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ENAYAT MODELS OF PEANO ARITHMETIC

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Abstract. Simpson [6] showed that every countable model $\mathcal{M} \models \mathsf{PA}$ has an expansion $(\mathcal{M}, X) \models \mathsf{PA}^*$ that is pointwise definable. A natural question is whether, in general, one can obtain expansions of a nonprime model in which the definable elements coincide with those of the underlying model. Enayat [1] showed that this is impossible by proving that there is $\mathcal{M} \models \mathsf{PA}$ such that for each undefinable class X of $\mathcal{M}$, the expansion $(\mathcal{M}, X)$ is pointwise definable. We call models with this property Enayat models. In this article, we study Enayat models and show that a model of PA is Enayat if it is countable, has no proper cofinal submodels and is a conservative extension of all of its elementary cuts. We then show that, for any countable linear order γ, if there is a model $\mathcal{M}$ such that $\mathrm{Lt}(\mathcal{M}) \cong \gamma$, then there is an Enayat model $\mathcal{M}$ such that $\mathrm{Lt}(\mathcal{M}) \cong \gamma$.

§1. Introduction. Given a model $\mathcal{M}$ of PA, a subset $X \subseteq M$ is called *inductive* if $(\mathcal{M}, X) \models \mathsf{PA}^*$. In other words, X is inductive if the structure $(\mathcal{M}, X)$ satisfies the induction schema for all formulas in the expanded language with a predicate symbol for X. A set $X \subseteq M$ is called a *class* if, for each $a \in M$, $\{x \in X : x < a\} \in \mathrm{Def}(\mathcal{M})$; that is, X is a class if every initial segment of X is definable with parameters in $\mathcal{M}$. Every inductive subset of a model of PA is a class. Simpson [6] showed that every countable model $\mathcal{M} \models \mathsf{PA}$ has an undefinable inductive subset X such that every element of M is definable in $(\mathcal{M}, X)$. Simpson's argument uses arithmetic forcing. One may ask whether arithmetic forcing can be used to find an undefinable, inductive set $X \subseteq M$ so that no new elements are definable in $(\mathcal{M}, X)$. Enayat [1] showed that this is impossible: for every completion T of PA, there are $2^{\aleph_0}$ nonisomorphic models $\mathcal{M} \models T$ with the property that for any undefinable class $X \subseteq M$, the expansion $(\mathcal{M}, X)$ is pointwise definable. Enayat's result inspires the following definition:

Definition 1.1. Let $\mathcal{M} \models \mathsf{PA}$ be countable. If $\mathcal{M}$ is not prime and, for every undefinable class X of M, $(\mathcal{M}, X)$ is pointwise definable, then $\mathcal{M}$ is called an *Enayat model*.

If $\mathcal{M} \prec \mathcal{N}$, we say that $\mathcal{N}$ is a *minimal* extension of $\mathcal{M}$ if whenever $\mathcal{M} \preccurlyeq \mathcal{K} \preccurlyeq \mathcal{N}$, then either $\mathcal{K} = \mathcal{M}$ or $\mathcal{K} = \mathcal{N}$. Given a model $\mathcal{M} \models \mathsf{PA}$ and a set $X \subseteq M$, the Skolem closure of X, denoted $\mathrm{Scl}^{\mathcal{M}}(X)$ is the smallest elementary submodel of $\mathcal{M}$ containing X. We often suppress the reference to the larger model $\mathcal{M}$ and write $\mathrm{Scl}(X)$. An elementary extension $\mathcal{M} \prec \mathcal{N}$ is called superminimal if, for all $a \in N \setminus M$, $\mathcal{N} = \mathrm{Scl}(a)$; it is clear that superminimal extensions are also minimal extensions.

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An extension $\mathcal{M} \prec \mathcal{N}$ is conservative, denoted $\mathcal{M} \prec_{\text{cons}} \mathcal{N}$ if, for all $X \in \text{Def}(\mathcal{N})$, $X \cap M \in \text{Def}(\mathcal{M})$. $\mathcal{N}$ is an elementary end extension of $\mathcal{M}$, denoted $\mathcal{M} \prec_{\text{end}} \mathcal{N}$, if, for each $b \in N \setminus M$ and $a \in M$, $\mathcal{N} \models a < b$. In such a case, $\mathcal{M}$ is a *cut* of $\mathcal{N}$. $\mathcal{N}$ is a cofinal elementary extension of $\mathcal{M}$, written $\mathcal{M} \prec_{\text{cof}} \mathcal{N}$, if for each $a \in N$ there is $b \in M$ such that $\mathcal{N} \models a < b$. Conservative elementary extensions are always end extensions.

Enayat [1] showed that, for each completion T of PA, any minimal conservative extension of the prime model of T is Enayat. By a similar proof, if α is a countable ordinal, then the union of an elementary chain of superminimal conservative extensions of length α is Enayat. Such models exist because every countable model of PA has a superminimal conservative extension ([3, Corollary 2.2.12]).

The work in this article is based in large part on the discussion of substructure lattices of models of PA given in [3, Chapter 4]. We will repeat some definitions and results here.

Given $\mathcal{M} \models \text{PA}$, the set of all $\mathcal{K} \prec \mathcal{M}$ forms a lattice under inclusion, called the *substructure lattice* of $\mathcal{M}$ and denoted $\text{Lt}(\mathcal{M})$. Given $\mathcal{M} \prec \mathcal{N}$, the *interstructure lattice*, denoted $\text{Lt}(\mathcal{N}/\mathcal{M})$ is the set of all $\mathcal{K}$ such that $\mathcal{M} \preccurlyeq \mathcal{K} \preccurlyeq \mathcal{N}$. Given a lattice L, $a \in L$ is *compact* if whenever $X \subseteq L$ and $a \leq \bigvee X$, then there is a finite $Y \subseteq X$ such that $a \leq \bigvee Y$. L is *algebraic* if it is complete and each $a \in L$ is a supremum of a set of compact elements. If κ is a cardinal, then L is κ-algebraic if it is algebraic and each compact $a \in L$ has less than κ compact predecessors. If $\mathcal{M} \models \text{PA}$, then $\text{Lt}(\mathcal{M})$ is $\aleph_1$-algebraic.

Section 2 of this article characterizes which finite lattices can be realized as the substructure lattice of an Enayat model. Section 3 contains the first main result of this article, Theorem 3.2, which states that a countable model of PA that is a conservative extension of all its submodels, and contains no proper cofinal submodel is Enayat. Section 4 contains the second main result, Theorem 4.1, which shows that any countable linear order that can be the substructure lattice of a model of PA can be the substructure lattice of an Enayat model. We conclude with some open problems in Section 5.

§2. Enayat models with finite substructure lattices.

The ultimate goal of this project is to give a complete characterization of Enayat models in terms of better-known model-theoretic properties. So far, we can identify a few such properties. First we show that Enayat models cannot have proper cofinal submodels.

Lemma 2.1. *Let $\mathcal{M} \models \text{PA}$ be countable and suppose $\mathcal{K} \prec_{cof} \mathcal{M}$ is a proper submodel. Then $\mathcal{M}$ is not Enayat.*

Proof. Because $\mathcal{K}$ is countable, it must have an undefinable inductive subset X. Such an X can be found using arithmetic forcing (see [6], for example, or [3, Chapter 6]). Recalling that inductive sets are classes, we can extend this X to $Y \subseteq M$ as follows: for each $a \in K$, there is a formula $\phi_a(x)$ (possibly using parameters from K) which defines

$$\{x \in K : (\mathcal{K}, X) \models x \leq a \wedge x \in X\}.$$

Let $Y = \bigcup_{a \in K} \{x \in M : \mathcal{M} \models \phi_a(x)\}$, and one can show that $(\mathcal{K}, X) \prec (\mathcal{M}, Y)$. Since $\text{Scl}^{(\mathcal{M},Y)}(0) \subseteq \mathcal{K}$, $\mathcal{M}$ is not Enayat. ⊣

In the above proof, the construction of Y given an inductive set $X \subseteq K$ is due to Kotlarski and Schmerl independently; see [3, Theorem 1.3.7].

Lemma 2.1 gives us an easy characterization of which finite lattices can appear as the substructure lattices of an Enayat model. To state this characterization, we use the "lattice sum" notation. Given two lattices L_1 and L_2, if L_1 has a top element and L_2 has a bottom element, then the lattice $L = L_1 \oplus L_2$ is the lattice formed by identifying the top element of L_1 with the bottom element of L_2. In particular, for any lattice L, $L \oplus \mathbf{2}$ is the lattice formed by adding one new element above the top element of L. As an example, if $\mathcal{N}$ is a superminimal elementary extension of $\mathcal{M}$, then $\mathrm{Lt}(\mathcal{N}) \cong \mathrm{Lt}(\mathcal{M}) \oplus \mathbf{2}$.

The proof of the next result relies on Theorems 4.5.21 and 4.5.22 from [3] which involve n-CPP representations of lattices. The following definition and these results are originally due to Schmerl and can be found in [3, Section 4.5]. We include the definition and the results here for the sake of completeness. If A is a set, the lattice $\mathrm{Eq}(A)$ is the lattice of equivalence relations on A, with a maximum element $\mathbf{1}_A$ being the trivial relation and a minimum element $\mathbf{0}_A$ the discrete relation. If $B \subseteq A$ and $\alpha : L \to \mathrm{Eq}(A)$ is any function, the function $\alpha|B : L \to \mathrm{Eq}(B)$ is defined by $(\alpha|B)(r) = \alpha(r) \cap B^2$.

DEFINITION 2.2. Let L be a finite lattice and A a set. $\alpha : L \to \mathrm{Eq}(A)$ is a *representation* if it is an injection, $\alpha(0) = \mathbf{1}_A$, $\alpha(1) = \mathbf{0}_A$, and, for each $r, s \in L$, $\alpha(r \vee s) = \alpha(r) \wedge \alpha(s)$. α is a 0-CPP representation if, for each $r > 0$, $\alpha(r)$ has more than two classes. α is an $(n+1)$-CPP representation if for all $\Theta \in \mathrm{Eq}(A)$, there is $r \in L$ and $B \subseteq A$ such that $\Theta \cap B^2 = \alpha(r) \cap B^2$ and $\alpha|B$ is an n-CPP representation.

If L is a finite lattice, there is a Σ_1 formula $cpp(L, x)$ such that if $n < \omega$, the sentence $cpp(L, n)$ is true in $\mathbb{N}$ if and only if L has an n-CPP representation. Moreover, if $\mathbb{N} \models cpp(L, n)$, then $\mathsf{PA} \vdash cpp(L, n)$.

THEOREM 2.3 ([3, Theorem 4.5.21]). *Let L be a finite lattice and suppose $\mathcal{M} \prec \mathcal{N}$. If $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$, then $\mathcal{M} \models cpp(L, n)$ for each $n < \omega$. $\mathcal{M}$ be a countable nonstandard model of* PA *and let L be a finite lattice. If $\mathcal{M} \models cpp(L, n)$ for each $n < \omega$, then $\mathcal{M}$ has a cofinal extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$.*

THEOREM 2.4 ([3, Theorem 4.5.22]). *Let L be a finite lattice and suppose $\mathcal{M} \prec \mathcal{N}$. If $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$, then $\mathcal{M} \models cpp(L, n)$ for each $n < \omega$.*

Combining these results, if L is a finite lattice and $\mathcal{M} \prec \mathcal{N}$ is such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$, then $\mathcal{M}$ has a cofinal extension $\mathcal{N}_1$ such that $\mathrm{Lt}(\mathcal{N}_1/\mathcal{M}) \cong L$. Furthermore, if $\mathcal{M}$ is a model of TA such that $\mathrm{Lt}(\mathcal{M}) \cong L$, then for any completion $T \neq \mathsf{TA}$ of PA, there is $\mathcal{M} \models T$ which is a cofinal extension of its minimal submodel, such that $\mathrm{Lt}(\mathcal{M}) \cong L$.

COROLLARY 2.5.

1. *Let $\mathcal{M} \models \mathsf{PA}$ be an Enayat model. If $\mathrm{Lt}(\mathcal{M})$ is finite, then it is of the form $L \oplus \mathbf{2}$ where L is some finite lattice.*
2. *Let L be a finite lattice, T a completion of* PA *and $T \neq \mathsf{TA}$. If there is $\mathcal{N} \models T$ such that $\mathrm{Lt}(\mathcal{N}) \cong L$, then there is an Enayat $\mathcal{M} \models T$ such that $\mathrm{Lt}(\mathcal{M}) \cong L \oplus \mathbf{2}$.*

PROOF. To prove (1), all we need to show here is that the top element of $\mathrm{Lt}(\mathcal{M})$ cannot have more than one immediate predecessor. Suppose there are two: $\mathcal{K}_1$

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and $\mathcal{K}_2$. Notice that, since these are immediate predecessors of $\mathcal{M}$, the extensions $\mathcal{K}_i \prec \mathcal{M}$ are minimal. By Gaifman's Splitting Theorem ([2]), there is $\bar{\mathcal{K}_i}$ such that $\mathcal{K}_i \preccurlyeq_{\text{cof}} \bar{\mathcal{K}_i} \preccurlyeq_{\text{end}} \mathcal{M}$. By minimality, for each i, either $\mathcal{K}_i = \bar{\mathcal{K}_i}$ or $\bar{\mathcal{K}_i} = \mathcal{M}$. So either $\mathcal{K}_i \prec_{\text{end}} \mathcal{M}$ or $\mathcal{K}_i \prec_{\text{cof}} \mathcal{M}$. Suppose neither $\mathcal{K}_1$ nor $\mathcal{K}_2$ is cofinal in $\mathcal{M}$, and therefore they are both cuts. Because $\mathcal{K}_1$ and $\mathcal{K}_2$ are incomparable in $\text{Lt}(\mathcal{M})$, there are $a \in K_1 \setminus K_2$ and $b \in K_2 \setminus K_1$. Then either $\mathcal{M} \models a < b$ or $\mathcal{M} \models b < a$. Because the $\mathcal{K}_i$ are cuts, in the former case, that means $a \in K_2$ and in the latter case, $b \in K_1$. These are both contradictions, so one of the $\mathcal{K}_i$ must be a cofinal submodel of $\mathcal{M}$. If $\mathcal{M}$ is Enayat, by Lemma 2.1 it has no proper cofinal submodels, so this cannot be the case; therefore, the top element of $\text{Lt}(\mathcal{M})$ does not have more than one immediate predecessor. Since $\text{Lt}(\mathcal{M})$ is finite, the top element in $\text{Lt}(\mathcal{M})$ has exactly one immediate predecessor, and so $\text{Lt}(\mathcal{M}) \cong L \oplus \mathbf{2}$ for some finite lattice L.

For the proof of (2), let $\mathcal{M}_T \models T$ be a prime model of T. Since there is $\mathcal{N} \models T$ with $\text{Lt}(\mathcal{N}) \cong L$, then by Theorems 4.5.21 and 4.5.22 in [3], there is a cofinal extension $\mathcal{K}$ of $\mathcal{M}_T$ such that $\text{Lt}(\mathcal{K}) \cong L$. Let $\mathcal{M}$ be a superminimal conservative extension of $\mathcal{K}$. Theorem 2.2.13 in [3] shows that this $\mathcal{M}$ must be Enayat. ⊣

Corollary 2.5 characterizes the finite lattices which can appear as the substructure lattice of an Enayat model. To see this, we note that if L is a finite lattice that is the substructure lattice of a model of TA, then, for any completion $T \neq$ TA of PA, there is a cofinal extension $\mathcal{M}$ of the prime model $\mathcal{M}_T$ such that $\text{Lt}(\mathcal{M}) \cong L$. To get such an extension, we again appeal to Theorems 4.5.21 and 4.5.22 in [3].

There is a strong caveat to the preceding paragraph which should be mentioned here: the question of which finite lattices are isomorphic to $\text{Lt}(\mathcal{M})$ for some $\mathcal{M} \models$ PA is very much open. In particular, it is unknown if there are any finite lattices which are not isomorphic to a substructure lattice. See [3, Chapter 4] for more information on this problem.

Even granting the above caveat, the following remains open:

QUESTION 2.6. *Which finite lattices can be realized as the substructure lattice of an Enayat model of* TA?

We can modify the proof of Corollary 2.5(2) to get that, for a finite lattice L, if there is a model $\mathcal{M} \models$ TA such that $\text{Lt}(\mathcal{M}) \cong \mathbf{2} \oplus L$, then there is an Enayat model of TA whose substructure lattice is $\mathbf{2} \oplus L \oplus \mathbf{2}$. This is done in much the same way: first we find a minimal, conservative extension $\mathcal{M}$ of $\mathbb{N}$, and then find a cofinal extension $\mathcal{M}_1$ of $\mathcal{M}$ such that $\text{Lt}(\mathcal{M}_1) \cong \mathbf{2} \oplus L$ and so that the greatest common initial segment between $\mathcal{M}$ and $\mathcal{M}_1$ contains a nonstandard element. Then a superminimal conservative extension of $\mathcal{M}_1$ is Enayat.

Other Enayat models of TA can be found using results in the next section. As an example, there is an Enayat model of TA whose substructure lattice is isomorphic to $\mathbf{B}_2 \oplus \mathbf{2}$, showing that substructure lattices of models of TA need not be isomorphic to a lattice of the form $\mathbf{2} \oplus L \oplus \mathbf{2}$ for some finite lattice L. To find such a model, let $p(x)$ be a minimal type over TA and let a and b be two elements realizing it. Then if $\mathcal{M}$ is a superminimal conservative extension of $\text{Scl}(a, b)$, it is Enayat and $\text{Lt}(\mathcal{M}) \cong \mathbf{B}_2 \oplus \mathbf{2}$.

Corollary 2.5 implies that there are Enayat models of PA whose substructure lattice is isomorphic to $\mathbf{N}_5 \oplus \mathbf{2}$. It is unknown whether there is an Enayat model of TA whose substructure lattice is isomorphic to this lattice; more generally, it is unknown if there are Enayat models which are not conservative over all their elementary cuts. If $\mathcal{M} \models \mathsf{TA}$ is such that $\mathrm{Lt}(\mathcal{M}) \cong \mathbf{N}_5 \oplus \mathbf{2}$, then $\mathcal{M}$ is not a conservative extension of $\mathbb{N}$.

§3. Characterizing Enayat models. In this section, we show our first main result: a model is Enayat if it has no proper cofinal submodel and is a conservative extension of each of its elementary cuts. First, we prove a lemma which will be needed for this result. This lemma is very similar to [3, Theorem 2.2.13]. If $\mathfrak{A}$ is any first order structure with universe A, and $Y \subseteq A$, the definable closure of Y in $\mathfrak{A}$, denoted $\mathrm{dcl}^{\mathfrak{A}}(Y)$, is the set of those $x \in A$ such that $\{x\}$ is definable in $\mathfrak{A}$ using parameters from Y.

LEMMA 3.1. *Suppose $\mathcal{N} \models \mathsf{PA}$, X is an undefinable class of $\mathcal{N}$, $\mathcal{M} \prec_{cons} \mathcal{N}$, and C is a cofinal subset of $\mathcal{M}$. Then there is $b \in N \setminus M$ such that $b \in \mathrm{dcl}^{(\mathcal{M},X)}(C)$.*

PROOF. Recall that conservative extensions are end extensions. Let $c > M$. Since X is a class, the set $\{x \in X : x < c\} \in \mathrm{Def}(\mathcal{N})$. By conservativity, we have, for some $b \in M$:

$$X \cap M = \{x \in M : \mathcal{M} \models \phi(x, b)\}.$$

Because C is cofinal in $\mathcal{M}$, there is some $a \in C$ such that $b < a$. Consider the set

$$Y = \{z \in N : (\mathcal{N}, X) \models \exists y < a\, \forall x < z\, (\phi(x, y) \leftrightarrow x \in X)\}.$$

This set contains $\mathcal{M}$. It must also be bounded, since, if it were not, then $Y = N$, and there would be some $b < a$ such that

$$X = \{x \in N : \mathcal{N} \models \phi(x, b)\}.$$

However, since X is undefinable, there can be no such b. Let b be the maximum of Y. Clearly b is a definable element in $(\mathcal{N}, X)$ using only parameters from C, and is above $\mathcal{M}$. ⊣

Let $\mathcal{M} \models \mathsf{PA}$ and $X \subseteq M$. Then $\sup(X) = \{x : \exists a \in X(\mathcal{M} \models x \leq a)\}$. If $\mathcal{K} \prec \mathcal{M}$, then $\mathcal{K} \preccurlyeq_{\mathrm{cof}} \sup(\mathcal{K}) \preccurlyeq_{\mathrm{end}} \mathcal{M}$. This is another form of Gaifman's Splitting Theorem ([2]).

THEOREM 3.2. *Suppose $\mathcal{M}$ is countable, has no proper cofinal submodel, and is a conservative extension of each of its elementary cuts. Then $\mathcal{M}$ is Enayat.*

PROOF. Let $X \subseteq M$ be an undefinable class and let C be the set of all elements definable in $(\mathcal{M}, X)$. Since C is closed under Skolem terms, it is an elementary submodel of $\mathcal{M}$. To be more precise, the structure $\mathcal{C} = (C, + \restriction C, \times \restriction C, 0, 1) \prec \mathcal{M}$. Let $\mathcal{K} = \sup(C)$. Then $\mathcal{K} \preccurlyeq_{\mathrm{cons}} \mathcal{M}$. If C is bounded in $\mathcal{M}$, then $\mathcal{K}$ is properly contained in $\mathcal{M}$, and by Lemma 3.1, there is $c \in M \setminus K$ definable in $(\mathcal{M}, X)$ using a parameter from C. Since each element of C is a definable element in $(\mathcal{M}, X)$, c is also a definable element. This is a contradiction; therefore C must be cofinal in $\mathcal{M}$. Since $\mathcal{M}$ has no proper cofinal submodels, $C = M$. ⊣

We can find many examples of Enayat models as a result of this theorem. As mentioned before, Corollary 2.2.12 of [3] states that every countable model of PA has a superminimal conservative extension. This means we can form countable elementary chains of superminimal conservative extensions, which, by Theorem 3.2, are Enayat models. That is, if α is a countable ordinal, $\mathcal{N} = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$, where $\mathcal{M}_0$

is prime, $\mathcal{M}_{\beta+1}$ is a superminimal conservative extension of $\mathcal{M}_\beta$, and $\mathcal{M}_\lambda = \bigcup_{\beta<\lambda} \mathcal{M}_\beta$ whenever λ is a limit ordinal, then $\mathcal{N}$ is Enayat.

Corollary 2.5 characterized the finite lattices which can be the substructure lattices of an Enayat model. For infinite lattices, we do not have a complete characterization; however, we note the following corollary of Theorem 3.2.

Corollary 3.3. *Let $T \neq \mathsf{TA}$ be a completion of* PA, *$\mathcal{M}_T \models T$ a prime model of T, and L a lattice. Suppose there is a countable $\mathcal{N} \succ_{cof} \mathcal{M}_T$ such that* $\mathrm{Lt}(\mathcal{N}) \cong L$. *Then there is an Enayat model $\mathcal{M} \models T$ such that* $\mathrm{Lt}(\mathcal{M}) \cong L \oplus \mathbf{2}$.

Proof. Let $\mathcal{M}$ be a superminimal conservative extension of $\mathcal{N}$. $\mathcal{N}$ is the only proper elementary cut of $\mathcal{M}$ and $\mathcal{M}$ has no proper cofinal submodels. By Theorem 3.2, $\mathcal{M}$ is Enayat. ⊣

Many examples of lattices can be realized as substructure lattices of Enayat models in this way. Paris [4] proved if L is a countable algebraic distributive lattice, then for any completion T of PA, there is $\mathcal{M} \models T$ with $\mathrm{Lt}(\mathcal{M}) \cong L$. This proof can be modified (see [3, Theorem 4.7.3]) to obtain the following: if L is a countable algebraic distributive lattice, then every countable nonstandard $\mathcal{M} \models \mathsf{PA}$ has a cofinal extension $\mathcal{N}$ such that $\mathrm{Lt}(\mathcal{N}/\mathcal{M}) \cong L$. A more general result is shown in [5], which covers a class of lattices properly including all distributive lattices. For such lattices L, there is an Enayat model whose substructure lattice is isomorphic to $L \oplus \mathbf{2}$.

§4. Linearly ordered substructure lattices. In this section, we show that if γ is a linear order such that there is some model of PA whose substructure lattice is isomorphic to γ, then there is a model $\mathcal{M} \models \mathsf{PA}$ whose substructure lattice is isomorphic to γ with the property that for each $\mathcal{K} \in \mathrm{Lt}(\mathcal{M})$, $\mathcal{M}$ is a conservative elementary extension of $\mathcal{K}$. If, in addition, γ is countable, then such an $\mathcal{M}$ is Enayat. Recall that if $\mathcal{M} \models \mathsf{PA}$, then the lattice $\mathrm{Lt}(\mathcal{M})$ is $\aleph_1$-algebraic; in other words, it is complete, compactly generated, and each compact element has countably many compact predecessors. If $\mathrm{Lt}(\mathcal{M})$ is a linear order, then the compact elements are successors, and there can only be countably many compact elements.

Theorem 4.1. *Let T be a completion of* PA *and let γ be an $\aleph_1$-algebraic linear order. There is $\mathcal{M} \models T$ such that* $\mathrm{Lt}(\mathcal{M}) \cong \gamma$ *and, for each* $\mathcal{K} \in \mathrm{Lt}(\mathcal{M})$, $\mathcal{K} \prec_{cons} \mathcal{M}$.

The *cofinality* quantifier C is defined so that $\mathsf{C}x\phi(x)$ is shorthand for $\forall w \exists x > w\phi(x)$. It is understood that the variable w does not appear in ϕ. The *cobounded* quantifier C^* is the dual of C; that is, $\mathsf{C}^* x\phi(x)$ is $\neg \mathsf{C}x\neg\phi(x)$. It can be thought of as shorthand for $\exists w \forall x > w\phi(x)$ (where w does not appear in ϕ).

We extend C and C^* to apply to n-tuples $\bar{x} = x_0, x_1, \ldots, x_{n-1}$ so that $\mathsf{C}\bar{x}$ is $\mathsf{C}x_0\mathsf{C}x_1 \ldots \mathsf{C}x_{n-1}$, and similarly for C^*. We note that the order is important here.

Fix T a completion of PA.

Definition 4.2. If $1 \leq n < \omega$, an n-ary formula $\theta(x_0, x_1, \ldots, x_{n-1})$ is *big* if $T \vdash \mathsf{C}\bar{x}\theta(\bar{x})$.

The 1-ary formula $x = x$ is big. The following lemma is a simple observation which allows us to extend big n-ary formulas to big $(n+1)$-ary formulas.

LEMMA 4.3. *If $\theta(x_0, x_1, \ldots, x_{n-1})$ is a big n-ary formula and $i < n$, and x' is a new free variable, the formula $\phi(x_0, x_1, \ldots, x_i, x', x_{i+1}, \ldots, x_{n-1})$ defined as $\theta(x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_{n-1})$ is a big $(n+1)$-ary formula.*

As an example, if $\theta(x)$ is a big 1-ary formula, then it is not hard to see that $T \vdash \mathsf{C}y\mathsf{C}x\theta(x)$ and $T \vdash \mathsf{C}x\mathsf{C}y\theta(x)$.

LEMMA 4.4. *Let $n \geq 1$. If $\mathcal{M}_0 \prec_{end} \mathcal{M}_1 \prec_{end} \cdots \prec_{end} \mathcal{M}_n$ are models of T, then the n-ary formula $\theta(\bar{x})$ is big if and only if there are $c_i \in M_{i+1} \setminus M_i$ such that $\mathcal{M}_n \models \theta(\bar{c})$.*

PROOF. Let $n = 1$ and let $\mathcal{M}_0 \prec_{\text{end}} \mathcal{M}_1$. First suppose $\theta(x)$ is big. Then $\mathcal{M}_1 \models \forall w \exists x > w\theta(x)$. Let $w \in M_1 \setminus M_0$, and let $c > w$ be such that $\mathcal{M}_1 \models \theta(c)$. Since $\mathcal{M}_1$ is an end extension of $\mathcal{M}_0$, $c \in M_1 \setminus M_0$. Conversely, if there is $c \in M_1 \setminus M_0$ such that $\mathcal{M}_1 \models \theta(c)$, then, for each $w \in M_0$, $\mathcal{M}_1 \models \exists x > w\theta(x)$. By elementarity, this statement is also true in $\mathcal{M}_0$, so $\mathcal{M}_0 \models \mathsf{C}x\theta(x)$, and so θ is big.

If $n > 1$, let $\mathcal{M}_0 \prec_{\text{end}} \cdots \prec_{\text{end}} \mathcal{M}_n$. First suppose $\theta(\bar{x})$ is big and let $\phi(x_0, \ldots, x_{n-2})$ be the formula $\mathsf{C}x_{n-1}\theta(\bar{x})$. Since $\theta(\bar{x})$ is big, $\phi(\bar{x})$ is a big $(n-1)$-ary formula. By induction, there are $c_i \in M_{i+1} \setminus M_i$, for each $i < n-1$, such that $\mathcal{M}_{n-1} \models \mathsf{C}x_{n-1}\theta(c_0, \ldots, c_{n-2}, x_{n-1})$. By elementarity, this statement is true in $\mathcal{M}_n$. Let $w \in M_n \setminus M_{n-1}$, and let $c_{n-1} > w$ be such that $\mathcal{M}_n \models \theta(c_0, \ldots, c_{n-2}, c_{n-1})$. Since $\mathcal{M}_n$ is an end extension of $\mathcal{M}_{n-1}$, $c_{n-1} \in M_n \setminus M_{n-1}$.

Lastly, suppose $c_i \in M_{i+1} \setminus M_i$ are such that $\mathcal{M}_n \models \theta(\bar{c})$. Let $\phi(x_0, \ldots, x_{n-2})$ be the formula $\mathsf{C}x_{n-1}\theta(\bar{x})$. Since, for each $w \in M_{n-1}$,

$$\mathcal{M}_n \models \exists x_{n-1} > w\theta(c_0, \ldots, c_{n-2}, x_{n-1}),$$

by elementarity $\mathcal{M}_{n-1} \models \exists x_{n-1} > w\theta(c_0, \ldots, c_{n-2}, x_{n-1})$. Therefore, $\mathcal{M}_{n-1} \models \phi(c_0, \ldots, c_{n-2})$. By induction, ϕ is a big $(n-1)$-ary formula, and so by the definition of ϕ, θ is a big n-ary formula. ⊣

DEFINITION 4.5. Suppose $1 \leq n < \omega$, $t(u, \bar{x})$ is an $(n+1)$-ary Skolem term and $\theta(\bar{x})$ is an n-ary formula. We say that $\theta(\bar{x})$ *handles* $t(u, \bar{x})$ if:

$$\forall u \bigvee_{i \leq n} \mathsf{C}^*\bar{x}\mathsf{C}^*\bar{y}[(\theta(\bar{x}) \wedge \theta(\bar{y})) \rightarrow (t(u, \bar{x}) = t(u, \bar{y}) \leftrightarrow \bigwedge_{j<i} x_j = y_j)].$$

To motivate this definition, notice that if $t(u, \bar{x})$ is an $(n+1)$-ary Skolem term, we can define a family of equivalence relations Θ_u on n-tuples by letting $(\bar{x}, \bar{y}) \in \Theta_u$ iff $t(u, \bar{x}) = t(u, \bar{y})$. In addition, for each $i \leq n$, there is a canonical equivalence relation $\widetilde{\Theta_i}$ given by $((x_0, \ldots, x_{n-1}), (y_0, \ldots, y_{n-1})) \in \widetilde{\Theta_i}$ iff $x_j = y_j$ for all $j < i$. $\widetilde{\Theta_0}$ is the trivial equivalence relation, $\widetilde{\Theta_n}$ is the discrete relation, and, for each $i < n$, $\widetilde{\Theta_{i+1}}$ refines $\widetilde{\Theta_i}$. If an n-ary formula $\theta(\bar{x})$ handles an $(n+1)$-ary Skolem term, then for each u there is some $i \leq n$ for which Θ_u is eventually equal to $\widetilde{\Theta_i}$ on the set defined by θ. In particular, if $\theta(x)$ is a 1-ary formula which handles $t(u, x)$, then, for each u, the function $t(u, \cdot)$ is eventually one to one or constant on the set defined by θ.

The following lemma states that every Skolem term can be handled.

LEMMA 4.6. *If $\theta(\bar{x})$ is a big n-ary formula and $t(u, \bar{x})$ is an $(n+1)$-ary Skolem term, then there is a big n-ary formula $\theta'(\bar{x})$ such that $T \vdash \forall \bar{x}[\theta'(\bar{x}) \rightarrow \theta(\bar{x})]$ and $\theta'(\bar{x})$ handles $t(u, \bar{x})$.*

PROOF. We prove the lemma by induction on n. First suppose $n = 1$. Let $\mathcal{M}_0$ be the prime model of T and $\mathcal{M}_0 \prec \mathcal{M}_1$ a superminimal conservative extension. Since $\theta(x)$ is big, by Lemma 4.4, there must be $c_0 \in M_1 \setminus M_0$ such that $\mathcal{M}_1 \models \theta(c_0)$.

Let $F = \{\langle u, m\rangle : \mathcal{M}_1 \models t(u, c_0) = m\}$. By conservativity, $F \cap M_0 \in \mathrm{Def}(\mathcal{M}_0)$, so there is a partial $\mathcal{M}_0$-definable function f such that $f(u) = m$ if and only if $\langle u, m\rangle \in F$ for all $u, m \in M_0$. Let D be the domain of f, and let

$$X = \{x : \mathcal{M}_0 \models \theta(x) \wedge \forall u \in D(t(u, x) = f(u))\}.$$

Since $\mathcal{M}_1 \models c_0 \in X$ (that is, c_0 satisfies the definition for X as interpreted in $\mathcal{M}_1$), X must be unbounded by Lemma 4.4.

D is the set of those u for which $t(u, c_0) \in M_0$. The set $M_0 \setminus D$ is the set of those u for which $t(u, c_0) > M_0$. Enumerate $M_0 \setminus D$ as $u_0, u_1, \dots$ (we can assume this set is infinite; if it is finite, the argument is similar). Let x_0 be the least $x \in X$. Given $x_0, \dots, x_i$, let x_{i+1} be the least $x \in X$ such that $x > x_i$ and

$$\forall j, k \leq i(t(u_j, x_k) \neq t(u_j, x)).$$

Let $Y = \{x_i : i \in M_0\}$. This set is definable; informally, $x \in Y$ iff there are k and s such that s codes a finite sequence of length more than k, $(s)_0 = x_0$, $(s)_{i+1}$ is the least element of X satisfying the conditions above (for each $i < k$), and $(s)_k = x$. Y is unbounded since each x_i is different for all $i \in M_0$ (in other words, the function $i \mapsto x_i$ is a definable, injective function with unbounded domain, so its image must be unbounded). Let θ' be the formula defining Y. θ' is big and handles the term $t(u, x)$.

Let $n > 0$ and assume that the lemma holds for $(n-1)$-ary big $\theta(\bar{x})$ and n-ary Skolem terms $t(u, \bar{x})$. Let $\theta(\bar{x})$ be a big n-ary formula and $t(u, \bar{x})$ an $(n+1)$-ary Skolem term. We again let $\mathcal{M}_0$ be the prime model of T and let $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots \prec \mathcal{M}_n$ be a chain of superminimal, conservative extensions. Since $\theta(\bar{x})$ is big, there are $c_i \in M_{i+1} \setminus M_i$ such that $\mathcal{M}_n \models \theta(\bar{c})$.

We again let $F = \{\langle u, m\rangle : \mathcal{M}_n \models t(u, \bar{c}) = m\}$. By conservativity, the set $F \cap M_{n-1}$ is definable by a formula $\phi(u, m, a)$, where $a \in \mathcal{M}_{n-1}$. By superminimality, we can find a formula $\psi(u, m, x_0, \dots, x_{n-2})$ such that

$$\mathcal{M}_{n-1} \models \forall u, m[\psi(u, m, c_0, \dots, c_{n-2}) \iff \langle u, m\rangle \in F].$$

That is, ψ defines a partial $\mathcal{M}_0$-definable function $f(u, x_0, \dots, x_{n-2})$. Let $D = \{u : \mathcal{M}_{n-1} \models \exists m[\langle u, m\rangle \in F]\}$, and by conservativity, $D \cap M_0$ is definable without parameters. We again call this set D. D is the set of those $u \in M_0$ such that $t(u, \bar{c}) \in M_{n-1}$.

Let $\phi(x_0, \dots, x_{n-2})$ be the formula

$$\forall w \mathsf{C} x_{n-1} \theta(\bar{x}) \wedge \forall u \in D[t(u, \bar{x}) = f(u, x_0, \dots, x_{n-2})] \wedge \forall u \notin D[t(u, \bar{x}) > w].$$

Since $\mathcal{M}_{n-2} \models \phi(c_0, \dots, c_{n-2})$, ϕ is a big $(n-1)$-ary formula. By induction, there is a big $(n-1)$-ary formula θ_0 that handles f, such that $T \vdash \forall \bar{x}[\theta_0(\bar{x}) \to \phi(\bar{x})]$. Let $d_0 \in M_1, \dots, d_{n-2} \in M_{n-1}$ be such that $\mathcal{M}_{n-1} \models \theta_0(\bar{d})$. Since $\mathcal{M}_{n-1} \models \phi(\bar{d})$, there is $d_{n-1} \in M_n \setminus M_{n-1}$ such that

$$\mathcal{M}_n \models \theta(\bar{d}) \wedge \forall u \in D[t(u, \bar{d}) = f(u, d_0, \dots, d_{n-2})],$$

and, if $u \notin D$, $t(u, \bar{d}) > M_{n-1}$.

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Since $\mathrm{Scl}(d_{n-1}) = \mathcal{M}_n$, we also have a Skolem term g such that $\mathcal{M}_n \models g(d_{n-1}) = \langle d_0, d_1, \ldots, d_{n-2}\rangle$. Given $\bar{x}$ an $(n-1)$-tuple, we let

$$Y_{\bar{x}} = \{x : \mathcal{M}_0 \models g(x) = \bar{x} \wedge \forall u \in D[t(u, \bar{x}, x) = f(u, \bar{x})]\}.$$

Notice that $\mathcal{M}_{n-1} \models \theta_0(\bar{d}) \wedge \mathsf{C}x[g(x) = \bar{d}]$, and so the formula $\theta_0(\bar{x}) \wedge \mathsf{C}x[g(x) = \bar{x}]$ is a big $(n-1)$-ary formula which handles f. We abuse notation by referring to this formula as θ_0 again. Now if $\mathcal{M}_0 \models \theta_0(\bar{x})$, then $Y_{\bar{x}}$ is unbounded.

Similar to the above proof, we enumerate $M_0 \backslash D$ as $u_0, u_1, \ldots$ and we will construct a sequence $y_0, y_1, \ldots$ as follows. Enumerate those $(n-1)$-tuples $\bar{x}$ such that $\mathcal{M}_0 \models \theta_0(\bar{x})$ as $\bar{x}_0, \bar{x}_1, \ldots$, so that each such $\bar{x}$ appears infinitely often. Let y_0 be the least element of $Y_{\bar{x}_0}$. Given $y_0, \ldots, y_i$, let y_{i+1} be the least $y > y_i$ such that $y \in Y_{\bar{x}_{i+1}}$ and

$$\forall j \leq i, k \leq i[t(u_j, \bar{x}_k, y_k) \neq t(u_j, \bar{x}_{i+1}, y)].$$

Such a y exists because, if $u \notin D$, then for each tuple $(x_0, \ldots, x_{n-2})$ such that $\mathcal{M}_0 \models \theta_0(\bar{x})$, the function $h(x)$ defined as $h(x) = t(u, x_0, \ldots, x_{n-2}, x)$ has unbounded range on the set $Y_{\bar{x}}$. Let $Y = \{y_i : i \in M_0\}$; this is definable in a similar manner as the set Y in the case $n = 1$ above. Let $\theta'(x_0, \ldots, x_{n-1})$ be the formula

$$\theta_0(x_0, \ldots, x_{n-2}) \wedge x_{n-1} \in Y \wedge g(x_{n-1}) = \langle x_0, \ldots, x_{n-2}\rangle.$$

θ' is as required. To see that θ' handles t, first note that, if $u \notin D$, then for large enough j, k, $t(u, \bar{x}_j, y_j) \neq t(u, \bar{x}_k, y_k)$. If $u \in D$, then

$$\mathcal{M}_0 \models \forall x_0 \ldots \forall x_{n-1}[\theta'(\bar{x}) \rightarrow t(u, \bar{x}) = f(u, x_0, \ldots, x_{n-2})].$$

Since θ_0 handles f, θ' handles t. ⊣

Proof of Theorem 4.1. Let $\mathcal{M}_0 \models T$ be the prime model. Let $t_0(u, \bar{x})$, $t_1(u, \bar{x}), \ldots$ be an enumeration of all Skolem terms so that each t_n has at most $(n+1)$ free variables. Let $s_0, s_1, \ldots$ be an enumeration of the (countably many) compact elements of γ. Given $n < \omega$, let π_n be the permutation of $\{0, \ldots, n\}$ such that $s_{\pi_n(0)} < s_{\pi_n(1)} < \cdots < s_{\pi_n(n)}$. For an $(n+1)$-ary formula $\theta(\bar{x})$, by $\theta(\pi_n(\bar{x}))$ we mean $\theta(x_{\pi_n(0)}, \ldots, x_{\pi_n(n)})$. Similarly, for an $(n+2)$-ary Skolem term $t(u, \bar{x})$, by $t(u, \pi_n(\bar{x}))$ we mean $t(u, x_{\pi_n(0)}, \ldots, x_{\pi_n(n)})$.

Using Lemmas 4.3 and 4.6, we construct a sequence of formulas $\theta_0(x_0), \theta_1(x_0, x_1), \ldots$ such that, for each $n < \omega$:

- $\theta_n(\pi_n(\bar{x}))$ is big and
$$T \vdash \forall \bar{x}(\theta_{n+1}(\bar{x}) \rightarrow \theta_n(\bar{x})),$$
- If t_n is $(m+1)$-ary, then there is an m-ary formula $\theta(\bar{x})$ such that $\theta(\pi_{m-1}(\bar{x}))$ handles $t_n(u, \pi_{m-1}(\bar{x}))$ and
$$T \vdash \forall \bar{x}(\theta_n(\bar{x}) \rightarrow \theta(\bar{x})).$$

The set $\{\theta_n(\bar{x}) : n \in \omega\}$ determines a complete, consistent type: if $\theta(u, \bar{x})$ is any formula, then the corresponding Skolem term $t(u, \bar{x})$ (defined as $t(u, \bar{x}) = 0$ iff $\theta(u, \bar{x})$ and $t(u, \bar{x}) = 1$ otherwise) is handled at some stage n. Let $c_0, c_1, \ldots$ be elements realizing this type and let $\mathcal{M} \models T$ be generated by these elements. We claim that $\mathcal{M}$ is as desired.

First, we show that $\mathrm{Lt}(\mathcal{M}) \cong \gamma$. Let s_i and s_j be compact elements of γ. We show that $s_i \leq s_j \iff \mathrm{Scl}(c_i) \preccurlyeq \mathrm{Scl}(c_j)$. Suppose $s_i \leq s_j$. Let m be the maximum of i and j and let $t(u, x_0, \ldots, x_m)$ be the term defined as $t(u, x_0, \ldots, x_m) = x_u$ if $u \leq m$ and $t(u, x_0, \ldots, x_m) = 0$ otherwise. There is $\theta(x_0, \ldots, x_m)$ such that $\theta(\pi_m(\bar{x}))$ handles $t(u, \pi_m(\bar{x}))$. Let X be the set defined by θ; that is,

$$X = \{\langle x_0, \ldots x_m \rangle : \mathcal{M}_0 \models \theta(x_0, \ldots, x_m)\}.$$

Then, for large enough x, if there are $\langle x_0, \ldots, x_{m-1} \rangle, \langle y_0, \ldots, y_{m-1} \rangle$ such that $\langle x_0, \ldots, x_{m-1}, x \rangle \in X$ and $\langle y_0, \ldots, y_{m-1}, x \rangle \in X$, then $x_k = y_k$ for each $k \leq m-1$. Let $t_k(x)$ be the Skolem term defined (on all such large enough x) as that unique such x_k. Then, if k is such that $\pi_m(k) = i$, $\mathcal{M} \models t_k(c_j) = c_i$.

Conversely, suppose $\mathrm{Scl}(c_i) \preccurlyeq \mathrm{Scl}(c_j)$. There is a term f such that $\mathrm{Scl}(c_j) \models f(c_j) = c_i$. Let n be the maximum of i and j. We claim that if k_i and k_j are such that $\pi_n(k_i) = i$ and $\pi_n(k_j) = j$, then $k_i \leq k_j$. To see this, let $t(u, \bar{x}) = f(x_{k_j})$, and then suppose $\theta(\pi_n(\bar{x}))$ is big and handles $t(u, \pi_n(\bar{x}))$. Then if $k_j < k_i$, if we fix $x_0, \ldots, x_{k_j}$, there would be infinitely many different x_{k_i} such that $f(x_{k_j}) = x_{k_i}$, which is impossible.

Next, we show that if $b \in M$, then there is some c_i such that $\mathrm{Scl}(b) = \mathrm{Scl}(c_i)$. Let $n < \omega$, $m \in \mathcal{M}_0$, and $t(u, x_0, \ldots, x_{n-1})$ be such that

$$\mathcal{M} \models t(m, \pi_{n-1}(\bar{c})) = b.$$

Then there is some $\theta(\bar{x})$ such that $\theta(\pi_{n-1}(\bar{x}))$ handles $t(u, \pi_{n-1}(\bar{x}))$. Let $i \leq n$ be such that, in $\mathcal{M}_0$, the following statement holds:

$$\mathcal{M}_0 \models \mathsf{C}^* \bar{x} \mathsf{C}^* \bar{y} [\theta(\pi_{n-1}(\bar{x})) \wedge \theta(\pi_{n-1}(\bar{y})) \rightarrow (t(m, \pi_{n-1}(\bar{x})) = t(m, \pi_{n-1}(\bar{y})) \leftrightarrow \bigwedge_{j \leq i} x_{\pi_{n-1}(j)} = y_{\pi_{n-1}(j)})].$$

Therefore, there are Skolem functions f and g so that

$$\mathcal{M} \models f(c_{\pi_{n-1}(0)}, \ldots, c_{\pi_{n-1}(i)}) = b$$

and $\mathcal{M} \models g(b) = \langle c_{\pi_{n-1}(0)}, \ldots, c_{\pi_{n-1}(i)} \rangle$. Combining this with the argument above, we have that $\mathrm{Scl}(b) = \mathrm{Scl}(c_{\pi_{n-1}(i)})$.

This means that all the finitely generated substructures of $\mathcal{M}$ are the $\mathrm{Scl}(c_i)$ for each $i < \omega$ and therefore that $\mathrm{Lt}(\mathcal{M}) \cong \gamma$.

Lastly, we show that $\mathrm{Scl}(c_i) \prec_{\mathrm{cons}} \mathcal{M}$ for each $i < \omega$. Let $X \subseteq M$ be defined as

$$X = \{u : \mathcal{M} \models \phi(u, \pi_{n-1}(\bar{c}))\}.$$

Then if c_j is such that $c_0, \ldots, c_{n-1} \in \mathrm{Scl}(c_j)$, clearly $X \cap \mathrm{Scl}(c_j) \in \mathrm{Def}(\mathrm{Scl}(c_j))$. Suppose i is such that there are some $k < n$ such that $c_k \notin \mathrm{Scl}(c_i)$. Let $t(u, \bar{x})$ be the Skolem term such that $t(u, \bar{x}) = 0$ iff $\phi(u, \bar{x})$ and $t(u, \bar{x}) = 1$ otherwise. If $\theta(\pi_{n-1}(\bar{x}))$ is big and handles $t(u, \pi_{n-1}(\bar{x}))$, it must be the case that, for each u,

$$\mathsf{C}^* \bar{x} (\theta(\pi_{n-1}(\bar{x})) \rightarrow \phi(u, \pi_{n-1}(\bar{x}))) \vee \mathsf{C}^* \bar{x} (\theta(\pi_{n-1}(\bar{x})) \rightarrow \neg \phi(u, \pi_{n-1}(\bar{x}))).$$

Let θ be such that $\theta(\pi_{n-1}(\bar{x}))$ handles $t(u, \pi_{n-1}(\bar{x}))$, and let $c_{\pi_{n-1}(0)}, \ldots, c_{\pi_{n-1}(j)} \in \mathrm{Scl}(c_i)$. Then $u \in X \cap \mathrm{Scl}(c_i)$ if and only if

$$\text{Scl}(c_i) \models \mathsf{C}^* x_{j+1} \cdots \mathsf{C}^* x_{n-1}\, \theta(c_{\pi_{n-1}(0)}, \ldots, c_{\pi_{n-1}(j)}, x_{j+1}, \ldots, x_{n-1}) \\ \wedge\, \phi(u, c_{\pi_{n-1}(0)}, \ldots, c_{\pi_{n-1}(j)}, x_{j+1}, \ldots, x_{n-1}). \qquad \dashv$$

COROLLARY 4.7. *Suppose γ is a countable algebraic linear order. Then there is an Enayat model $\mathcal{M} \models \mathsf{PA}$ such that $\text{Lt}(\mathcal{M}) \cong \gamma$.*

§5. Open problems.

From Theorem 3.2, if a model of PA is countable, has no proper cofinal submodel and is a conservative extension of each of its elementary cuts, then it is Enayat. Additionally, Lemma 2.1 shows that models with proper cofinal submodels cannot be Enayat. A negative answer to the following problem would complete the classification of Enayat models:

PROBLEM 5.1. *Suppose $\mathcal{M} \models \mathsf{PA}$ is countable but is not a conservative extension of (at least) one of its proper elementary cuts. Can $\mathcal{M}$ be Enayat?*

Corollary 2.5 characterizes the finite lattices which can appear as a substructure lattice of an Enayat model. We do not have such a characterization for countable lattices, though Corollary 3.3 provides a first attempt.

PROBLEM 5.2. *Suppose L is a countable lattice such that there is $\mathcal{M} \models \mathsf{PA}$ with $\text{Lt}(\mathcal{M}) \cong L$. Under what circumstances is there an Enayat model $\mathcal{M}$ such that $\text{Lt}(\mathcal{M}) \cong L$?*

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