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Anisotropic Gauss curvature flows and their associated Dual Orlicz-Minkowski problems

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In this paper we study a normalized anisotropic Gauss curvature flow of strictly convex, closed hypersurfaces in the Euclidean space. We prove that the flow exists for all time and converges smoothly to the unique, strictly convex solution of a Monge-Ampère type equation and we obtain a new existence result of solutions to the Dual Orlicz-Minkowski problem for smooth measures, especially for even smooth measures.

Keywords: Gauss curvature flow; convex hypersurface; Monge-Ampère type equation

2010 *Mathematics subject classification:* Primary 35J96
Secondary 53C45

1. Introduction

The Gauss curvature flow was introduced by Firey [15] to model the shape change of worn stones. The first celebrated result was proved by Andrews in [3], where Firey's conjecture that convex surfaces in $\mathbb{R}^3$ moving by their Gauss curvature became spherical as they contracted to points was proved. Guan-Ni [16] proved that convex hypersurfaces in $\mathbb{R}^{n+1}$ contracting along the Gauss curvature flow converged (after rescaling to fixed volume) to a smooth strictly convex self-similar solution of the flow. Soon, Andrews-Guan-Ni [7] extended the results in [16] to the flow by powers of the Gauss curvature K^α with $\alpha > (1/n+2)$. Recently, Brendle-Choi-Daskalopoulos [9] proved that round spheres were the only closed, strictly convex self-similar solutions to the K^α-flow with $\alpha > (1/n+2)$. Therefore, the generalized Firey's conjecture proposed by Andrews in [1] was completely solved, that was, the solutions of the flow by powers of the Gauss curvature converged to spheres for $\alpha > (1/n+2)$. We also refer to [2, 5, 6, 14] and the references therein.

As a natural extension of Gauss curvature flows, anisotropic Gauss curvature flows have attracted considerable attention and they provide alternative proofs for the existence of solutions to elliptic PDEs arising in geometry and physics, especially for the Minkowski type problem. For example an alternative proof based on the logarithmic Gauss curvature flow was given by Chou-Wang in [13] for the classical Minkowski problem, in [22] for a prescribing Gauss curvature problem. Bryan-Ivaki-Scheuer in [10] have given an unified flow approach to smooth, even L_p-Minkowski problems. Using a contracting Gauss curvature flow, Li-Sheng-Wang [18] have provided a parabolic proof in the smooth category for the classical Aleksandrov and dual Minkowski problems. Recently, two kinds of normalized anisotropic Gauss curvature flow have been used to prove the L_p dual Minkowski problems by Chen-Huang-Zhao [11] and Chen-Li [12], respectively. These results are major source of inspiration for us.

Let $\mathcal{M}_0$ be a strictly convex, closed and smooth hypersurface in $\mathbb{R}^{n+1}$ enclosing the origin given by

$$X_0 : \mathcal{M} \to \mathbb{R}^{n+1},$$

where $\mathcal{M}$ is an n-dimensional closed smooth Riemannian manifold. In this paper, we study the long-time behavior of the following normalized anisotropic Gauss curvature flow which is a family of hypersurfaces $\mathcal{M}_t = X(\mathcal{M}, t)$ given by smooth maps $X : \mathcal{M} \times (0, T) \to \mathbb{R}^{n+1}$ satisfying the initial value problem

$$\begin{cases} \dfrac{\partial X}{\partial t} = -\theta(t) f(\nu) \dfrac{r^{n+1}}{\varphi(r)} K\nu + X, & \text{on } \mathcal{M} \times (0, T), \\ X(\cdot, 0) = X_0, & \text{on } \mathcal{M}, \end{cases} \tag{1.1}$$

where ν is the unit outer vector of $\mathcal{M}_t$ at X, K denotes the Gauss curvature of $\mathcal{M}_t$ at X, $r = |X|$ denotes the distance form X to the origin, $f \in C^\infty(\mathbb{S}^n)$ with $f > 0$, $\varphi \in C^\infty(0, +\infty)$ is a positive smooth function, and

$$\theta(t) = \int_{\mathbb{S}^n} \varphi(r(\xi, t)) \mathrm{d}\xi \left[\int_{\mathbb{S}^n} f(x) \mathrm{d}x \right]^{-1}.$$

Here we parametrize the radial function r as a function from $\mathbb{S}^n$ to $\mathbb{R}$. Both $d\xi$ and dx are the spherical measures on $\mathbb{S}^n$.

The reason that we study the flow (1.1) is to explore the existence of the smooth solutions to the dual Orlicz-Minkowski problem introduced by Zhu-Xing-Ye [24], which is equivalent to solve the following Monge-Ampère type equation

$$\frac{u\ \varphi(r)}{r^{n+1}} \cdot \det(u_{ij} + u\ \delta_{ij}) = \lambda f(x) \quad \text{on} \quad \mathbb{S}^n, \tag{1.2}$$

where $r = \sqrt{|Du|^2 + u^2}$ and λ is a positive constant. In deed, let $\mathcal{K}_0$ be the set of all convex bodies in $\mathbb{R}^{n+1}$ which contain the origin in their interiors, $\varphi : (0, +\infty) \to (0, +\infty)$ be a continuous function. Zhu-Xing-Ye [24] have introduced the dual Orlicz curvature measure of $K \in \mathcal{K}_0$

$$\widetilde{C}_\varphi(K, E) = \frac{1}{n+1} \int_{\alpha_K^*(E)} \varphi(r_K(\xi)) \mathrm{d}\xi$$

for each Borel set $E \subset \mathbb{S}^n$, where α_K^* is the reverse radial Gauss image on $\mathbb{S}^n$, r_K is the radial function of K, $\mathrm{d}\xi$ is the spherical measure on $\mathbb{S}^n$, see [24] for more details. They posed the following dual Orlicz-Minkowski problem in [24]:

PROPOSITION 1.1 Dual Orlicz-Minkowski problem. *Under what conditions on* φ *and a nonzero finite Borel measure* μ *on* $\mathbb{S}^n$*, there exists a constant* $\lambda > 0$ *and* $K \in \mathcal{K}_0$ *such that* $\lambda \cdot \mu = \widetilde{C}_\varphi(K, \cdot)$*?*

When μ has a density f, this kind of Minkowski problem is equivalent to solve the Monge-Ampère type equation (1.2). When $\varphi(r) = r^q$, it becomes the dual Minkowski problem for the q-th dual curvature measure considered by Huang-Lutwak-Yang-Zhang [**17**]. It is worth pointing out that they also proved the existence of symmetric solutions for the case $q \in (0, n+1)$ under some conditions. For $q = n+1$, the dual Minkowski problem becomes the logarithmic Minkowski problem which was studied in [**8**]. For $q < 0$, the existence and uniqueness of weak solutions were obtained by Zhao [**23**].

It is to be expected that the flow (1.1) converges to a solution of the equation (1.2). We obtain the following result for the flow (1.1).

THEOREM 1.2. *Suppose that* $f \in C^\infty(\mathbb{S}^n)$ *is a positive and even function and* $\varphi \in C^\infty(0, +\infty)$ *is a positive function satisfying*

$$\int_0^1 \frac{\varphi(s)}{s} \mathrm{d}s < +\infty. \tag{1.3}$$

Let $\mathcal{M}_0 \subset \mathbb{R}^{n+1}$ *be an origin-symmetric, strictly convex, closed and smooth hypersurface which contains the origin in its interior. Then,*

(i) *the normalized flow* (1.1) *has a unique smooth solution, which exists for any time* $t \in [0, \infty)$*;*

(ii) *for each* $t \in [0, \infty)$*,* $\mathcal{M}_t = X(\mathcal{M}, t)$ *is an origin-symmetric, strictly convex, closed and smooth hypersurface which also contains the origin in its interior;*

(iii) *the support function* $u(x, t)$ *of* $\mathcal{M}_t = X(\mathcal{M}, t)$ *converges smoothly, as* $t \to \infty$*, to a positive, strictly convex and smooth solution of the equation* (1.2) *with* $\lambda = \lim_{t_i \to \infty} \theta(t_i) > 0$.

As a corollary of theorem 1.2, we get the following existence of solutions to the dual Orlicz-Minkowski problem (1.2).

THEOREM 1.3. *Suppose that* $f \in C^\infty(\mathbb{S}^n)$ *is a positive and even function and* $\varphi \in C^\infty(0, +\infty)$ *is a positive function satisfying* (1.3)*. Then there exists a positive constant* λ *and a positive, smooth and even function* u *satisfying the equation* (1.2).

For the special case $\varphi(r) = r^q$, the condition (1.3) in theorem 1.2 is equivalent to $q > 0$. Thus, theorem 1.2 recovers a parabolic proof in the smooth category for the existence of solutions to the even dual Minkowsi problem for $q > 0$ which is given in [**18**]. Recently, Liu-Lu [**19**] also used the flow method to study the dual Orlicz-Minkowski problem and they obtained the existence result under the condition that

$$\lim_{r\to+\infty}\varphi(r) < f(x) < \lim_{r\to 0^+}\varphi(r), \quad \forall x\in\mathbb{S}^n,$$

which means $q<0$ if $\varphi(r)=r^q$.

Our proof of theorem 1.2 is inspired by [**11**] and [**18**]. We need to obtain uniform positive upper and lower bounds for the support function and principal curvatures along the flow to derive its long-time existence. The difficulty of these a priori estimates for the flow (1.1) lies in the inhomogeneous term $\varphi(r)$. So we need to choose proper auxiliary functions and do more delicate computations. Then the long-time existence follows by standard arguments.

The organization of this paper is as follows. In § 2 we start with some preliminaries. The C^0, C^1 and C^2 estimates are given in § 3. In § 4 we prove theorem 1.2.

2. Preliminaries

2.1. Basic properties of convex hypersurfaces

We first recall some basic properties of convex hypersurfaces. Let $\mathcal{M}$ be a smooth, closed and strictly convex hypersurface in $\mathbb{R}^{n+1}$. Assume that $\mathcal{M}$ is parametrized by the inverse Gauss map

$$X:\mathbb{S}^n\to\mathcal{M}.$$

The support function $u:\mathbb{S}^n\to\mathbb{R}$ of $\mathcal{M}$ is defined by

$$u(x)=\sup\{\langle x,y\rangle : y\in\mathcal{M}\}.$$

The supreme is attained at a point y such that x is the outer normal of $\mathcal{M}$ at X. It is easy to check that

$$X=u(x)x+Du(x),$$

where D is the covariant derivative with respect to the standard metric σ_{ij} of the sphere $\mathbb{S}^n$. Hence,

$$r=|X|=\sqrt{u^2+|Du|^2}, \tag{2.1}$$

and

$$u=\frac{r^2}{\sqrt{r^2+|Dr|^2}}. \tag{2.2}$$

The second fundamental form of $\mathcal{M}$ is given by, see e.g. [**4**, **21**],

$$h_{ij}=u_{ij}+u\sigma_{ij}, \tag{2.3}$$

where $u_{ij}=D_iD_ju$ denotes the second-order covariant derivative of u with respect to the spherical metric σ_{ij}. By Weingarten's formula,

$$\sigma_{ij}=\left\langle\frac{\partial\nu}{\partial x^i},\frac{\partial\nu}{\partial x^j}\right\rangle=h_{ik}g^{kl}h_{jl}, \tag{2.4}$$

where g_{ij} is the metric of $\mathcal{M}$ and g^{ij} is its inverse. It follows from (2.3) and (2.4) that the principal radii of curvature of $\mathcal{M}$, under a smooth local orthonormal frame on $\mathbb{S}^n$, are the eigenvalues of the matrix

$$b_{ij} = u_{ij} + u\delta_{ij}.$$

In particular, the Gauss curvature is given by

$$K = \frac{1}{\det(u_{ij} + u\delta_{ij})}.$$

2.2. Geometric flow and its associated functional

Recall the normalized anisotropic Gauss curvature flow (1.1)

$$\begin{cases} \dfrac{\partial X}{\partial t} = -\theta(t) f(\nu) \dfrac{r^{n+1}}{\varphi(r)} K\nu + X, \\ X(\cdot, 0) = X_0, \end{cases}$$

where

$$\theta(t) = \int_{\mathbb{S}^n} \varphi(r(\xi, t)) \mathrm{d}\xi \left[\int_{\mathbb{S}^n} f(x) \mathrm{d}x \right]^{-1}.$$

By the definition of support function, we know $u(x, t) = \langle x, X(x, t)\rangle$. Hence,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x, t) = -\theta(t) \dfrac{f(x) r^{n+1}}{\varphi(r)} K + u(x, t), \\ u(\cdot, 0) = u_0. \end{cases} \tag{2.5}$$

The normalized flow (1.1) can be also described by the following scalar equation for the radial function $r(\cdot, t)$

$$\begin{cases} \dfrac{\partial r}{\partial t}(\xi, t) = -\theta(t) \dfrac{f(x) r^{n+2}}{\varphi(r) u} K + r(\xi, t), \\ r(\cdot, 0) = r_0, \end{cases} \tag{2.6}$$

where we use the following relation (see § 3 in [**12**] for the proof) to get the above equation

$$\frac{1}{r(\xi, t)} \frac{\partial r(\xi, t)}{\partial t} = \frac{1}{u(x, t)} \frac{\partial u(x, t)}{\partial t}.$$

For a convex body $\Omega \subset \mathbb{R}^{n+1}$ which contains the origin in its interior, we define φ-volume of Ω as

$$V_\varphi(\Omega) = \int_{\mathbb{S}^n} \mathrm{d}\xi \int_0^{r(\xi)} \frac{\varphi(s)}{s} \mathrm{d}s,$$

where r is the radial function of Ω. When $\varphi(s) = s^q$, $V_\varphi(\Omega)$ is the q-volume of the convex body $\Omega \subset \mathbb{R}^{n+1}$, see [**11, 12**]. Under the condition (1.3), V_φ is well defined.

We will show below that $V_\varphi(\Omega_t)$ is unchanged along the flow (1.1), where Ω_t is a compact convex body in $\mathbb{R}^{n+1}$ with the boundary $\mathcal{M}_t$.

LEMMA 2.1. *Let $X(\cdot,t)$ be a strictly convex solution to the flow (1.1) which encloses the origin for $t\in(0,T)$, then under the condition (1.3), we obtain*

$$V_\varphi(\Omega_t)=V_\varphi(\Omega_0).$$

Proof. By a direction calculation, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_\varphi(\Omega_t)&=\int_{\mathbb{S}^n}\frac{\varphi(r)}{r}\frac{\partial r}{\partial t}\mathrm{d}\xi\\&=\int_{\mathbb{S}^n}\frac{\varphi(r)}{r}\Big(-\theta(t)\frac{f(x)r^{n+2}}{\varphi(r)u}K+r(\xi,t)\Big)\mathrm{d}\xi\\&=-\theta(t)\int_{\mathbb{S}^n}\frac{f(x)r^{n+1}}{u}K\mathrm{d}\xi+\int_{\mathbb{S}^n}\varphi(r)\mathrm{d}\xi\\&=0,\end{aligned}$$

where we use the integration by substitution (see e.g. [**12**, **17**]) to get the last inequality

$$\frac{\mathrm{d}x}{\mathrm{d}\xi}=\frac{r^{n+1}K}{u}.$$

We complete the proof. □

Next, we define the entropy functional along the flow (1.1)

$$\mathcal{J}_\varphi(X(\cdot,t))=\int_{\mathbb{S}^n}\log u(x,t)\cdot f(x)\mathrm{d}x.$$

The following lemma shows that the functional $\mathcal{J}_\varphi$ is non-increasing along the flow (1.1).

LEMMA 2.2. *Assume the condition (1.3) holds and let $X(\cdot,t)$ be a strictly convex solution to the flow (1.1) which encloses the origin for $t\in(0,T)$. For any $\varphi\geqslant 0$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_\varphi(X(\cdot,t))\leqslant 0$$

and the equality holds if and only if X_t satisfies the elliptic equation (1.2) with $\lambda=\theta(t)$.

Proof. We have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_\varphi(X(\cdot,t))\\&\quad=\int_{\mathbb{S}^n}\frac{1}{u}\frac{\partial u(x,t)}{\partial t}f(x)\mathrm{d}x\\&\quad=\int_{\mathbb{S}^n}\frac{1}{u}\Big(-\theta(t)\frac{f(x)r^{n+1}}{\varphi(r)}K+u(x,t)\Big)f(x)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \left[\int_{\mathbb{S}^n} f(x)\mathrm{d}x\right]^{-1}\left\{-\int_{\mathbb{S}^n}\frac{u\varphi(r)}{r^{n+1}K}\mathrm{d}x\int_{\mathbb{S}^n}\frac{r^{n+1}K}{u\varphi(r)}f^2\mathrm{d}x+\int_{\mathbb{S}^n} f\mathrm{d}x\int_{\mathbb{S}^n} f\mathrm{d}x\right\}\\
&= \left[\int_{\mathbb{S}^n} f(x)\mathrm{d}x\right]^{-1}\left\{-\int_{\mathbb{S}^n}\frac{u\varphi(r)}{fr^{n+1}K}\mathrm{d}\sigma\int_{\mathbb{S}^n}\frac{r^{n+1}K}{u\varphi(r)}f\mathrm{d}\sigma+\int_{\mathbb{S}^n}\mathrm{d}\sigma\int_{\mathbb{S}^n}\mathrm{d}\sigma\right\}\\
&\leqslant 0
\end{aligned}
$$

in view of

$$
\left(\int_{\mathbb{S}^n}\mathrm{d}\sigma\right)^2\leqslant\int_{\mathbb{S}^n}\frac{u\varphi(r)}{fr^{n+1}K}\mathrm{d}\sigma\int_{\mathbb{S}^n}\frac{r^{n+1}K}{u\varphi(r)}f\mathrm{d}\sigma,
$$

which is implied by Hölder inequality and where $\mathrm{d}\sigma = f(x)\mathrm{d}x$. Clearly, the equality holds if and only if

$$
\frac{f(x)r^{n+1}K}{u\varphi(r)}=\frac{1}{c(t)}.
$$

Thus, $X(\cdot,t)$ satisfies the elliptic equation (1.2) with $\lambda=\theta(t)$. □

3. A priori estimates

In this section, we will derive uniform positive upper and lower bounds for the support function and principal curvatures along the flow (1.1). The key is the lower bound of u. The difficulty of the proof lies in dealing with the inhomogeneous term $\varphi(r)$.

3.1. C^0 and C^1 estimates

In this subsection, we will derive C^0 and C^1 estimates for the support function along the flow (1.1).

Lemma 3.1. *Suppose the condition (1.3) holds and let $X(\cdot,t)$ be an origin-symmetric and strictly convex solution to the flow (1.1) which encloses the origin for $t\in(0,T)$, then we have*

$$
\frac{1}{C}\leqslant u(x,t)\leqslant C,\quad \forall(x,t)\in\mathbb{S}^n\times(0,T), \tag{3.1}
$$

and

$$
|Du|(x,t)\leqslant C,\quad \forall(x,t)\in\mathbb{S}^n\times(0,T). \tag{3.2}
$$

Proof. Assume $u(\cdot,t)$ attains its spatial maximum at a point x_t. Since $\mathcal{M}_t$ is origin-symmetric, we have by the definition of support function that

$$
u(x,t)\geqslant u(x_t,t)|\langle x,x_t\rangle|,\quad \forall x\in S^n.
$$

Thus, we know from lemma 2.2

$$\begin{aligned} C &\geqslant \int_{\mathbb{S}^n} \log u(x,t) \cdot f(x)\mathrm{d}x \\ &\geqslant \log u(x_t,t) \int_{\mathbb{S}^n} f(x)\mathrm{d}x + \int_{\mathbb{S}^n} \log |\langle x, x_t\rangle| f(x)\mathrm{d}x \\ &\geqslant C_1 \log u(x_t,t) - C_2, \end{aligned}$$

which implies

$$C \geqslant \max_{\mathbb{S}^n} u(\cdot,t).$$

This yields the first inequality in (3.1). By (2.1), we conclude

$$\max_{\mathbb{S}^n} |Du(\cdot,t)|^2 \leqslant \max_{\mathbb{S}^n} r^2(\cdot,t) = \max_{\mathbb{S}^n} u^2(\cdot,t).$$

So

$$\max_{\mathbb{S}^n} |Du|(\cdot,t) \leqslant \max_{\mathbb{S}^n} u(\cdot,t),$$

leading to the inequality (3.2).

Next we will derive a positive lower bound of u. Here we use the idea in [**11**] to complete our proof by contradiction. Assume $u(x,t)$ is not uniformly bounded away from 0 which means there exists t_i such that

$$\inf_{x\in\mathbb{S}^n} u(x,t_i) \to 0$$

as $i \to \infty$, where $t_i \in (0,T)$. Recall that Ω_t is the origin-symmetric convex body containing the origin satisfying $\partial\Omega_t = \mathcal{M}_t$. Thus, using the Blaschke selection theorem, we can say that $\{\Omega_{t_i}\}_{i=1,2,\ldots}$ (after choosing a subsequence) converge to a origin-symmetric convex body Ω_0. Then, the support function u_{Ω_0} of Ω_0 satisfies

$$\inf_{\xi\in\mathbb{S}^n} u_{\Omega_0}(x) = 0.$$

So, there exists $x_0 \in \mathbb{S}^n$ such that $u_{\Omega_0}(x_0) = 0$ and thus $u_{\Omega_0}(-x_0) = 0$, which implies that Ω_0 is contained in a lower-dimensional subspace. This means that

$$r(\xi,t_i) \to 0$$

as $i \to \infty$ almost everywhere with respect to the spherical Lebesgue measure. Under the condition (1.3), we can use the bounded convergence theorem to get

$$\int_{\mathbb{S}^n} \mathrm{d}\xi \int_0^{r(\xi,0)} \frac{\varphi(s)}{s}\mathrm{d}s = \int_{\mathbb{S}^n} \mathrm{d}\xi \int_0^{r(\xi,t_i)} \frac{\varphi(s)}{s}\mathrm{d}s \to 0$$

as $i \to \infty$, which is a contradiction. So, we complete our proof. □

Clearly, C^0 and C^1 estimates of u imply the corresponding C^0 and C^1 estimates of r by (2.1) and (2.2).

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COROLLARY 3.2. *Under the same assumptions in lemma 3.1, we have*

$$\frac{1}{C} \leqslant r(\xi,t) \leqslant C, \quad \forall(\xi,t) \in \mathbb{S}^n \times (0,T),$$
$$|Dr|(\xi,t) \leqslant C, \quad \forall(\xi,t) \in \mathbb{S}^n \times (0,T),$$

and

$$\frac{1}{C} \leqslant \theta(t) \leqslant C, \quad \forall t \in (0,T).$$

3.2. C^2-estimates

In this subsection we establish the uniformly upper bound of Gauss curvature, and uniformly positive lower bounds for the principle curvatures for the normalized flow (1.1). We first use the technique introduced by Tso [**20**] to derive the upper bound of the Gauss curvature along the flow (1.1), see also the proof of lemma 4.1 in [**18**] and lemma 5.1 in [**11**].

LEMMA 3.3. *Let $X(\cdot,t)$ be a strictly convex solution to the flow (1.1) which encloses the origin for $t \in (0,T)$. Then, there exists a positive constant C depending only φ, $\max_{\mathbb{S}^n \times (0,T)} u$ and $\min_{\mathbb{S}^n \times (0,T)} u$, such that*

$$\max_{\mathbb{S}^n} K(\cdot,t) \leqslant C, \quad \forall t \in (0,T).$$

Proof. We apply the maximum principle to the following auxiliary function defined on the unit sphere $\mathbb{S}^n$,

$$W(x,t) = \frac{1}{\theta(t)} \frac{-u_t + u}{u - \varepsilon_0} = \frac{f(x)}{\varphi(r)} r^{n+1} \frac{K}{u - \varepsilon_0},$$

where

$$\varepsilon_0 = \frac{1}{2} \min_{(x,t)\in\mathbb{S}^n \times (0,T)} u(x,t) > 0.$$

For any fixed $t \in (0,T)$, we assume the maximum of W is attained at $x_0 \in \mathbb{S}^n$. Then, we have at (x_0,t)

$$0 = \theta(t) W_i = \frac{-u_{ti} + u_i}{u - \varepsilon_0} + \frac{u_t - u}{(u - \varepsilon_0)^2} u_i, \tag{3.3}$$

and

$$0 \geqslant \theta(t) D^2_{ij} W = \frac{-u_{tij} + u_{ij}}{u - \varepsilon_0} + \frac{(u_t - u)u_{ij}}{(u - \varepsilon_0)^2}, \tag{3.4}$$

where (3.3) is used in deriving (3.4). The inequality (3.4) should be understood in sense of positive semi-definite matrix. Hence,

$$u_{tij} + u_t \delta_{ij} \geqslant \theta(t)(-b_{ij} + \varepsilon_0 \delta_{ij}) W + b_{ij}.$$

Thus,

$$K_t = -Kb^{ij}(u_{tij} + u_t\delta_{ij}) \leqslant -nK - \theta(t)KW(-n + \varepsilon_0 H),$$

where H denotes the mean curvature of $X(\cdot, t)$. Notice that $H \geqslant nK^{\frac{1}{n}}$, we obtain

$$K_t \leqslant CW(1 + W) - CW^{2+\frac{1}{n}}.$$

Using the equation (2.5) and the inequality above, we have

$$\begin{aligned} W_t &= \left[\frac{f(x)}{\varphi(r)}\frac{r^{n+1}}{u-\varepsilon_0}\right]_t K + \left[\frac{f(x)}{\varphi(r)}\frac{r^{n+1}}{u-\varepsilon_0}\right] K_t \\ &\leqslant CW^2 + CW - CW^{2+\frac{1}{n}}, \end{aligned}$$

in view of

$$u_t \approx CW + C, \quad r_t = \frac{uu_t + u^k u_{kt}}{r} \approx CW + C.$$

Without loss of generality we assume that $K \approx W \gg 1$, which implies that

$$W_t \leqslant 0.$$

Therefore, we arrive at $W \leqslant C$ for some positive constant C depending on the C^1-norm of r and ε_0. Thus, the upper bound of K follows consequently. □

Next, we show the principle curvatures of $X(\cdot, t)$ are bounded from below along the flow (1.1). The proof is similar to lemma 4.2 in [**18**] and lemma 5.1 in [**11**].

LEMMA 3.4. *Let $X(\cdot, t)$ be a strictly convex solution to the flow (1.1) which encloses the origin for $t \in (0, T)$. Then, there exists a positive constant C depending only on φ, $\max_{\mathbb{S}^n\times(0,T)} u$ and $\min_{\mathbb{S}^n\times(0,T)} u$, such that the principle curvatures of $X(\cdot, t)$ are bounded from below*

$$\kappa_i(x, t) \geqslant C, \quad \forall (x, t) \in \mathbb{S}^n \times (0, T), \text{ and } i = 1, 2 \ldots, n. \tag{3.5}$$

Proof. We consider the auxiliary function

$$\widetilde{\Lambda}(x, t) = \log \lambda_{\max}(\{b_{ij}\}) - A \log u + B|Du|^2,$$

where A and B are positive constants which will be chosen later, and $\lambda_{max}(\{b_{ij}\})$ denotes the maximal eigenvalue of $\{b_{ij} = u_{ij} + u\delta_{ij}\}$. For convenience, we write $\{b^{ij}\}$ for $\{b_{ij}\}^{-1}$.

For any fixed $t \in (0, T)$, we assume the maximum $\widetilde{\Lambda}$ is achieved at some point $x_0 \in \mathbb{S}^n$. By rotation, we may assume $\{b^{ij}(x_0, t)\}$ is diagonal and $\lambda_{\max}(\{b_{ij}\})(x_0, t) = b_{11}(x_0, t)$. Thus, it is sufficient to prove $b_{11}(x_0, t) \leqslant C$. Then, we define a new auxiliary function

$$\Lambda(x,t) = \log b_{11} - A\log u + B|Du|^2,$$

which attains the local maximum at x_0 for fixed t. Thus, we have at x_0

$$0 = D_i\Lambda = b^{11}b_{11;i} - A\frac{u_i}{u} + 2B\sum_k u_k u_{ki} \tag{3.6}$$

and

$$0 \geqslant D_iD_j\Lambda = b^{11}b_{11;ij} - (b^{11})^2 b_{11;i}b_{11;j} - A\left(\frac{u_{ij}}{u} - \frac{u_iu_j}{u^2}\right) + 2B\sum_k \left(u_{kj}u_{ki} + u_k u_{kij}\right). \tag{3.7}$$

We can rewrite equation (2.5) as

$$\log(u - u_t) = -\log\det(b_{ij}) + \alpha(x,t), \tag{3.8}$$

where

$$\alpha(x,t) = \log\left(\theta(t)\frac{f(x)r^{n+1}}{\varphi(r)}\right).$$

Differentiating (3.8), we have

$$\frac{u_k - u_{kt}}{u - u_t} = -b^{ij}b_{ij;k} + D_k\alpha \tag{3.9}$$

and

$$\frac{u_{11} - u_{11t}}{u - u_t} = \frac{(u_1 - u_{1t})^2}{(u - u_t)^2} - b^{ii}b_{ii;11} + b^{ii}b^{jj}(b_{ij;1})^2 + D_1D_1\alpha. \tag{3.10}$$

Recall the Ricci identity

$$b_{ii;11} = b_{11;ii} - b_{11} + b_{ii},$$

by taking it into (3.10) we have

$$\frac{u_{11} - u_{11t}}{u - u_t} = \frac{(u_1 - u_{1t})^2}{(u - u_t)^2} - b^{ii}b_{11;ii} + \sum_i b^{ii}b_{11} - n + b^{ii}b^{jj}(b_{ij;1})^2 + D_1D_1\alpha. \tag{3.11}$$

So, we have

$$\begin{aligned}\frac{\partial_t\Lambda}{u - u_t} &= b^{11}\left(\frac{u_{11t} - u_{11}}{u - u_t} + \frac{u_{11} + u - u + u_t}{u - u_t}\right) - A\frac{1}{u}\frac{u_t - u + u}{u - u_t} + 2B\frac{u^k u_{kt}}{u - u_t}\\ &= b^{11}\left[-\frac{(u_1 - u_{1t})^2}{(u - u_t)^2} + b^{ii}b_{11;ii} - \sum_i b^{ii}b_{11} - b^{ii}b^{jj}(b_{ij;1})^2 - D_1D_1\alpha\right]\\ &\quad + \frac{1 - A}{u - u_t} + \frac{A}{u} + 2B\frac{\sum_k u_k u_{kt}}{u - u_t} + (n-1)b^{11}.\end{aligned} \tag{3.12}$$

We know from (3.7) and (3.9)

$$
\begin{aligned}
0 \geqslant\ & b^{11}[b^{ii}b_{11;ii} - b^{ii}b^{11}(b_{i1;1})^2] - A\frac{n}{u} + A\sum_i b^{ii} + Ab^{ii}\frac{u_iu_i}{u^2} \\
& + 2B\left[b^{ii}(b_{ii}-u)^2 + \sum_k u_k\left(D_k\alpha - \frac{u_k - u_{kt}}{u-u_t}\right) - b^{ii}u_iu_i\right] \\
\geqslant\ & b^{11}[b^{ii}b_{11;ii} - b^{ii}b^{jj}(b_{ij;1})^2] - A\frac{n}{u} + A\sum_i b^{ii} + Ab^{ii}\frac{u_iu_i}{u^2} \\
& + 2B\left[\sum_i b^{ii}(b_{ii}^2 - 2ub_{ii}) + \sum_k u_k\left(D_k\alpha - \frac{u_k - u_{kt}}{u-u_t}\right) - b^{ii}u_iu_i\right] \\
\geqslant\ & b^{11}[b^{ii}b_{11;ii} - b^{ii}b^{jj}(b_{ij;1})^2] - A\frac{n}{u} + A\sum_i b^{ii} + Ab^{ii}\frac{u_iu_i}{u^2} \\
& + 2B\left[\sum_i b_{ii} - 2nu + \sum_k u_k\left(D_k\alpha - \frac{u_k - u_{kt}}{u-u_t}\right) - b^{ii}u_iu_i\right].
\end{aligned}
$$

Thus, plugging the inequality above into (3.12), we have

$$
\begin{aligned}
\frac{\partial_t\Lambda}{u-u_t} \leqslant\ & -b^{11}D_1D_1\alpha - 2B\sum_k u_kD_k\alpha + \frac{1-A+2B|Du|^2}{u-u_t} \\
& + \frac{(n+1)A}{u} + (n-1)b^{11} + (2B|Du| - A - 1)\sum_i b^{ii} \\
& - Ab^{ii}\frac{u_iu_i}{u^2} - 2B\sum_i b_{ii} + 4nBu.
\end{aligned} \tag{3.13}
$$

Now, we need to estimate the first two terms in the right hand of the inequality. Clearly, a direct calculation results in

$$
r_i = \frac{uu_i + \sum_k u_ku_{ki}}{r} = \frac{u_ib_{ii}}{r}
$$

and

$$
r_{ij} = \frac{uu_{ij} + u_iu_j + \sum_k u_ku_{kij} + \sum_k u_{kj}u_{ki}}{r} - \frac{u_iu_ib_{ii}b_{jj}}{r^3}.
$$

Hence, we obtain by lemma 3.1 and corollary 3.2

$$
\begin{aligned}
& -b^{11}D_1D_1\alpha - 2B\sum_k u_kD_k\alpha \\
& = -b^{11}\left[\frac{f_{11}}{f} - \frac{f_1^2}{f^2} - (n+1)\frac{r_1^2}{r^2} + \frac{(\varphi')^2r_1^2}{\varphi^2} - \frac{\varphi''r_1^2}{\varphi}\right] \\
& \quad - b^{11}\left[(n+1)\frac{1}{r} - \frac{\varphi'}{\varphi}\right]r_{11} - 2B\sum_k u_k\left(\frac{f_k}{f} + [(n+1)\frac{1}{r} - \frac{\varphi'}{\varphi}]r_k\right)
\end{aligned}
$$

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$$\leqslant Cb^{11}(1+b_{11})+CB-\left[(n+1)\frac{1}{r}-\frac{\varphi'}{\varphi}\right](b^{11}r_{11}+2Bu_kr_k)$$
$$\leqslant Cb^{11}(1+b_{11}+b_{11}^2)+CB-\left[(n+1)\frac{1}{r}-\frac{\varphi'}{\varphi}\right]\left(b^{11}\frac{u_ku_{k11}}{r}+2B\frac{u_ku_ku_{kk}}{r}\right).$$

Then, using (3.6), we have

$$\begin{aligned}
&-b^{11}D_1D_1\alpha-2B\sum_k u_kD_k\alpha\\
&\quad\leqslant Cb^{11}(1+b_{11}+b_{11}^2)+CB-\left[(n+1)\frac{1}{r}-\frac{\varphi'}{\varphi}\right]\frac{u_k}{r}\left(A\frac{u_k}{u}-b^{11}u_1\delta_{k1}\right)\\
&\quad\leqslant Cb^{11}(1+b_{11}+b_{11}^2)+CB+CA.
\end{aligned}$$

Thus, using the inequality above, we conclude from (3.13)

$$\begin{aligned}
\frac{\partial_t\Lambda}{u-u_t}&\leqslant C(b^{11}+1+b_{11})+CB+CA+\frac{1-A+2B|Du|^2}{u-u_t}+\frac{(n+1)A}{u}\\
&\qquad+(n-1)b^{11}+(2B|Du|-A-1)\sum_i b^{ii}-Ab^{ii}\frac{u_iu_i}{u^2}-2B\sum_i b_{ii}+4nBu\\
&\quad<0,
\end{aligned}$$

provided $b_{11}\gg 1$ and if we choose $A\gg B$. So we complete the proof. □

4. The convergence of the normalized flow

With the help of a prior estimates in the section above, we show the long-time existence and asymptotic behavior of the normalized flow (1.1) which complete theorem 1.2.

Proof. Since the equation (2.5) is parabolic, we have the short time existence. Let T be the maximal time such that $u(\cdot,t)$ is a positive, smooth and strictly convex solution to (2.5) for all $t\in(0,T)$. Since $\mathcal{M}$ is origin-symmetric and f is even, $X(\cdot,t)$ is an origin-symmetric and strictly convex solution to the flow (1.1) which encloses the origin for $t\in(0,T)$. Thus, lemmas 3.1, 3.3 and corollary 3.2 enable us to apply lemma 3.4 to equation (2.5) and thus we can deduce a uniformly lower estimate for the biggest eigenvalue of $\{(u_{ij}+u\delta_{ij})(x,t)\}$. This together with lemma 3.4 implies

$$C^{-1}I\leqslant(u_{ij}+u\delta_{ij})(x,t)\leqslant CI,\quad\forall(x,t)\in\mathbb{S}^n\times(0,T),$$

where $C>0$ depends only on n,φ,f and u_0. This shows that the equation (2.5) is uniformly parabolic. Using Evans-Krylov estimates and Schauder estimates, we obtain

$$|u|_{C^{l,m}_{x,t}(\mathbb{S}^n\times(0,T))}\leqslant C_{l,m}$$

for some $C_{l,m}$ independent of T. Hence $T=\infty$. The uniqueness of the smooth solution $u(\cdot,t)$ follows by the parabolic comparison principle.

By the monotonicity of $\mathcal{J}_\varphi$ (see lemma 2.2), and noticing that

$$|\mathcal{J}_\varphi(X(\cdot,t))| \leqslant C, \quad \forall t \in (0,\infty),$$

we conclude that

$$\int_0^\infty |\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_\varphi(X(\cdot,t))| \leqslant C.$$

Hence, there is a sequence $t_i \to \infty$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_\varphi(X(\cdot,t_i)) \to 0.$$

In view of lemma 2.2, we see that $u(\cdot,t_i)$ converge smoothly to a positive, smooth and strictly convex u_∞ solving (1.2) with $\lambda = \lim_{t_i\to\infty}\theta(t_i)$. $\square$

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