all polynomials in ζ_n over the (positive or negative) integers $\mathbb{Z}$. See for example [1], [2], or the short proof in [3]. Integer combinations of the fourth roots of unity ± 1, $\pm i$ form a square lattice in $\mathbb{C}$, and integer combinations of cube roots or sixth roots of unity also form lattices. The same cannot be said of fifth roots of unity, one reason for the crystallographic restriction that lattices do not possess fivefold rotational symmetry. In fact, it is well known in algebraic number theory that integer combinations of fifth roots of unity are dense in $\mathbb{C}$. Indeed, all rings of cyclotomic integers $\mathbb{Z}(\zeta_n)$ where $n \neq 1, 2, 3, 4, 5, 6$ are dense in the plane. A short proof is given in [4, Corollary 12].

Sums of distinct powers

The primitive fifth root of unity

$$\zeta = e^{2\pi i/5} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} + i\sqrt{\frac{5+\sqrt{5}}{8}}$$

satisfies the equation

$$1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0. \tag{1}$$

There are 32 sums of powers ζ^k as k runs through a subset of $\{0, 1, 2, 3, 4\}$, but the empty sum and the sum of all five powers are both zero. The relation (1) generates all integer linear relations between these powers, so 31 of these sums are distinct. Figure 1 shows their location in $\mathbb{C}$. For reasons of space, sums of four terms are replaced by minus the fifth, so for example $1 + \zeta^2 + \zeta^3 + \zeta^4$ is replaced by $-\zeta$. (We retain sums of three terms

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because the geometry is clearer that way.) The thin grey lines are construction lines indicating addition of the corresponding power of ζ. For example the line from ζ^2 to $\zeta + \zeta^2$ is parallel to the line from the origin to and they have the same length.

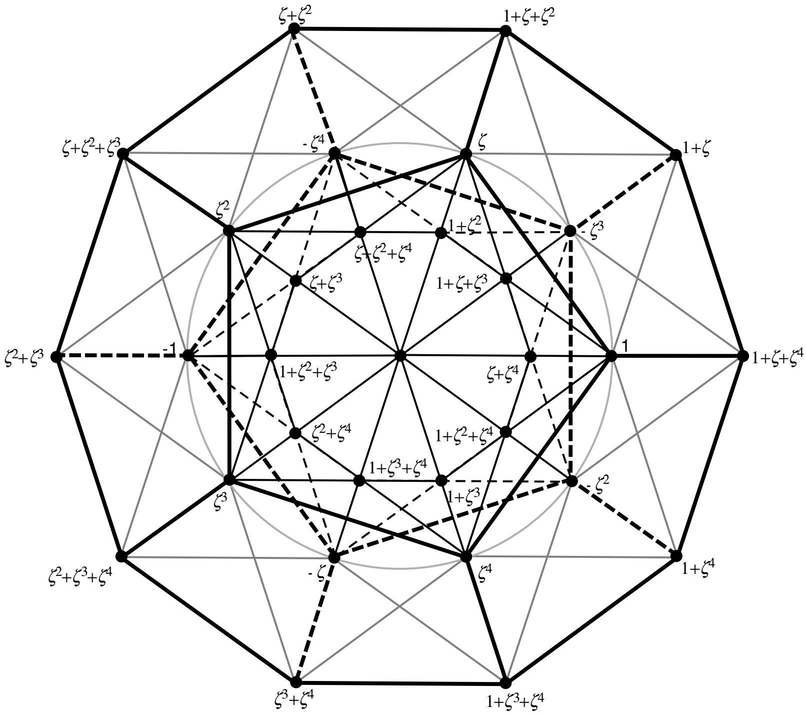

FIGURE 1: The 31 sums of distinct fifth roots of unity. Thick black lines (solid and dashed) resemble a projection of a regular dodecahedron. The grey circle is the unit circle in $\mathbb{C}$.

If we do not distinguish solid and dashed lines, the symmetry group of Figure 1 is the dihedral group $\mathbb{D}_{10}$ of order 10, the symmetry group of the regular decagon formed by the 10 outermost points. This group is generated by two rigid motions: rotation ρ by $\pi / 5$ anticlockwise and reflection in the real line. If we distinguish solid and dashed lines, the symmetry group is $\mathbb{D}_5$, generated by ρ^2 and σ. These symmetries play a crucial role later.

Figure 1 has additional structure that requires explanation, which is what motivated this Article. Anyone familiar with the regular solids can hardly fail to notice that the thick lines (solid and dashed) bear a striking resemblance to an orthogonal projection of a regular dodecahedron. (This is easier to see in Figure 2 (left) below, where extraneous points and lines have been removed.) The main aim of this article is to prove that this resemblance

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is exact. To do this, let V be this set of 20 points in $\mathbb{C}$. We will prove:

Theorem: The thick lines of Figure 1 are an orthogonal projection of the edges of a regular dodecahedron, whose vertices project to the set V in Figure 1.

The remaining 11 points in Figure 1 are the origin and the innermost ring of 10 points (containing, for example, $\zeta + \zeta^4$).

Coxeter [5, Section 3.7] derives Cartesian coordinates for the vertices of the regular solids (and more). The Theorem leads to a different set of coordinates derived from fifth roots of unity, which must be equivalent to Coxeter's up to rotation and dilation. Other interesting geometric figures are also visible in Figure 1, such as the kites and darts that can be used in a Penrose tiling (see Gardner [6]). We leave these for further investigation.

There are two ways to approach the Theorem :

(a) Set up an appropriate regular dodecahedron and calculate how it projects.

(b) Consider 'lifting' the projected figure to produce a regular dodecahedron.

Here we choose method (b).

Rings and Levels

We begin by separating the points of the projection into four *rings* $R_1 - R_4$ as in Table 1.

R_1	1	ζ	ζ^2	ζ^3	ζ^4
R_2	$1 + \zeta + \zeta^4$	$1 + \zeta + \zeta^2$	$\zeta + \zeta^2 + \zeta^3$	$\zeta^2 + \zeta^3 + \zeta^4$	$1 + \zeta^3 + \zeta^4$
R_3	$1 + \zeta$	$\zeta + \zeta^2$	$\zeta^2 + \zeta^3$	$\zeta^3 + \zeta^4$	$1 + \zeta^4$
R_4	-1	$-\zeta$	$-\zeta^2$	$-\zeta^3$	$-\zeta^4$

TABLE 1: Coordinates in $\mathbb{C}$ of points in the four rings $R_1 - R_4$

Each ring is symmetric under ρ^2, rotation by $2\pi/5$; that is, under multiplication by ζ. Moreover, taking (1) into account,

$$R_j = -R_{5-j} = \{-z : z \in R_j\} \qquad 1 \leqslant j \leqslant 5.$$

We now construct a (usually irregular) dodecahedron D whose vertices project to the 20 points V by 'lifting' each ring R_j to a new position in 3-dimensional space, in a direction perpendicular to $\mathbb{C}$, while keeping it horizontal. We call the resulting set of points the corresponding *level* L_j. Figure 2 shows the geometry for one choice of levels. On the left is projection onto $\mathbb{C}$, which corresponds to the thick black lines in Figure 1. (The figure is rotated slightly compared to Figure 1.) On the right is a 3-dimensional dodecahedron (which at this stage may not be regular) resting on $\mathbb{C}$.

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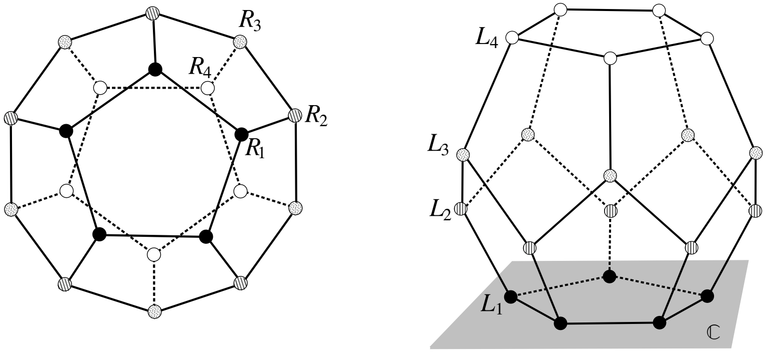

FIGURE 2: *Left*: Geometry of rings R_i in $\mathbb{C}$. Points in a given ring indicated by shading of vertices: L_1 (black); L_2 (striped); L_3 (dotted); L_4 (white). *Right*: A typical lifted (irregular) dodecahedron showing corresponding levels L_i.

We displace the four levels at right-angles to $\mathbb{C}$ as shown in Figure 3. Here a, b are two distances, to be calculated below. We assume top/bottom symmetry in the spacings because the corresponding levels of a regular dodecahedron are symmetrically located.

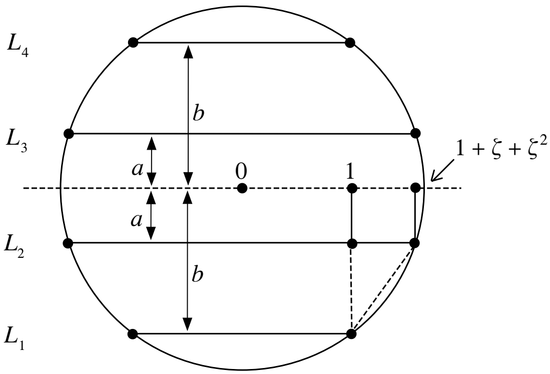

FIGURE 3: Cross-section of presumed circumscribing sphere S, showing the four levels. See text for dashed triangle at top right.

We now choose a and b to make the lifted dodecahedron D regular. It turns out that three conditions suffice:

(a) All 20 vertices of D lie on a sphere S with centre the origin.

(b) All edges of D have the same length, which is $l = \sqrt{3 - \tau}$, where $\tau = \frac{1}{2}\left(1 + \sqrt{5}\right)$ is the golden number.

(c) The radius of S equals the radius of the circumsphere of a regular dodecahedron of side l.

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We take these conditions in turn before explaining why they complete the proof of the Theorem. Observe that condition (a) fails for the levels illustrated in Figure 2 (right), which is deliberately drawn to illustrate this failure.

Condition (a).

Let S be the sphere with centre at the origin of $\mathbb{C} \times \mathbb{R}$ and radius r. Displace the rings R_j to levels L_j as in Figure 3, which shows the cross-section of S vertically through the real line. Both 1 and $1 + \zeta + \zeta^4$ lie on the real line, so the points $(1, -b)$ and $(1 + \zeta + \zeta^4, -a)$ lie in this cross-section. Therefore they lie on S provided that

$$b^2 + 1 = r^2 \qquad a^2 + \left(1 + \zeta + \zeta^4\right)^2 = r^2. \tag{2}$$

Now $1 + \zeta + \zeta^4 = \tau$ and $\tau^2 = 1 + \tau$. Therefore

$$b^2 + 1 = r^2 \qquad a^2 + (1 + \tau) = r^2$$

so

$$b^2 = a^2 + \tau. \tag{3}$$

To fix a and b we also want the vertices at $(1, -b)$ and $(\zeta, -b)$ to be the same distance apart as those at $(1, -b)$ and $\left(1 + \zeta + \zeta^4, -a\right)$. Thus (dashed lines at lower right in Figure 3) we also require

$$(a - b)^2 + (\tau - 1)^2 = |1 - \zeta|^2 = 3 - \tau$$

so

$$(a - b)^2 = 3 - \tau - (\tau - 1)^2 = 3 - \tau - \tau^2 + 2\tau - 1 = 2 + \tau - \tau^2 = 1$$

and $b - a = 1$. Now (3) implies that $(a + 1)^2 = a^2 + \tau$, so $2a + 1 = \tau$, so $a = \frac{1}{2}(\tau - 1)$. Since $b = a + 1$, we obtain

$$a = \frac{\tau - 1}{2} \qquad b = \frac{\tau + 1}{2}. \tag{4}$$

Symmetry by ρ^2 now implies that, for these choices of a and b, all vertices of the lifted dodecahedron lie on this sphere.

Condition (b).

The symmetries ρ^2 and σ in $\mathbb{D}_5$ extend from Figure 1 to any lift D constructed as above. With symmetrically spaced levels as in Figure 2, the symmetry also extends to D, but now rotation by $\pi/5$ must be combined with a top-bottom flip.

Condition (b) is trivial for levels L_1 and L_4. The symmetries just discussed, together with the conditions already imposed, reduce this calculation to that of the length of the edge between $\left(1 + \zeta, -a\right)$ and $\left(1 + \zeta + \zeta^4, a\right)$. If this length is d then

$$d^2 = \left|\zeta^4\right|^2 + (2a)^2 = 1 + 4a^2 = 1 + (\tau - 1)^2 = 3 - \tau$$

so $d = l$ as required.

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Condition (c)

By (2) the radius r of the sphere S is

$$r = \sqrt{b^2 + 1} = \sqrt{\frac{(\tau + 1)^2}{4} + 1}$$

so

$$\frac{r}{l} = \frac{\sqrt{\frac{(\tau+1)^2}{4} + 1}}{\sqrt{3 - \tau}}$$

which simplifies to

$$\frac{r}{l} = \frac{\sqrt{3}}{4}\left(1 + \sqrt{5}\right).$$

This is known to be the radius of the circumsphere of a regular dodecahedron of side-length 1; see [5, Table 1 p.292] or [7]. (Coxeter uses edge-length $2l$, not l.) By similarity, r is the circumradius of a regular dodecahedron of side l.

This completes the proof that conditions (a), (b) and (c) are satisfied.

We now explain why the Theorem follows from these conditions.

In Figure 1 there are five lines passing through the origin and a vertex in V, namely the real axis and its anticlockwise rotations by ρ, ρ^2, ρ^3, ρ^4. These are mirror lines for the corresponding reflections of the figure. They are the projections of five vertical planes in $\mathbb{C} \times \mathbb{R}$, which we call $P_0 - P_4$. Every vertex of any lifted dodecahedron lies in one of these planes P_j. Conditions (a), (b), (c) imply that the vertices of D are the *unique* points that:

(a) lie on S at the appropriate level,

(b) lie on one of the mirror planes P_j,

(c) are distance l from their immediate neighbours along the edges of D.

The points of a regular dodecahedron also satisfy these conditions. By uniqueness, these two sets of vertices coincide, so the lifted dodecahedron is is regular and the Theorem is proved.

Cartesian coordinates

We end by listing the Cartesian coordinates of the vertices of the regular dodecahedron according to this construction. We write the coordinates in the unorthodox form (z, v), where z is a complex coordinate on the horizontal plane $\mathbb{C}$ and v is a real one on the vertical axis $\mathbb{R}$. Now levels L_2 and L_3 lie in planes $v = \pm a$ and L_1 and L_4 lie in planes $v = \pm(a + b)$, where a, b are as in (4). The coordinates of the vertices are therefore:

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$(1, -\tau)\ (\zeta, -\tau)\ \ (\zeta^2, -\tau)\ (\zeta^3, -\tau)\ (\zeta^4, -\tau)$
$(1 + \zeta + \zeta^4, 1 - \tau)\ (1 + \zeta + \zeta^2, 1 - \tau)\ (\zeta + \zeta^2 + \zeta^3, 1 - \tau)$
$\quad (\zeta^2 + \zeta^3 + \zeta^4, 1 - \tau)\ (1 + \zeta^2 + \zeta^4, 1 - \tau)$
$(1 + \zeta, \tau - 1)\ (\zeta + \zeta^2, \tau - 1)\ (\zeta^2 + \zeta^3, \tau - 1)$
$\quad (\zeta^3 + \zeta^4, \tau - 1)\ (1 + \zeta^4, \tau - 1)$
$(-1, \tau)\ (-\zeta, \tau)\ (-\zeta^2, \tau)\ (-\zeta^3, \tau)\ (-\zeta^4, \tau)$

Alternative Approaches

Instead of appealing to the known formula for the radius of the circumsphere of a regular dodecahedron, we can start from the above list of coordinates and prove by direct calculation that they are the vertices of a regular dodecahedron. We already know that all edge-lengths are equal, so all that remains to be proved is that the pentagons corresponding to faces are planar. This can be done by using three points to find the equation of the plane in which they lie, and verifying that the other two points also lie on that plane. Symmetries reduce the number of calculations to just one such face (other than the top or bottom face).

Another possibility is to use the coordinates above to show that there are additional symmetries of the dodecahedron D. For example, that there is an orthogonal matrix fixing the points $\{1, \zeta\}$ at level L_1 and sending the points $\{\zeta^2, \zeta^3, \zeta^4\}$ at level L_1 to $\{1 + \zeta + \zeta^2,\ 1 + \zeta,\ 1 + \zeta + \zeta^4\}$ at level L_2. (This reflects the L_1 pentagon leaving the edge from 1 to ζ fixed.)

As presented, our results arise from a series of calculations that just happen to give the required answer. The same remark applies to the alternative proofs just sketched. This surely cannot be the full story. In particular, none of this gives a transparent representation of the full symmetry group of the dodecahedron. Presumably this is hidden in the above list of coordinates.

What is really going on?

Acknowledgements

We thank the referee, who corrected several errors and suggested a proof of the Theorem that improved on our original version. This led to a radical rethink, which eventually led to the proof given above, which differs from both.

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