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THE WIGNER PROPERTY OF SMOOTH NORMED SPACES

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Abstract

We prove that every smooth complex normed space X has the Wigner property. That is, for any complex normed space Y and every surjective mapping $f : X \to Y$ satisfying

$$\{\|f(x) + \alpha f(y)\| : \alpha \in \mathbb{T}\} = \{\|x + \alpha y\| : \alpha \in \mathbb{T}\}, \quad x, y \in X,$$

where $\mathbb{T}$ is the unit circle of the complex plane, there exists a function $\sigma : X \to \mathbb{T}$ such that $\sigma \cdot f$ is a linear or anti-linear isometry. This is a variant of Wigner's theorem for complex normed spaces.

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1. Introduction

Let X and Y be normed spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively. Denote $\mathbb{T} = \{\alpha \in \mathbb{F} : |\alpha| = 1\}$. A function $\sigma : X \to \mathbb{T}$ whose values are of modulus one is called a *phase function* on X. A mapping $f : X \to Y$ is said to be *phase equivalent* to another mapping $g : X \to Y$ if there exists a phase function $\sigma : X \to \mathbb{T}$ such that $f = \sigma \cdot g$, that is, $f(x) = \sigma(x)g(x)$ for $x \in X$.

The celebrated Wigner's unitary–anti-unitary theorem is particularly important in the mathematical foundations of quantum mechanics. It states that for inner product spaces $(X, \langle\cdot,\cdot\rangle)$ and $(Y, \langle\cdot,\cdot\rangle)$ over $\mathbb{F}$, a mapping $f : X \to Y$ satisfies

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle|, \quad x, y \in X \tag{1.1}$$

if and only if f is phase equivalent to a linear or anti-linear isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear isometry in the case $\mathbb{F} = \mathbb{R}$. There are several proofs of this result, see [1, 2, 4, 6, 13, 18, 22] to list just some of them. For further generalisations of this

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fundamental result, we mention the papers [3, 5, 15, 17]. Wigner's theorem is very important and therefore worthy of study from various points of view.

A mapping $f : X \to Y$ between normed spaces over $\mathbb{F}$ is called a *phase-isometry* if it satisfies the functional equation

$$\{\|f(x) + \alpha f(y)\| : \alpha \in \mathbb{T}\} = \{\|x + \alpha y\| : \alpha \in \mathbb{T}\}, \quad x, y \in X. \tag{1.2}$$

It is worth noting that if X and Y are inner product spaces, then $f : X \to Y$ satisfies (1.1) if and only if f satisfies (1.2). Indeed, with the substitution $y = x$, we deduce from either (1.1) or (1.2) that f is norm-preserving. Squaring the norms on both sides of (1.2), it follows that (1.2) holds if and only if

$$\{\mathrm{Re}(\alpha\langle f(x), f(y)\rangle) : \alpha \in \mathbb{T}\} = \{\mathrm{Re}(\alpha\langle x, y\rangle) : \alpha \in \mathbb{T}\}, \quad x, y \in X,$$

which happens if and only if (1.1) holds. Due to Wigner's theorem, a mapping between inner product spaces is a phase-isometry if and only if it is phase equivalent to a linear or anti-linear isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear isometry in the case $\mathbb{F} = \mathbb{R}$.

When X and Y are normed spaces, one can easily see that if $f : X \to Y$ is phase equivalent to a linear or anti-linear isometry, then f is a phase-isometry. For instance, if $f = \sigma \cdot U$, where U is a linear isometry and $\sigma : X \to \mathbb{T}$ is a phase function, then for $x, y \in X$ and $\alpha \in \mathbb{T}$,

$$\begin{aligned}\|f(x) + \alpha f(y)\| &= \|\sigma(x)U(x) + \alpha\sigma(y)U(y)\| = \|U(\sigma(x)x + \alpha\sigma(y)y)\| \\ &= \|\sigma(x)x + \alpha\sigma(y)y\| = \|x + \alpha\overline{\sigma(x)}\sigma(y)y\|\end{aligned}$$

and then

$$\|x + \alpha y\| = \|x + (\alpha\sigma(x)\overline{\sigma(y)})\overline{\sigma(x)}\sigma(y)y\| = \|f(x) + \alpha\sigma(x)\overline{\sigma(y)}f(y)\|.$$

Similar reasoning applies when U is an anti-linear isometry. Therefore, a natural problem posed by Maksa and Páles [13] (the case $\mathbb{F} = \mathbb{R}$), and Wang and Bugajewski [23] (the case $\mathbb{F} = \mathbb{C}$), can be restated as the following problem.

PROBLEM 1.1. Under what conditions is every phase-isometry between two normed spaces over $\mathbb{F}$ phase equivalent to a linear or anti-linear isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear isometry in the case $\mathbb{F} = \mathbb{R}$?

A normed space X over $\mathbb{F}$ is said to have the *Wigner property* if for any normed space Y over $\mathbb{F}$, every surjective phase-isometry $f : X \to Y$ is phase equivalent to a linear or anti-linear isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear isometry in the case $\mathbb{F} = \mathbb{R}$.

There have been several recent papers considering Problem 1.1 or the Wigner property in the case $\mathbb{F} = \mathbb{R}$. For relevant results, please refer to [7–9, 11–13, 19–21, 23]. In particular, Tan and Huang [19] proved that smooth real normed spaces have the Wigner property. Further, Ilišević *et al.* [9] proved that any real normed spaces have the Wigner property. However, to the best of our knowledge, apart from the case where X and Y are inner product spaces, there has been no progress in addressing Problem 1.1 in the case $\mathbb{F} = \mathbb{C}$. The aim of this paper is to give a partial solution for the case

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$\mathbb{F} = \mathbb{C}$. Specifically, we show that every smooth complex normed space has the Wigner property. As a by-product, we give a Figiel-type result for phase-isometries. Although our paper is interesting in its own right, we hope that it will serve as a stepping stone to show that all complex normed spaces have the Wigner property.

2. Results

In the remainder of this paper, unless otherwise specified, all the normed spaces are over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Although the real case has been solved, for the sake of brevity and universality, we will present our lemmas, theorems and proofs in the united form $\mathbb{F}$ rather than the single form $\mathbb{C}$. For a normed space X, we use the notation S_X, B_X and X^* to represent the unit sphere, closed unit ball and dual space of X, respectively. The set of positive integers is denoted by $\mathbb{N}$.

We start this section with a simple and frequently-used property of phase-isometries between two normed spaces.

LEMMA 2.1. *Let X and Y be normed spaces and $f : X \to Y$ a phase-isometry. Then f is a norm-preserving map. Moreover, if f is surjective, then*

$$\{f(\alpha x) : \alpha \in \mathbb{T}\} = \{\alpha f(x) : \alpha \in \mathbb{T}\}, \quad x \in X.$$

PROOF. With the substitution $y = x$, it follows from (1.2) that

$$2\|f(x)\| = \max\{\|f(x) + \alpha f(x)\| : \alpha \in \mathbb{T}\} = \max\{\|x + \alpha x\| : \alpha \in \mathbb{T}\} = 2\|x\|,$$

which shows that f is norm-preserving.

Now suppose that f is surjective. Let us take a nonzero $x \in X$ and $\alpha \in \mathbb{T}$. The surjectivity guarantees that there exists some $y \in X$ such that $f(y) = \alpha f(x)$. Then

$$\min\{\|y + \beta x\| : \beta \in \mathbb{T}\} = \min\{\|f(y) + \beta f(x)\| : \beta \in \mathbb{T}\} = 0,$$

which implies that

$$\{\alpha f(x) : \alpha \in \mathbb{T}\} \subset \{f(\alpha x) : \alpha \in \mathbb{T}\}.$$

Moreover, we conclude from (1.2) that

$$\min\{\|f(\alpha x) + \beta f(x)\| : \beta \in \mathbb{T}\} = \min\{\|\alpha x + \beta x\| : \beta \in \mathbb{T}\} = 0,$$

which shows that

$$\{f(\alpha x) : \alpha \in \mathbb{T}\} \subset \{\alpha f(x) : \alpha \in \mathbb{T}\}.$$

This competes the proof. □

From [19, Lemma 2], it follows that every surjective phase-isometry between two real normed spaces is injective. The following example shows that a surjective phase-isometry between two complex normed spaces may not be injective.

EXAMPLE 2.2. Let X be a complex normed space and $x_0 \in X\backslash\{0\}$. Define $f : X \to X$ by $f(\alpha x_0) = \alpha^2 x_0$ for all $\alpha \in \mathbb{T}$ and $f(x) = x$ otherwise. Then f is a surjective phase-isometry, but it is not injective since $f(-x_0) = x_0 = f(x_0)$.

In Example 2.2, f is phase equivalent to the identity mapping, letting the phase function σ be $\sigma(\alpha x_0) = \alpha$ for all $\alpha \in \mathbb{T}$ and $\sigma(x) = 1$ otherwise.

Recall that a *support functional* ϕ at $x \in X\backslash\{0\}$ is a norm-one linear functional in X^* such that $\phi(x) = \|x\|$. Denote by $D(x)$ the set of all support functionals at $x \neq 0$, that is,

$$D(x) = \{\phi \in S_{X^*} : \phi(x) = \|x\|\}.$$

The Hahn–Banach theorem implies that $D(x) \neq \emptyset$ for every $x \in X\backslash\{0\}$. A normed space X is said to be *smooth* at $x \neq 0$ if there exists a unique supporting functional at x, that is, $D(x)$ consists of only one element. If X is smooth at every $x \neq 0$, then X is said to be *smooth*. It follows from [14, Proposition 5.4.20] that each subspace of a smooth normed space is smooth.

Recall also the concept of Gateaux differentiability. Let X be a normed space, $x, y \in X$. Define

$$G_+(x, y) := \lim_{t\to 0^+, t\in\mathbb{R}} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t\to +\infty, t\in\mathbb{R}} (\|tx + y\| - \|tx\|)$$

and

$$G_-(x, y) := \lim_{t\to 0^-, t\in\mathbb{R}} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t\to +\infty, t\in\mathbb{R}} (\|tx\| - \|tx - y\|).$$

It is known [14, 16] that both $G_+(x, y)$ and $G_-(x, y)$ exist for each $x, y \in X$ and

$$G_+(x, y) = \max\{\mathrm{Re}\,\phi(y) : \phi \in D(x)\}, \quad G_-(x, y) = \min\{\mathrm{Re}\,\phi(y) : \phi \in D(x)\}.$$

We say that the norm of X is *Gateaux differentiable* at $x \neq 0$ whenever $G_+(x, y) = G_-(x, y)$ for all $y \in X$, in which case the common value of $G_+(x, y)$ and $G_-(x, y)$ is denoted by $G(x, y)$. It is easy to see that a normed space X is smooth at x if and only if the norm is Gateaux differentiable at x.

A point $\phi \in S_{X^*}$ is said to be a *w^*-exposed point of B_{X^*}* provided that ϕ is the only supporting functional for some smooth point $u \in S_X$. Recently, Tan and Huang [19] showed that for every phase-isometry f of a real normed space X into another real normed space Y and every w^*-exposed point ϕ of B_{X^*}, there exists $\varphi \in S_{Y^*}$ such that $\phi(x) = \pm\varphi(f(x))$ for all $x \in X$. This result can be viewed as an extension of Figiel's theorem, which plays an important role in the study of isometric embedding. We will present a similar result for a phase-isometry between two normed spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

LEMMA 2.3. *Let X and Y be normed spaces and $f : X \to Y$ a phase-isometry. Then for every w^*-exposed point ϕ of B_{X^*}, there exists $\varphi \in S_{Y^*}$ such that*

$$|\phi(x)| = |\varphi(f(x))|, \quad x \in X.$$

PROOF. Let $u \in S_X$ be a smooth point such that $\phi(u) = 1$. For every $n \in \mathbb{N}$, the Hahn–Banach theorem guarantees the existence of $\varphi_n \in S_{Y^*}$ such that

$$\varphi_n(f(nu)) = \|f(nu)\| = \|nu\| = n.$$

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For $t \in [0, n]$, there exists some $\alpha_{t,n} \in \mathbb{T}$ such that

$$\|f(nu) - \alpha_{t,n} f(tu)\| = \|nu - tu\| = n - t.$$

Consequently, we deduce that

$$\begin{aligned} 2n &= |\varphi_n(f(nu) - \alpha_{t,n} f(tu)) + \varphi_n(f(nu) + \alpha_{t,n} f(tu))| \\ &\le |\varphi_n(f(nu) - \alpha_{t,n} f(tu))| + |\varphi_n(f(nu) + \alpha_{t,n} f(tu))| \\ &\le \|f(nu) - \alpha_{t,n} f(tu)\| + \|f(nu) + \alpha_{t,n} f(tu)\| \\ &\le (n - t) + (n + t) = 2n, \end{aligned}$$

which implies that $\varphi_n(\alpha_{t,n} f(tu)) = t$. This means that for each $t \in (0, n]$, there exists a unique $\alpha_{t,n} \in \mathbb{T}$ such that $\varphi_n(f(tu)) = \overline{\alpha_{t,n}} t$. By Alaoglu's theorem, the sequence $\{\varphi_n\}$ has a cluster point $\varphi \in S_{Y^*}$ in the w^* topology. It follows that for each $t > 0$, there exists $\alpha_t \in \mathbb{T}$ depending only on t such that $\varphi(f(tu)) = \alpha_t t$.

For each $x \in X$, there exist $\alpha_x, \beta_x \in \mathbb{T}$ such that $\alpha_x \phi(x) = |\phi(x)|$ and $\beta_x \varphi(f(x)) = |\varphi(f(x))|$. For each $n \in \mathbb{N}$, there exists $\alpha_{x,n}, \beta_{x,n} \in \mathbb{T}$ such that

$$\begin{aligned} \|nu - \alpha_x x\| = \|f(nu) - \alpha_{x,n}\alpha_n f(x)\| &\ge |\varphi(f(nu)) - \alpha_{x,n}\alpha_n \varphi(f(x))| \\ &= |\alpha_n n - \alpha_{x,n}\alpha_n \varphi(f(x))| = |n - \alpha_{x,n}\varphi(f(x))| \end{aligned}$$

and

$$\begin{aligned} |n + \beta_x \varphi(f(x))| = |\alpha_n n + \alpha_n \beta_x \varphi(f(x))| &= |\varphi(f(nu)) + \alpha_n \beta_x \varphi(f(x))| \\ &\le \|f(nu) + \alpha_n \beta_x f(x)\| = \|nu + \beta_{x,n} x\|. \end{aligned}$$

Given that $\mathbb{T}$ is compact, there must be a strictly increasing sequence $\{n_j : j \in \mathbb{N}\}$ in $\mathbb{N}$ and $\alpha'_x, \beta'_x \in \mathbb{T}$ such that $\lim_{j\to\infty} \alpha_{x,n_j} = \alpha'_x$ and $\lim_{j\to\infty} \beta_{x,n_j} = \beta'_x$. Since ϕ is the only supporting functional at u,

$$\begin{aligned} |\phi(x)| = \mathrm{Re}\, \phi(\alpha_x x) &= \lim_{j\to\infty} (\|n_j u\| - \|n_j u - \alpha_x x\|) \\ &\le \lim_{j\to\infty} (n_j - |n_j - \alpha_{x,n_j}\varphi(f(x))|) = \lim_{j\to\infty} (n_j - |n_j - \alpha'_x \varphi(f(x))|) \\ &= \mathrm{Re}\,(\alpha'_x \varphi(f(x))) \le |\varphi(f(x))| \end{aligned}$$

and

$$\begin{aligned} |\varphi(f(x))| = \mathrm{Re}\,(\beta_x \varphi(f(x))) &= \lim_{j\to\infty} (|n_j + \beta_x \varphi(f(x))| - n_j) \\ &\le \lim_{j\to\infty} (\|n_j u + \beta_{x,n_j} x\| - \|n_j u\|) = \lim_{j\to\infty} (\|n_j u + \beta'_x x\| - \|n_j u\|) \\ &= \mathrm{Re}\, \phi(\beta'_x x) \le |\phi(x)|. \end{aligned}$$

This completes the proof. □

Let V be a vector space. For $M \subset V$, $[M]$ denotes the subspace generated by M. If $x, y \in V$, then we write $[x] := [\{x\}]$ and $[x, y] := [\{x, y\}]$ for simplicity.

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LEMMA 2.4. *Let X and Y be normed spaces with X being smooth. Suppose that $f : X \to Y$ is a surjective phase-isometry. Then for every $x \in X$,*

$$f([x]) = [f(x)].$$

PROOF. We first prove that $[f(x)] \subset f([x])$ for each $x \in X$. Assume, for a contradiction, that $tf(x) \notin f([x])$ for some nonzero $x \in X$ and $t \in \mathbb{F}$. Since f is surjective, there exists $y \in X$ such that $f(y) = tf(x)$. The function $s \mapsto \|y - sx\|$ is continuous and its value tends to infinity when $|s|$ tends to infinity. Hence, there is at least one point $s_0 \in \mathbb{F}$ such that

$$d := d(y, [x]) = \min\{\|y - sx\| : s \in \mathbb{F}\} = \|y - s_0 x\| > 0.$$

Set $E := [x, y]$. By the Hahn–Banach theorem, there exists $\phi \in S_{E^*}$ which satisfies $\phi(y) = d$ and $\phi(x) = 0$. Note that E being a two-dimensional subspace of X is reflexive. This guarantees the existence of some $z \in S_E$ such that $\phi(z) = 1$. Since X is smooth, so is its subspace E. Therefore, ϕ is the only supporting functional at $z \in S_E$. We apply Lemma 2.3 to $f|_E : E \to Y$ to obtain $\varphi \in S_{Y^*}$ such that $|\phi| = |\varphi \circ f|$ on E. Then

$$0 < d = |\phi(y)| = |\varphi(f(y))| = |\varphi(tf(x))| = |t||\varphi(f(x))| = |t||\phi(x)| = 0,$$

which is a contradiction. This proves $[f(x)] \subset f([x])$.

Conversely, fix a nonzero $x \in X$. For each $r \in (0, +\infty)$, by the above inclusion and the norm preserving property of f, there exists some $\alpha_r \in \mathbb{T}$ such that $r^{-1} f(rx) = f(\alpha_r x)$. For each $\alpha \in \mathbb{T}$, by Lemma 2.1, there exist $\beta_{r,\alpha}, \alpha'_r \in \mathbb{T}$ such that

$$f(r\alpha x) = \beta_{r,\alpha} f(rx) = \beta_{r,\alpha} r f(\alpha_r x) = r\beta_{r,\alpha}\alpha'_r f(x),$$

which implies that $f([x]) \subset [f(x)]$. The proof is complete. □

Note that the conclusion of Lemma 2.4 is equivalent to

$$\{f(r\alpha x) : \alpha \in \mathbb{T}\} = \{r\alpha f(x) : \alpha \in \mathbb{T}\}, \quad x \in X, \ r \in [0, +\infty).$$

LEMMA 2.5. *Let X and Y be normed spaces with X being smooth. Suppose that $f : X \to Y$ is a surjective phase-isometry. Then for every $x, y \in X$,*

$$\{G_+(f(x), \alpha f(y)) : \alpha \in \mathbb{T}\} = \{G(x, \alpha y) : \alpha \in \mathbb{T}\} = \{G_-(f(x), \alpha f(y)) : \alpha \in \mathbb{T}\}.$$

PROOF. We only prove the first equality, the second being similar. Let $x, y \in X$ be nonzero and $\alpha \in \mathbb{T}$. For each $n \in \mathbb{N}$, Lemma 2.4 and (1.2) imply that there exist $\alpha_n, \beta_n, \gamma_n \in \mathbb{T}$ such that $f(nx) = \alpha_n n f(x)$ and

$$\|f(nx) + \alpha_n \alpha f(y)\| = \|nx + \beta_n y\|, \quad \|f(nx) + \alpha_n \gamma_n f(y)\| = \|nx + \alpha y\|.$$

By the compactness of $\mathbb{T}$, there is a strictly increasing sequence $\{n_j : j \in \mathbb{N}\}$ in $\mathbb{N}$ and $\beta, \gamma \in \mathbb{T}$ such that $\lim_{j\to\infty} \beta_{n_j} = \beta$ and $\lim_{j\to\infty} \gamma_{n_j} = \gamma$. Then

$$
\begin{aligned}
G_+(f(x), \alpha f(y)) &= \lim_{j\to\infty}(\|n_j f(x) + \alpha f(y)\| - \|n_j f(x)\|) \\
&= \lim_{j\to\infty}(\|f(n_j x) + \alpha_{n_j}\alpha f(y)\| - \|n_j f(x)\|) \\
&= \lim_{j\to\infty}(\|n_j x + \beta_{n_j} y\| - \|n_j x\|) = \lim_{j\to\infty}(\|n_j x + \beta y\| - \|n_j x\|) = G(x, \beta y)
\end{aligned}
$$

and

$$
\begin{aligned}
G(x, \alpha y) &= \lim_{j\to\infty}(\|n_j x + \alpha y\| - \|n_j x\|) \\
&= \lim_{j\to\infty}(\|f(n_j x) + \alpha_{n_j}\gamma_{n_j} f(y)\| - \|f(n_j x)\|) \\
&= \lim_{j\to\infty}(\|n_j f(x) + \gamma_{n_j} f(y)\| - \|n_j f(x)\|) \\
&= \lim_{j\to\infty}(\|n_j f(x) + \gamma f(y)\| - \|n_j f(x)\|) = G_+(f(x), \gamma f(y)).
\end{aligned}
$$

The proof is complete. □

LEMMA 2.6. *Let X and Y be normed spaces with X being smooth. Suppose that $f : X \to Y$ is a surjective phase-isometry. Then Y is smooth.*

PROOF. Let $x \in X$ be a nonzero element with the unique supporting functional $\phi_x \in D(x)$. It suffices to prove that $D(f(x))$ is a singleton set. Let $\varphi, \psi \in D(f(x))$ and $f(y) \in \ker\varphi$. For each $\alpha \in \mathbb{T}$, Lemma 2.5 implies that there exists $\beta, \gamma \in \mathbb{T}$ such that

$$\mathrm{Re}(\alpha\phi_x(y)) = \mathrm{Re}\phi_x(\alpha y) = G(x, \alpha y) = G_+(f(x), \beta f(y)) \geq \mathrm{Re}\varphi(\beta f(y)) = 0$$

and

$$\mathrm{Re}(\alpha\psi(f(y))) = \mathrm{Re}\psi(\alpha f(y)) \leq G_+(f(x), \alpha f(y)) = G(x, \gamma y) = \mathrm{Re}\phi_x(\gamma y).$$

Using the arbitrariness of $\alpha \in \mathbb{T}$ twice gives $\phi_x(y) = 0$ by the first inequality and therefore $\psi(f(y)) = 0$ by the second inequality. This shows that $\ker\varphi \subset \ker\psi$. Thus, $\psi = \lambda\varphi$ for some $\lambda \in \mathbb{F}$. Considering that $\psi, \varphi \in D(f(x))$, we find that $\lambda = 1$. This implies that $\psi = \varphi$, which completes the proof. □

Recently, Ilišević and Turnšek [10, Theorem 2.2 and Remark 2.1] generalised Wigner's theorem to smooth normed spaces via semi-inner products. This can be translated into the following theorem in the language of supporting functionals.

THEOREM 2.7. *Let X and Y be smooth normed spaces over $\mathbb{F}$ and $f : X \to Y$ a surjective mapping satisfying, for all nonzero $x, y \in X$,*

$$|\phi_{f(x)}(f(y))| = |\phi_x(y)|.$$

Then f is phase equivalent to a linear or anti-linear surjective isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear surjective isometry in the case $\mathbb{F} = \mathbb{R}$.

Combining the above results gives our main theorem.

THEOREM 2.8. *Every smooth normed space has the Wigner property.*

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PROOF. Let X and Y be normed spaces with X being smooth. Suppose that $f : X \to Y$ is a surjective phase-isometry. By Lemma 2.6, Y is smooth. Then Lemma 2.5 implies that for all nonzero $x, y \in X$,

$$\{\mathrm{Re}\phi_{f(x)}(\alpha f(y)) : \alpha \in \mathbb{T}\} = \{\mathrm{Re}\phi_x(\alpha y) : \alpha \in \mathbb{T}\}.$$

Taking the maximum on both sides, for all nonzero $x, y \in X$,

$$|\phi_{f(x)}(f(y))| = |\phi_x(y)|.$$

By Theorem 2.7, f is phase equivalent to a linear or anti-linear surjective isometry in the case $\mathbb{F} = \mathbb{C}$ and to a linear surjective isometry in the case $\mathbb{F} = \mathbb{R}$. This completes the proof. □

It is well known that $L^p(\mu)$ is a smooth normed space, where μ is a measure and $1 < p < \infty$. The following corollary is immediate.

COROLLARY 2.9. *$L^p(\mu)$ has the Wigner property, where μ is a measure and $1 < p < \infty$.*

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