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MD SURVEY

GENERALIZATIONS OF OPTIMAL GROWTH THEORY: STOCHASTIC MODELS, MATHEMATICS, AND METASYNTHESIS

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1. INTRODUCTION

In previous papers [Spear and Young (2014, 2015)], we surveyed the origins, evolution, and dissemination of optimal growth, two-sector and turnpike models up to the early 1970s. Regarding subsequent developments in growth theory, a number of prominent observers, such as Fischer (1988), Stern (1991), and McCallum (1996), maintained that after significant progress in the 1950s and 1960s, economic growth theory "received relatively little attention for almost two decades" [Fischer (1988, p. 329)], and that "by the late 1960s early 1970s, research on the theory of growth more or less stopped" [Stern (1991, p. 259)]. Stern went on to say "the latter half of the 1980s saw a rekindling of growth theory, particularly in the work of Romer . . . and Lucas" (1991, p. 259), that is to say, in the form of "endogenous growth" models. McCallum, for his part, wrote (1996, p. 41), "After a long period of quiescence, growth economics has in the last decade (1986–1995) become an extremely active area of research." Moreover, Brock and Mirman's (1972b) paper was the sole "extension" of Ramsey–Cass–Koopmans to a "stochastic environment" mentioned by McCallum (1996, 49).

This paper deals with the evolution of the "classical" growth research program of Ramsey–Cass–Koopmans vintage via its stochastic "variants" and "generalizations" [Samuelson (1976, note 1)]. Thus, here we trace the origins and impact of the stochastic generalization that brought about a paradigm shift in modern economics, and still generates significant research in the form of "quantitative macroeconomics," that is to say, "real business cycle theory" (RBC henceforth), and its metamorphosis into the dynamic stochastic general equilibrium (DSGE) approaches of both new classical and New Keynesian vintage. The evolution of

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endogenous growth approaches and "new" and "unified" growth models will be dealt with in a separate paper.[1]

Our focus, then, is on the origins and development of optimal stochastic growth models in *continuous-time* and *discrete-time* forms. The paper is divided into three sections. The first section deals with unpublished and published papers by Phelps (1960a, 1960b, 1961, 1962a, 1962b), and Mirrlees (1965, 1966). Phelps's unpublished Cowles Foundation Papers on both continuous-time and discrete-time stochastic optimal growth (1960b, 1961) are also dealt with in this context—the former never published, the latter the basis for his 1962 *Econometrica* paper.

We then deal with Mirrlees' unpublished papers, dating from 1965, which had a *significant* impact on subsequent work in the area of stochastic optimal growth, such as on the contributions of Merton, Mirman, and Brock and Mirman. Mirrlees's use of the conceptual and mathematical tools provided by Wiener, Doob, and Ito is also dealt with, as they still influence the financial economics developed by Merton based upon them. Merton's contributions (1969, 1975) to the continuous-time approach are also dealt with in this section.

The second section deals with the application of the dynamic programming approach of Bellman and Blackwell over the period 1952–1970, its application to economic planning and growth models, especially by Radner, over the period 1963–1974, and cross-fertilization between Radner, Brock, and Mirman and Radner's Ph.D. student, Jeanjean. The third section tells the story of how Brock and Mirman developed their watershed approach over the period 1970–1973. It surveys the development of their 1972 *JET* (1972b) and 1973 *IER* papers from their origins in their early joint work and Mirman's thesis (1970b), through conference presentation, and finally publication. This section also deals with the important, albeit little-known *third* Brock–Mirman paper, that is, their 1971 conference paper published in the volume *Techniques of Optimization* (1972a).

2. PHELPS, MIRRLEES, AND MERTON: UNPUBLISHED AND PUBLISHED PAPERS, 1960–1975

2.1. Phelps: 1960–1962

In his June 1960 RAND paper "Optimal inventory policy for serviceable and reparable stocks," Phelps applied dynamic programming to the problem of determining "a unique stockage policy" regarding serviceable and reparable materials that would correspond to specific "decision regions" (1960b). Phelps wrote that "the model developed to treat this sequential decision problem, in being one of the comparatively few two-dimensional dynamic programing models for which the structure of the optimal policy has been ascertained, may be of some methodological interest" (1960b, 4). He went on to apply Bellman's "principle of optimality" to "current" and "future decisions," such that "all future decisions must be optimal"

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to achieve "overall optimality" (1960b, 7). In dealing with "the infinite-stage program," he applied the "fundamental theorems of dynamic programming for decision processes" outlined by Bellman [Phelps (1960b, 11)].

In December 1960, Cowles Foundation Discussion Paper [CFDP] 101, entitled "Capital risk and household consumption path: A sequential utility analysis," by Phelps appeared. The paper presented what Phelps called "a stochastic process of capital growth" (1960a, 1) in "a continuous time formulation" (1961, 1). Phelps's CFDP 101 *was never published*. In our view, the paper is important in three respects. First, it was perhaps the earliest paper to apply an ostensibly continuous-time approach to optimal stochastic growth, although Phelps's 1960 approach will be shown to be problematic, to say the least. Second, it was also one of the earliest papers to apply a dynamic programming approach and the Bellman (1957) "principle of optimality" to stochastic optimal growth (1960a, 9). Third, Phelps's 1960 CFDP 101 provided the basis for the discrete-time extension of his approach in the form of his subsequent February 1961 CFDP 109, entitled "The accumulation of risky capital: a discrete-time sequential utility analysis." Now, although CFDP 101 (1960a) *was* cited by Phelps in his 1961 CFDP (1961, 1–3, 33), *only* CFDP 109 was mentioned by him in a note in his 1962 *Econometrica* paper, albeit with its title referred to incorrectly as being "identical" to that of the *Econometrica* paper (1962b, 733, note 6). This may explain why CDFP 101 has gone uncited and CFDP 109 sparsely cited.

But more is involved here than the fact that Phelps's CFDPs have gone virtually unnoticed until now. In correspondence regarding CFDP 101, Phelps wrote (23 September 2014), "I don't recall any reaction to CF 101 at all . . . I did not continue with the work that started in CF 101 because it was not clear to me that I had the analytical tools to push it any farther."

With regard to the relationship between CFDP 101, CFDP 109 and his *Econometrica* 1962 paper, he wrote (23 September 2014),

> I did not really "switch" to the discrete-time framework. I had already done all or most of the work for it in my first post-doctoral job at the RAND Corporation.[2] (The great Richard Bellman was there, as you very likely know, so I finally showed it to him. "That's trivial", he exclaimed. "The capital stock goes to infinity!" Of course the whole exercise was aimed at characterizing that path, solving for the consumption function, etc.) When to my surprise I ended right back at Yale in September 1960, I worked on the Golden Rule and what became CF 101. Then, frustrated by how hard CF 101 was, I prepared the discrete-time paper for what became the CF 109.

Turning first to Phelps's 1960 CDFP 101, a number of points stand out. Phelps described what he was analyzing as a "continuous time formulation" of a "stochastic process" (1960a, 8–10; 1961, 1). He then applied dynamic programming and the optimality principle (Bellman 1957) as the analytical basis for his approach (1960a, 9–17). As Phelps put it (1960a, 9, 16–17), "In what follows we take our inspiration from Chapter 9 of Bellman . . . our continuous-time process can be viewed as the limiting case of a discrete-time process in which the length of each period goes to zero while the number of discrete periods goes to infinity."

A close reading of Phelps's 1960 CFDP 101 reveals the difficulties Phelps faced in attempting to apply and develop his ostensibly "continuous" approach. First of all, he stated that in his model "capital gains and losses occur in unit amounts . . . fluctuations in the capital stock occur at random times . . . Thus capital grows according to a *discontinuous* Markov process" (1960a, 2) [our emphasis]. In other words, this meant that the expected time path of wealth shocks was continuous, even though the underlying random shocks were discontinuous, that is to say, a "*mixed model*" rather than a pure continuous-time approach.

Second, the result Phelps outlined in his 1960 CFDP 101 was that a consumer could end up "ruined" (1960a, 3) because of "absorbing" states and "cyclic" states, although he noted that there could be both low and high capital persistence states that occur with positive probability, given the finite time horizon he assumed (1960a, 21–22). If, however, Phelps had pushed this through for an infinite horizon setting, he would have ended up with an *ergodic* distribution of the recurrent states, and the high or low capital "traps" (1960a, 21) would not occur, because there would be no absorbing states. Now, although Phelps does in fact consider what happens when the time horizon goes to infinity, and therefore the "ruin" probability goes to zero, he does not discuss this in terms of ergodic theory.

We think that the reason for Phelps abandoning the ostensibly continuous-time setting of his 1960 CFDP 101 for the discrete-time framework of his 1961 CFDP 109 may have been to get away from the awkwardness of mixing a discrete stochastic process with a continuous-time model. We surmise that he could have chosen to work with a continuous Markov process, as the works of Wiener (1923), Ito (1951), and Doob (1953)—i.e., *stochastic* calculus—were available at the time, but the move to a continuous stochastic process would, in turn, *rule out* purely transient and cyclic states, and thereby "trap" behavior, even with finite time horizons, as the continuity of the random shock process would make things look as though there were infinitely many instants of time, so that ergodic behavior would come into play, something Phelps may not have wanted in the context of what he was trying to do in his 1960 CFDP 101.

As noted, Phelps CFDP 109, entitled "The accumulation of risky capital: A discrete-time sequential utility analysis," appeared in February 1961. In the opening paragraph, Phelps wrote (1961, 1), "A continuous-time formulation of the same problem was presented in a previous Cowles Foundation Discussion paper." His October 1962 *Econometrica* paper, based on CFDP 109, was entitled "The accumulation of risky capital: A sequential utility analysis." And, although not cited in the references, Phelps did mention CFDP 109 as an "earlier version of this paper" (733, note 8). Indeed, there were many additions, changes, and elisions made in the text of CFDP 109 as published in *Econometrica*.

But perhaps the most important addition seen in the 1962 version is the statement made by Phelps, which set out the "vehicle of analysis" for all further work done on discrete optimal stochastic growth models. As he put it in his opening paragraph (1962b, 729), "The vehicle of analysis is a stochastic, discrete-time dynamic programming model that postulates an expected lifetime utility function to be maximized."

2.2. Mirrlees: 1964–1974

According to Fischer and Merton, "Following the early unpublished work by Mirrlees (1965), Brock and Mirman (1972[*sic*], 1973), Bourguignon (1974), and Merton (1975) among others, extended the neoclassical growth model to include uncertainty about technological progress and demographics" (1984, 58). Over a decade earlier, in their seminal paper, Brock and Mirman wrote (1972b, 481),

> The only work that is known to the authors which attempts to generalize deterministic optimal growth models to uncertainty . . . is the work of Mirrlees [1965]. Works by Phelps [1962] and Levhari and Srinivasan [1969] are somewhat related to optimal growth under uncertainty. . . .

They went on to also cite Merton (1969) as using "techniques similar to those employed by Mirrlees" (1972b, 482). We will deal with the Brock–Mirman interpretation of these authors in the following. At this point, we note that Olson and Roy, in their survey of stochastic optimal growth, wrote (2006, 298, note 1), "there is a large literature on stochastic growth in continuous time that built on Merton's . . . early work [RES 1975]," and also referred to Brock and MacGill (1979). They did not, however, refer to the seminal, albeit *unpublished* paper by Mirrlees entitled "Optimum accumulation under uncertainty" (1965, 1966), which was cited by many who dealt with the stochastic optimal growth model, as will be seen in the following.

In his contribution to the IEA conference volume entitled *Allocation under Uncertainty*, edited by Dreze, and published in 1974, Mirrlees wrote (1974, 36), "In the theory of optimum growth it has been found that models with discrete time are easier to treat rigorously than models with continuous time. But continuous-time models often have the advantage of providing simpler results."

Mirrlees worked with both nonstochastic and stochastic models. After his Cambridge Ph.D. dissertation entitled "Optimum planning for a dynamic economy" (Mirrlees, 1964b) he attended a number of Econometric Society and other meetings. At the September Zurich 1964 Econometric Society meeting, he presented a paper entitled "The structure of optimum policies in a macro-economic model with technical change" (1964c), which later became the core of his 1967 RES paper "Optimum growth when technology is changing" (1967, 96, 124), in essence a continuous nonstochastic extension of Cass's approach, with *exogenous* technical change.

Two months earlier, in July 1964, Mirrlees attended the Rochester conference "Mathematical models of economic growth," sponsored by the SSRC, and led by McKenzie. At this conference, according to McKenzie's account (1998, 5), Mirrlees presented two papers: a two-sector extension of Uzawa (1964), and an attempt "to extend the Ramsey model to the case of uncertainty." McKenzie wrote

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that in this paper, Mirrlees "apparently met a snag he was unable to overcome," and that although McKenzie later tried to discover "what the difficulty was," Mirrlees "could not recall, and had lost the paper" (1998, 5). When asked about his Rochester presentation, Mirrlees replied (4 Sept 2014), "I recollect that at the Rochester Conference I was trying to deal with the discrete-time model, also with the assumption of a random factor multiplying output, not stationary as in the later paper published in the Dreze book. I didn't succeed after the conference any better than I had in Rochester." In a subsequent communication (30 December 2014), Mirrlees also recalled the title of his Rochester paper to be "Optimum economic policies under uncertainty"(1964a), and that in "this unfinished paper" he "was trying to work out optimal saving under uncertainty for a discrete-time infinite-horizon model, and finding it remarkably difficult to get any results."

Mirrlees also attended the July 1965 Stanford MSSB Conference on "Optimal Economic Growth," conducted by Arrow, and initiated by McKenzie himself "as the economics member of the MSSB." According to McKenzie's recollections, Mirrlees presented a paper on "planning in mixed economies with 'surplus labor,'" and also led a roundtable discussion "on population and growth" (1998, 6–7).

Mirrlees's paper entitled "Optimum accumulation under uncertainty" was presented at the First World Congress of the Econometric Society, held in Rome, in September 1965 (1965), and revised and circulated in December 1965 (1966). And, as noted, this paper, *albeit unpublished,* was cited by leading growth theorists and many of those who developed stochastic optimal growth.

A comparison of the abstract of the September 1965 version of Mirrlees's paper (*Econometrica*, Supplementary Issue, 1966) with parallel text and equations in the December 1965 version shows only very minor differences.

Two interesting points emerge from Mirrlees's December 1965 paper. The first is its use of Bellman's dynamic programming in a stochastic setting [(1966, 27); Bellman (1957)]. The second is that it also contains a section entitled "Orders of magnitude," which comments on the "quantitative implications" of the theory he presents in the paper, linking it to "the interesting values of the parameters and the capital–output ratio" he dealt with in his 1964 Zurich paper (1965b, 44). Mirrlees' unpublished December 1965 paper was cited by most optimal growth theorists and others [McKenzie (1968); Levhari and Srinivasan (1969); Merton (1969); Stiglitz (1969); Dobell (1970); Sandmo (1970); Brock and Mirman (1972a, 1972b); Merton (1973, 1975); Leland (1974); Samuelson (1976); Fischer and Merton (1984)]. What is strange is that two references said that the paper was "forthcoming" in *Econometrica* [Dobell (1970); Sandmo (1970)].

When asked about this, Mirrlees replied (28 June 2014),

> *Econometrica* more or less (I don't remember precisely) accepted the paper subject to revision. When I came to revise it, I found an error in the existence proof, and didn't succeed in correcting it. Later I convinced myself that the existence claim was false, and it is old unfinished business for me to sort out the fundamental existence problem with the model used in the paper. The worrying point is that utility discounting seemed to me not enough to deal with it. By the time I realized

> the problem, my interests had moved well away from optimum accumulation, but I am ashamed of not having sorted the problem out. I have never seen a paper that does.

Mirrlees also attended the July 1971 IEA Bergen conference, where he gave a paper entitled "optimal growth with uncertainty" [Merton (1975)], which he revised and expanded in May 1972, changing the title to "optimum growth and uncertainty," and eventually publishing it in the 1974 IEA conference volume edited by Dreze, *Allocation under Uncertainty*, with the title "Optimum accumulation under uncertainty: The case of stationary returns to investment" [Mirrlees (1974, 36)].

2.3. Merton 1969–1975

Merton submitted his MIT Ph.D. thesis, supervised by Samuelson, and entitled "Analytical optimal control theory as applied to stochastic and non-stochastic economics," in September 1970. The thesis consisted of five chapters, three of which were published in 1969. Chapter II, entitled "Lifetime portfolio selection under uncertainty: The continuous-time case," was published in August 1969 in the *Review of Economics and Statistics*. In the introduction, Merton wrote, "Phelps (1962) has a model used to determine the optimal consumption rule for a multiperiod example where income is partly generated by an asset with an uncertain return. Mirrlees (1965) has developed a continuous time optimal consumption model of the neoclassical type with technical progress a random variable" (1969, 247). In the concluding section, Merton wrote (1969, 256–257),

> A more general production function of the neoclassical type could be introduced to replace the simple linear one of this model. Mirrlees (1965) has examined this case in the context of the growth model with . . . technical progress a random variable. His equations (19) and (20) correspond to my equations (35) and (37) with the obvious proper substitution for variables.
>
> Thus, the technique employed for this model can be extended to a wide class of economic models. However, because the optimality equations involve a partial differential equation, computational solution of even a slightly generalized model may be quite difficult.

Two things should be pointed out here. First, Merton was referring to equations in the *unpublished* December 1965 paper by Mirrlees. It would seem, then, that a number of leading economists, and especially growth theorists, were familiar with this paper. Indeed, in the copy of the Mirrlees's paper provided to the authors by Merton, there are marginal notes made by Samuelson, who seems to have given a copy of the paper to Merton. Second, regarding his "equation 19," Mirrlees wrote (1966, 13),

> It is also interesting to note that (19) generalizes what is often called the Keynes–Ramsey formula for optimum policy . . . In that case, of course, (19) solves the

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> problem, provided the side conditions are satisfied. Mathematically (19) and (20) are a fairly decent pair of partial differential equations. The trouble is that the side conditions are of such an odd kind, and that creates serious computational difficulties.

In the 1971 published version in *Journal of Economic Theory* of Chapter 5 of his thesis—which had earlier been presented at the Second World Conference of the Econometric Society—entitled "Optimum consumption and portfolio rules in a continuous time model," Merton wrote (1971, 412), "By the introduction of Ito's Lemma and the Fundamental Theorem of Stochastic Dynamic Programming . . . we have shown how to construct systematically and analyze optimal continuous-time dynamic models under uncertainty."

Merton's seminal paper on growth, "An asymptotic theory of growth under uncertainty," was published in the *Review of Economic Studies* in July 1975. It started as a paper presented "in various forms" at venues such as the December 1971 Yale NBER growth conference, the March 1973 Rochester mathematical economics seminar, and the April 1973 mathematical economics seminar at Columbia [Merton (1975, 375)]. It was subsequently circulated as MIT Sloan School working paper 673-73 in August 1973 [Merton (1973)]. The paper was submitted to the *Review of Economic Studies* a month later, and finally accepted in May 1974 [Merton (1975)].

A comparison between the 1973 working paper and the 1975 published version shows a number of significant additions in the published paper in the form of explanatory notes, a result both of editorial suggestion (1973, 11; 1975, 383, note 1) and of Merton's efforts to explain his use of various methods and issues in the paper. For example, he added notes explaining his use of the Bellman approach and optimality in solving dynamic programming problems in continuous time, boundary conditions, and generalizations of maximization to uncertainty (1973, 14,16; 1975, 384, notes 1 and 2, 386, note 1).

In the introduction to his paper (1973, 1; 1975, 375), Merton cited works that dealt with "capital accumulation under uncertainty" and "the optimal consumption–savings decision under uncertainty," based upon "a given linear production technology" [Phelps (1962b), Levhari and Srinivasan (1969), among others]. He also cited the unpublished December 1965 and the early version of the 1964 conference volume paper by Mirrlees, as examples of dealing with "the stochastic Ramsey problem and a continuous time neoclassical one-sector model subject to uncertainty about technical progress." He went on to describe the work of Brock and Mirman (1972b) and Mirman (1973) as "important contributions," albeit having "little to say about the specific structure" of "steady state" or "asymptotic distributions" regarding the "capital–labor ratio" when "outcomes are uncertain." And as against the model he proposed one "where the dynamics of the capital–labor ratio" is "described by a diffusion-type stochastic process."

The Brock–Mirman assessment of Mirrlees–Merton will be dealt with later, in the section on the development of the work of Brock and Mirman from 1970 onward. At this point, suffice it to say that Merton (1969, 248; 1971, 412;

1973, 6; 1975, 377), like Mirrlees before him (1965, 3) had utilized an approach based upon "a generalized theory of stochastic differential equations developed by Ito," and extended by Ito and McKean, "which is applicable to diffusion processes."

Ito calculus is a variety of stochastic calculus, which extends the normal operations of calculus—differentiation and integration—to stochastic processes. Unlike smooth (i.e., continuous and continuously differentiable) functions, stochastic processes can be discontinuous and nondifferentiable, manifesting sudden random jumps in value.

At the core, stochastic calculus operations follow conventional operations, with the difference being the recognition that over even small intervals of time, nonvanishing changes in the value of the stochastic process can occur, and hence need to be included in computations of things such as rates of change, in the case of differentiation, or weighted suitably in taking the limits of partitions that define the Riemann integral.

The specific version of the stochastic calculus applied by Mirrlees and Merton, Ito's calculus, was formulated between 1938 and 1945 by the Japanese mathematician Kiyoshi Ito while he was working at the National Statistical Office [Ito (1942, 1951, 1960); Ito and McKean (1964)].

3. DYNAMIC PROGRAMMING, ECONOMIC PLANNING, AND GROWTH: 1948–1973

3.1. Bellman, Karlin, and Blackwell

As mentioned in the preceding, in their survey on stochastic optimal growth, Olson and Roy (2006) did not cite Mirrlees's seminal, albeit unpublished, paper (1966). They also did not cite Bellman's famous book *Dynamic Programming* (1957), although they did cite Blackwell's paper "Discounted dynamic programming" (1965). *What is important to recall here is that Bellman's 1957 book was based upon his work at RAND and elsewhere, and published and unpublished papers, from 1948 onward,* and this according to Bellman's own recollections and accounts [Bellman (1984); Bellman and Lee (1984, 24)].

Bellman first visited RAND to attend its 1948 summer program, which was also visited by Morgenstern. Among the other attendees were Dantzig, Karlin, Tukey, Blackwell, Arrow, and Shubik [Assad (2011, 422)]. Bellman wrote that his introduction to Dantzig's linear programming algorithm while there was his "first exposure to effective numerical solutions," which, as he recalled, "subsequently became a central theme" of his research program [Bellman (1984, 135)]. He then went to Stanford to take up a position as associate professor in its Mathematics Department. In the summer of 1949, Bellman returned to visit RAND again. This time, at the suggestion of a colleague at RAND, he shifted the focus of his research to "multistage decision processes" [Bellman (1984, 157); Assad (2011, 424)]. During this period, according to his recollections, he had to essentially "find a name for multistage decision processes" [Bellman (1984, 159)].

In a July 1951 RAND paper entitled "On a general class of problems involving sequential analysis," Bellman set out the basis for what he subsequently called "dynamic programming." As he put it (1951, 1),

> We wish to discuss a general class of multi-stage problems involving a sequence of operations . . . This class of problems is characterized by the fact that at each time the problem may be described by a set of parameters which change from operation to operation, which is to say that each operation performs a mapping of the parameter space upon itself, and secondly, that the purpose of the operations is to optimize according to a criterion which has the important property that after any initial number of operations, starting from the state one finds oneself in, one optimizes according to the same criterion. . . . this last point . . . allows a mathematical formulation by means of recurrence relations which are very useful both theoretically and computationally.

Over the period 1952–1957, Bellman produced a significant number of papers on dynamic programming and on its applications to many areas, including problems in mathematical economics, as will be seen. The central message of these papers, as manifest in what he called "the principle of optimality," and the method he developed, as reflected in what became known as the "Bellman equation," eventually appeared in his 1957 monograph. Indeed, as one reviewer said, Bellman's 1957 book brought "under one cover the introduction and development of the theory of dynamic programing, which to a great extent has appeared previously in many papers scattered throughout many journals and pamphlets" [Newhouse (1958, 788)].

Now, one of the problems emanating from the fact that Bellman's 1957 book was, in essence, a *compilation* of his previously published papers and RAND reports is that the focus has been on his 1957 book rather than on his earlier work. This, in turn, has led to some misunderstanding regarding the origin of his central message, that is to say, the principle of optimality. For example, Puterman (1994, 155) noted that his book "presented the optimality equations and the principle of optimality together with references to his earlier papers (dating back to 1952) which introduced and illustrated many of the key ideas of dynamic programming."

Acemoglu, for his part, wrote (2008, 222), "the basic ideas of dynamic programming, including the principle of optimality, were introduced by Richard Bellman in his famous monograph (Bellman, 1957)." And Acemoglu, like Puterman, noted that Karlin (1955) had provided "a formal mathematical structure for the analysis of dynamic programing models" (Puterman 1994, 155) and "a simple formal proof to the principle of optimality" (Acemoglu 2008, 222).

Both Puterman and Acemoglu, however, overlooked the fact that Bellman's earliest published contribution specifically regarding dynamic programming. entitled "On the theory of dynamic programming," appeared in the 1952 *Proceedings of the National Academy of Sciences*, "communicated by Von Neumann in June 1952" [Bellman (1952, 716)], in which he acknowledges that dynamic programming was "intimately related to the theory of sequential analysis due to Wald [1950]" (1952, 717).[3] Had they also looked carefully at Karlin's 1955 paper they would have seen

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that Karlin (1955, 285) pointed to the fact that the principle of optimality appeared in Bellman's *Econometrica* paper, "Some problems in the theory of dynamic programming," published in January 1954 (1954a, 47). Moreover, in July 1954, in his RAND paper P-550, entitled "The theory of dynamic programming," Bellman again presented the principle of optimality (1954b, 4), as in the published version of this RAND paper, which appeared in the November 1954 issue of the *Bulletin of the American Mathematical Society* (1954c, 504).

Another interesting aspect of the Bellman–Karlin nexus is that prior to Karlin's paper, published in the December 1955 issue of the *Naval Research Logistics Quarterly*, Bellman had discussed Karlin's paper "Some aspects of dynamic programming," presented on 1 September 1955, at the Ann Arbor meeting of the Econometric Society, at a session also attended by Koopmans [Report of 1955 Ann Arbor Meeting (1956, 208)].

What is also important to mention here is that in a number of papers that appeared both before and after his 1957 book, Bellman specifically dealt with the application of dynamic programming to mathematical economics. In 1956, his RAND paper "Dynamic programming and its application to variational problems in mathematical economics" appeared (1956). It was subsequently presented at the Eighth American Mathematical Society's Symposium in Applied Mathematics, held at the University of Chicago in April 1956, and was eventually published in the symposium *Proceedings* in 1958 (1958). He wrote (1958, 115),

> The purpose of this paper is to discuss some variational problems arising from mathematical economics, and some of the methods that can be used to treat these questions both analytically and computationally.
>
> Since the range of mathematical economics is so extensive—and indeed the subject possesses no precise boundaries—and since the array of mathematical techniques which have been borrowed, begged, stolen, or improvised to cope with this field is so imposing, we cannot hope to present any adequate survey in any reasonably sized article. In consequence, we have restricted our attention to two important and interesting classes of processes, allocation and smoothing processes, and to a discussion of the application of the theory of dynamic programming to these processes.

Later, in March 1963, Bellman's RAND paper, "Dynamic programming and mathematical economics" (1963) appeared. Here, he toned down his language a bit, and wrote (1963, Preface, iii),

> In this memorandum, the author describes the uses and contributions of the mathematical theory known as dynamic programming to certain problems in economics. Examples of these are the optimal allocation of resources, and multistage decision processes that involve planning and learning in the face of uncertainty (i.e., adaptive control processes).

He went on to say (1963, Introduction, 1),

> The functions of a mathematical theory in the scientific field are to furnish the systematic means of formulation of classes of problems, to indicate various

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techniques for their analysis, and to provide methods for obtaining numerical answers to numerical questions.

At one point in the nineteenth century, serious doubt was expressed that problems in the field of economics could ever be handled mathematically. The introduction of the digital computer changed the situation drastically. Nevertheless, much remains to be done, and many new approaches must be devised, before we can consider ourselves to have a firm hold in the domain of mathematical economics.

In this memorandum we outline briefly some of the principal contributions of the theory of dynamic programming the formulation, analysis, and computational treatment of economic processes.

In a series of papers from 1961 onward, Blackwell also focused on dynamic programming (1961, 1962, 1964a, 1964b, 1965). Interestingly enough, as early as 1952, Blackwell had assisted Bellman in obtaining a solution to at least one of the fundamental theorems of Bellman's dynamic programming approach [Bellman (1952, Theorem 7, 719)]. Later, in his 1961 paper, "On the functional equation of dynamic programming," Blackwell demarcated stable, optimal, and stable optimal policy (1961, 274) and extended Bellman's1957 treatment to the case of *policy switching* and its effects on optimal policy and stability (1961, 274). In a subsequent paper entitled "Discrete dynamic programming" (1962), Blackwell turned to simplifying the results obtained by Howard (1960) regarding the introduction of discount factors less than and equal to unity into "the general dynamic programming problem" set out in the works, as he put it, of "Dvoretzky, Kiefer and Wolfowitz (1957), Karlin (1955) and Bellman (1957)."

In his 1965 paper "Discounted dynamic programming," Blackwell wrote (1965, 2–6),

Soon after the appearance of Wald's work on sequential analysis, Richard Bellman recognized the broad applicability of the methods of sequential analysis, named this body of methods of dynamic programming, and applied the methods to many problems... The first development of a general theory underlying these methods is due to Karlin (1955), and a rather complete analysis of the finite case was given by Howard (1960) . . . Our formulation of the dynamic programming problem is somewhat narrower than Bellman's.

3.2. Bellman and Blackwell

We now try to explicate the contributions of Bellman and Blackwell in the context of a discrete-time model, but the extensions to continuous time are relatively straightforward. Let us consider a multisector capital model with a single infinitely lived representative agent. The objective function for the planning problem is to maximize an infinite discounted sum of utilities of consumption in each period subject to the constraints that output is produced using labor (fixed) and capital (variable) inputs and that output allocations of consumption goods must be allocated feasibly. Further, assume that the period utility functions are strictly concave, and production functions exhibit constant returns to scale.

Now, look at a subproblem where we want to do the maximization over some finite number of periods T. In this case, because the objective function is a sum of concave functions, and constraint sets are convex, the standard results from concave programming theory will be applicable, once we recognize that we need a terminal condition on what the period-(T+1) vector of capital stocks should be. One approach, which Lucas and Stokey use to get the standard transversality condition, is to set the terminal stock to zero. If we are interested, however, in looking instead at short-run efficient allocations, we should maximize the sum of utilities up to T plus $K(T+1)$. In between, we can specify whatever terminal condition we like. Once we do this, concave programming theory says that the first-order conditions (more generally, the Kuhn–Tucker conditions) will be necessary and sufficient to characterize the solution.

To get an answer to our original question, we need to let T go to infinity. Doing this formally in a way that answers the original question then requires the techniques of Pontryagin for dealing with the terminal condition and getting the right limiting transversality condition. This is what Cass did in Chap. 1 of his thesis (1965). What would not be correct, though, would be to claim that in the T = infinity case, we have an infinite discounted sum of concave utilities, so that we are in the world of concave programming and the first-order conditions (i.e., the Euler equations) are necessary and sufficient. Indeed, they are not, as Malinvaud's counterexample to Koopmans showed (1964), even though they are for any finite T. Something is very different in the infinite-dimensional case.

To see what is different, one needs to go back to how the concave programming (Kuhn–Tucker theory) result is proven, and what we find is that it relies, not surprisingly, on applying the separating hyperplane theorem (i.e., Minkowski's theorem). Now, Minkowski's theorem holds in very general spaces, but it does have one stringent requirement: one of the two disjoint, convex sets being separated must have a nonempty interior. One of things that John Tukey (1942) is famous for is having provided a counterexample to the claim that the nonemptiness requirement in Minkowski's theorem can be dispensed with. He constructed two disjoint convex sets in ℓ_2 (classic Hilbert space), neither of which had an interior, and showed that there was no linear functional that would separate them.

Because the discounted sum of utilities specification of the capital model necessarily involves looking at something in the positive orthant of an infinite-dimensional space, the interiority criterion becomes relevant, and a direct application of the Minkowski theorem is not available. So we are left with the situation in which the Euler equations are necessary, but not sufficient. What Pontryagin et al. (1962) demonstrate is how to augment this with the right specification of the terminal conditions to pick out the one correct trajectory satisfying the Euler equations, so that the two conditions together are necessary and sufficient.

Now, the forward dynamic programming (DP) algorithm gets around the problem of terminal conditions by embedding the optimization problem in a recursive time structure where the future looks like a steady-state extension of the past. Specifically, given a function specifying the continuation value of the DP given the values of the state variables one period hence, we are left with a simple, finite-dimensional programming problem. If all of the functions involved are concave, we have a simple, finite-dimensional concave programming problem, and, of course, the FOCs here are necessary and sufficient.

The genius of Bellman and Blackwell was to recognize that they were not actually getting rid of the infinities that cause problems in the control theory setting; rather, they were converting the problem of the infinite time horizon into that of finding one among an infinity of possible continuation functions. In the problems that involve discounting (or other setups that satisfy the Blackwell conditions for a contraction), the functional equation that replaces the infinite discounted sum of utilities turns out to be a contraction mapping, and the Banach theorem therefore applies, yielding a fixed point to the Bellman equation, and, via the obvious recursion, a solution to the original optimization problem. Again, because the Bellman equation evaluated at the fixed-point value function is a finite-dimensional concave programming problem, the FOCs are necessary and sufficient.

The solution that comes out of the DP approach can also be used to demonstrate the sufficiency of the transversality condition in the original problem. The ascendance of the DP approach in growth theory is clearly due to the availability of the contraction mapping results, and their computational tractability. We think this also explains why the profession brushed aside the whole question of the moral implications of discounting so widely discussed in the Vatican Conference volume (1964), at least until the issue was resurrected in the contemporary discussion of global warming.

3.3. Cross Fertilization: Radner, Brock–Mirman, and Jeanjean

As noted in the preceding, Phelps was perhaps the first to introduce Bellman's dynamic programming approach into economics in both continuous- and discrete-time settings in his work between 1960 and 1962. Over the period 1963–1974, Roy Radner also advanced the application of dynamic programming in economics. For example, his ONR-supported 1963 technical report entitled "Notes on the Theory of Planning" utilized dynamic programming, which, as he wrote (1963, 2), "is relatively new to the theory of economic planning." He gave accounts of "the dynamic programming valuation function for various functions and programs" (1963, Lecture 4, Sects. 2, 3). In February 1964, Radner's Berkeley Center for Research in Management Science Technical Report Number 17 (also supported by the ONR), entitled "Dynamic programming of economic growth," appeared. The abstract of the paper read as follows:

> A class of problems of optimal economic growth is formulated in terms of the functional equation approach of dynamic programming (Bellman, 1957). A study is made of the continuity and concavity properties of the state valuation function,

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i.e., the function indicating a maximum total discounted welfare (utility) that can be achieved starting from a given initial state of the economy. Under suitable conditions this function is characterized by a certain functional equation. Both the cases of a finite and an infinite planning horizon are treated, the latter case being discussed under the assumption of constant technology and tastes. Here iteration of a certain transformation associated with the functional equation is shown to provide convergence to the state valuation function. Exact solutions are given for the case of linear-logarithmic production and welfare functions.

Radner's 1964 Technical Report 17 went virtually unnoticed. It was only cited in his 1966 *International Economic Review* paper "Optimal growth in a linear-logarithmic economy" [Radner (1966)]. An abridged version of the report was published in 1967 under the same title, that is, "Dynamic programming of economic growth" (1967, 111–141). Now, the introductory paragraph in the 1967 abridged version was identical to the abstract of his 1964 Technical Report. However, in contrast to the "relatively new" description of dynamic programming he used in his 1963 Technical Report 9, in his 1967 paper Radner wrote (1967, 111),

The techniques used are familiar to workers in the field of dynamic programming, although the particular assumptions appropriate to a study of optimal economic growth differ from those commonly encountered in other applications (e.g. inventory and replacement theory) [such as in Phelps, 1960].

Radner continued,

The interest in such an approach must ultimately derive from its power, if any, in producing new characteristics of optimal growth and/ or new and more efficient computational techniques.

Over the period 1971–1973, Radner both presented, at various meetings, and published two important papers relating "optimal steady-state" programs and stochastic production. One was presented at the Mathematical Social Science Board (MSSB) workshop at the University of California, Berkeley, July–August 1971, and published in the *Journal of Economic Theory* in February 1973. The other was presented at the symposium on mathematical analysis of economic systems at the Fall 1971 meeting of the Society for Industrial and Applied Mathematics (SIAM), held at the University of Wisconsin–Madison, 11–13 October 1971, and published in *Mathematical Topics in Economic Theory and Computation* (1972). In both his 1972 and 1973 papers, Radner used the *identical* "bibliographic note," in which he cited *two* papers by Brock and Mirman, one a paper they presented at the summer 1971 Berkeley MSSB workshop, the other their now famous 1972 *Journal of Economic Theory* paper. Radner wrote (1972, 89; 1973, 110–111),

W.A. Brock and L.J. Mirman (1971), (1972) have studied optimal growth under uncertainty in a model with one commodity and a sequence of independent and identically distributed states of the environment. In

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particular, in the second paper they considered the problem of optimal stationary programs.

We will return to the MSSB meeting later.

Another work cited by Radner in his *SIAM* (1972) and *JET* (1973) papers was the unpublished September 1972 Berkeley Ph.D. thesis of Jeanjean, supervised by Radner, entitled *Optimal Growth with Stochastic Technology in a Multi-sector Economy*. In May 1971 Jeanjean had circulated his Berkeley Center for Research in Management Science working paper 332, entitled "Optimal growth with stochastic technology in a closed economy."

Interestingly enough, in 1974, Jeanjean published two additional papers, one in French, which was, in effect, the translation of his thesis (1974a). The other was a paper published in *JET* entitled "Optimal development programs under uncertainty: The undiscounted case" (1974b). In the French translation of his thesis (1974a, 98), Jeanjean cited an unpublished paper by Mirman entitled "The steady state behavior of a class of one sector growth models with uncertain technology" (1970a). This paper was later published by Mirman in the June 1973 issue of *JET* under the same title. According to Mirman (9 November 2014, personal communication to authors), he gave the 1970 paper, and other papers, to Radner at the MSSB meeting. This cross fertilization continued, with Jeanjean's thesis being cited by Brock and Mirman in their paper in the volume *Techniques of Optimization* (1972a), and in their 1973 *International Economic Review* paper "Optimal economic growth and uncertainty: The no discounting case." They reported that Jeanjean had "extended some" of their "results to multisector models" (1972a, 418) and that "Recently our results have been generalized to the multisector case by Jeanjean" (1973, 572), citing his unpublished thesis accordingly (1972a, 418; 1973, 573). The summer 1971 Berkeley MSSB workshop on "Theory of Markets and Uncertainty" was conducted by Radner. The participants were William Brock, Peter Diamond, Jerry Green, Theodore Groves, Werner Hildebrand, Leonard Mirman, Michael Rothschild, Jose Scheinkman, Michael Spence, Bernt Stigum, and Shmuel Zamir [Cutler (1973)]. Now, according to McKenzie (1999 [1998], 10), *two* papers by Brock and Mirman on growth with uncertainty, dealing with the undiscounted and discounted cases, were given at Radner's workshop. There are, however, some problems regarding McKenzie's account. First, only *one* Brock–Mirman paper is ever cited as having been presented and discussed at the MSSB meeting, that is, their paper entitled "Optimal economic growth and uncertainty: The Ramsey–Weizacker case," that is to say, the "undiscounted case." This was later published by Brock and Mirman in the October 1973 issue of *IER* under the title "optimal economic growth and uncertainty: The no discounting case," according to Mirman (9 November 2014, personal communication to authors). Second, the "discounted" case, that is to say, the now classic 1972 Brock–Mirman *JET* paper, was "received June 28, 1971" by the journal (1972b, 479), or in other words, *immediately prior* to the July–August 1971 MSSB conference. Finally, as will be seen later, the Brock–Mirman "discounted" case paper was first circulated as a Rochester/Cornell mimeo in 1970, and also presented at an NBER conference at Yale led by Stiglitz and at the North American Meeting of the Econometric Society in December 1970. And thus, it is to the intriguing story of the evolution and impact of the Brock–Mirman approach that we now turn.

4. BROCK AND MIRMAN: FROM THESIS TO META-SYNTHESIS, 1970–1973

Brock and Mirman's 1972 *JET* and 1973 *IER* papers are well known, and the former is widely cited. What is less well known is that they also published another joint paper on the stochastic growth model. As Mirman wrote (personal communication, 13 June 2014), "there is a third Brock-Mirman paper, which mirrors the discounted case paper" [*JET* 1972]. This paper, albeit not widely cited, was presented at an October 1971 conference on "optimization techniques," and will be discussed in the following. All *three* Brock–Mirman papers were the outcome of Mirman's 1970 thesis and their collaboration in developing, as they put it, "the unification" of previous approaches to growth (1972a, 483) or, in other words, their *meta-synthesis*. But before proceeding to a discussion of the evolution of the joint work, we deal with the development of Mirman's approach to uncertainty and growth, his 1970 thesis, and early papers.

4.1. Mirman, Uncertainty, and Growth: Chicago and Rochester

Mirman's interest in issues surrounding uncertainty and growth started as a graduate student. He recalled (personal communication,14 June 2015) that his interest in the study of uncertainty in economics "was a big problem for me early on, so many people—including Hirofumi Uzawa and David Cass—disparaged my interest in uncertainty." Mirman recalled that in 1967 he went to the summer workshop on growth at University of Chicago organized by Uzawa, also attended by Ethier, Calvo, and Razin, among others. Mirman also wrote (personal communication 15 June 2014),

> I was a graduate student, at the end of my second year, just finishing my course work, when I was sent by the department (or by McKenzie) to a summer workshop on growth at the University of Chicago run by Hiro Uzawa. In my early discussions with Hiro about my work he was dead set against my working on uncertainty and growth, I had some very preliminary ideas but needed much more thought and work and help before anything could develop from these ideas. I remember he thought that putting uncertainty in a growth model was too hard and not enough was known about certainty and growth to waste time on uncertainty. He didn't let me work on uncertainty all summer, he had me working on a project he found interesting but I had no idea what he was talking about. At the end at the summer I was at wits end because I needed a second year paper for Rochester. Luckily the only thing that made sense to me then was uncertainty and I was able to write a paper on an uncertain Von Neumann growth model. It mimicked the work McKenzie was doing and it began to lay the foundation for my thesis. In any case, I had met Cass at the conferences run by Uzawa and discussed my ideas with him. In my view both Cass (who was a student of Uzawa) and Uzawa were (are) brilliant economists. In fact, Uzawa was

> a brilliant mentor, too bad he had no use for uncertainty. But lucky for me that was my only idea.

4.2. Mirman's Thesis and Early Papers

In their *JET* paper, "Optimal growth and uncertainty: The discounted case," Brock and Mirman wrote (1972b, 482), "The basic framework for the paper was developed by Mirman in [9, 10], for discrete time one-sector stochastic growth models." Reference 9 was to Mirman's 1970 Ph.D. thesis, entitled "Two Essays on Uncertainty and Economics"(Mirman, 1970b); reference 10 was to Mirman's as yet "unpublished" 1970 paper, entitled "The steady state behavior of a class of one sector growth models with uncertain technology" (1970a), which emanated from his thesis, and was later published in *JET* in 1973.

With regard to his thesis and influences on it, Mirman recalled (personal communication 9 February 2007),

> Brock and I overlapped at Rochester, he as an assistant professor and I as a graduate student, for one semester, and he told me he had no idea what I was doing, but we were friends. I turned in my thesis the next spring—I was at Cornell—and he was assigned to read it. My advisor was McKenzie—Brock and Zable were on my committee from economics—but the biggest help I got was from a mathematician named Kemperman, who was also on my committee. A statistician—who taught me stochastic processes—was also on the committee, Keilson. I was really lucky to be surrounded by a group of great scholars.

Another paper Mirman wrote at the time was his 1971 *Econometrica* paper, entitled "Uncertainty and Optimal Consumption Decisions," received in March 1969, the final revision dated November 1969. In the first note to the paper, Mirman acknowledged "the encouragement and advice of Prof. J.H.B. Kemperman" (1971, 179).

In January 1971, Mirman sent his second thesis paper, entitled "On the existence of steady state measures for one sector growth models with uncertain technology," for publication to *IER*, and after revision in September 1971, it was published in June 1972. Now, in his *IER* paper, Mirman *cited* his then "unpublished" 1970 thesis paper "The steady state behavior of a class of one-sector growth models with uncertain technology" as having introduced "a stochastic generalization of the concept of a steady state equilibrium for a model of economic growth" (1972, 271). However, in his *IER* paper, Mirman *did not cite his own 1970 thesis* [*as against its citatio***n** in Brock and Mirman (1972b)], and in the *IER* paper, Mirman cited his *JET* paper *with Brock* as "forthcoming" (1972, 286).

Mirman's *1970 paper*, "The steady state behavior of a class of one sector growth models with uncertain technology," *finally appeared* in the June 1973 issue of *JET*. When asked about the differential citations, Mirman replied [personal communication 8 February 2007], "This is easy to answer. I think my thesis paper and the . . . *IER* paper are almost exactly the same so there was no need to quote the thesis, but then I needed to quote *the not yet forthcoming JET paper* [our emphasis, as Mirman is talking here about the June 1973 *JET* version of his 1970 paper]. But the 1973 *JET* paper and the thesis paper are different—the referee insisted that I change the proof."

And indeed, in the introductory note to the June 1973 *JET* version of his 1970 paper, Mirman wrote, "This paper contains results reported in my Ph.D. thesis . . . However, the organization of the paper and the proofs of the main theorem have undergone considerable change." In the references to his 1973 *JET* paper, which, as Mirman wrote (1973, 219), was based on his 1970 Ph.D. thesis, both Cass (1965) and Koopmans (1965) are cited, as is Radner (1971). Brock and Mirman (1972b), however, cite only Cass and Koopmans and a paper by Brock entitled "Sensitivity of optimal growth paths with respect to a change in final stocks," actually published [Brock (1971)] in the same conference volume as Radner (1971).

4.3. Brock–Mirman *Papers*: Recollections, Presentations, Meta-synthesis

Over the period 1970–1973, the interaction between Brock and Mirman took place on two levels. The first was the relationship emanating from Brock being Mirman's thesis examiner and intellectual catalyst for extending his approach. The second was as partners in presenting their joint papers at conferences and publishing them to achieve the widest possible audience for what they described as "the unification" of the approaches to growth their framework provided.

In the preceding, we discussed the 1973 and 1975 versions of Merton's paper "An asymptotic theory of growth under uncertainty." In *both* versions, he cited a paper by Brock and Mirman entitled "The stochastic modified golden rule in a one-sector model of economic growth with uncertain technology" as being published in the June 1972 issue of *JET* (1973, 33; 1975, 392). Moreover, Samuelson (1976, 491) also cited the 1972 *JET* Brock–Mirman paper under the same title used by Merton, "The stochastic modified golden rule in a one-sector model of economic growth with uncertain technology." The title of the oft-cited 1972 Brock–Mirman 1972 *JET* paper, however, was "Optimal economic growth and uncertainty: The discounted case."

In their citations, Merton and Samuelson actually referred to the title of the *earlier versions* of Brock and Mirman's watershed paper, which was initially circulated and presented at conferences in 1970. The paper, under its original title, first appeared as a "mimeo" emanating from "Rochester and Cornell"; Mirman by then at Cornell, Brock at Rochester.

Recollections. In a series of queries from the authors, and replies from Brock and Mirman, they recounted the evolution of their 1972 *JET* and 1973 *IER* papers, which they identified as "Brock–Mirman I" and "II," respectively (1973, 560 footnote 2). When asked about the origins and dissemination of the 1970 "Cornell/Rochester mimeo"(1970a) and the first version of the 1972 *JET* paper, Mirman recalled (personal communications 12 June and 10 August 2014),

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"I do remember that we wrote, what is from hindsight, a preliminary version of the [*1972 JET*] . . . The first version of the paper was sent out to a rather wide audience and we did get comments on it" (12 June). He went on to say (10 August),

> I believe that my first communication with Brock . . . was in the spring of 1970. I was to defend my thesis in May at Rochester and he was a member of the committee and had just read the thesis. He told me he had an idea for extending my thesis, which was for a positive growth model under uncertainty, to the optimal case. My work used the notion of stochastic technical progress, which was taken from the work of Mirrlees that inspired me . . . to study the question of growth under uncertainty.

Brock, for his part, recalled (personal communication 12 June 2014),

> Rochester was my first job. I had just arrived there. I was reading Len's thesis and I got very excited about the possibility of generalizing his work on the stochastic Solow type model to the infinite horizon optimization case. I recall not only talking to Len by phone about this problem, but also going to Cornell to visit him, where we worked on this problem and probably the undiscounted one too . . . I recall it [the 1970 version of the 1972 *JET* paper] being well known to researchers in growth before it actually was published.

In a subsequent communication (13 June 2014), Mirman expanded his account and recalled,

> I was an assistant professor at Cornell and just finished my defense, when I received a call from Brock who was relatively new in the department at Rochester. He told me that he had been assigned to read my dissertation and was upset because he had just finished reading a dissertation that was very boring and uninteresting. He thought mine would be the same. He told me he was very surprised that my dissertation was very interesting; in fact he was sure that we can generalize it to the optimal case, and he asked McKenzie why he had never told him about my work. My dissertation studied the steady state behavior of a stochastic growth model, similar to what Solow did in his classic paper for the deterministic growth model. So Brock and I did the optimal case, both in the discounted case and in the non-discounted case, the latter never got the audience of the former.

When asked about the citations by Samuelson and Merton and impact of the published paper, Mirman replied (personal communication 14 June 2014),

> I do remember that Brock got a short note from Samuelson. He praised the work and introduced his student Merton, who was working on similar problem in continuous time. Actually, over time, it turns out that not many people actually read the published version, for several reasons.

Mirman continued, relating the paper to his later work with Zilcha,[4]

> There, I think, are two main reasons which are related. The first is that the paper is well known and taught a lot so people think they understand and know very well what's in it. The role of the other is that the books of Sargent cover it in a way that people don't think they need to read it. For example, our paper does not contain any examples. But most people think that a Brock–Mirman model is with log utility and Cobb Douglas production. This is from my paper with Zilcha, and is presented by

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> Sargent as the Brock–Mirman model. So many people I've spoken with have never read the original.

When asked about conference presentations of their 1972 *JET* paper, Mirman recalled (personal communication, 10 August 2014),

> Brock came down to Cornell, where we wrote (what is now) a barebones version of our paper. This is the paper, "The stochastic modified golden rule in a one-sector model of economic growth with uncertain technology." In the fall . . . we went to a conference on growth theory at Yale at which Brock gave our paper, in a pretty rough form.

Mirman continued,

> In attendance was among others Karl Shell and David Cass. Karl was at MIT the time and I believe that is how the paper got to Samuelson (but I conjecture here) . . . In the meantime Brock and I corresponded by mail and the paper started to expand with many of the cracks and holes filled in. I would be remiss not to mention that every time he sent me the paper I would send it back with the key inequalities backwards, which drove him crazy . . . In the meantime Brock and I worked on the paper.

Mirman went on to say,

> My next recollection is a visit by Brock to Cornell, I was writing a proposal for the NSF and he suggested that the introductory material for the grant application be part of the paper. But at the same time Brock had an idea for a second paper, the no discounted case. We realized at this time that the work was more general than just the stochastic technical progress case and that the dynamics played an important part. So we decided that the reference to modified golden rule was too narrow. We then changed the title to take account of the generality of the paper and to link it to the second paper. We continued to fill cracks and holes, as well as added the appropriate diagrams. I think that it was in this form with the name change that we sent it to *JET*.

Presentations. In the following, we complement the recollections of Brock and Mirman by reference to the published record of conference presentations of the *three* Brock–Mirman papers.

Brock–Mirman I. With regard to their 1972 *JET* paper, they recalled that it was first presented at Yale in fall 1970 at a conference on "growth theory." An examination of the record of 1970 conferences shows that the paper was actually presented in November 1970 at the NBER conference on econometrics and mathematical economics funded by the NSF, and organized by Joseph Stiglitz (NBER Record, 1970).

The 1970 version of their 1972 *JET* paper was presented to a wider audience at the December 1970 North American regional conference of the Econometric Society held in Detroit (Report of North American Regional Conference, *Econometrica*, July 1971, 300) at Session 15, "Growth Models," chaired by Stiglitz. The abstract of their paper, "The stochastic modified golden rule in a one-sector model

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of economic growth with uncertain technology" (1970b), read as follows (Report of North American Regional Conference, *Econometrica,* July 1971, 345–346):

> In a discrete time one sector model of economic growth with uncertain technology we show that the distribution functions of optimal consumption and capital stock at time *t* (which are random variables) converge pointwise to limit distributions. The limit distribution, *F*, of optimal capital stock is a natural generalization of the modified golden rule. Hence our result is an extension of the Cass, "Optimal Growth in an Aggregative Model of Capital Accumulation," *RES* 1965; Koopmans "On the Concept of Optimal Economic Growth," *Pontificae Academiae Scientiarum Scripta Varia*, 1965, results to the case where technology is uncertain. All of our assumptions are similar to Cass–Koopmans except for the random technology and the planning objective which is assumed to be maximization of the capital value of the discounted sum of utilities. We assume that technology can be represented by *F(K, L, r)* where *F* satisfies the usual assumptions for each value of the random variable *r*, and increases in *r* for each *K, L*. We have found an elementary set of techniques to deal with this rather difficult problem. First we follow Levhari and Srinivasan, *RES* 1969, and use dynamic programming to establish the existence of time invariant policy functions, giving optimal consumption and capital stock at time $t+1$ as a function of capital in existence at time *t*. Hence the evolution of optimal capital stock is given by a stochastic process. This stochastic process is shown to converge in distribution by exploitation of the necessary conditions of optimality.

Brock–Mirman II. In his 1973 *JET* paper "Optimal stationary consumption with stochastic production and resources," Radner cited a paper by Brock and Mirman entitled "Optimal economic growth and uncertainty: The Ramsey–Weizacker case" as Working Paper number 7, emanating from the Mathematical Social Science Board Workshop on "The theory of markets and uncertainty," held at the Department of Economics, University of California, Berkeley, in 1971. As noted earlier, this paper became their October 1973 *IER* Paper "Optimal economic growth and uncertainty: The no discounting case," which was received at that journal in January 1972, and revised in November 1972 (1973, 560).

Brock–Mirman III. In October 1971, what Brock and Mirman have called "Brock–Mirman III" was presented by Brock at the Fourth International Federation for Information Processing colloquium on optimization techniques, Los Angeles, 19–22 October. This paper was entitled, "A one-sector model of economic growth with uncertain technology: An example of steady state analysis in a stochastic optimal control problem." It was published in the 1972 conference volume on techniques of optimization edited by Balakrishnan [Brock and Mirman (1972b, 407–419)].

What is interesting about this sparsely cited paper is that it is what could be considered an "executive summary" of their more extensive 1972 *JET* paper. This is evident in the introduction, literature survey, and concluding remarks of the paper. For example, in the introduction to Brock–Mirman III, they pointed to their extension of Cass–Koopmans to the case of stochastic "output or technical progress," while departing from the Cass–Koopmans approach, as Brock and Mirman deal with "uncertain technology or technological progress," and this "in

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discrete time," thereby establishing "a stochastic analog" of steady state convergence, which they termed "the modified golden rule." They then described their approach as "essentially a blend of dynamic programming and discrete-time stochastic optimal control theory" (1972a, 407).

In their literature review, they cited the work of Mirman [1970, Ph.D. thesis] and Mirrlees [1965] as "most relevant." However, they went on to say that "Mirrlees operates in continuous time where the theory of stochastic processes is messy," counterpointing this by then saying, "we avoid the messy mathematics by working in discrete time" (1972a, 408).

In their concluding remarks, they cited Mirman's 1971 University of Massachusetts paper "The steady state behavior of a class of one-sector growth models with uncertain technology," which became his 1973 *JET* paper, and their own summer 1971 MSSB Working Paper 7, which became their 1973 *IER* Paper on the "non-discounted case." They also referred to Jeanjean's 1971 Ph.D. thesis as extending some of "their results to multisector models." Finally, they concluded by saying that in Brock–Mirman III "they chose to take the self-contained route in order to delineate the ideas and to reach a wider audience" (1972a, 418).

Meta-synthesis. A close reading of Brock and Mirman's seminal 1972 *JET* paper reveals three major issues they dealt with: (i) the dynamics of optimal processes and steady states; (ii) the use of dynamic programming; and (iii) the synthesis of approaches to optimal growth. The first and second issues were set out by Mirman in a communication to the authors (12 August 2014). He wrote that in the 1972 Brock–Mirman paper there were

> Two issues that needed to be dealt with. The first is the dynamics of the optimal process and its corresponding steady state. Mirrlees [1965] dealt with an economy that had a concave technology and thus, although not done, could deal with the stochastic dynamics and corresponding steady state as, say, Cass [1965] does in the deterministic case. The work of Levhari and Srinivasan [1969] and Phelps [1962] deals with a linear technology. Hence, although it might have, the issue of the dynamics and steady state does not arise . . . The second issue is the use of dynamic programming. Mirrlees does not use dynamic programming techniques. He takes a deterministic "Euler conditions" and linearizes them . . . Both Levhari–Srinivasan and Phelps use dynamic programming techniques, in a very rudimentary form to get at the results, which were done in a linear technology setting. Hence their method at getting at the optimal program is foundationally similar to ours.

The third issue relates to Brock and Mirman's "unification," that is to say, "meta-synthesis" of previous approaches to optimal growth. This is expressed in what we take to be the *central message* of their 1972 *JET* paper, in a paragraph that, in our view, has been overlooked by most observers up to now, possibly reflecting the situation that they may not have actually read the full text of the paper, as Mirman noted in his recollections cited earlier. They wrote (1972b, 483),

> The model used in this paper is analogous to the Mirrlees and Mirman model of a one sector economy under uncertainty, which is essentially the generalization of

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> the Cass–Koopmans model with a random variable in the production function. *In fact, our methods unify the structure of growth theory*. The dynamic programming formulation makes the Cass–Koopmans results somewhat easier to obtain. It is thus seen that this paper represents a nontrivial extension and unification of the work of Cass, Koopmans, Mirman and Solow. [Our emphasis]

But let us leave the last word regarding the impact of their "unification" to Lucas, who recognized the Brock–Mirman approach as one of the starting points for Kydland and Prescott's own "meta-synthesis" that led to quantitative or empirical macroeconomics. Indeed, as Lucas put it, "technically the immediate ancestor of Kydland and Prescott" was the Brock–Mirman 1972 *JET* paper [Lucas (1987, 32, note 1)]. Just how Brock and Mirman's approach influenced subsequent developments in growth and cycle theory is another story [see Young (2014, Chap. 1)].

5. CONCLUSION

With the unification of growth theory around the three elements of optimization, dynamic programming, and stochastic control, the modern development of the neoclassical growth model reaches completion in the work of Brock and Mirman. The model that now bears their name has become a workhorse in real business cycle theory and is the basis for models of repeated games and optimal taxation. The basic techniques of stochastic dynamic programming have spun off from the Brock–Mirman nexus into applied microeconomics, finance, contract theory, and even dynamic game theory. But the underlying theory encapsulated by the model has not changed in the forty-plus years since the Brock–Mirman paper appeared.

It is worth reflecting, then, on what the model and theory deliver, because it is only against the backdrop of these results that we can begin to understand why the neoclassical framework has fallen short as a theory not just of growth but also of microfounded macroeconomics. The key results we associate with the neoclassical model are as follows:

- optimality—the equilibria generated by the model satisfy the first welfare theorem, generating Pareto-optimal outcomes;
- determinacy—the application of dynamic programming converts the model into one amenable to analysis via concave programming, so that optimal trajectories of the model have a saddlepath property that implies the solutions are unique;
- ergodicity—under reasonable specifications of the discount factor, the deterministic steady state of the model is locally stable; hence the stochastic extension of the model will exhibit ergodic behavior asymptotically.

Although these features seem eminently reasonable (and seem to reflect the fundamental results one obtains from the static Arrow–Debreu model of general equilibrium), when we confront these results with empirical facts, the shortcomings of the model become apparent.

On the question of optimality, particularly in the context of real business cycle theory, the optimality of equilibria means that there can be no such thing as involuntary unemployment of input resources. During the period from the 1980s through the first decade of the 2000s, the so-called "Great Moderation" in the world macroeconomy relegated this issue to a back burner, because spells associated with downturns in the business cycle were short, and the world economy, for the most part, spent most of the time near the full-employment threshold. It was also easy to ignore the experience of Japan in the 1990s, blaming their woes on demographic effects or the peculiarities of government regulation. But the financial crisis of 2008 and the subsequent long-lived downturns in the United States and Europe saw the economics profession arguing with itself (as it had in the 1930s) over whether or not there was anything to be done about the slump. Proponents of real business cycle theory stated clearly that the observed unemployment resulting from the crash was entirely due to a marked reduction in productivity that made continuing to work undesirable. This, of course, is part and parcel of the so-called freshwater–saltwater divide in macroeconomics, but the distinctly unmoderated effects of the '08 crash and its aftermath have brought the optimality implication of the RBC model and its neoclassical underpinnings to the fore.

A second issue that RBC macro has pushed to center stage is the question of what gives rise to aggregate shocks. Although the Brock–Mirman assumption was, as a theoretical construct, entirely acceptable, confronting the mechanism with actual data in the calibrated versions of Brock–Mirman pioneered by Kydland and Prescott has posed problems. Specifically, actual shocks large enough to impact the economy as whole (particularly large economies such as the United States or European Union) do not occur at anything close to business-cycle frequencies. Sectoral shocks do occur more frequently, but the connectedness of input–output relationships between different sectors leads to the conclusion that the law of large numbers should dampen the overall effect of these shocks on the economy as a whole. So, despite the success of RBC models in explaining many macroeconomic comovements, the question remains of what actually drives business cycles.

Finally, as a model of economic growth (independent of any other applications of the model or its methodology), the neoclassical growth model never actually moves beyond Solow's original work and its finding that, except for population growth, nothing in the neoclassical model explains the economic growth experienced since the onset of industrialization in the mid-1700s (i.e., what we now routinely refer to as the Solow residual). This problem is particularly galling, because it means that all of the work that went into the development of optimal growth theory cannot actually explain growth, optimal or not. And, in this problem, we find the seeds for the development of a new theory of growth that ultimately leads to the unraveling of the key features of the neoclassical growth model, as optimality gives way to equilibrium in environments in which the perfect competition and complete markets assumptions of the static Arrow–Debreu model and its dynamic extension in the neoclassical model must give way to increasing returns and

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knowledge externalities. We pursue this topic in our next paper on the endogenous growth revolution.

NOTES

1. It should be noted that the relatively new approach manifest in stochastic *endogenous* growth models will not be surveyed here, nor will the von Neumann–Gale model in its deterministic and stochastic versions; the evolution and development of these models will be dealt with elsewhere.

2. Phelps had utilized Bellman's dynamic programming approach in his RAND papers on "Optimal inventory policy for serviceable and replaceable stocks" (1960b), as indicated in the preceding, and in his paper "Optimal decision rules for procurement, repair or disposable spare parts" (1962a).

3. There are both priority and multiple discovery issues regarding Bellman's approach. These were raised by colleagues at RAND, in the early 1950s, and recently, by historians of mathematics. On these issues, see Pesch (2012) and Pesch and Plail (2009, 2012).

4. The Mirman–Zilcha growth model (1975) is based upon log utility and Cobb–Douglas production with exponential uncertainty. In three seminal papers, Mirman and Zilcha (1975, 1976, 1977) extended and amended the original Brock–Mirman model. Here, we only deal with the origins of Brock–Mirman, and thus we do not discuss the Mirman–Zilcha papers. However, the Mirman–Zilcha papers deserve careful reading by those applying the Brock–Mirman framework.

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