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A new complemented subspace for the Lorentz sequence spaces, with an application to its lattice of closed ideals

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Abstract. We show that every Lorentz sequence space $d(\mathrm{w}, p)$ admits a 1-complemented subspace Y distinct from ℓ_p and containing no isomorph of $d(\mathrm{w}, p)$. In the general case, this is only the second nontrivial complemented subspace in $d(\mathrm{w}, p)$ yet known. We also give an explicit representation of Y in the special case $\mathrm{w} = (n^{-\theta})_{n=1}^{\infty}$ $(0 < \theta < 1)$ as the ℓ_p-sum of finite-dimensional copies of $d(\mathrm{w}, p)$. As an application, we find a sixth distinct element in the lattice of closed ideals of $\mathcal{L}(d(\mathrm{w}, p))$, of which only five were previously known in the general case.

1 Introduction

Little is known about the complemented subspace structure of Lorentz sequence spaces $d(\mathbf{w}, p)$. Until recently, the only nontrivial complemented subspace discussed in the literature was ℓ_p [ACL73]. Then, in [Wa20], it was shown that for certain weights $\mathbf{w}$ (see Theorem 2.2 below), the space $d(\mathbf{w}, p)$ contains a 1-complemented subspace isomorphic to $(\bigoplus_{n=1}^{\infty} \ell_{\infty}^{n})_p$. Up to now, these were the only nontrivial complemented subspaces known to exist.

In this short note, we show that each Lorentz sequence space admits a 1-complemented subspace Y distinct from ℓ_p (Section 2). We also give an explicit representation of Y for the case $\mathbf{w} = (n^{-\theta})_{n=1}^{\infty}$ $(0 < \theta < 1)$, as the ℓ_p-sum of finite-dimensional copies of $d(\mathbf{w}, p)$ (Section 3). Note that this choice of $\mathbf{w} = (n^{-\theta})_{n=1}^{\infty}$ corresponds to the classical Lorentz sequence spaces $\ell_{q,p}$ with $p/q = 1 - \theta$. Finally, as an application, we find a sixth distinct element in the lattice of closed ideals in the operator algebra $\mathcal{L}(d(\mathbf{w}, p))$, where only five were previously known in the general case (Section 4).

Let us set up the main notation we need to use. We begin by fixing $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Denote by Π the set of all permutations of $\mathbb{N}$, and denote by $\mathbb{W}$ the set of all sequences $\mathbf{w} = (w_n)_{n=1}^{\infty} \in c_0 \setminus \ell_1$ satisfying

$$1 = w_1 \geqslant w_2 \geqslant w_3 \geqslant \cdots > 0.$$

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Fix $1 \leqslant p < \infty$ and $\mathbf{w} \in \mathbb{W}$. For each $(a_n)_{n=1}^\infty \in \mathbb{K}^\mathbb{N}$, we set

$$\|(a_n)_{n=1}^\infty\|_{d(\mathbf{w},p)} := \sup_{\pi\in\Pi}\left(\sum_{n=1}^\infty |a_{\pi(n)}|^p w_n\right)^{1/p},$$

and let $d(\mathbf{w},p)$ denote the linear space of all $(a_n)_{n=1}^\infty \in \mathbb{K}^\mathbb{N}$ with $\|(a_n)_{n=1}^\infty\|_{d(\mathbf{w},p)} < \infty$ endowed with the norm $\|\cdot\|_{d(\mathbf{w},p)}$, called a *Lorentz sequence space*. Recall that if $(a_n)_{n=1}^\infty \in c_0$ then there exists a "decreasing rearrangement" $(\hat{a}_n)_{n=1}^\infty$ of $(|a_n|)_{n=1}^\infty$. In this case, the rearrangement inequality gives us

$$\|(a_n)_{n=1}^\infty\|_{d(\mathbf{w},p)} = \left(\sum_{n=1}^\infty \hat{a}_n^p w_n\right)^{1/p} \quad \text{for all } (a_n)_{n=1}^\infty \in c_0. \tag{1.1}$$

Because $d(\mathbf{w},p) \subset c_0$ as linear spaces (although not as normed spaces), this represents an alternative formulation of the Lorentz sequence space norm.

For each $i, k \in \mathbb{N}$, we define

$$W_k := \sum_{n=1}^k w_n \quad \text{and} \quad w_i^{(k)} := \frac{1}{W_k}\sum_{n=(i-1)k+1}^{ik} w_n,$$

and $\mathbf{w}^{(k)} := (w_i^{(k)})_{i=1}^\infty$. It is readily apparent that $\mathbf{w}^{(k)} \in \mathbb{W}$. When p is clear from context, we also set

$$d_i^{(k)} := \frac{1}{W_k^{1/p}}\sum_{n=(i-1)k+1}^{ik} d_n,$$

where $(d_n)_{n=1}^\infty$ is the canonical unit vector basis for $d(\mathbf{w},p)$. It is routine to verify that $(d_i^{(k)})_{i=1}^\infty$ is a normalized basic sequence isometrically equivalent to the $d(\mathbf{w}^{(k)},p)$ basis. If necessary, we may sometimes abuse this notation; for instance, if $(j_k)_{k=1}^\infty$ is a sequence in $\mathbb{N}$, then we could write $((d_i^{(j_k)})_{i=1}^k)_{k=1}^\infty$ for appropriately translated successive normalized constant-coefficient blocks of lengths j_k.

Our main tool for finding complemented subspaces of $d(\mathbf{w},p)$ is the fact that every constant-coefficient block basic sequence of a symmetric basis spans a 1-complemented subspace (cf., e.g., [LT77, Proposition 3.a.4]). We will use this well-known fact freely and without further reference.

2 Lorentz sequence spaces contain at least two nontrivial complemented subspaces

The first discovery of a nontrivial complemented subspace in $d(\mathbf{w},p)$ came almost half a century ago, with the following result.

Theorem 2.1 ([ACL73, Lemma 1]) *Fix $1 \leqslant p < \infty$ and $\mathbf{w} \in \mathbb{W}$, and let*

$$x_i = \sum_{n=p_i}^{p_{i+1}-1} a_n d_n, \quad i \in \mathbb{N},$$

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form a seminormalized block basic sequence in $d(\mathbf{w}, p)$. If $a_n \to 0$, then $(x_i)_{i=1}^\infty$ admits a subsequence equivalent to ℓ_p and complemented in $d(\mathbf{w}, p)$.

By taking sufficiently long constant-coefficient blocks, it follows that $d(\mathbf{w}, p)$ contains a 1-complemented copy of ℓ_p. Much later was shown the following.

Theorem 2.2 ([Wa20, Theorem 4.3]) *Let $1 \leqslant p < \infty$ and $\mathbf{w} = (w_n)_{n=1}^\infty \in \mathbb{W}$. If*

$$\inf_{k\in\mathbb{N}} \frac{\sum_{n=1}^{2k} w_n}{\sum_{n=1}^{k} w_n} = 1,$$

then $d(\mathbf{w}, p)$ admits a 1-complemented subspace spanned by constant-coefficient blocks and isomorphic to $(\bigoplus_{n=1}^\infty \ell_\infty^n)_p$.

Thanks in large part to the ideas of William B. Johnson, our main result in this section is to generalize this to all Lorentz sequence spaces, as follows.

Theorem 2.3 *Let $1 \leqslant p < \infty$ and $\mathbf{w} \in \mathbb{W}$. Then, there exists an increasing sequence $(N_k)_{k=1}^\infty \in \mathbb{N}^\mathbb{N}$ such that $((d_i^{(k)})_{i=1}^{N_k})_{k=1}^\infty$ spans a 1-complemented subspace Y which contains no isomorph of $d(\mathbf{w}, p)$ and which is not isomorphic to ℓ_p.*

To prove it, we need a few preliminaries.

Lemma 2.4 *If $1 < p < \infty$, then every complemented subspace of $L_p[0,1]$ with a subsymmetric basis $(x_n)_{n=1}^\infty$ is isomorphic to either ℓ_p or ℓ_2.*

Proof The case $p = 2$ is trivial, because every complemented subspace of $L_2[0,1]$ is isomorphic to ℓ_2. For the case $p > 2$, recall from [KP62, Corollary 6] that every seminormalized basic sequence in $L_p[0,1]$, $p \in (2,\infty)$, admits a subsequence equivalent to ℓ_p or ℓ_2, and so because $(x_n)_{n=1}^\infty$ is also subsymmetric, then it is in fact equivalent to ℓ_p or ℓ_2. In case $1 < p < 2$, because $(x_n)_{n=1}^\infty$ is complemented in $L_p[0,1]$, its corresponding sequence of biorthogonal functionals $(x_n^*)_{n=1}^\infty$ is contained in $L_{p'}[0,1]$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Because $p' > 2$, a subsequence of $(x_n^*)_{n=1}^\infty$ is equivalent to $\ell_{p'}$ or ℓ_2, whence by subsymmetry $(x_n)_{n=1}^\infty$ is equivalent to ℓ_p or ℓ_2. ∎

Lemma 2.5 *Let X be a Banach space whose canonical isometric copy in X^{**} is complemented. Then, for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the canonical copy of X in $X^\mathcal{U}$ is complemented in $X^\mathcal{U}$.*

Proof Let $q : X \to X^{**}$ denote the canonical embedding, and define the norm-1 linear operator $V : \ell_\infty(X) \to X^{**}$ by the rule

$$V(x_n)_{n=1}^\infty = \text{weak}^*\text{-}\lim_{\mathcal{U}} qx_n,$$

which exists by the weak$*$-compactness of $B_{X^{**}}$ together with the fact that if K is a compact Hausdorff space, then for each $(k_n)_{n=1}^\infty \in K^\mathbb{N}$, the (unique) limit $\lim_\mathcal{U} k_n$

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exists in K. Note that if $\lim_{\mathcal{U}} x_n = 0$, then $V(x_n)_{n=1}^\infty = 0$, and so V induces an operator $\widehat{V} : X^{\mathcal{U}} \to X^{**}$ which agrees with V along the diagonal. In particular, $\widehat{V}$ sends the canonical copy of X in $X^{\mathcal{U}}$ isomorphically to the canonical copy of X in X^{**}. ∎

Theorem 2.6 *Fix $1 \leqslant p < \infty$, and let $(x_n)_{n=1}^\infty$ be a basis for a Banach space X whose canonical copy in X^{**} is complemented. If the finite-dimensional spaces $[x_n]_{n=1}^N$, $N \in \mathbb{N}$, are uniformly complemented in $L_p(\mu)$ for some measure μ, then X is complemented in $L_p[0,1]$.*

Proof Let $X_N = [x_n]_{n=1}^N$ and $P_N : X \to X_N$ the projection onto X_N. By uniform complementedness of X_N, we can find uniformly bounded linear operators $A_N : X_N \to L_p(\mu)$ and $B_N : L_p(\mu) \to X_N$ such that $B_N A_N$ is the identity on X_N. Let $\mathcal{U}$ be any free ultrafilter on $\mathbb{N}$. Define the bounded linear operators $A : X \to L_p(\mu)^{\mathcal{U}}$ by the rule $Ax = (A_N P_N x)_{\mathcal{U}}$, and $B : L_p(\mu)^{\mathcal{U}} \to X^{\mathcal{U}}$ by $B(y_N)_{\mathcal{U}} = (B_N y_N)_{\mathcal{U}}$. Let $x \in \operatorname{span}(x_n)_{n=1}^\infty$, so that, for some $k \in \mathbb{N}$,

$$BAx = (P_1 x, \dots, P_k x, x, x, x, \dots)_{\mathcal{U}} = x^{\mathcal{U}}.$$

By continuity, BA is the canonical injection of X into $X^{\mathcal{U}}$. Because its range is complemented by Lemma 2.5, we have the identity on X factoring through $L_p(\mu)^{\mathcal{U}}$.

It was proved in [He80, Theorem 3.3] that ultrapowers preserve L_p lattice structure, and in particular $L_p(\mu)^{\mathcal{U}}$ is isomorphic to $L_p(\nu)$ for some measure ν. Although $L_p(\nu)$ itself is nonseparable, we could pass to the closed sublattice generated by AX to find a space isomorphic to a separable L_p containing a complemented copy of X. Due mostly to a famous result of Lacey and Wojtaszczyk, it is known that separable and infinite-dimensional L_p spaces are isomorphic to either ℓ_p or $L_p[0,1]$ [JL01, Section 4, p. 15]. This means an isomorph of X is complemented in $L_p[0,1]$. ∎

An immediate corollary to Lemma 2.4 and Theorem 2.6 is as follows.

Corollary 2.7 *Let $1 < p < \infty$ and $\mathbf{w} \in \mathbb{W}$. Then, no $L_p(\mu)$ space contains uniformly complemented copies of $[d_n]_{n=1}^N$, $N \in \mathbb{N}$.*

Now, we are ready to prove the main result of this section, Theorem 2.3.

Proof Fix $k \in \mathbb{N}$, and note that $(d_i^{(k)})_{k=1}^\infty$ is isometric to the $d(\mathbf{w}^{(k)}, p)$ basis. Consider the case where $p = 1$. Then, we can choose the N_k's large enough that each $(d_i^{(k)})_{i=1}^{N_k}$ fails to be k-equivalent to $\ell_1^{N_k}$, and hence $((d_i^{(k)})_{i=1}^{N_k})_{k=1}^\infty$ fails to be equivalent to ℓ_1. As ℓ_1 has a unique unconditional basis by a result of Lindenstrauss and Pełczyński, it follows that Y is not isomorphic to ℓ_1.

Next, consider the case where $1 < p < \infty$. By Corollary 2.7, we can select N_k's large enough that $[d_i^{(k)}]_{i=1}^{N_k}$ fails to be k-complemented in ℓ_p. As $[d_i^{(k)}]_{i=1}^{N_k}$'s are all 1-complemented in Y, that means Y is not isomorphic to ℓ_p.

It remains to show that Y contains no isomorph of $d(\mathbf{w}, p)$. Suppose toward a contradiction that it does. As $(d_n)_{n=1}^\infty$ is weakly null (cf., e.g., [ACL73, Proposition 1]), we can use the gliding hump method together with symmetry to find a normalized

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block sequence of $((d_i^{(k)})_{i=1}^{N_k})_{k=1}^{\infty}$ equivalent to $(d_n)_{n=1}^{\infty}$. However, every such block sequence is also a block sequence w.r.t. $(d_n)_{n=1}^{\infty}$ with coefficients tending to zero. By Theorem 2.1, it follows that $(d_n)_{n=1}^{\infty}$ admits a subsequence equivalent to ℓ_p, which is impossible. ■

3 A special case

In this section, we show that when $\mathbf{w} = (n^{-\theta})_{n=1}^{\infty}$ for some fixed $0 < \theta < 1$, the space Y described in Theorem 2.3 can be chosen to be isomorphic to the space

$$Y_{\mathbf{w},p} := \left(\bigoplus_{N=1}^{\infty} D_N \right)_p ,$$

where $D_N := [d_n]_{n=1}^{N}$, for each $N \in \mathbb{N}$. As usual, we require some preliminaries.

Lemma 3.1 *Let $0 < \theta < 1$ and $j, k \in \mathbb{N}$. Then,*

$$\left(\frac{j+1}{k}+1\right)^{1-\theta} - \left(\frac{j+1}{k}\right)^{1-\theta} \leqslant \frac{\sum_{n=j+1}^{j+k} n^{-\theta}}{\sum_{n=1}^{k} n^{-\theta}} \leqslant \frac{(j/k+1)^{1-\theta} - (j/k)^{1-\theta}}{2^{1-\theta}-1}.$$

Proof Observe that the map

$$f(t) = (1+1/t)^{1-\theta} - (1/t)^{1-\theta}$$

is increasing on $[1,\infty)$ and hence has a minimum $f(1) = 2^{1-\theta} - 1$. Hence,

$$\begin{aligned}
\left(\frac{j+1}{k}+1\right)^{1-\theta} - \left(\frac{j+1}{k}\right)^{1-\theta} &\leqslant \frac{(j+k+1)^{1-\theta} - (j+1)^{1-\theta}}{k^{1-\theta} - \theta} \\
&= \frac{\int_{j+1}^{j+k+1} t^{-\theta}\, dt}{1 + \int_1^k t^{-\theta}\, dt} \\
&\leqslant \frac{\sum_{n=j+1}^{j+k} n^{-\theta}}{\sum_{n=1}^{k} n^{-\theta}} \\
&\leqslant \frac{\int_j^{j+k} t^{-\theta}\, dt}{\int_1^{k+1} t^{-\theta}\, dt} \\
&= \frac{(j+k)^{1-\theta} - j^{1-\theta}}{(k+1)^{1-\theta} - 1} \\
&= \frac{(j/k+1)^{1-\theta} - (j/k)^{1-\theta}}{(1+1/k)^{1-\theta} - (1/k)^{1-\theta}} \\
&\leqslant \frac{(j/k+1)^{1-\theta} - (j/k)^{1-\theta}}{2^{1-\theta} - 1}.
\end{aligned}$$

■

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Lemma 3.2 *Let $0 < \theta < 1$ and $\mathbf{w} = (w_n)_{n=1}^{\infty} = (n^{-\theta})_{n=1}^{\infty} \in \mathbb{W}$. Then,*

$$\frac{1-\theta}{2} \cdot w_i \leqslant w_i^{(k)} \leqslant \frac{2-2^{\theta}}{2^{1-\theta}-1} \cdot w_i \quad \text{for all } i, k \in \mathbb{N}.$$

In particular, if $1 \leqslant p < \infty$, then there is a constant $C \in [1, \infty)$, depending only on θ, such that

$$(d_n)_{n=1}^{\infty} \approx_C (d_i^{(k)})_{i=1}^{\infty} \quad \text{for all } k \in \mathbb{N}.$$

Proof We can assume $i, k \geqslant 2$. Observe that

$$t \mapsto t - (t-1)^{1-\theta} \cdot t^{\theta}$$

is decreasing on $[2, \infty)$, and hence has the maximum $2 - 2^{\theta}$. Furthermore, the function

$$t \mapsto t - (t-1/2)^{1-\theta} \cdot t^{\theta}$$

is decreasing on $[2, \infty)$ and hence has infimum

$$\lim_{t \to \infty} \left(t - (t-1/2)^{1-\theta} \cdot t^{\theta} \right) = \frac{1-\theta}{2}.$$

Thus, by the above, and applying Lemma 3.1 with $j = k(i-1)$,

$$\begin{aligned}
\frac{1-\theta}{2} \cdot i^{-\theta} &\leqslant \left(i - (i-1/2)^{1-\theta} \cdot i^{\theta} \right) i^{-\theta} \\
&= i^{1-\theta} - (i-1/2)^{1/\theta} \\
&\leqslant (i+1/k)^{1-\theta} - (i-1+1/k)^{1/\theta} \\
&\leqslant \frac{\sum_{n=(i-1)k+1}^{ik} n^{-\theta}}{\sum_{n=1}^{k} n^{-\theta}} && \left(\text{which is equal to } w_i^{(k)}\right) \\
&\leqslant \frac{i^{1-\theta} - (i-1)^{1-\theta}}{2^{1-\theta}-1} \\
&= \frac{i - (i-1)^{1-\theta} \cdot i^{\theta}}{2^{1-\theta}-1} \cdot i^{-\theta} \\
&\leqslant \frac{2-2^{\theta}}{2^{1-\theta}-1} \cdot i^{-\theta}.
\end{aligned}$$

■

Remark 3.3 Suppose $x = \sum_{n \in A} a_n d_n$ and $y = \sum_{n \in B} b_n d_n$ for finite and disjoint sets $A, B \subset \mathbb{N}$, where $(a_n)_{n \in A}$ and $(b_n)_{n \in B}$ are sequences of scalars. Then,

$$\|x + y\|^p \leqslant \|x\|^p + \|y\|^p.$$

Lemma 3.4 *Let $(j_k)_{k=1}^{\infty}$ be a sequence of positive integers, and, for each k, set*

$$J_k = j_1 + 2j_2 + 3j_3 + \cdots + kj_k.$$

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Suppose that there are constants $A, B \in (0, \infty)$ such that

$$w_i^{(j_k)} \leqslant A w_i \tag{3.1}$$

and

$$B w_i \leqslant \frac{1}{W_{j_k}} \sum_{n=J_{k-1}+(i-1)j_k+1}^{J_{k-1}+ij_k} w_n, \tag{3.2}$$

for all $i = 1, \ldots, k$ and all $k \in \mathbb{N}$. Then, $((d_i^{(j_k)})_{i=1}^k)_{k=1}^\infty$ is equivalent to the canonical $Y_{w,p}$ basis.

Proof Due to (3.1), we have $(d_i^{(j_k)})_{i=1}^k \lesssim_A d(\mathbf{w}, p)^k$. Now, using Remark 3.3, for any finitely supported scalar sequence $((a_i^{(k)})_{i=1}^k)_{k=1}^\infty$,

$$\begin{aligned} \left\| \sum_{k=1}^\infty \sum_{i=1}^k a_i^{(k)} d_i^{(j_k)} \right\|^p &\leqslant \sum_{k=1}^\infty \left\| \sum_{i=1}^k a_i^{(k)} d_i^{(j_k)} \right\|^p \\ &\leqslant A^p \sum_{k=1}^\infty \left\| (a_i^{(k)})_{i=1}^k \right\|_{d(\mathbf{w},p)}^p \\ &= A^p \left\| ((a_i^{(k)})_{i=1}^k)_{k=1}^\infty \right\|_{Y_{\mathbf{w},p}}^p . \end{aligned}$$

For the reverse inequality, let $(\hat{a}_i^{(k)})_{i=1}^k$ denote the decreasing rearrangement of $(|a_i^{(k)}|)_{i=1}^k$. Then, applying (3.2),

$$\begin{aligned} \left\| \sum_{k=1}^\infty \sum_{i=1}^k a_i^{(k)} d_i^{(j_k)} \right\|^p &= \left\| \sum_{k=1}^\infty \sum_{i=1}^k \frac{a_i^{(k)}}{W_{j_k}^{1/p}} \sum_{n=J_{k-1}+(i-1)j_k+1}^{J_{k-1}+ij_k} d_n \right\|^p \\ &\geqslant \sum_{k=1}^\infty \sum_{i=1}^k \frac{\hat{a}_i^{(k)p}}{W_{j_k}} \sum_{n=J_{k-1}+(i-1)j_k+1}^{J_{k-1}+ij_k} w_n \qquad \text{(from (1.1))} \\ &\geqslant B \sum_{k=1}^\infty \sum_{i=1}^k \hat{a}_i^{(k)p} w_i \\ &= B \left\| ((a_i^{(k)})_{i=1}^k)_{k=1}^\infty \right\|_{Y_{\mathbf{w},p}}^p . \end{aligned}$$

■

Theorem 3.5 *Let $(j_k)_{k=1}^\infty$ and $(J_k)_{k=1}^\infty$ be as in Lemma 3.4. Suppose there is $M \in [1, \infty)$ such that*

$$\frac{J_{k-1}}{j_k} \leqslant M, \quad \textit{for all } k = 2, 3, 4, \ldots .$$

Then, $((d_i^{(j_k)})_{i=1}^k)_{k=1}^\infty$ is equivalent to the canonical $Y_{w,p}$ basis.

Proof Due to Lemma 3.4, it suffices to show that (3.2) and (3.1) both hold. To do this, fix an arbitrary $k \in \mathbb{N}$. We may assume, without loss of generality, that $j_k \geqslant 2$.

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Now, by Lemma 3.1,

$$\begin{aligned}
\frac{1}{W_{j_k}}\sum_{n=J_{k-1}+(i-1)j_k+1}^{J_{k-1}+ij_k} w_n &\geqslant \left(\frac{J_{k-1}+(i-1)j_k+1}{j_k}+1\right)^{1-\theta}-\left(\frac{J_{k-1}+(i-1)j_k+1}{j_k}\right)^{1-\theta}\\
&=\left(\frac{J_{k-1}}{j_k}+i+\frac{1}{j_k}\right)^{1-\theta}-\left(\frac{J_{k-1}}{j_k}+i-1+\frac{1}{j_k}\right)^{1-\theta}\\
&\geqslant\left(\frac{J_{k-1}}{j_k}+i\right)^{1-\theta}-\left(\frac{J_{k-1}}{j_k}+i-1+\frac{1}{2}\right)^{1-\theta}\\
&=i^{\theta}\left[\left(\frac{J_{k-1}}{j_k}+i\right)^{1-\theta}-\left(\frac{J_{k-1}}{j_k}+i-\frac{1}{2}\right)^{1-\theta}\right]w_i.
\end{aligned}$$

Applying the Mean Value Theorem to the function $x \mapsto (\phi+x)^{1-\theta}$, $\phi\in[1,\infty)$, we can find $x_\phi\in(-1/2,0)$ such that

$$\phi^{1-\theta}-(\phi-1/2)^{1-\theta}=\frac{(1-\theta)(\phi+x_\phi)^{-\theta}}{2}\geqslant\frac{(1-\theta)\phi^{-\theta}}{2}.$$

Hence, letting $\phi=J_{k-1}/j_k+i$, we have

$$\begin{aligned}
i^{\theta}\left[\left(\frac{J_{k-1}}{j_k}+i\right)^{1-\theta}-\left(\frac{J_{k-1}}{j_k}+i-\frac{1}{2}\right)^{1-\theta}\right] &\geqslant i^{\theta}\left[\frac{(1-\theta)(J_{k-1}/j_k+i)^{-\theta}}{2}\right]\\
&=\frac{1-\theta}{2}\left(\frac{i}{J_{k-1}/j_k+i}\right)^{\theta}\\
&\geqslant\frac{1-\theta}{2}\left(\frac{1}{M+1}\right)^{\theta}.
\end{aligned}$$

This proves (3.2), and (3.1) follows immediately from Lemma 3.2. ∎

Taking inductively $j_1=1$ and $j_{k+1}=J_k$, the following is now immediate.

Corollary 3.6 *Let $1\leqslant p<\infty$, $0<\theta<1$, and $\mathbf{w}=(w_n)_{n=1}^\infty=(n^{-\theta})_{n=1}^\infty\in\mathbb{W}$. Then, $d(\mathbf{w},p)$ admits a 1-complemented subspace isomorphic to $Y_{\mathbf{w},p}$.*

4 Application to the lattice of closed ideals

In [KPSTT12], it was shown (among other results) that the lattice of closed ideals for the operator algebra $\mathcal{L}(d(\mathbf{w},p))$ can be put into a chain:

$$\{0\}\subsetneq\mathcal{K}(d(\mathbf{w},p))\subsetneq\mathcal{SS}(d(\mathbf{w},p))\subsetneq\mathcal{S}_{d(\mathbf{w},p)}(d(\mathbf{w},p))\subsetneq\mathcal{L}(d(\mathbf{w},p)).$$

Here, $\mathcal{K}$ denotes the compact operators, $\mathcal{SS}$ the strictly singular operators, and $\mathcal{S}_{d(\mathbf{w},p)}$ the ideal of operators which fail to be bounded below on any isomorph of $d(\mathbf{w},p)$. While, in [Wa20, Corollary 2.7], for the special case where $1<p<2$ and $\mathbf{w}\in\mathbb{W}\cap\ell_{2/(2-p)}$, a chain of distinct closed ideals with cardinality of the continuum were identified lying between $\mathcal{K}(d(\mathbf{w},p))$ and $\mathcal{SS}(d(\mathbf{w},p)$, for the general case, the only distinct elements known were those of the above chain.

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For an operator T, let $\mathcal{J}_T$ denote the class of operators factoring through T. If Z is any Banach space, we then set $\mathcal{J}_Z = \mathcal{J}_{Id_Z}$. By Theorem 4.3 below, we can extend the chain above as follows:

$$\{0\} \subsetneq \mathcal{K}(d(\mathbf{w},p)) \subsetneq \mathcal{SS}(d(\mathbf{w},p)) \subsetneq (\overline{\mathcal{J}_{\ell_p}} \vee \mathcal{SS})(d(\mathbf{w},p)) \\ \subsetneq \mathcal{S}_{d(\mathbf{w},p)}(d(\mathbf{w},p)) \subsetneq \mathcal{L}(d(\mathbf{w},p)).$$

Furthermore, by [KPSTT12, Corollary 3.2 and Theorem 5.3] together with the fact that $d(\mathbf{w},p)$ has the approximation property, any additional distinct closed ideals in the above chain must lie between $\mathcal{K}(d(\mathbf{w},p))$ and $\mathcal{SS}(d(\mathbf{w},p))$, or else between $(\overline{\mathcal{J}_{\ell_p}} \vee \mathcal{SS})(d(\mathbf{w},p))$ and $\mathcal{S}_{d(\mathbf{w},p)}(d(\mathbf{w},p))$, although there may be other ideals in the lattice which are not a part of the chain.

To prove Theorem 4.3, we need a couple of preliminary results.

Proposition 4.1 *Let X and Z be an infinite-dimensional Banach spaces such that $Z^2 \approx Z$, and X fails to be isomorphic to a complemented subspace of Z. Then, $\overline{\mathcal{J}_Z}(X)$ is a proper ideal in $\mathcal{L}(X)$. Furthermore, if $P \in \mathcal{L}(X)$ is a projection with image isomorphic to Z, then*

$$\mathcal{J}_P(X) = \mathcal{J}_Z(X).$$

Proof Because $Z^2 \approx Z$, [KPSTT12, Lemma 2.2] guarantees that $\mathcal{J}_Z(X)$ is an ideal in $\mathcal{L}(X)$. Suppose toward a contradiction that $Id_X \in \mathcal{J}_Z(X)$. Then, $Id_X = AB$ for operators $A \in \mathcal{L}(Z,X)$ and $B \in \mathcal{L}(X,Z)$. By [KPSTT12, Lemma 2.1], BX is complemented in Z and isomorphic to X, which contradicts our hypotheses. It follows that $\mathcal{J}_Z(X)$ is a proper ideal in $\mathcal{L}(X)$. Recall that the closure of a proper ideal in a unital Banach algebra is again proper; in particular, $\overline{\mathcal{J}_Z}(X)$ is a proper ideal in $\mathcal{L}(X)$.

To prove the "furthermore" part, assume $A \in \mathcal{L}(Z,X)$ and $B \in \mathcal{L}(X,Z)$. Let $Q : Z \to X$ be the canonical embedding, so that $PQ = Id_Z$ and hence $AB = APQB \in \mathcal{J}_P(X)$. It follows that $\mathcal{J}_Z(X) \subseteq \mathcal{J}_P(X)$, and the reverse inclusion is even more obvious. ■

For the next result, $\mathcal{F}$ denotes the class of finite-rank operators and $\mathcal{E}$ the class of inessential operators. Recall also that a basis $\mathcal{B}$ is called *semispreading* whenever every subsequence of $\mathcal{B}$ is dominated by $\mathcal{B}$ itself. In particular, the unit vector basis of ℓ_p is semispreading.

Proposition 4.2 ([LLR04, Corollary 3.8]) *Let Z be a Banach space with a semispreading basis (z_n), and let X be a Banach space with basis (x_n) such that any seminormalized block sequence of (x_n) contains a subsequence equivalent to (z_n) and spanning a complemented subspace of X. Then,*

$$\{0\} \subsetneq \overline{\mathcal{F}}(X) = \mathcal{K}(X) = \mathcal{SS}(X) = \mathcal{E}(X) \subsetneq \overline{\mathcal{J}_Z}(X),$$

and any additional distinct closed ideals must lie between $\overline{\mathcal{J}_Z}(X)$ and $\mathcal{L}(X)$.

In the proof of what follows, we use the fact that if $\mathcal{I}$ and $\mathcal{J}$ are ideals in $\mathcal{L}(X)$, then $\overline{\mathcal{I}} \vee \overline{\mathcal{J}} = \overline{\mathcal{I} + \mathcal{J}}$.

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Theorem 4.3 *Fix $1 \leqslant p < \infty$ and $\mathbf{w} \in \mathbb{W}$. Let Y be as in Theorem 2.3, and $P_Y \in \mathcal{L}(d(\mathbf{w}, p))$ any continuous linear projection onto Y. Then,*

$$P_Y \in \mathcal{S}_{d(\mathbf{w},p)}(d(\mathbf{w}, p)) \setminus (\overline{\mathcal{J}_{\ell_p}} \vee \mathcal{SS})(d(\mathbf{w}, p)).$$

Proof Let $P_{\ell_p} \in \mathcal{L}(d(\mathbf{w}, p))$ be any projection onto an isomorphic copy of ℓ_p spanned by basis vectors of Y. (Such a copy exists by Theorem 2.1.) By Theorem 2.3, Y contains no isomorph of $d(\mathbf{w}, p)$ and hence $P_Y \in \mathcal{S}_{d(\mathbf{w},p)}(d(\mathbf{w}, p))$. Because $\mathcal{S}_{d(\mathbf{w},p)}(d(\mathbf{w}, p))$ is the unique maximal ideal in $\mathcal{L}(d(\mathbf{w}, p))$, and $\mathcal{J}_{P_{\ell_p}}(d(\mathbf{w}, p)) = \mathcal{J}_{\ell_p}(d(\mathbf{w}, p))$ by Proposition 4.1, it is sufficient to prove that $P_Y \notin (\overline{\mathcal{J}_{P_{\ell_p}}} \vee \mathcal{SS})(d(\mathbf{w}, p))$.

Next, we claim that $P_Y \in (\overline{\mathcal{J}_{P_{\ell_p}}} \vee \mathcal{SS})(d(\mathbf{w}, p))$ only if $Id_Y \in (\overline{\mathcal{J}_{\ell_p}} \vee \mathcal{SS})(Y)$. To prove it, fix $\varepsilon > 0$, and suppose there are $A, B \in \mathcal{L}(d(\mathbf{w}, p))$ and $S \in \mathcal{SS}(d(\mathbf{w}, p))$ such that

$$\left\| AP_{\ell_p}B + S - P_Y \right\| < \varepsilon.$$

Let $J_Y : Y \to d(\mathbf{w}, p)$ be an embedding satisfying $P_Y J_Y = J_Y$, or $P_Y J_Y = Id_Y$ when viewed as an operator in $\mathcal{L}(Y)$. Composing P_Y on the left and J_Y on the right, we have

$$\left\| P_Y AP_{\ell_p} BJ_Y + P_Y SJ_Y - Id_Y \right\|_{\mathcal{L}(Y)} < \|P_Y\| \cdot \varepsilon \cdot \|J_Y\|.$$

On the other hand, because $AP_{\ell_p} = A|_Y P_{\ell_p}$ and $P_{\ell_p} = P_{\ell_p} P_Y$, we have

$$P_Y AP_{\ell_p} BJ_Y = (P_Y A|_Y) P_{\ell_p} (P_Y BJ_Y),$$

and hence

$$\left\| (P_Y A|_Y) P_{\ell_p} (P_Y BJ_Y) + P_Y SJ_Y - Id_Y \right\|_{\mathcal{L}(Y)} < \|P_Y\| \cdot \varepsilon \cdot \|J_Y\|.$$

Because $\mathcal{J}_{\ell_p}(Y) = \mathcal{J}_{P_{\ell_p}}(Y)$ by Proposition 4.1, where P_{ℓ_p} is likewise viewed as an operator in $\mathcal{L}(Y)$, from the above together with the ideal property of $\mathcal{SS}$, the claim follows.

Let $\mathcal{B}_Y = ((d_i^{(k)})_{i=1}^{N_k})_{k=1}^{\infty}$ denote the canonical basis of Y from Theorem 2.3. Note that because $\mathcal{B}_Y$ is made up of constant coefficient blocks of (d_n) of increasing length, any seminormalized blocks of $\mathcal{B}_Y$ will contain a subsequence equivalent to ℓ_p by Theorem 2.1. In fact, in [CL74, Lemma 15], this result was refined to show that we can choose that subsequence to span a complemented subspace of $d(\mathbf{w}, p)$, and hence of Y itself. We can therefore apply Theorem 4.2 to conclude that $\mathcal{SS}(Y) \subset \overline{\mathcal{J}_{\ell_p}}(Y)$. Meanwhile, again by Proposition 4.1, $\overline{\mathcal{J}_{\ell_p}}(Y)$ is a proper ideal in $\mathcal{L}(Y)$, which means $Id_Y \notin \overline{\mathcal{J}_{\ell_p}}(Y)$. Hence, $P_Y \notin (\overline{\mathcal{J}_{P_{\ell_p}}} \vee \mathcal{SS})(d(\mathbf{w}, p))$ as desired. ■

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