Can. J. Math.Vol. 43 (4), 1991 pp. 770–791

PERIODS AND THE ASYMPTOTICS OF A DIOPHANTINE PROBLEM II

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Introduction Let $P(z_1, \ldots, z_n)$ be a polynomial with positive coefficients. For positive x define

$$N_1(x) = \sum_{\{m \in \mathbb{N}^n : P(m) \leq x\}} 1.$$

A classical diophantine problem is to describe the asymptotic behavior of $N_1(x)$ as $x \longrightarrow \infty$. More generally, one can introduce a second polynomial φ satisfying the condition

(0.1) Sign $\varphi(m)$ is constant for all m outside at most a finite subset of $\mathbb{N}^n$.

Now define

$$N_\varphi(x) = \sum_{\{m \in \mathbb{N}^n : P(m) \leq x\}} \varphi(m).$$

One can also ask about the asymptotic behavior of N_φ as $x \longrightarrow \infty$. This is an example of a *weighted* diophantine problem, each lattice point m weighted by $\varphi(m)$.

The answer to such questions has been given by Sargos [Sa-2], as described in Theorem A below. The analytic method used to study N_φ is based upon the functional properties of the series

$$D_P(s, \varphi) = \sum_{m \in \mathbb{N}^n} \frac{\varphi(m)}{P(m)^s}.$$

One knows from [Li-1,Sa-1] that D_P is analytic in a halfplane $Re(s) > B(\varphi)$ and admits a meromorphic extension to $\mathbb{C}$ with rational poles (by [Sa-1, Theorem 1.2], also cf. [Li-3]) of order at most n. That there exists *at least one* pole follows from a well known theorem of Landau which says that a Dirichlet series with almost every coefficient positive (or negative) must have a real pole, lying on the boundary of the domain of absolute convergence of the series.

Order the poles as $\rho_0(\varphi) > \rho_1(\varphi) > \ldots$. For each j, set

$$\mathrm{Pol}_{s=\rho_j(\varphi)}\left(\frac{D_P(s, \varphi)}{s}\right) = \sum_{i=1}^{n} \frac{A_{i,j}(\varphi)}{\left(s - \rho_j(\varphi)\right)^i}$$

to be the principal part at $\rho_j(\varphi)$. Define

$$N_j(x, \varphi) = x^{\rho_j(\varphi)} \sum_{i=1}^{n} A_{i,j}(\varphi) \log^{i-1} x.$$

Let D =degree of P. Define the index k by the condition $\rho_0(\varphi) > \rho_1(\varphi) \cdots > \rho_k(\varphi) > \rho_0(\varphi) - 1/D \geq \rho_{k+1}(\varphi) > \cdots$. Sargos has shown [Sa-2],

Received by the editors October 2, 1989 .

THEOREM A.

$$N_\varphi(x) = \sum_{j=0}^{k} N_j(x, \varphi) + O_\epsilon(x^{\rho_0(\varphi)-1/D+\epsilon}).$$

Define the dominant term $\hat{N}_\varphi(x)$ of $N_\varphi(x)$ to equal $N_0(x, \varphi)$. Evidently, this term contains most of the important information about $N_\varphi(x)$ for large x.

Now let φ_1, φ_2 be two polynomials satisfying (0.1). An analytic question that is natural to ask here is this:
Under what conditions does

$$\hat{N}_{\varphi_1}(x) = \hat{N}_{\varphi_2}(x) \text{ for all } x \gg 1? \tag{0.2}$$

The main result of this article gives a *sufficient* condition, answering (0.2), and which is *cohomological* in nature. This is Theorem 4 (Section 3). The reader should note that the proof of Theorem 4 is, in fact, a simple consequence of the solution of a more general problem that has nothing to do, *a priori,* with the diophantine problem discussed above. This is the following. Assume P is a polynomial on $\mathbb{C}^n$ that satisfies the growth condition

$$\lim_{\substack{\|x\|\to\infty \\ x\in[1,\infty)^n}} |P(x)| = \infty.$$

Thus, one can define

$$I_P(s, \varphi) = \int_{[1,\infty)^n} (1/P)^s \varphi \, dz_1 \cdots dz_n.$$

With the growth condition on P, it is well known that $I_P(s, \varphi)$ is analytic in a halfplane $\operatorname{Re}(s) > B(\varphi)$ and admits an analytic continuation to $\mathbb{C}$ as a meromorphic function with rational poles. Order the poles as $\rho_0'(\varphi) > \rho_1'(\varphi) > \cdots$. For each pole $\rho_j'(\varphi)$ denote the principal part of $I_P(s, \varphi)$ by $\operatorname{Pol}_{s=\rho_j'(\varphi)} I_P(s, \varphi)$.

PROBLEM. Find additional conditions to impose upon P which imply the existence of a finite number of simple arcs $\kappa(i)$ in $\mathbb{C}$ and, for each i, a continuous family of bounded $n-1$ cycles $\gamma_t(i)$ over $\kappa(i)$ such that for each j

$$\operatorname{Pol}_{s=\rho_j'(\varphi)} I_P(s, \varphi) = \operatorname{Pol}_{s=\rho_j'(\varphi)} \Big\{ \sum_i \int_{\kappa(i)} t^{-s} \Big(\int_{\gamma_t(i)} \varphi \, dz_1 \cdots dz_n / dP \Big) \, dt \Big\}. \tag{0.3}$$

Here, a simple arc means an unbounded real analytic arc with no self-intersections that is contained in a region $\{\operatorname{Arg}(t) \in (a, b)\}$ with $b - a < 2\pi$.

A solution of this problem is given in the proof of Theorem 3, whose proof takes up most of Section 3.

To relate the solution of this problem to Dirichlet series, one introduces the notion of a *good pair.*

DEFINITION 1. The pair (P, φ) is good if:
(1) $\rho_0(\varphi) = \rho_0'(\varphi)$:

When equality holds, one denotes the common value by $\rho(\varphi)$.

(2) One has

$$\mathrm{Pol}_{s=\rho(\varphi)} D_P(s,\varphi) = \mathrm{Pol}_{s=\rho(\varphi)} I_P(s,\varphi). \tag{0.4}$$

Section 1 recalls the definitions of 2 classes of good pairs from [Sa-3, Li-1].

The key point in the proof of Theorem 3 is the ability to describe the (homology class of the) chain $[1,\infty)^n$ as a *finite* linear combination of (homology classes of) *Lefschetz thimbles* (cf.(2.2)(d)). The thimbles are constructed by transporting an $(n-1)$-cycle in some smooth fiber $\{P = t_0\}$ over a simple arc in the t plane. They are therefore n chains which are fibered by P. Using standard Mellin transform arguments, the expression for $\mathrm{Pol}_{s=\rho_0'(\varphi)} I_P(s,\varphi)$ is then determined by the asymptotic expansion of periods like those appearing in the right side of (0.3). It is well known that in a neighborhood of infinity such periods are Nilsson functions, whose algebro-geometric significance has been explained in [De, Gr, Nil]. In this way, when (P,φ) is good, one sees that $\hat{N}_\varphi(x)$ is also determined by the behavior of certain periods in a neighborhood of infinity. This implies that $\hat{N}_\varphi(x)$ is a *cohomological invariant* in the sense that its value for $x \gg 1$ only depends upon the cohomology class of the residue form $\varphi\, dz_1 \cdots dz_n / dP$ in the generic fiber of P. This is the essential point of Theorem 4. Earlier results of Cassou-Nogues [C-N] had also suggested that one might be able to give such a meaning, at least to $\hat{N}_\varphi(x)$, for much more general classes of (P,φ). It would seem to be interesting to investigate further this phenomenom and its implications.

The arguments needed for the finite homological description of $[1,\infty)^n$ are based on earlier work of Broughton [Bro-1,2,3], which is briefly recalled in Section 2, and especially Pham [P-1,2], whose motivation was the work of Malgrange [Ma-1,2]. Part of Section 3 extends, in a straightforward way, Pham's ideas to a *relative* homology with supports setting, which is more appropriate for the problem addressed here. The homological methods for general n, used here, are to be contrasted with the concrete geometric construction in [Li-2], where question (0.2) was answered when $n = 2$. It appears to be difficult to give a similar geometric construction when $n > 2$.

I am grateful to Prof. Lê, for helpful conversations.

1. **Good Pairs.** The analytical description of $\mathrm{Pol}_{s=\rho_0(\varphi)} D_P(s,\varphi)$ can be quite complicated (cf. [Sa-1,Li-1]). However, there are two classes of good pairs that have been discovered so far. These will now be defined.

CLASS (I). Define $Q, \Phi \in \mathbb{R}[x_1,\ldots,x_n]$ by the following equations and condition.

$$P(1/x_1,\ldots,1/x_n) = \frac{Q(x_1,\ldots,x_n)}{x_1^{M_1}\cdots x_n^{M_n}}, \tag{1.1}$$

$$\varphi(1/x_1,\ldots,1/x_n)d(1/x_1)\cdots d(1/x_n) = \frac{\Phi(x_1,\ldots,x_n)}{x_1^{m_1}\cdots x_n^{m_n}}\,dx_1\cdots dx_n,$$

such that

$$\gcd(Q, x_1\cdots x_n) = \gcd(\Phi, x_1\cdots x_n) = 1.$$

For any polynomial $H = \sum_I a_I x^I, I \in \mathbb{Z}_+^n$, define

$$\operatorname{supp}(H) = \{I : a_I \neq 0\}.$$

One assigns to H its Newton polyhedron at the origin, denoted $\Gamma_0(H)$, and defined to be the boundary of the convex hull of $\bigcup_{\{I\in\operatorname{supp}(H)\}}(I+\mathbb{R}_+^n)$. Class (I) uses the nondegeneracy condition for polynomials with given polyhedron at the origin defined in [Var, sec. 9].

Impose these conditions upon Q, Φ.

(1.2)
- i) Each coefficient of Q is positive;
- ii) $Q(0,\dots,0) = 0$;
- iii) Q is a convenient polynomial in the sense of [Ku];
- iv) Φ is nondegenerate with respect to $\Gamma_0(\Phi)$;
- v) On $[0,1]^n$ either $\Phi \geq 0$ or $\Phi \leq 0$.

It is clear that P has positive coefficients iff Q has positive coefficients.

NOTE 1. The proof of Theorem 2 [Li-1] shows that if (1.2)(i–v) are satisfied, then (P,φ) is a good pair. ■

CLASS (II). The advantages of Class (II) are these. First, one finds good pairs without forcing P to have positive coefficients. Secondly, the characterization is formulated without compactifying $\mathbb{C}^n$ as in (1.1).

Write $P(z_1,\dots,z_n) = \sum_I b_I z^I$. Define the Newton polyhedron of P at infinity, denoted $\Gamma_\infty(P)$, to be the boundary of the convex hull of $\bigcup_{\{I\in\operatorname{supp}(P)\}}(I-\mathbb{R}_+^n)$. Let $\mathcal{S}(P) = \{I \in \operatorname{supp}(P)\} \cap \Gamma_\infty(P)$. Define the principal part of P,

$$P_{\Gamma_\infty}(z_1,\dots,z_n) = \sum_{I\in\mathcal{S}(P)} z^I.$$

DEFINITION 2. $P \in \mathbb{R}[z_1,\dots,z_n]$ is non-degenerate with respect to $\Gamma_\infty(P)$ on $[1,\infty)^n$ if there exists $C > 0$ such that for $(z_1,\dots,z_n) \in [1,\infty)^n$

$$P(z_1,\dots,z_n) \geq C P_{\Gamma_\infty}(z_1,\dots,z_n). \tag{1.3}$$

REMARKS 1.4.

(1) If P has positive coefficients, then P is evidently non-degenerate with respect to $\Gamma_\infty(P)$.

(2) If P is nondegenerate with respect to $\Gamma_\infty(P)$, then the polynomial Q, defined in (1.1), is nondegenerate with respect to $\Gamma_0(Q)$ [Gin, Theorem 1.5].

(3) For $\alpha > 0$, define the cone

$$\Gamma(\alpha) = \{(x_1,y_1,\dots,x_n,y_n) \in \mathbb{R}^{2n} : |y_i| \leq \alpha(x_i-1),\ x_i \geq 1, \text{ for each i}\}.$$

Using a toroidal modification of $\mathbb{C}^n$, determined by dualizing $\Gamma_\infty(P)$, [Sa-1, Da-Kh], Sargos (cf. [Sa-3, Lemma 5.2]) shows that if P is non-degenerate, then there exists $\alpha, C, C', \theta > 0$ such that for all $(z_1,\dots,z_n) = (x_1,y_1,\dots,x_n,y_n) \in \Gamma(\alpha)$

$$|\operatorname{Re}(P)(z_1,\dots,z_n)| \geq C P_{\Gamma_\infty}(|z_1|,\dots,|z_n|) \geq C'|z_1\cdots z_n|^\theta.$$ ■

Given a monomial $z_1^{d_1} \dots z_n^{d_n}$, one sets

$$\Delta(d_1, \dots, d_n) = \{ t \cdot (d_1, \dots, d_n) : t \geq 0 \}.$$

Let $\tau(d_1, \dots, d_n)$ denote the value t_0 for which

$$\{ t_0 \cdot (d_1, \dots, d_n) \} = \Gamma_\infty(P) \cap \Delta(d_1, \dots, d_n).$$

Let $\bar{1} = (1, \dots, 1)$.

If $\varphi(z_1, \dots, z_n) = \sum_I c_I z^I$ set

$$\begin{aligned} \tau(\varphi) &= \min\{ \tau(I + \bar{1}) : c_I \neq 0 \}, \\ \mathcal{L}(\varphi) &= \{ I \in \operatorname{supp}(\varphi) : \tau(I + \bar{1}) = \tau(\varphi) \}, \end{aligned} \tag{1.5}$$

and

$$\varphi_\Gamma(z_1, \dots, z_n) = \sum_{I \in \mathcal{L}(\varphi)} c_I z^I.$$

DEFINITION 3. φ is well-situated with respect to $\Gamma_\infty(P)$ if each closed face (of dimension at most $n-1$) of $\Gamma_\infty(P)$ that intersects $\bigcup_{I \in \mathcal{L}(\varphi)} \Delta(I + \bar{1})$ is compact (that is, does not intersect any face parallel to any coordinate plane). P is well situated if $\varphi \equiv 1$ is well situated with respect to $\Gamma_\infty(P)$.

Impose these conditions upon (P, φ).

(1.6)
- i) P is non-degenerate with respect to $\Gamma_\infty(P)$;
- ii) φ is non-degenerate with respect to $\Gamma_\infty(\varphi)$;
- iii) φ is well-situated with respect to $\Gamma_\infty(P)$.

NOTE 2. The proof of Theorem 5.4 [Sa-3] shows that if (1.6)(i–iii) are satisfied, then (P, φ) is a good pair. ∎

REMARK 1.7. The non-degeneracy condition used in (1.2) resp. (1.6) is an open condition on the space of polynomials g with given Newton polyhedron $\Gamma_0(g)$ resp. $\Gamma_\infty(g)$. ∎

EXAMPLE 1.8. A simple class of well-situated polynomials is the following.

Let $\nu_1, \dots, \nu_n \in \mathbb{N}$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$. To each j assign the vector $\nu_j e_j = (0, \dots, 0, \nu_j, 0, \dots, 0)$.

If $\emptyset \neq J \subset \{1, \dots, n\}$, set $J = \{j_1, \dots, j_\ell\}$. Set $\boldsymbol{\nu}(J)$ to be the $(\ell - 1)$-simplex spanned by $\nu_{j_1} e_{j_1}, \dots, \nu_{j_\ell} e_{j_\ell}$. If $I = (i_1, \dots, i_n)$, set $I_J = (i_{j_1}, \dots, i_{j_\ell})$. One says that I_J lies on or below the simplex $\boldsymbol{\nu}(J)$ if $\sum_{u=1}^{\ell} \frac{i_{j_u}}{\nu_{j_u}} \leq 1$. Now set

$$\begin{aligned} \mathcal{J} = \{ I = (i_1, \dots, i_n) : |I| > 1 \text{ and for each } \emptyset \neq J \subset \{1, \dots, n\}, \\ I_J \text{ lies on or below } \nu(J) \}. \end{aligned}$$

Set

$$P(z_1, \dots, z_n) = \sum_{i=1}^{n} a_i z_i^{\nu_i} + \sum_{I \in \mathcal{J}} a_I z^I,$$

where $a_1 \cdots a_n \neq 0$. It is easy to see that P is well-situated. ∎

REMARK 1.9. If P belongs to class (I) or (II), a simple expression for $\rho(\varphi)$ is known to be the following, using the notation from (1.5):

$$\rho(\varphi) = 1/\tau(\varphi).$$ ∎

REMARK 1.10. It is clear that the following property holds. If (P, φ) belongs either to class (I) or (II) and if η is a sufficiently small positive number, then $(P + \eta, \varphi)$ also belongs to the same class. This trivial observation will be used in Section 3. ∎

2. **Tame Polynomials.** A condition that will be important for Theorems 3, 4 is the following (cf. [Bro-1]).

DEFINITION 4. A polynomial $P(z_1, \ldots, z_n)$ is tame if there is a compact neighborhood U of the singular locus Σ_P of P and a constant $B > 0$ such that

$$\|\operatorname{grad} P(x)\| \geq B$$

for all $x \in \mathbb{C}^n - U$.

REMARK 2.1. The singular locus of a tame polynomial consists only of finitely many critical points. This follows immediately from the fact that Σ_P is a compact algebraic set in $\mathbb{C}^n$. Thus, the class of tame polynomials is the global analogue to the class of germs of analytic functions with isolated critical points. ∎

A numerical criterion that is equivalent to tameness was found by Broughton [Bro-2]. For the polynomial P and $w \in \mathbb{C}^n$ set

$$P^w(z_1, \ldots, z_n) = P(z_1, \ldots, z_n) + \sum_{i=1}^{n} w_i z_i.$$

Also, if Q is any polynomial let

$$\mu(Q) = \dim_{\mathbb{C}} \mathbb{C}[z_1, \ldots, z_n]/JQ$$

where JQ is the Jacobean ideal of Q.

THEOREM B. *P is tame iff $\mu(P) < \infty$ and $\mu(P^w) = \mu(P)$ for all w sufficiently close to $\bar{0}$.*

NOTE 3. In this paper we will only consider the tame polynomials P for which $\mu(P) > 0$. ∎

REMARKS 2.2. The following topological constructions/properties will be used below.

(a) (Verdier [Ve]) For any polynomial mapping P, there is a finite set $\Sigma \subset \mathbb{C}$ such that $P\colon \mathbb{C}^n - P^{-1}(\Sigma) \longrightarrow \mathbb{C} - \Sigma$ is a locally trivial fibration. Set $\mathbb{C}^* = \mathbb{C} - \Sigma$, and define $P^* = P\big|_{P^{-1}(\mathbb{C}^*)}$. For $t \in \mathbb{C}^*$ set $X_t = P^{-1}(t)$. Let $\mathbb{H}^{n-1}$ denote the flat vector bundle on $\mathbb{C}^*$ with fiber at t equal to the finite dimensional vector space $H^{n-1}(X_t, \mathbb{C})$. Let

$\mathcal{H}^{n-1} = \mathbb{H}^{n-1} \otimes \mathcal{O}_{\mathbb{C}^*}$ be the sheaf of germs of analytic sections of $\mathbb{H}^{n-1}$. Any rational differential n-form ω determines an analytic section of $\mathcal{H}^{n-1}$, defined as

$$\sigma(\omega)\colon t \longrightarrow [\omega / dP|_{X_t}]$$

where $\omega / dP\big|_{X_t} = \operatorname{Res}\left(\frac{\omega}{P-t}\right)\Big|_{X_t}$.

Similarly, P^* determines a homology bundle $\mathbb{H}_{n-1}$, whose fiber at t equals $H_{n-1}(X_t, \mathbb{C})$. If t_0 is any point of $\mathbb{C}^*$ and $\gamma_0 \in H_{n-1}(X_{t_0}, \mathbb{C})$, one can construct a continuous (multi-valued) section of $\mathbb{H}_{n-1}$ by using parallel transport in the fibers of P^*. This section is denoted γ_t. If κ is any simple arc from t_0 to ∞, then there exists a unique continuous section of $\mathbb{H}_{n-1}$ over κ which equals γ_0 at t_0.

Proofs of the following can be found in [Bro-1,2].

(b) Let $\Sigma_P = \{t_1, \ldots, t_d\}$ denote the set of critical values of P. Assume P is tame. Let $y \in \Sigma_P$ and $U(y)$ a closed disc such that $U(y) \cap \Sigma_P = \{y\}$. Then $P^{-1}(y)$ is a deformation retract of $P^{-1}\,U(y)$.

(c) Let x be a singular point of P and μ_x the local Milnor number at x. Set $\mu(y) = \sum_{x \in P^{-1}(y)} \mu_x$. If $y \notin \Sigma_P$, $\mu(y) = 0$. Set $\mu = \sum_{y \in \mathbb{C}} \mu(y)$. If P is tame, then $\mu(y) < \infty$. For any $z \in \mathbb{C}$ one then has

$$\tilde{H}_q\big(P^{-1}(z), \mathbf{Z}\big) = \begin{Bmatrix} 0 & q \neq n-1 \\ \mathbf{Z}^{\mu-\mu(z)} & q = n-1 \end{Bmatrix}.$$

(d) There is a geometric description of (c) as follows. For each i, let U_i be a closed disc centered at $t_i \in \Sigma_P$ such that $U_i \cap U_j = \emptyset$, $i \neq j$. Let σ_i be a smooth path from a fixed $z \notin \Sigma_P$ to t_i. Set $|\sigma_i|$ to denote the points on the path. Choose the paths σ_i so that

i) $|\sigma_i| \cap |\sigma_j| = \{z\}$;

ii) $|\sigma_i| \cap U_j = \emptyset \quad i \neq j$;

iii) $|\sigma_i| \not\subset U_i$ and $|\sigma_i| \cap U_i$ has cardinality 1.

Define $Y(i) = P^{-1}(U_i \cup |\sigma_i|)$ and $Y_z = P^{-1}(z)$. The inclusion $\varphi(i,z)\colon Y_z \hookrightarrow Y(i)$ induces the homomorphisms $\varphi_*(i,z)\colon H_{n-1}(Y_z) \longrightarrow H_{n-1}(Y(i))$ and the relative homology group $H_n(Y(i), Y_z)$. Set $\psi_{n-1}(i) = \ker[\varphi_*(i,z)\colon H_{n-1}(Y_z) \longrightarrow H_{n-1}(Y(i))]$ for each i. This is the vanishing homology (along σ_i). It is generated by Lefschetz thimbles [P-1].

(e) By (a), (b), there is a map for each i

$$\delta_i\colon Y_z \longrightarrow Y_{t_i}.$$

Then

$$\psi_{n-1}(i) = \ker[(\delta_i)_*\colon H_{n-1}(Y_z) \longrightarrow H_{n-1}(Y_{t_i})].$$

In addition, there are isomorphisms for each i

$$\psi_{n-1}(i) \cong H_n\big(Y(i), Y_z\big) \text{ and } \tilde{H}_{n-1}(Y_z) \cong \bigoplus_{i=1}^{d} H_n\big(Y(i), Y_z\big).$$

3. **Main results** One will assume that $P \in \mathbb{C}[z_1, \ldots, z_n]$ satisfies the following properties.

(1) P is a tame polynomial.

(2) There exist a cone $\Gamma(\alpha)$ and $\theta > 0$ (cf. (1.4)(3)) such that

$$(z_1, \ldots, z_n) \in \Gamma(\alpha) \text{ implies } |\operatorname{Re} P(z_1, \ldots, z_n)| \gg |z_1 \cdots z_n|^{\theta}. \tag{3.1}$$

Evidently (2) says that $P|_{\Gamma(\alpha)}$ is proper. Now define

$$D = \bigcup_{i=1}^{n} \{z_i = 1\} \text{ and } D_\alpha = D \cap \Gamma(\alpha).$$

It is clear that one may view the chain $[1, \infty)^n$ as an n-chain with boundary in D_α.

Now define the following collection of *closed* sets, forming a support family for homology groups,

a) $\mathcal{P} = \{A \subset \mathbb{C}^n : P\big|_A \text{ is proper}\}$

b) $\Phi = \{A :$ There exists a semi-analytic set B, $A \subset B$, so that $\big|P|_B(z)\big| \longrightarrow \infty$ as $|z| \longrightarrow \infty\}$.

The collection of sets which will actually be used in Theorem 3 is

c) $[\Phi] = \{A : A$ is semi-analytic and $\big|P|_A(z)\big| \longrightarrow \infty$ as $|z| \longrightarrow \infty\}$.

REMARK 3.2. It is clear that $[1, \infty)^n$ is an element of $[\Phi]$. This is different from (i.e. more restrictive than) the situation considered by Pham in [P-1,2], in which no growth condition was imposed upon P on the n-chain $\mathbb{R}^n$. Our analysis can therefore follow Pham's with one evident modification—that in the situation of interest here, one deals with an n-chain with boundary. In [ibid.], the chain of interest had no boundary. Because the boundary of $[1, \infty)^n$ is contained inside D_α, on which P also satisfies (3.1), it will be fairly easy to show that relative (to D_α) homology with supports and absolute homology with supports are isomorphic. Nonetheless, we have described the topological decomposition of the chain $[1, \infty)^n$ (Theorem 2) with a fair amount of supplementary material not given or briefly sketched in [P-1,2]. This is because the methods appear to be both interesting and useful for other problems and so, seemingly deserving of attention. It also appears that this type of discussion is not so easy to find detailed in the literature, so perhaps an example can be useful. ∎

To begin the discussion, leading to Theorem 2, one first observes that the collection $[\Phi]$ is not a support family for a homology theory—not every closed subset of a semi-analytic set is semi-analytic. So one cannot work so easily with this collection alone. However, one wants to show that the Lefschetz thimbles into which $[1, \infty)^n$ will be decomposed (at the level of homology classes mod D_α) do in fact lie in $[\Phi]$. This is needed because one will have to deal analytically with relative cycles. A very useful tool will then be a version of Stokes' theorem due to Herrera [He] that will allow one to kill off the contribution from the boundary. Now, there is a boundary map between chains of semi-analytic sets of given dimension q whose image is semi-analytic of dimension $q - 1$. This is discussed in [ibid.]. Thus, one *can* form a chain complex on $\mathbb{C}^n$ whose cycle class representatives have support in $[\Phi]$. A natural goal is to show that this complex is quasi-isomorphic to the complex which uses chains with support in Φ. This is proved in Proposition 2.

The following construction will be of basic use. Given the polynomial P, one can compactify $\mathbb{C}^n$ by an open embedding $\iota\colon \mathbb{C}^n \hookrightarrow Z$ such that the following properties hold.

(3.3)
- a) Z is a smooth algebraic variety
- b) Let $H_\infty = Z - \iota(\mathbb{C}^n)$ and $\bar{D} =$ closure of $\iota(D)$ in Z. Then $H_\infty \cup \bar{D}$ is a divisor with smooth irreducible components and locally normal crossings.
- c) P extends to give a regular map $\hat{P}\colon Z \longrightarrow P^1(\mathbb{C})$ such that $Y_\infty = \hat{P}^{-1}(\infty) \subset H_\infty$.
- d) Define
  - i) $W_\infty = H_\infty \cap \bar{D}$
  - ii) $C = \{$ irreducible components E of $H_\infty \cup \bar{D}$ such that $E \not\subseteq Y_\infty\}$

  and set $X_\infty = \cup_{E \in C} E$.

Then

$$H_\infty \cup \bar{D} = W_\infty \cup X_\infty \cup Y_\infty \cup \iota(D).$$

In the following, set $\bar{D}_\alpha = \overline{\iota(D_\alpha)}$ in $\bar{D}$. Then $\bar{D}_\alpha$ is a semi-algebraic set in Z.

REMARK 3.4. It will follow from the following discussion that the constructions below will not depend upon the choice of Z so that (3.3)(a–d) hold. ■

Now define these collections of *closed* sets in Z.

a) $\Phi' = \{A' :$ there exists a semi- analytic set $B', A' \subset B'$, and $A' \cap \mathbb{C}^n \in \Phi\}$.

b) $[\Phi'] = \{A' : A'$ is semi- analytic and $A' \cap \mathbb{C}^n \in [\Phi]\}$.

Let F resp. F' denote either Φ or $[\Phi]$ resp. Φ' or $[\Phi']$. For any subset $Y \subset Z$ define $F'|Y = \{C \in F' : C \subset Y\}$. Set $C_q^{F'|Y}(Y)$ to be the free $\mathbb{Z}$-module generated by q-dimensional chains with support in $F'|Y$. There is a boundary morphism (cf. [He] for $[\Phi']$)

$$\partial_q\colon C_q^{F'|Y}(Y) \longrightarrow C_{q-1}^{F'|Y}(Y).$$

Let $H_q^{F'|Y}(Y)$ denote the q^{th} homology group of the complex $(C^{F'|Y}(Y), \partial.)$.

NOTATION. For simplicity, in the following, if Y resp. Y' is a subset of $\mathbb{C}^n$ resp. Z the groups $H_q^{F|Y}(Y)$ resp. $H_q^{F'|Y'}(Y')$ will be denoted $H_q^F(Y)$ resp. $H_q^{F'}(Y')$. ■

Given a triple $(X \subset Y \subset Z)$ there is the long exact sequence in relative homology with supports in F'.

$$(3.5)\qquad \cdots \longrightarrow H_q^{F'}(Y,X) \longrightarrow H_q^{F'}(Z,X) \longrightarrow H_q^{F'}(Z,Y) \longrightarrow H_{q-1}^{F'}(Y,X) \longrightarrow \cdots$$

The basic objects of study are:

(1) The long exact sequence (3.5) for the triple $(H_\infty, H_\infty \cup \bar{D}_\alpha, Z)$ at $q = n$:

$$\cdots \xrightarrow{\partial} H_n^{\Phi'}(H_\infty \cup \bar{D}_\alpha, H_\infty) \longrightarrow H_n^{\Phi'}(Z, H_\infty) \longrightarrow H_n^{\Phi'}(Z, H_\infty \cup \bar{D}_\alpha) \xrightarrow{\partial} \cdots;$$

and

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(2) the commutative diagram induced from the inclusion of triples $(\emptyset, D_\alpha, \mathbb{C}^n) \xhookrightarrow{\iota}$ $(H_\infty, H_\infty \cup \bar{D}_\alpha, Z)$ and the definitions of Φ, Φ':

$$
(3.6)\qquad
\begin{array}{ccccccccc}
\xrightarrow{\partial} & H_q^{\Phi}(D_\alpha) & \longrightarrow & H_q^{\Phi}(\mathbb{C}^n) & \longrightarrow & H_q^{\Phi}(\mathbb{C}^n, D_\alpha) & \xrightarrow{\partial} & \cdots \\
\vdots & \downarrow & & \downarrow & & \downarrow & \vdots & \cdots \\
\xrightarrow{\partial} & H_q^{\Phi'}(H_\infty \cup \bar{D}_\alpha, H_\infty) & \longrightarrow & H_q^{\Phi'}(Z, H_\infty) & \longrightarrow & H_q^{\Phi'}(Z, H_\infty \cup \bar{D}_\alpha) & \xrightarrow{\partial} & \cdots
\end{array}
$$

The first important result is

THEOREM 1. *For each* $q = 1, 2, \ldots,$
(a) $H_q^{\Phi'}(H_\infty \cup \bar{D}_\alpha, H_\infty) \simeq H_q^{\Phi'}(\bar{D}_\alpha, \bar{D}_\alpha \cap H_\infty) \simeq H_q^{\Phi}(D_\alpha) = 0.$
(b) $H_q^{\Phi'}(Z, H_\infty) \simeq H_q^{\Phi}(\mathbb{C}^n).$

Thus, Theorem 1 and (3.6) imply immediately with $q = n$

COROLLARY 1. $H_n^{\Phi'}(Z, H_\infty \cup \bar{D}_\alpha) \simeq H_n^{\Phi}(\mathbb{C}^n, D_\alpha)$ *and* $H_n^{\Phi}(\mathbb{C}^n) \simeq H_n^{\Phi}(\mathbb{C}^n, D_\alpha)$.

The proof of Theorem 1 uses the following result. In the situation of (3.3), let A be a semi-algebraic subset of $\mathbb{C}^n$ such that

$$\lim_{\substack{|z|\to\infty \\ z\in A}} |P(z)| = \infty.$$

Let $\bar{A}$= closure of A in Z. Set $P_A = P|_A$ and $\hat{P}_A = \hat{P}|_{\bar{A}}$, the extension. Evidently, $\hat{P}_A^{-1}(\infty) = \bar{A} \cap H_\infty$ and is a closed semi-algebraic subset of $\bar{A}$.

PROPOSITION 1. *There exists a closed disc D centered at ∞ in $P^1(\mathbb{C})$ such that $\hat{P}_A^{-1}(\infty)$ is a deformation retract of $\hat{P}_A^{-1}(D)$.*

PROOF. For any semi-algebraic closed disc U about ∞, $\hat{P}_A^{-1}(U)$ is a semi-algebraic subset of $\bar{A}$. Thus the pair $\big(\hat{P}_A^{-1}(U), \hat{P}_A^{-1}(\infty)\big)$ is an N.D.R pair (neighborhood deformation retract). This follows from the triangulability of semi-algebraic sets. Secondly, there exits a closed disc D about ∞ such that

$$\hat{P}_A : \hat{P}_A^{-1}(D) - \hat{P}_A^{-1}(\infty) \longrightarrow D - \{\infty\}$$

is a locally trivial fibering. This is because P_A is a proper map.

There is a standard retraction lemma in homotopy theory which can now be applied [Bro-3, p. 94]:

LEMMA 1. *Let $p: X \longrightarrow Y$ be a proper map onto the paracompact space Y. Let B be a closed subspace of Y and set $A = p^{-1}(B)$. Suppose that (X, A) is an N.D.R. pair and that $p: X - A \longrightarrow Y - B$ is a locally trivial fibering. Suppose further that B has a cofinal system of neighborhoods $\mathcal{U}$ such that if $U \in \mathcal{U}$ then ∂U is a deformation retract of $\overline{Y - U}$. Then A is a deformation retract of X.*

Proposition 1 now follows. ∎

PROOF OF THEOREM 1. By (3.3)(b), it is easy to construct a closed neighborhood $\mathcal{T}$ of $H_\infty \cap \bar{D}_\alpha$ in H_∞ such that $H_\infty \cap \bar{D}_\alpha$ is a deformation retract of $\mathcal{T}$. The homotopy

$h\colon [0,1] \times \mathcal{T} \longrightarrow \mathcal{T}$ is definable so that for each t, h_t leaves invariant $\Phi'|H_\infty \cup \bar{D}_\alpha$. Thus, excising $H_\infty - \mathcal{T}$, one has for each q

$$H_q^{\Phi'}(H_\infty \cup \bar{D}_\alpha, H_\infty) \simeq H_q^{\Phi'}(\bar{D}_\alpha, \bar{D}_\alpha \cap H_\infty).$$

To show the second isomorphism in (a), set $P_\alpha = P\big|_{D_\alpha}$. By (3.1), one notes that the extension $\hat{P}_\alpha = \hat{P}\big|_{\bar{D}_\alpha}$ satisfies the condition

$$\hat{P}_\alpha^{-1}(\infty) = \bar{D}_\alpha \cap H_\infty. \tag{3.7}$$

Let $U_\epsilon(\infty)$ be the closed disc of radius ϵ centered at ∞ in $P^1(\mathbb{C})$. By Proposition 1, the contraction of $U_\epsilon(\infty)$ to ∞ can be lifted over $\hat{P}_\alpha$ to give a deformation retract of $\hat{P}_\alpha^{-1}\big(U_\epsilon(\infty)\big)$ to $\bar{D}_\alpha \cap H_\infty$. Since (3.1) implies (3.7), one observes that the homotopy leaves invariant $\Phi'|\hat{P}_\alpha^{-1}\big(U_\epsilon(\infty)\big)$. Thus, one concludes

$$H_q^{\Phi'}(\bar{D}_\alpha, \bar{D}_\alpha \cap H_\infty) \simeq H_q^{\Phi'}\Big(\bar{D}_\alpha, \hat{P}_\alpha^{-1}\big(U_\epsilon(\infty)\big)\Big). \tag{3.8}$$

By excision, one has

$$H_q^{\Phi'}\Big(\bar{D}_\alpha, \hat{P}_\alpha^{-1}\big(U_\epsilon(\infty)\big)\Big) \simeq H_q^{\Phi'}\Big(\bar{D}_\alpha - \hat{P}_\alpha^{-1}\big(U_{\epsilon/2}(\infty)\big), \hat{P}_\alpha^{-1}\big(U_\epsilon(\infty) - U_{\epsilon/2}(\infty)\big)\Big).$$

Now let D_r denote the disc of radius r centered at 0 in the finite chart $\mathbb{C}$ of $P^1\mathbb{C}$.
By (3.1) and the definition of Φ' one notes that

$$\begin{aligned} H_q^{\Phi'}\Big(\bar{D}_\alpha - \hat{P}_\alpha^{-1}\big(U_{\epsilon/2}(\infty)\big), \hat{P}_\alpha^{-1}\big(U_\epsilon(\infty) - U_{\epsilon/2}(\infty)\big)\Big) \\ \simeq H_q\big(D_\alpha - P_\alpha^{-1}(D^c_{2/\epsilon}), P_\alpha^{-1}(D^c_{1/\epsilon} - D^c_{2/\epsilon})\big) \\ \simeq_{\text{excision}} H_q\big(D_\alpha,\ P_\alpha^{-1}(D^c_{1/\epsilon})\big), \text{ for each } q. \end{aligned}$$

The family of groups $H_q\big(D_\alpha, P_\alpha^{-1}(D^c_{1/\epsilon})\big)$ forms a direct system via the homomorphism induced by inclusion when $\epsilon_1 < \epsilon_2$

$$\big(D_\alpha, P_\alpha^{-1}(D^c_{1/\epsilon_1})\big) \hookrightarrow \big(D_\alpha, P_\alpha^{-1}(D^c_{1/\epsilon_2})\big).$$

The above isomorphisms show that for ϵ sufficiently small this direct system stabilizes, that is,

$$H_q\big(D_\alpha, P_\alpha^{-1}(D^c_{1/\epsilon})\big) \simeq H_q\big(D_\alpha, P_\alpha^{-1}(D^c_{1/\epsilon'})\big)$$

for $\epsilon, \epsilon' \ll 1$.

This implies that the chain complexes

$$\big\langle C_\bullet\big(D_\alpha, P_\alpha^{-1}(D^c_{1/\epsilon_1})\big), \partial\big\rangle, \big\langle C_\bullet\big(D_\alpha, P_\alpha^{-1}(D^c_{1/\epsilon_2})\big), \partial\big\rangle$$

are homotopic for $\epsilon_1, \epsilon_2 \ll 1$.

Now, one has by definition

$$\langle C_\bullet^\Phi(D_\alpha), \partial\rangle = \varinjlim_{\epsilon \to 0} \big\langle C_\bullet\big(D_\alpha, P_\alpha^{-1}(D^c_{1/\epsilon})\big), \partial\big\rangle.$$

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By definition

$$H_q^{\Phi}(D_\alpha) = H_q\big(\langle C_\bullet^{\Phi}(D_\alpha), \partial\rangle\big).$$

So, one concludes, for each q,

$$\begin{aligned}(3.9)\qquad H_q^{\Phi}(D_\alpha) &\simeq \varinjlim_{\epsilon\to 0} H_q\Big(\big\langle C_\bullet\big(D_\alpha, P_\alpha^{-1}(D_{1/\epsilon}^c)\big), \partial\big\rangle\Big) \\ &= H_q\big(D_\alpha, P_\alpha^{-1}(D_{1/\epsilon}^c)\big)\end{aligned}$$

for any $\epsilon \ll 1$.

Thus, $H_q^{\Phi}(D_\alpha) = H_q^{\Phi'}(\bar{D}_\alpha, \bar{D}_\alpha \cap H_\infty)$.

To show

$$(3.10)\qquad H_q^{\Phi}(D_\alpha) = 0 \text{ for each } q$$

it suffices to observe that D_α is contractible to the point $(a, \dots, a)$ by means of a homotopy that preserves $\Phi | D_\alpha$. This is because (3.1) applies to all $z \in D_\alpha$.

Note that using all of D in place of D_α would not allow one to conclude $H_q^{\Phi}(D) = 0$. This is because (3.1) is only assumed to apply within $\Gamma(\alpha)$. Combining (3.8)–(3.10) completes the proof of part (a). ∎

Implicit in part (b) is a situation different from that in (a). One cannot *a priori* assert $H_q^{\Phi}(\mathbb{C}^n) = 0$ simply because $\mathbb{C}^n$ is contractible. This is because the homotopy need not preserve Φ.

The proof of (b) is, however, a straightforward modification of Pham's arguments in [P-2, p. 42–44]. The choice of Φ here, however, simplifies certain parts of his argument. A brief sketch of the proof is presented for the convenience of the reader.

Pham used the family of supports Ψ defined as follows. For fixed θ and any $c > 0$ define

$$S_c^- = \{t \in \mathbb{C} : \operatorname{Re}\langle t, e^{-i\theta}\rangle \le c\}.$$

Set

$$\Psi = \{A \subset \mathbb{C}^n : A \text{ is closed and for each } c > 0,\ A \cap P^{-1}(S_c^-) \text{ is compact}\}.$$

Given a compactification Z of $\iota(\mathbb{C}^n)$, as in (3.3), and extension $\hat{P}: Z \longrightarrow P^1(\mathbb{C})$ of P, one defines the family

$$\Psi' = \{A, \text{ closed in } Z : A \cap \iota(\mathbb{C}^n) \in \Psi\}.$$

In the notation of (3.3)(d), set

$$C' = \{\text{irreducible components } E \text{ of } H_\infty \text{ such that } E \not\subseteq Y_\infty\},$$

and

$$X'_\infty = \bigcup_{E \in C'} E.$$

Then $H_\infty = X'_\infty \cup Y_\infty$.

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Since X'_∞ is a normal crossing divisor, one can easily construct a closed neighborhood $\mathcal{V}$ of $X'_\infty \cap Y_\infty$ in X'_∞ for which there exists a deformation retract onto $X'_\infty \cap Y_\infty$.

Set $S = X'_\infty - \mathcal{V}$, $Z' = Z - S$, $Y = Z' - \big(X'_\infty - (X'_\infty \cap Y_\infty)\big) = \iota(\mathbb{C}^n) \cup Y_\infty$, and $H'_\infty = H_\infty - S$.

Then one has for $F = \Psi'$ or Φ' and for each q

$$H_q^F(Z, H_\infty) = H_q^{F|Z'}(Z', H'_\infty) = H_q^{F|Y}(Y, Y_\infty).$$

Consider now $\hat{P}_Y = \hat{P}|_Y$. Let $U_\epsilon(\infty)$ be a closed disc as in part (a). For ϵ sufficiently small the contraction of $U_\epsilon(\infty)$ onto ∞ can be lifted over $\hat{P}_Y$ to give a deformation retract of $\hat{P}_Y^{-1}\big(U_\epsilon(\infty)\big)$ onto Y_∞. This follows from the retraction Lemma 1.

When $F = \Psi'$ the retraction may not preserve $\Psi'|Y$. However the retraction *will* preserve $\Phi'|Y$. This follows directly from the fact that for a set $A \subset Y$ only the growth of $|\hat{P}_Y(z)|$, as $z \in A$ converges to a point in H_∞, determines if $A \in \Phi'|Y$. So, an additional argument is required in [P-2, p. 44] but not here. One concludes by an excision argument as in (a) that for ϵ sufficiently small and each q

$$H_q^{\Phi'}(Y, Y_\infty) = H_q^{\Phi'}\Big(Y - \hat{P}_Y^{-1}(U_{\epsilon/2}(\infty)),\, \hat{P}_Y^{-1}\big(U_\epsilon(\infty) - U_{\epsilon/2}(\infty)\big)\Big).$$

Let $c =$ family of compact sets (of a space X). Then by definition

$$\Phi'|Y - \hat{P}_Y^{-1}\big(U_{\epsilon/2}(\infty)\big) = c|\mathbb{C}^n - P^{-1}(D^c_{2/\epsilon}).$$

Thus, for any $\epsilon \ll 1$,

$$\begin{aligned} H_q^{\Phi'}(Y, Y_\infty) &= H_q^c\big(\mathbb{C}^n - P^{-1}(D^c_{2/\epsilon}),\, P^{-1}(D^c_{1/\epsilon} - D^c_{2/\epsilon})\big) \\ &=_{\text{excision}} H_q^c\big(\mathbb{C}^n,\, P^{-1}(D^c_{1/\epsilon})\big), \end{aligned}$$

which says

$$H_q^{\Phi'}(Y, Y_\infty) = \varinjlim_{\epsilon \to 0} H_q^c\big(\mathbb{C}^n, P^{-1}(D^c_{1/\epsilon})\big).$$

As in (a), one now considers the complex of chains in $\mathbb{C}^n$ with support in Φ, $\langle C_\bullet^\Phi(\mathbb{C}^n), \partial\rangle$. The inclusions $\big(\mathbb{C}^n, P^{-1}(D^c_{1/r})\big) \hookrightarrow \big(\mathbb{C}^n, P^{-1}(D^c_{1/s})\big)$ are homotopy equivalences when $r < s \ll 1$. Moreover, by definition

$$\begin{aligned} \langle C_\bullet^\Phi(\mathbb{C}^n), \partial\rangle &= \varinjlim_{\epsilon \to 0} \langle C_\bullet\big(\mathbb{C}^n, P^{-1}(D^c_{1/\epsilon})\big), \partial\rangle \\ &= \varinjlim_{\epsilon \to 0} \langle C_\bullet^c\big(\mathbb{C}^n, P^{-1}(D^c_{1/\epsilon})\big), \partial\rangle. \end{aligned}$$

Thus one concludes

$$H_q^\Phi(\mathbb{C}^n) = H_q^{\Phi'}(Y, Y_\infty),$$

and completes the proof of (b). ∎

As pointed out at the beginning of the section, for purposes of the analytical argument in Theorem 3, it is necessary to identify the groups $H_n^\Phi(\mathbb{C}^n)$, $H_n^\Phi(\mathbb{C}^n, D_\alpha)$ with $H_n^{[\Phi]}(\mathbb{C}^n)$, $H_n^{[\Phi]}(\mathbb{C}^n, D_\alpha)$. To this end, it evidently suffices to show

PROPOSITION 2. *For each* q

i) $H_q^{[\Phi]}(\mathbb{C}^n) \simeq H_q^{\Phi}(\mathbb{C}^n)$.

ii) $H_q^{[\Phi]}(D_\alpha) \simeq H_q^{\Phi}(D_\alpha) = 0$.

REMARK 3.11. Before beginning the proof of Proposition 2, it is helpful to recall a theorem of Lojasiewicz. The main result of [Lo] showed that semi-analytic sets admit triangulations by semi-analytic sets. The significance of this for homology with supports in $[\Phi]$ is seen in a standard result from homology theory. As discussed in [Spa., p. 178, Thm. 14], any homology theory of a compact space X that can be defined with (support of) chains in a collection $\mathcal{S}$ of closed sets with the following property (3.12) is necessarily isomorphic to the singular homology of X. In (3.12),the words *subdivision* and *small* are used in the sense of simplicial/singular homology [ibid., p. 173 ff.]

(3.12) For any chain σ, one can find arbitrarily small divisions of σ by chains τ with $|\tau|$ also an element of $\mathcal{S}$.

Now, a simple argument shows that the family $\mathcal{S} = [\Phi']$ satisfies (3.12). This is because by [Lo] any chain σ with support in $[\Phi']$ can be triangulated by semi-analytic chains τ. Evidently, each τ in any such semi-analytic refinement either satisfies

$$|\tau| \subset H_\infty \quad \text{or} \quad |\tau| \cap \iota(\mathbb{C}^n) \neq \emptyset.$$

If the latter, then $|\tau| \subset |\sigma|$ implies $\tau \in [\Phi']$ by the definition of $[\Phi']$. If the former, then it is clear that $\tau \in [\Phi']$. Thus, $[\Phi']$ satisfies (3.12). ■

PROOF OF PROPOSITION 2. By the above remark one concludes in particular the following two properties.

i) If W_1, W_2 are two semi-analytic subsets of Z which are homotopic by a homotopy preserving $[\Phi']$ then for each q

$$H_q^{[\Phi]}(W_1) \simeq H_q^{[\Phi]}(W_2).$$

ii) If $A \subset W$ are semi-analytic subsets of Z, then one can use excision on $H_q^{[\Phi]}(W, A)$.

One now notes that i), ii) were the two essential properties used in the proof of part (b) of Proposition (1). It follows that one can then mimic this proof to show for each q and $\epsilon \ll 1$

$$\begin{aligned} H_q^{[\Phi']}(Z, H_\infty) &\simeq H_q^{[\Phi']}(Y, Y_\infty) \\ &\simeq H_q^{[\Phi]}\big(\mathbb{C}^n - P^{-1}(D^c_{2/\epsilon}), P^{-1}(D^c_{1/\epsilon} - D^c_{2/\epsilon})\big) \\ &\simeq H_q^{[\Phi]}\big(\mathbb{C}^n, P^{-1}(D^c_{1/\epsilon})\big) \simeq H_n^c\big(\mathbb{C}^n, P^{-1}(D^c_{1/\epsilon})\big). \end{aligned}$$

As in proposition (1) one has

$$H_q^{[\Phi]}(\mathbb{C}^n) \simeq \varinjlim_{r \to \infty} H_q^{[\Phi]}\big(\mathbb{C}^n, P^{-1}(D^c_r)\big).$$

So one concludes

$$H_q^{[\Phi]}(\mathbb{C}^n) \simeq H_q^{\Phi}(\mathbb{C}^n).$$

Exactly as in part (a) of Proposition (1), one also concludes by (i) that

$$H_q^{[\Phi]}(D_\alpha) \simeq H_q^{\Phi}(D_\alpha) = 0.$$

This completes the proof of Proposition 2. ∎

Thus, one concludes

COROLLARY 1. *For each q,*

$$H_q^{[\Phi]}(\mathbb{C}^n, D_\alpha) \simeq H_q^{[\Phi]}(\mathbb{C}^n) \simeq H_q^{\Phi}(\mathbb{C}^n).$$

It suffices now to determine the structure of $H_n^{\Phi}(\mathbb{C}^n)$. This was done by Pham when P is a tame polynomial. Let $t_1, \ldots, t_d$ be the critical values of P. There exists a direction ℓ such that the following holds. Let $\delta_1(\tau), \ldots, \delta_d(\tau)$ be the rays in the direction ℓ and starting at $t_1, \ldots, t_d$. Then

(3.13)
i) no two of the arcs $\delta_i(\tau), \delta_j(\tau)$ should intersect,
ii) there exists $B > 0$ so that $\inf_{i,j} |\delta_i(\tau) - \delta_j(\tau)| > B$ for all τ, and
iii) $\delta_j(\tau) \neq 0$ for all $\tau \neq 0$ and each j.

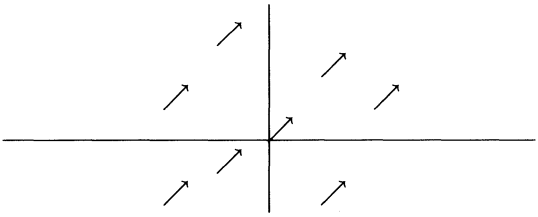

FIGURE 1

It is clear that such a direction vector ℓ exists. One can then enclose each ray δ_i within a closed semi-analytic neighbourhood Ξ_i such that $\Xi_i \cap \Xi_j = \emptyset$ if $i \neq j$. Pham shows [P-2 p. 14, P-1, p. 322].

PROPOSITION 3. *Let $z_i(1), \ldots, z_i(m_i)$ be the singular points with critical value t_i. Let $\mathcal{T}_i(j)$ be the Lefschetz thimble ((2.2)(d)) determined by $z_i(j)$ and path δ_i. For each i*

$$H_n^{[\Phi]}\big(P^{-1}(\Xi_i)\big) = \bigoplus_{j=1}^{m_i} \mathbb{Z}\,[\mathcal{T}_i(j)].$$

Let Δ denote the singular n-chain in $[\Phi]$ whose support is $[1, \infty)^n$. By Proposition 3, one now concludes

THEOREM 2. *There exist integers $n_{i,j}$ such that*

$$[\Delta] = \sum_{i,j} n_{i,j} [\mathcal{T}_i(j)] \tag{3.14}$$

in $H_n^{[\Phi]}(\mathbb{C}^n, D_\alpha)$.

To use (3.14) analytically, one makes precise what it implies at the chain level. Let $\Phi_\alpha = \Phi|D_\alpha$. By definition, if $\partial.$ denotes the differential for the complex $C_.^{[\Phi]}(\mathbb{C}^n)/C_.^{[\Phi_\alpha]}(D_\alpha)$, one has

$$H_n^{[\Phi]}(\mathbb{C}^n, D_\alpha) = \frac{\ker\left\{\partial_n : C_n^{[\Phi]}(\mathbb{C}^n)/C_{n-1}^{[\Phi_\alpha]}(D_\alpha) \longrightarrow C_{n-1}^{[\Phi]}(\mathbb{C}^n)/C_{n-1}^{[\Phi_\alpha]}(D_\alpha)\right\}}{\operatorname{Im}\left\{\partial_{n+1} : C_{n+1}^{[\Phi]}(\mathbb{C}^n)/C_{n+1}^{[\Phi_\alpha]}(D_\alpha) \longrightarrow C_n^{[\Phi]}(\mathbb{C}^n)/C_n^{[\Phi_\alpha]}(D_\alpha)\right\}}.$$

Thus,

$$\Delta - \sum n_{i,j}\mathcal{T}_i(j) = \partial_{n+1}(\xi) + \omega$$

where $\omega \in C_n^{[\Phi_\alpha]}(D_\alpha)$ and $\xi \in C_{n+1}^{[\Phi]}(\mathbb{C}^n)$.

Define the integrand

$$\Theta(z,s) = (1/P)^s \varphi\, dz_1 \cdots dz_n.$$

In $(z_1, \ldots, z_n)$ it is a closed $(n,0)$ form.

LEMMA 2. *Assume P satisfies (3.1). Impose the following condition upon P:*

(3.15) *0 is not a critical value of P.*

Then there exists $B(\varphi) \in \mathbb{R}$ such that for any i,j

(1)

$$\int_\Delta \Theta(z,s) \text{ and } \int_{\mathcal{T}_i(j)} \Theta(z,s)$$

converge absolutely and uniformly in compact subsets of the halfplane $\operatorname{Re}(s) > B(\varphi)$;

(2) Set $\varphi dz = \varphi\, dz_1 \cdots dz_n$. There exists a continuous family of $n-1$ cycles $\gamma_i(j,t)$ such that the following identity holds for $\operatorname{Re}(s) > B(\varphi)$ *and each i,j:*

$$\int_{\mathcal{T}_i(j)} (1/P)^s \varphi\, dz_1 \cdots dz_n = \int_{\delta_i(\tau)} t^{-s}\left(\int_{\gamma_i(j,t)} \varphi dz/dP\right) dt.$$

PROOF OF (1). Using (3.1) and the fact that there exists a positive integer N so that

$$\varphi(z_1, \ldots, z_n) = O(|z_1 \cdots z_n|^N) \text{ as } \|(z_1, \ldots, z_n)\| \longrightarrow \infty, (z_1, \ldots, z_n) \in [1,\infty)^n,$$

it is simple to find $B_1(\varphi) > 0$ so that (1) holds for the first integral.

The argument for each $\mathcal{T}_i(j)$ is a little different. In the following, the notation from (2.2)(d) will be used.

Because P fibers each $\mathcal{T}_i(j)$ over the path δ_i, defined in (3.13), for any *regular* value $t = \delta_i(\tau)$ of P there is a compact cycle $\gamma_i(j,t)$ with support in Y_t and so that $[\gamma_i(j,t)]$ equals the image of $[\mathcal{T}_i(j)]$ under the boundary morphism $\partial : H_n\big(Y(i), Y_t\big) \longrightarrow H_{n-1}(Y_t)$. If $\tau_2 > \tau_1 > 0$ and t runs over the bounded segment $[\delta_i(\tau_1), \delta_i(\tau_2)]$,of the path δ_i, the representative cycle of $\partial\big([\mathcal{T}_i(j)]\big)$ in any fiber Y_t is produced from $\gamma_i\big(j, \delta_i(\tau_1)\big)$ by parallel transport over the segment $[\delta_i(\tau_1), \delta_i(\tau_2)]$. This forms a continuous family of compact cycles. Because the chain $\mathcal{T}_i(j) \cap P^{-1}\big([\delta_i(\tau_1), \delta_i(\tau_2)]\big)$ is compact and $\{t = 0\} \not\subseteq |\delta_i|$, one concludes from Fubini's theorem that for any $\tau_2 > \tau_1 > 0$

$$\int_{\mathcal{T}_i(j)\cap P^{-1}\left([\delta_i(\tau_1),\delta_i(\tau_2)]\right)} \Theta(z,s) = \int_{[\delta_i(\tau_1),\delta_i(\tau_2)]} t^{-s}\left(\int_{\gamma_i(j,t)} \varphi\, dz/\,dP\right) dt.$$

Now there are two cases to consider. First, one fixes τ_1 and lets $\tau_2 \longrightarrow \infty$. Secondly, one fixes τ_2 and lets $\tau_1 \longrightarrow 0$.

For the first case, one uses the fact [Gr-Nil] that there exist $\alpha_1(i,j) > \alpha_2(i,j) > \cdots$ such that

$$\int_{\gamma_i(j,t)} \varphi\, dz/\,dP = \sum_{u=1}^{\infty}\sum_{k=0}^{n} c_{\alpha_u(i,j),k}\, t^{\alpha_u(i,j)} \log^k t \tag{3.16}$$

as $t \longrightarrow \infty$. This is true because each period satisfies a differential equation with regular singular point at $t = \infty$. So it can blow up at infinity at a polynomial rate only. Moreover, the series in (3.16) converges uniformly on the ray from $\delta_i(\tau_1)$ to the point at infinity in the t plane, when τ_1 is sufficiently large. Denote this ray by $[\delta_i(\tau_1), \infty]$. Thus, if $\mathrm{Re}(s) > \alpha_1(i,j) + 2$, one concludes that

$$\lim_{\tau_2\to\infty} \int_{\mathcal{T}_i(j)\cap P^{-1}\left([\delta_i(\tau_1),\delta_i(\tau_2)]\right)} \Theta(z,s) = \int_{[\delta_i(\tau_1),\infty]} t^{-s}\left(\int_{\gamma_i(j,t)} \varphi\, dz/\,dP\right) dt$$

converges absolutely and uniformly on compact subsets of this halfplane. This implies the following identity for $\mathrm{Re}(s) > \alpha_1(i,j) + 2$.

$$\int_{\mathcal{T}_i(j)\cap P^{-1}\left([\delta_i(\tau_1),\infty]\right)} \Theta(z,s) = \int_{[\delta_i(\tau_1),\infty]} t^{-s}\left(\int_{\gamma_i(j,t)} \varphi\, dz/\,dP\right) dt. \tag{3.17}$$

The somewhat artificial assumption (3.15) is needed at this point of the proof.

Since no critical value t_i equals zero, and by construction, δ_i does not pass through 0, one can introduce the coordinates $(x_1, \ldots, x_n)$ from (1.1) and analyze the behavior as $\tau \longrightarrow 0$ of $\int_{\mathcal{T}_i(j)\cap[\delta_i(\tau),\delta_i(\tau_1)]} \Theta(z,s)$ in these coordinates. Write

$$R(x_1, \ldots, x_n) = 1/\,P(1/\,x_1, \ldots, 1/\,x_n),$$
$$u(x_1, \ldots, x_n) = 1/\,x_1^{m_1} \cdots x_n^{m_n}.$$

For τ_1 fixed and $0 < \tau < \tau_1$ arbitrary, one evidently has for any s

$$\int_{\mathcal{T}_i(j)\cap[\delta_i(\tau),\delta_i(\tau_1)]} \Theta(z,s) = \int_{\mathcal{T}_i(j)\cap[\delta_i(\tau),\delta_i(\tau_1)]} R^s(u\,\Phi)\, dx_1 \cdots dx_n.$$

By construction, the chain $\mathcal{T}_i(j) \cap [\delta_i(0), \delta_i(\tau_1)]$ is compact in the $(z_1, \ldots, z_n)$ coordinate chart of $P^1(\mathbb{C}^n)$. Thus, $u\,\Phi$ and R are analytic in an open neighborhood of $\mathcal{T}_i(j) \cap [\delta_i(0), \delta_i(\tau_1)]$ in the $(x_1, \ldots, x_n)$ coordinate chart. Moreover, the chain evidently fibers over R. Thus, it is clear there exists $B_2(\varphi) > 0$ such that $\mathrm{Re}(s) > B_2(\varphi)$ implies

$$\lim_{\tau\to 0} \int_{\mathcal{T}_i(j)\cap[\delta_i(\tau),\delta_i(\tau_1)]} R^s(u\,\Phi)\, dx_1 \cdots dx_n = \int_{\mathcal{T}_i(j)\cap[\delta_i(0),\delta_i(\tau_1)]} R^s(u\,\Phi)\, dx_1 \cdots dx_n, \tag{3.18}$$

where the right side is analytic and absolutely convergent in s.

The only possible value for which this argument fails to apply is $t = 0$, which is excluded. Now define $B(\varphi) = \max_{i,j}\{\alpha_1(i,j) + 2, B_1(\varphi), B_2(\varphi)\}$. Because P is tame, this is a real number (i.e. not $+\infty$). Putting together (3.17),(3.18), one concludes that if $\mathrm{Re}(s) > B(\varphi)$, then

$$\int_{[1,\infty)^n} \Theta(z,s) \text{ and } \int_{\mathcal{T}_i(j)} \Theta(z,s) \text{ converge absolutely and,}$$

$$\int_{\mathcal{T}_i(j)} \Theta(z,s) = \int_{\delta_i(\tau)} t^{-s} \left(\int_{\gamma_i(j,t)} \varphi\, dz/\, dP \right) dt. \tag{3.19}$$

This completes the proof of Lemma 2. ∎

REMARK 3.20. Given any P satisfying the conditions in Lemma 2 except for (3.15), it is clear that for any sufficiently small $\eta > 0$, $P + \eta$ satisfies all the conditions in Lemma 2. It will turn out that this suffices for purposes of proving Theorem 4. ∎

One now observes that because $|\omega| \subset D_\alpha$ and $dz_1 \cdots dz_n\big|_{D_\alpha} = 0$ one also has for any s

$$\int_\omega \Theta(z,s) = 0.$$

One concludes from Lemma 2 that if (3.15) holds and $\mathrm{Re}(s) > B(\varphi)$ then one has the following identity:

$$\int_\Delta \Theta(z,s) - \sum_{i,j} n_{i,j} \int_{\delta_i(\tau)} t^{-s} \left(\int_{\gamma_i(j,t)} \varphi\, dz/\, dP \right) dt = \int_{\partial\xi} \Theta(z,s). \tag{3.21}$$

Now, the left side of (3.21) converges absolutely when $\mathrm{Re}(s) > B(\varphi)$. Thus, the right side of (3.21) converges absolutely when $\mathrm{Re}(s) > B(\varphi)$. One can then apply Herrera's version of Stokes Theorem [He-1, p. 172] to the integral over $\partial\xi$. Because the integrand is a closed $(n,0)$ form and ξ is a semi-analytic chain in $\mathbb{C}^n$, one concludes

$$\int_{\partial\xi} \Theta(z,s) = 0.$$

This proves the first main result of the paper.

THEOREM 3. *Let P be a tame polynomial which satisfies the growth condition (3.1). Assume 0 is not a critical value of P. Let φdz be a polynomial $(n,0)$-form. Let $\rho_0'(\varphi) > \rho_1'(\varphi) > \cdots$ denote the poles of $I_P(s,\varphi)$. Set $\{\delta_i\}$ to be a set of rays as in Figure 1. Then, for each k*

$$\mathrm{Pol}_{s=\rho_k'(\varphi)}\, I_P(s,\varphi) = \sum_{i,j} n_{i,j}\, \mathrm{Pol}_{s=\rho_k'(\varphi)} \int_{\delta_i(\tau)} t^{-s} \left(\int_{\gamma_{i,j}(t)} \varphi dz/\, dP \right) dt.$$

From Theorem 3, one concludes the following solution to the question (0.2), which is the main result of this paper.

THEOREM 4. *Given two polynomial $(n,0)$-forms $\varphi_1 dz$, $\varphi_2 dz$ on $\mathbb{C}^n$ and given the polynomial P on $\mathbb{C}^n$, assume the following conditions hold:*

(3.22)
(1) *P has positive coefficients;*
(2) *P is a tame polynomial on $\mathbb{C}^n$;*
(3) *(P,φ_1), (P,φ_2) are good pairs in the sense of Definition 1;*
(4) *For any $t \in \mathbb{C}^*$ (cf. (2.2)(a)),*

$$\sigma(\varphi_1 dz)(t) = \sigma(\varphi_2 dz)(t).$$

Then the following properties hold:
a) $\rho_0(\varphi_1) = \rho_0(\varphi_2)$;
b) $\hat{N}_{\varphi_1}(x) = \hat{N}_{\varphi_2}(x)$ *for all* $x \gg 1$.

PROOF. Let $\eta \geq 0$ and define $P_\eta = P + \eta$. The only condition η should satisfy is that P_η should not have 0 as a critical value. It is clear that such η exist and can be made arbitrarily small. Let $\rho_0(\eta,\varphi_i)$ denote the first pole of $D_{P_\eta}(s,\varphi_i)$, $i = 1,2$. By Remark (1.10), one has the identity, for i=1,2:

$$\mathrm{Pol}_{s=\rho_0(\eta,\varphi_i)} D_{P_\eta}(s,\varphi_i) = \mathrm{Pol}_{s=\rho_0(\eta,\varphi_i)} \int_{[1,\infty)^n} \frac{1}{P_\eta(z)^s}\varphi\, dz.$$

Evidently, (3.22)(4) implies that for any continuous family of $n-1$ cycles γ_t, in the sense of Lemma 2, one has

$$\int_{\gamma_t} \varphi_1\, dz/\, dP_\eta = \int_{\gamma_t} \varphi_2\, dz/\, dP_\eta, \text{ for all } t \in \mathbb{C}^*.$$

Thus, by Theorem 3 and (3.22)(2),(3) one has these two properties:

(1) $\rho_0(\eta,\varphi_1) = \rho_0(\eta,\varphi_2)$.
Denote the common value by $\rho_0(\eta)$.
(2) $\mathrm{Pol}_{s=\rho_0(\eta)} D_{P_\eta}(s,\varphi_1) = \mathrm{Pol}_{s=\rho_0(\eta)} D_{P_\eta}(s,\varphi_2)$.
A simple calculation that is left to the reader shows this:

LEMMA 3. *Assume that* $\mathrm{Pol}_{s=\rho_0(\eta)} D_{P_\eta}(s,\varphi_1) = \mathrm{Pol}_{s=\rho_0(\eta)} D_{P_\eta}(s,\varphi_2)$. *Then one concludes*

$$\mathrm{Pol}_{s=\rho_0(\eta)} \frac{D_{P_\eta}(s,\varphi_1)}{s} = \mathrm{Pol}_{s=\rho_0(\eta)} \frac{D_{P_\eta}(s,\varphi_2)}{s}.$$

For each i, set $N_{\varphi_i}(\eta,x) = \sum_{\{m: P_\eta(m)\leq x\}} \varphi_i(m)$. Denote the dominant term (cf. Introduction) of $N_{\varphi_i}(\eta,x)$ by $\hat{N}_{\varphi_i}(\eta,x)$. It is now clear that Lemma 3, (3.22)(1), and Theorem A imply that for all $x \gg 1$

$$\hat{N}_{\varphi_1}(\eta,x) = \hat{N}_{\varphi_2}(\eta,x).$$

Finally, one observes that $N_{\varphi_i}(\eta,x) = N_{\varphi_i}(x-\eta)$, $i = 1,2$. Thus, since Theorem A also applies to (P,φ_1), (P,φ_2), one concludes that

https://doi.org/10.4153/CJM-1991-044-9 Published online by Cambridge University Press

a) $\rho_0(\eta) = \rho_0(\varphi_1) = \rho_0(\varphi_2)$;

b) $\hat{N}_{\varphi_1}(x) = \hat{N}_{\varphi_2}(x)$ holds for all $x \gg 1$.

This completes the proof of Theorem 4. ∎

CONCLUDING REMARKS 3.23. (1) It is interesting to compare Theorem 4 with Corollary 3 of [Li-2]. Because the methods used here are homological and not strictly geometric, it is not necessary to include as a *hypothesis* the condition $\rho_0(\varphi_1) = \rho_0(\varphi_2)$. Instead, this equality is deduced as a consequence of the conditions (3.22)(1–4). The limitation of the geometric argument of [ibid.] is that this equality had to be assumed in order to conclude that the dominant terms in the $N_{\varphi_i}(x)$ agreed. Thus, Theorem 4 implies that the $\hat{N}_{\varphi}(x)$ are cohomological invariants in a sense that is stronger than that which follows from Corollary 3 [ibid.].

(2) In [Li-3], a *moderate growth condition* for $|D_P(s,\varphi)|$ was proved whenever P is a polynomial defined over $\mathbb{R}$ and hypoelliptic on $[a,\infty)^n$ for some $a \in (0,1)$. This implies the existence of a partial asymptotic expansion for $N_{\varphi}(x)$ similar to that in Theorem A. In particular, one has the dominant term $\hat{N}_{\varphi}(x)$, with an error term of strictly smaller order in x. The precise relationship between such hypoelliptic polynomials and nondegenerate polynomials in the sense of Definition 2 is not yet completely understood. On the other hand, it is clear that there exist hypoelliptic polynomials which are nondegenerate but do not have positive coefficients. Thus, the proof of Theorem 4 extends without change to polynomials defined over $\mathbb{R}$ and satisfying the following conditions:

(1) P is nondegenerate with respect to $\Gamma_\infty(P)$;

(2) P is hypoelliptic on $[a,\infty)^n$ for some $a \in (0,1)$;

(3) P is a tame polynomial;

However, if P is *not* nondegenerate with respect to $\Gamma_\infty(P)$, then there exists, so far, no known criterion for a pair (P,φ) to be good. Presumably, any such pair is good.

(3) The discussion in (2) should justify to the reader why the growth condition in (3.1) is used. However, if all that one wants to prove is Theorem 3, then it suffices to assume only these two conditions:

(1) $\lim_{\substack{\|x\|\to\infty \\ x\in[1,\infty)^n}} |P(x)| = \infty$;

(2) P is tame.

This can be seen easily by a standard application of Tarski-Seidenberg, cf. [Li-1, sec. 2]. Indeed, there exists a contractible closed set Γ containing $[1,\infty)^n$ and in which P satisfies the same growth condition as (1). One can then replace the cone $\Gamma(\alpha)$ everywhere in the proof of Theorem 3 by Γ. When combined with tameness, one can then argue exactly as in Section 3 to deduce the conclusion in Theorem 3. Details are left to the reader.

(4) The properties of the paths $\delta_i(\tau)$ (cf.(3.13)) can be weakened considerably. It suffices to impose these conditions

i′) $\mathrm{Re}\,\delta_i(\tau)$ is a monotone increasing function.

ii′) No two paths should intersect.

iii) Unless some $t_i = 0$ no $\delta_j(\tau)$ should contain 0.

One notes that the integral coefficients $n_{i,j}$ in (3.14) will in general depend upon the choice of paths $\delta_i(\tau)$. They can be interpreted as topological intersection numbers [P-2, p. 18 ff.].

(5) It would seem to be interesting to extend Theorem 3 to non-tame polynomials with some understood topology at infinity. The most reasonable class to consider are the stratified Morse mappings studied in [G-M].

(6) Theorem 4 should be compared to the form of the solution, when one exists, to the standard Waring's problem when $P \in \mathbb{N}[z_1, \ldots, z_n]$. For this problem one tries to describe the asymptotic behavior of the function

$$\mathcal{N}(k) = \operatorname{card}\{m \in \mathbb{N}^n : P(m) = k\}$$

by an expression, typically of the form,

$$\mathcal{N}(k) = A_P(k)k^\rho + O(k^{\rho-\tau}) \text{ as } k \longrightarrow \infty,$$

where $\rho \in \mathbb{Q}$ is an *expected* order of growth for $\mathcal{N}(k)$ and $A_P(k)$ is the singular series for $P = k$. When the circle method (cf. [Dav,Va]) and its refinements [Hoo] are able to show that $\mathcal{N}(k) \neq 0$ for all large k, it is the case that $A_P(k)$, as a function of k, is bounded from below and above by two positive numbers. Its order in k however is very difficult to determine. Moreover, when this series converges, it is usually an adelic quantity which reflects the strong global/local arithmetic properties of $\{P = k\}$ needed for $\mathcal{N}(k)$ to have such regular behavior for $k \gg 1$.

When the singular series diverges, a reasonable alternative subject to study is the asymptotic behavior of $N_1(x)$. To this end, one presumes that the global topology of P should replace the global arithmetic as the underlying mechanism by which one computes a dominant term for $N_1(x)$. It would then be interesting to ask if a topological analogue exists to the local-global arithmetic nature of the singular series. Is there a *local-global* decomposition to $\hat{N}_\varphi(x)$? According to Theorem 4, this question seems to be intimately tied to the problem of a geometric description of the vanishing homology at infinity (precisely, a compactification of P) for the mapping P. This should describe the local topology in a neighborhood of some component(s) of the singular locus of this fiber at infinity. As such, this could be one form of a topological analogue of the local/global principle that is reflected by the singular series.

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