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Stability to the dissipative Reissner–Mindlin–Timoshenko acting on displacement equation†

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In this paper, we show that there exists a critical number that stabilises the Reissner–Mindlin–Timoshenko system with frictional dissipation acting only on the equation for the transverse displacement. We identify that the Reissner–Mindlin–Timoshenko system has two speed characteristics $v_1^2 := K/\rho_1$ and $v_2^2 := D/\rho_2$ and we show that the system is exponentially stable if only if

$$v_1^2 = v_2^2.$$

In the general case, we prove that the system is polynomially stable with optimal decay rate. Numerical experiments using finite differences are given to confirm our analytical results. Our numerical results are qualitatively in agreement with the corresponding results from dynamical in infinite dimensional.

Key words: Reissner–Mindlin–Timoshenko system; wave propagation speed; asymptotic behaviour; finite difference.

1 Introduction

In this work, we consider the dissipative Reissner–Mindlin–Timoshenko system given by

$$\rho_1\omega_{tt} - K(\psi+\omega_x)_x - K(\varphi+\omega_y)_y + d_1\omega_t = 0, \text{ in } \Omega\times\mathbb{R}^+, \quad (1.1)$$

$$\rho_2\psi_{tt} - D\psi_{xx} - D\left(\frac{1-\mu}{2}\right)\psi_{yy} - D\left(\frac{1+\mu}{2}\right)\varphi_{xy} + K(\psi+\omega_x) = 0, \text{ in } \Omega\times\mathbb{R}^+, \quad (1.2)$$

$$\rho_2\varphi_{tt} - D\varphi_{yy} - D\left(\frac{1-\mu}{2}\right)\varphi_{xx} - D\left(\frac{1+\mu}{2}\right)\psi_{xy} + K(\varphi+\omega_y) = 0, \text{ in } \Omega\times\mathbb{R}^+. \quad (1.3)$$

Here $\rho_1 = \rho h$, $\rho_2 = \frac{\rho h^3}{12}$ where ρ is the (constant) mass per unit of surface area, h is the (uniform) plate thickness, μ is Poisson's ratio ($0 < \mu < 1/2$), $D = \frac{Eh^3}{12(1-\mu^2)}$ is the modulus of flexural rigidity, $K = \frac{kEh}{2(1+\mu)}$ is the shear modulus where E is the Young's modulus and k

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is the shear correction. The functions ω, ψ and φ depend on $(x, y, t) \in \Omega \times \mathbb{R}^+$ denote the transverse displacement of the plate and the rotational angles of a filament of the plate, respectively. Details on physical deduction of this hyperbolic system (for the undamped case) can be found in ref. [12,13].

We consider the initial data given by

$$\omega(x, y, 0) = \omega_0(x, y), \quad \omega_t(x, y, 0) = \omega_1(x, y), \text{ in } \Omega, \tag{1.4}$$

$$\psi(x, y, 0) = \psi_0(x, y), \quad \psi_t(x, y, 0) = \psi_1(x, y), \text{ in } \Omega, \tag{1.5}$$

$$\varphi(x, y, 0) = \varphi_0(x, y), \quad \varphi_t(x, y, 0) = \varphi_1(x, y), \text{ in } \Omega, \tag{1.6}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. The boundary conditions are

$$\omega = 0, \text{ on } \Gamma \times \mathbb{R}^+, \tag{1.7}$$

$$\psi = 0, \left(\frac{1-\mu}{2}\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\right), \frac{\partial\varphi}{\partial y} + \mu\frac{\partial\psi}{\partial x}\right) \cdot \nu = 0, \text{ on } \Gamma_1 \times \mathbb{R}^+, \tag{1.8}$$

$$\varphi = 0, \left(\frac{\partial\psi}{\partial x} + \mu\frac{\partial\varphi}{\partial y}, \frac{1-\mu}{2}\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\right)\right) \cdot \nu = 0, \text{ on } \Gamma_2 \times \mathbb{R}^+. \tag{1.9}$$

Here, we are considering $\Omega \subset \mathbb{R}^2$ as the rectangular configuration given by

$$\Omega := [0, L_1] \times [0, L_2], \quad \text{with} \quad L_1, L_2 > 0,$$

with boundary given by

$$\Gamma_1 := \{(x, y) : \ 0 < x < L_1, \ y = 0, \ L_2\},$$
$$\Gamma_2 := \{(x, y) : \ 0 < y < L_2, \ x = 0, \ L_1\},$$

satisfying $\Gamma := \overline{\Gamma}_1 \cup \overline{\Gamma}_2$.

The system (1.1)–(1.9) is damped by $d_1\omega_t$ where $d_1 > 0$ is the damping coefficient and, in this sense, its energy is decreasing with time t. It is interesting to know if the energy is controlled by an exponential or polynomial function. We therefore, focus on establishing necessary and sufficient conditions to obtain stability of this system.

One of the questions regarding stability of hyperbolic systems modelling mechanical deformations in beams and plates concerns the minimum dissipation needed to obtain exponential decay. The wave propagation speed play an important role in this respect. For one-dimensional cases of Timoshenko systems where few dissipative mechanisms act, exponential decay occurs if and only if

$$\frac{\kappa}{\rho_1} = \frac{b}{\rho_2},$$

where κ/ρ_1 and b/ρ_2 are the speeds of wave propagation to one-dimensional case. This is the case of the dissipative system given by

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \tag{1.10}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma\psi_t = 0, \tag{1.11}$$

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analysed by Muñoz Rivera and Racke [18]. There are a large number of publications concerning the stabilisation of Timoshenko systems where several types of dissipative mechanisms are considered [2–4, 10, 15, 17, 22, 23]. In this context, we can ask what are the speeds of wave propagation of the Reissner–Mindlin–Timoshenko. Then, keeping in mind the system (1.1)–(1.3) and taking into account that

$$\frac{1-\mu}{2} = 1 - \frac{1+\mu}{2}, \tag{1.12}$$

we can rewrite (1.1)–(1.3) as

$$\rho_1\omega_{tt} - K(\psi+\omega_x)_x - K(\varphi+\omega_y)_y + d_1\omega_t = 0, \text{ in } \Omega\times \mathbb{R}+, \tag{1.13}$$

$$\rho_2\psi_{tt} - D\Delta\psi + D\frac{1+\mu}{2}\psi_{yy} - D\frac{1+\mu}{2}\varphi_{xy} + K(\psi+\omega_x) = 0, \text{ in } \Omega\times \mathbb{R}+, \tag{1.14}$$

$$\rho_2\varphi_{tt} - D\Delta\varphi + D\frac{1+\mu}{2}\varphi_{xx} - D\frac{1+\mu}{2}\psi_{xy} + K(\varphi+\omega_y) = 0, \text{ in } \Omega\times \mathbb{R}+. \tag{1.15}$$

Now, since $h = \frac{\rho_1}{\rho}$, then we observe that $\frac{K}{\rho_1} = \frac{k}{2(1+\mu)}\frac{E}{\rho}$, where the ratio between the tension E and the density ρ has dimension of speed. Analogously, one also has $\frac{D}{\rho_2} = \frac{1}{(1-\mu^2)}\frac{E}{\rho}$ with dimension of speed. Therefore, we put

$$v_1^2 := K/\rho_1 \quad \text{and} \quad v_2^2 := D/\rho_2, \tag{1.16}$$

being the speeds of wave propagation for the Reissner–Mindlin–Timoshenko system. From the above, we ask: what happens regarding the stability of system (1.1)–(1.9) when

$$v_1^2 - v_2^2 = 0? \tag{1.17}$$

Some previous results concerning Reissner–Mindlin–Timoshenko systems should be mentioned. The most well known are due to Lagnese [12]. He addressed the question of uniform and strong stability of purely elastic plates due to boundary feedbacks. In particular, he considered a bounded domain Ω having a Lipschitz boundary Γ such that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, where Γ_0 and Γ_1 are relatively open, disjoint subsets of Γ with $\Gamma_1 \neq \emptyset$ and he considered the following boundary conditions

$$\omega = \psi = \varphi = 0 \quad \text{in } \Gamma_0, \tag{1.18}$$

$$K\left(\frac{\partial\omega}{\partial x} + \psi, \frac{\partial\omega}{\partial y} + \varphi\right)\cdot\nu = m_1 \quad \text{in } \Gamma_1, \tag{1.19}$$

$$D\left(\frac{\partial\psi}{\partial x} + \mu\frac{\partial\varphi}{\partial y}, \frac{1-\mu}{2}\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\right)\right)\cdot\nu = m_2 \quad \text{in } \Gamma_1, \tag{1.20}$$

$$D\left(\frac{1-\mu}{2}\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\right), \frac{\partial\varphi}{\partial y} + \mu\frac{\partial\psi}{\partial x}\right)\cdot\nu = m_3 \quad \text{in } \Gamma_1, \tag{1.21}$$

where $\nu := (\nu_1, \nu_2)$ is the unit exterior normal to Γ and $\{m_1, m_2, m_3\}$ signifies the linear boundary dissipations given by

$$(m_1, m_2, m_3)' = -F(\omega_t, \psi_t, \varphi_t)',$$

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with $F = [f_{ij}]$ a 3×3 matrix of real $L^\infty(\Gamma_1)$ functions such that F is symmetric and positive semi-definite on Γ_1. Lagnese proved that the problem (1.1)–(1.3) with $d_1 = 0$ and boundary conditions (1.18)–(1.21) is exponentially stable, without any restrictions on the coefficients of the system.

Muñoz Rivera and Portillo Oquendo [16] considered the Reissner–Mindlin–Timoshenko systems having boundary conditions of memory type and they proved exponential stability, provided that the kernels have exponential behaviour, and are polynomially stable for kernels of polynomial type. Similar dissipations have been used by Santos [21], where the author considered a Timoshenko model in $\Omega \subset \mathbb{R}^n$. On the other hand, when damping mechanisms act on the whole domain, we should note the work of Fernándes Sare [8]. He considered the Reissner–Mindlin–Timoshenko equations with damping acting only on the rotational angles ψ and φ. He proved, using a resolvent criterion, that the system is not exponentially stable independent of any relations between the coefficients, making this case different from the analogous one-dimensional case.

In this work, we are concerned with the stability of system (1.1)–(1.9) taking into account the wave propagation speeds (1.16). When then ask: what happens with the decay of the energy of this system when $v_1^2 = v_2^2$?

The method that we used to determine the asymptotic behaviour is based on *Gearhart–Herbst–Prüss–Huang* Theorem [9] for dissipative systems (see also [11, 20]). We now give two theorems in this direction.

Theorem 1.1 *Let $S(t) = e^{\mathscr{A}t}$ be a C_0-semigroup of contractions on a Hilbert space $\mathscr{H}$. Then, $S(t)$ is exponentially stable if and only if*

$$\rho(\mathscr{A}) \supseteq \{i\lambda : \lambda \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{1.22}$$

and

$$\overline{\lim}_{|\lambda|\to\infty} ||(i\lambda I - \mathscr{A})^{-1}||_{\mathscr{L}(\mathscr{H})} < \infty, \tag{1.23}$$

hold, where $\rho(\mathscr{A})$ is the resolvent set of the differential operator $\mathscr{A}$.

On the other hand, to show the polynomial stability we use the result due to Borichev and Tomilov [5].

Theorem 1.2 *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathscr{H}$ with generator $\mathscr{A}$ such that $i\mathbb{R} \subset \rho(\mathscr{A})$. Then*

$$\frac{1}{|\lambda|^\alpha}\|(i\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \leqslant C, \quad \forall \lambda \in \mathbb{R} \quad \Leftrightarrow \quad \|S(t)\mathscr{A}^{-1}\|_{\mathscr{L}(\mathscr{H})} \leqslant \frac{c}{t^{1/\alpha}}. \tag{1.24}$$

This paper is organised as follows. In Section 2 we discuss the existence, regularity and uniqueness of global solutions of (1.1)–(1.9). To do this, we use the semigroup technique (see [19]). In Section 3, we study the lack of exponential decay in accordance with a nice relationship between the wave propagation speeds (1.16). In Section 4, we study the

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exponential and polynomial decay of system (1.1)–(1.9). In general, we show that the Reissner–Mindlin–Timoshenko systems is polynomially stable giving an optimal decay rate. In Section 5, we show the numerical results by using finite difference method to confirm our analytical results. Finally, in Section 6, we conclude our work with some comments.

2 Semigroup setting

To give an accurate formulation for the Reissner–Mindlin–Timoshenko system, let $\Omega \subset \mathbb{R}^2$ denote the interior of a rectangle given by

$$\Omega := [0, L_1] \times [0, L_2], \quad L_1, L_2 > 0.$$

For the boundary $\Gamma = \partial\Omega$ of Ω, we define

$$\Gamma_1 := \{(x, y) : 0 < x < L_1, \ y = 0, \ L_2\},$$
$$\Gamma_2 := \{(x, y) : 0 < y < L_2, \ x = 0, \ L_1\}.$$

Thus, $\Gamma := \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. With the above hypothesis on Ω, let us consider the Hilbert space

$$\mathscr{H} := H_0^1(\Omega) \times L^2(\Omega) \times H_{\Gamma_1}^1(\Omega) \times L^2(\Omega) \times H_{\Gamma_2}^1(\Omega) \times L^2(\Omega),$$

where

$$H_{\Gamma_i}^1(\Omega) := \{\psi \in H^1(\Omega) \ : \ \psi = 0 \text{ on } \Gamma_i\}, \quad (i = 1, 2),$$

with inner product given by

$$\begin{aligned}
(U, V)_{\mathscr{H}} = {} & \rho_1 \int_\Omega u^2\overline{v^2}\, dxdy + \rho_2 \int_\Omega u^4\overline{v^4}\, dxdy + \rho_2 \int_\Omega u^6\overline{v^6}\, dxdy \\
& + K \int_\Omega (u^3 + u_x^1)(\overline{v^3 + v_x^1})\, dxdy + K \int_\Omega (u^5 + u_y^1)(\overline{v^5 + v_y^1})\, dxdy \\
& + D \int_\Omega u_x^3\overline{v_x^3}\, dxdy + D \int_\Omega u_y^5\overline{v_y^5}\, dxdy + D\left(\frac{1-\mu}{2}\right) \int_\Omega (u_y^3 + u_x^5)(\overline{v_y^3 + v_x^5})\, dxdy \\
& + D\mu \int_\Omega u_x^3\overline{v_y^5}\, dxdy + D\mu \int_\Omega u_y^5\overline{v_x^3}\, dxdy,
\end{aligned} \tag{2.1}$$

with norm given by

$$\begin{aligned}
||U||_{\mathscr{H}}^2 = {} & \rho_1 \int_\Omega |u^2|^2\, dxdy + \rho_2 \int_\Omega |u^4|^2\, dxdy \\
& + \rho_2 \int_\Omega |u^6|^2\, dxdy + K \int_\Omega |u^3 + u_x^1|^2\, dxdy + K \int_\Omega |u^5 + u_y^1|^2\, dxdy \\
& + D \int_\Omega |u_x^3|^2\, dxdy + D \int_\Omega |u_y^5|^2\, dxdy + D\left(\frac{1-\mu}{2}\right) \int_\Omega |u_y^3 + u_x^5|^2\, dxdy \\
& + D\mu \int_\Omega u_x^3\overline{u_y^5}\, dxdy + D\mu \int_\Omega u_y^5\overline{u_x^3}\, dxdy,
\end{aligned} \tag{2.2}$$

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where $U = (u^1, u^2, u^3, u^4, u^5, u^6)'$, $V = (v^1, v^2, v^3, v^4, v^5, v^6)'$ and $'$ denotes the transpose of the vector.

Let us denote by $\mathscr{V}$ the following Hilbert space

$$\mathscr{V} = H_0^1(\Omega) \times H_{\Gamma_1}^1(\Omega) \times H_{\Gamma_2}^1(\Omega).$$

The next lemma, which is a consequence of Korn's inequality, provides us an equivalence between the above norm on (2.2) and usual norm in $\mathscr{H}$ (see [12, 13]).

Lemma 2.1 *With the above notation we have that*

(a) There exists a constant $\alpha_0 > 0$ such that, for all $(\psi, \varphi) \in H_{\Gamma_1}^1(\Omega) \times H_{\Gamma_2}^1(\Omega)$,

$$\alpha_0 \left[||\psi||_{H^1}^2 + ||\varphi||_{H^1}^2 \right] \leqslant \int_\Omega \Big[D|\psi_x|^2 + D|\varphi_y|^2 + D\left(\frac{1-\mu}{2}\right)|\psi_y + \varphi_x|^2 + D\mu\psi_x\overline{\varphi}_y + D\mu\varphi_y\overline{\psi}_x \Big]\, dxdy;$$

(b) Moreover, for every $K_0 > 0$, there exists $\beta(K_0)$ such that, for all $K \geqslant K_0$ and $(\omega, \psi, \varphi) \in \mathscr{V}$,

$$\beta(K_0)||(\omega, \psi, \varphi)||_{\mathscr{V}}^2 \leqslant \int_\Omega \Big[K|\psi + \omega_x|^2 dxdy + K|\varphi + \omega_y|^2 dxdy + D|\psi_x|^2 + D|\varphi_y|^2 + D\left(\frac{1-\mu}{2}\right)|\psi_y + \varphi_x|^2 + D\mu\psi_x\overline{\varphi}_y + D\mu\varphi_y\overline{\psi}_x \Big]\, dxdy.$$

Now, if we write $U = (\omega, \omega_t, \psi, \psi_t, \varphi, \varphi_t)'$ and $U_0 = (\omega_0, \omega_1, \psi_0, \psi_1, \varphi_0, \varphi_1)'$ then the equations (1.1)–(1.9) can be rewritten as follows

$$\frac{dU}{dt} = \mathscr{A}U, \text{ for } t > 0, \tag{2.3}$$

$$U(0) = U_0, \tag{2.4}$$

where $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is the operator defined by

$$\mathscr{A} := \begin{pmatrix} 0 & I_d & 0 & 0 & 0 & 0 \\ \frac{K}{\rho_1}\Delta & -\frac{d_1}{\rho_1}I_d & \frac{K}{\rho_1}\partial_x & 0 & \frac{K}{\rho_1}\partial_y & 0 \\ 0 & 0 & 0 & I_d & 0 & 0 \\ -\frac{K}{\rho_2}\partial_x & 0 & \mathscr{B}_1 & 0 & \frac{D}{\rho_2}\left(\frac{1+\mu}{2}\right)\partial_{xy}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_d \\ -\frac{K}{\rho_2}\partial_y & 0 & \frac{D}{\rho_2}\left(\frac{1+\mu}{2}\right)\partial_{xy}^2 & 0 & \mathscr{B}_2 & 0 \end{pmatrix},$$

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where the operators $\mathscr{B}_i (i = 1, 2)$ are given by

$$\mathscr{B}_1 = \frac{D}{\rho_2}\left[\partial_x^2 + \left(\frac{1-\mu}{2}\right)\partial_y^2\right] - \frac{K}{\rho_2} I_d,$$
$$\mathscr{B}_2 = \frac{D}{\rho_2}\left[\left(\frac{1-\mu}{2}\right)\partial_x^2 + \partial_y^2\right] - \frac{K}{\rho_2} I_d.$$

Here I_d denotes the identity operator, and

$$\mathscr{D}(\mathscr{A}) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega) \times \left(H^2(\Omega) \cap H_{\Gamma_1}^1(\Omega)\right) \times H_{\Gamma_1}^1(\Omega) \times \left(H^2(\Omega) \cap H_{\Gamma_2}^1(\Omega)\right) \times H_{\Gamma_2}^1(\Omega).$$

Our result on existence and uniqueness of solutions follows:

Theorem 2.2 *The operator $\mathscr{A}$ generates a C_0-semigroup $S(t)$ of contraction on $\mathscr{H}$. Thus, for any initial data $U_0 \in \mathscr{H}$, problem* (1.1)–(1.9) *has a unique weak solution $U \in C^0([0,\infty), \mathscr{H})$. Moreover, if $U_0 \in \mathscr{D}(\mathscr{A})$, then U is strong solution of* (1.1)–(1.9)*, i.e., $U \in C^1([0,\infty), \mathscr{H}) \cap C^0([0,\infty), \mathscr{D}(\mathscr{A}))$.*

Proof It is easy to see that $\mathscr{D}(\mathscr{A})$ is dense in $\mathscr{H}$. On the other hand, for $U = (\omega, W, \psi, \Psi, \varphi, \Phi)' \in \mathscr{D}(\mathscr{A})$, a direct computation gives that

$$\mathscr{R}e(\mathscr{A}U, U)_{\mathscr{H}} = -d_1 \int_\Omega |W|^2 \, dxdy \leqslant 0, \tag{2.5}$$

from which it follows that $\mathscr{A}$ is a dissipative operator for $d_1 > 0$. Next, taking any $F = (f^1, f^2, f^3, f^4, f^5, f^6)' \in \mathscr{H}$ we solve the equation

$$\mathscr{A}U = F. \tag{2.6}$$

From (2.6), we can conclude that

$$W = f^1, \; \Psi = f^3, \; \Phi = f^5. \tag{2.7}$$

Substituting (2.7) into (2.6), we get

$$K(\psi + \omega_x)_x + K(\varphi + \omega_y)_y = \rho_1 f^2 + d_1 f^1, \tag{2.8}$$
$$D\left(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy} + \frac{1+\mu}{2}\varphi_{xy}\right) - K(\psi + \omega_x) = \rho_2 f^4, \tag{2.9}$$
$$D\left(\frac{1-\mu}{2}\varphi_{xx} + \varphi_{yy} + \frac{1+\mu}{2}\psi_{xy}\right) - K(\varphi + \omega_y) = \rho_2 f^6, \tag{2.10}$$

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from where we can define the bilinear form $a(\cdot,\cdot)$, with domain $\mathscr{V}\times\mathscr{V}$, given by

$$\begin{aligned} a(\Theta,\widetilde{\Theta}) :=&\ K\int_\Omega (u^3+u^1_x)(\overline{v^3+v^1_x})\,dxdy + K\int_\Omega (u^5+u^1_y)(\overline{v^5+v^1_y})\,dxdy \\ &+ D\int_\Omega u^3_x\overline{v^3_x}\,dxdy + D\int_\Omega u^5_y\overline{v^5_y}\,dxdy \\ &+ D\left(\frac{1-\mu}{2}\right)\int_\Omega (u^3_y+u^5_x)(\overline{v^3_y+v^5_x})\,dxdy \\ &+ D\mu\int_\Omega u^3_x\overline{v^5_y}\,dxdy + D\mu\int_\Omega u^5_y\overline{v^3_x}\,dxdy, \end{aligned} \tag{2.11}$$

where $\Theta=(u^1,u^3,u^5)$ and $\widetilde{\Theta}=(v^1,v^3,v^5)$. It is not difficult to see that $a(\cdot,\cdot)$ is continuous and coercive. Then, thanks to the Lax–Milgram theorem (see [6]), equation (2.6) admits a unique solution $U\in\mathscr{D}(\mathscr{A})$. Therefore, we deduce that $0\in\rho(\mathscr{A})$, where $\rho(\mathscr{A})$ is the resolvent set of $\mathscr{A}$. Then by the resolvent identity, for small $\lambda>0$, we have $R(\lambda I-\mathscr{A})=\mathscr{H}$ (see Theorem 1.2.4 in [14]). Finally, thanks to the Lumer–Phillips theorem (see [19], Theorem 1.4.3), the operator $\mathscr{A}$ generates a C_0-semigroup of contractions $e^{t\mathscr{A}}$ on $\mathscr{H}$. □

We introduce the energy functional of equations (1.1)–(1.9). It is given by

$$\begin{aligned} E(t) :=&\ \frac{1}{2}\int_\Omega \big[\rho_1|\omega_t|^2+\rho_2|\psi_t|^2+\rho_2|\varphi_t|^2+K|\psi+\omega_x|^2+K|\varphi+\omega_y|^2+D|\psi_x|^2 \\ &+D|\varphi_y|^2+D\left(\frac{1-\mu}{2}\right)|\psi_y+\varphi_x|^2+2D\mu\psi_x\varphi_y\big]\,dxdy, \quad \text{for} \quad t\geqslant 0. \end{aligned} \tag{2.12}$$

It is immediate that the energy functional (2.12) is a monotone decreasing function of the time t. Indeed, to see this we have the following Proposition:

Proposition 2.3 *Let $(\omega,\omega_t,\varphi,\varphi_t,\psi,\psi_t)$ be the solution of (1.1)–(1.9). Then, the instantaneous rate of change of energy of the system with respect to time t is given by*

$$\frac{d}{dt}E(t) = -d_1\int_\Omega |\omega_t|^2\,dxdy \ \leqslant 0,\ \forall t\geqslant 0. \tag{2.13}$$

Proof As usual, we can find that if we multiply formally equations (1.1)–(1.3) by ω_t, ψ_t and φ_t, respectively. Then using integration by parts, we obtain

$$\begin{aligned} &\frac{\rho_1}{2}\frac{d}{dt}\int_\Omega |\omega_t|^2\,dxdy + \frac{\rho_2}{2}\frac{d}{dt}\int_\Omega |\psi_t|^2\,dxdy + \frac{\rho_2}{2}\frac{d}{dt}\int_\Omega |\varphi_t|^2\,dxdy + K\int_\Omega (\psi+\omega_x)\omega_{xt}\,dxdy \\ &+K\int_\Omega(\varphi+\omega_y)\omega_{yt}\,dxdy + K\int_\Omega(\psi+\omega_x)\psi_t\,dxdy + K\int_\Omega(\varphi+\omega_y)\varphi_t\,dxdy \\ &+\frac{D}{2}\frac{d}{dt}\int_\Omega|\psi_x|^2\,dxdy + \frac{D}{2}\frac{d}{dt}\int_\Omega|\varphi_y|^2\,dxdy + \frac{D}{2}\left(\frac{1-\mu}{2}\right)\frac{d}{dt}\int_\Omega|\psi_y|^2\,dxdy \\ &+\frac{D}{2}\left(\frac{1-\mu}{2}\right)\int_\Omega \varphi_x\psi_{yt}\,dxdy + \frac{D}{2}\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_y\varphi_{xt}\,dxdy \end{aligned}$$

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\begin{aligned}
&+\frac{D}{2}\left(\frac{1-\mu}{2}\right)\frac{d}{dt}\int_{\Omega}|\varphi_x|^2\,dxdy+D\mu\int_{\Omega}\varphi_y\psi_{xt}\,dxdy+D\mu\int_{\Omega}\psi_x\varphi_{yt}\,dxdy\\
&-K\int_{\Gamma}(\psi+\omega_x)\nu_1\omega_t\,d\Gamma-K\int_{\Gamma}(\varphi+\omega_y)\nu_2\omega_t\,d\Gamma\\
&-D\int_{\Gamma_1}\left(\frac{1-\mu}{2}(\varphi_x+\psi_y),\varphi_y+\mu\psi_x\right)\cdot\nu\varphi_t\,d\Gamma_1\\
&-D\int_{\Gamma_2}\left(\psi_x+\mu\varphi_y,\frac{1-\mu}{2}(\varphi_x+\psi_y)\right)\cdot\nu\psi_t\,d\Gamma_2=-d_1\int_{\Omega}|\omega_t|^2\,dxdy.
\end{aligned}
$$

Thus, we arrive at

$$
\begin{aligned}
&\frac{\rho_1}{2}\frac{d}{dt}\int_{\Omega}|\omega_t|^2\,dxdy+\frac{\rho_2}{2}\frac{d}{dt}\int_{\Omega}|\psi_t|^2\,dxdy+\frac{\rho_2}{2}\frac{d}{dt}\int_{\Omega}|\varphi_t|^2\,dxdy\\
&\quad+\frac{K}{2}\frac{d}{dt}\int_{\Omega}|\psi+\omega_x|^2\,dxdy+\frac{K}{2}\frac{d}{dt}\int_{\Omega}|\varphi+\omega_y|^2\,dxdy\\
&\quad+\frac{D}{2}\frac{d}{dt}\int_{\Omega}|\psi_x|^2\,dxdy+\frac{D}{2}\frac{d}{dt}\int_{\Omega}|\varphi_y|^2\,dxdy+\frac{D}{2}\left(\frac{1-\mu}{2}\right)\frac{d}{dt}\int_{\Omega}|\psi_y+\varphi_x|^2\,dxdy\\
&\quad+2D\mu\int_{\Omega}\psi_x\varphi_y\,dxdy-\mathscr{F}=-d_1\int_{\Omega}|\omega_t|^2\,dxdy.
\end{aligned}
$$

where $\mathscr{F}$ is given by

$$
\begin{aligned}
\mathscr{F}=K&\int_{\Gamma}(\psi+\omega_x)\nu_1\omega_t\,d\Gamma+K\int_{\Gamma}(\varphi+\omega_y)\nu_2\omega_t\,d\Gamma\\
&+D\int_{\Gamma_2}\left(\psi_x+\mu\varphi_y,\frac{1-\mu}{2}(\varphi_x+\psi_y)\right)\cdot\nu\psi_t\,d\Gamma_2\\
&+D\int_{\Gamma_1}\left(\frac{1-\mu}{2}(\varphi_x+\psi_y),\varphi_y+\mu\psi_x\right)\cdot\nu\varphi_t\,d\Gamma_1.
\end{aligned}
$$

Therefore, from boundary conditions (1.7)–(1.9), we obtain that $\mathscr{F}=0$ and then

$$
\frac{d}{dt}E(t):=-d_1\int_{\Omega}|\omega_t|^2\,dxdy\leqslant 0,\ \forall t\geqslant 0, \tag{2.14}
$$

since $d_1>0$. Hence, we obtain the energy dissipation law

$$
E(t)\leqslant E(0),\quad \forall t\geqslant 0. \tag{2.15}
$$

It is clear that if $d_1=0$, we obtain the energy conservation law

$$
E(t)=E(0),\quad \forall t\geqslant 0. \tag{2.16}
$$

□

Remark 1 *An important physical property of the energy of the hyperbolic systems says respect to the positivity of the energy. In that direction, we note that $E(t)\geqslant 0$ for $0<\mu<1$.*

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Indeed, we have:

$$
\begin{aligned}
E(t) := \frac{1}{2}\int_{\Omega}\Big[&\rho_1|\omega_t|^2+\rho_2|\psi_t|^2+\rho_2|\varphi_t|^2+K|\psi+\omega_x|^2+K|\varphi+\omega_y|^2\\
&+D|\psi_x|^2+D|\varphi_y|^2+D\left(\frac{1-\mu}{2}\right)|\psi_y+\varphi_x|^2+2D\mu\psi_x\varphi_y\\
&+D\mu|\psi_x|^2+D|\varphi_y|^2-D\mu|\psi_x|^2-D|\varphi_y|^2\Big]\,dxdy\\
= \frac{1}{2}\int_{\Omega}\Big[&\rho_1|\omega_t|^2+\rho_2|\psi_t|^2+\rho_2|\varphi_t|^2+K|\psi+\omega_x|^2+K|\varphi+\omega_y|^2\\
&+D(1-\mu)(|\psi_x|^2+|\varphi_y|^2)+D\left(\frac{1-\mu}{2}\right)|\psi_y+\varphi_x|^2\\
&+D\mu|\psi_x+\varphi_y|^2\Big]\,dxdy \geqslant 0, \quad \forall t\geqslant 0.
\end{aligned}
$$

However, μ is the Poisson's ratio and then we take $0<\mu<1/2$.

3 Lack of exponential decay

Our starting point is to show that the semigroup associated with the Reissner–Mindlin–Timoshenko system (1.1)–(1.9) is not exponentially stable if $v_1^2 \neq v_2^2$ where v_1^2 and v_2^2 are defined in (1.16).

To do this we will argue by contradiction, that is, we will show that there exists a sequence of values $(\lambda_n) \subset \mathbb{R}$ with $\lim_{n\to\infty}|\lambda_n| = \infty$ and $U_n = (\omega_n, W_n, \psi_n, \Psi_n, \varphi_n, \Phi_n)' \subset \mathscr{D}(\mathscr{A})$ for $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n}, f_{5n}, f_{6n})' \subset \mathscr{H}$ such that

$$(i\lambda_n I - \mathscr{A})U_n = F_n, \tag{3.1}$$

where F_n is bounded in $\mathscr{H}$ but $\|U_n\|_{\mathscr{H}}$ tends to infinity. Rewriting the spectral equation in terms of its components, we have

$$i\lambda_n\omega_n - W_n = f_n^1, \tag{3.2}$$

$$i\lambda_n W_n - \frac{K}{\rho_1}(\psi_n+\omega_{nx})_x - \frac{K}{\rho_1}(\varphi_n+\omega_{ny})_y + \frac{d_1}{\rho_1}W_n = f_n^2, \tag{3.3}$$

$$i\lambda_n\psi_n - \Psi_n = f_n^3, \tag{3.4}$$

$$i\lambda_n\Psi_n - \frac{D}{\rho_2}\left[\psi_{nxx} + \left(\frac{1-\mu}{2}\right)\psi_{nyy} + \left(\frac{1+\mu}{2}\right)\varphi_{nxy}\right] + \frac{K}{\rho_2}(\psi_n+\omega_{nx}) = f_n^4, \tag{3.5}$$

$$i\lambda_n\varphi_n - \Phi_n = f_n^5, \tag{3.6}$$

$$i\lambda_n\Phi_n - \frac{D}{\rho_2}\left[\left(\frac{1-\mu}{2}\right)\varphi_{nxx} + \varphi_{nyy} + \left(\frac{1+\mu}{2}\right)\psi_{nxy}\right] + \frac{K}{\rho_2}(\varphi_n+\omega_{ny}) = f_n^6. \tag{3.7}$$

Now we are in a position to establish the principal result of this section.

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Theorem 3.1 *Let us suppose that*

$$v_1^2 - v_2^2 \neq 0. \tag{3.8}$$

Then, the semigroup associated with the system (1.1)–(1.9) *is not exponentially stable.*

Proof Let us take $F_n = (0, f_n^2, 0, f_n^4, 0, f_n^6)'$ with

$$\begin{aligned} f_n^2 &:= F^2 \sin(\delta\lambda_1 x)\sin(\delta\lambda_2 y),\\ f_n^4 &:= F^4 \cos(\delta\lambda_1 x)\sin(\delta\lambda_2 y),\\ f_n^6 &:= F^6 \sin(\delta\lambda_1 x)\cos(\delta\lambda_2 y), \end{aligned}$$

where

$$\lambda_j = \lambda_{j,n} := \frac{n\pi}{\delta L_j}, \quad j = 1, 2, \quad (n \in \mathbb{N}), \quad \delta := \sqrt{\frac{\rho_2}{D}}.$$

Here F^2, F^4 and F^6 are constants that will be chosen suitably. Now, we define

$$\lambda_n := \sqrt{\lambda_1^2 + \lambda_2^2}. \tag{3.9}$$

Taking into account the above, the equations (3.2)–(3.7) can be rewritten as

$$-\lambda_n^2\rho_1\omega_n - K(\psi_n + \omega_{nx})_x - K(\varphi_n + \omega_{ny})_y + i\lambda_n d_1\omega_n = \rho_1 f_n^2, \tag{3.10}$$

$$-\lambda_n^2\rho_2\psi_n - D\left[\psi_{nxx} + \left(\frac{1-\mu}{2}\right)\psi_{nyy} + \left(\frac{1+\mu}{2}\right)\varphi_{nxy}\right] + K(\psi_n + \omega_{nx}) = \rho_2 f_n^4, \tag{3.11}$$

$$-\lambda_n^2\rho_2\varphi_n - D\left[\left(\frac{1-\mu}{2}\right)\varphi_{nxx} + \varphi_{nyy} + \left(\frac{1+\mu}{2}\right)\psi_{nxy}\right] + K(\varphi_n + \omega_{ny}) = \rho_2 f_n^6. \tag{3.12}$$

Now, we choose

$$\begin{aligned} \omega_n(x,y) &:= A\sin(\delta\lambda_1 x)\sin(\delta\lambda_2 y),\\ \psi_n(x,y) &:= B\cos(\delta\lambda_1 x)\sin(\delta\lambda_2 y),\\ \varphi_n(x,y) &:= C\sin(\delta\lambda_1 x)\cos(\delta\lambda_2 y), \end{aligned}$$

where A, B and C depend on λ_n and will be determined explicitly in what follows. Note that this choice is just compatible with boundary conditions (1.7)–(1.9). Therefore, the solutions of system (3.10)–(3.12) is equivalent to finding A, B and C such that

$$\left[-\lambda_n^2\rho_1 + K\delta^2\left(\lambda_1^2 + \lambda_2^2\right) + i\lambda_n d_1\right]A + K\delta\lambda_1 B + K\delta\lambda_2 C = \rho_1 F^2, \tag{3.13}$$

$$\begin{aligned} &K\delta\lambda_1 A + \left[-\lambda_n^2\rho_2 + D\delta^2\lambda_1^2 + D\delta^2\lambda_2^2 - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_2^2 + K\right]B\\ &\quad + D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2 C = \rho_2 F^4, \end{aligned} \tag{3.14}$$

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$$K\delta\lambda_2 A + D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2 B + \Big[-\lambda_n^2\rho_2 + D\delta^2\lambda_1^2 + D\delta^2\lambda_2^2 - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1^2 + K\Big]C = \rho_2 F^6. \tag{3.15}$$

Choosing $F^2 = 0$, $F^4 = F^6 = 1$ and taking account the definition of λ_n given by (3.9) and δ, we get

$$\left[-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right]A + K\delta\lambda_1 B + K\delta\lambda_2 C = 0, \tag{3.16}$$

$$K\delta\lambda_1 A + \left[-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_2^2 + K\right]B + D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2 C = \rho_2, \tag{3.17}$$

$$K\delta\lambda_2 A + D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2 B + \left[-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1^2 + K\right]C = \rho_2. \tag{3.18}$$

Solving (3.16), we obtain that A is

$$A = -\frac{K\delta\lambda_1 B + K\delta\lambda_2 C}{-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1}. \tag{3.19}$$

Substituting (3.19) into (3.17) and (3.18), respectively, we arrive at

$$\begin{aligned}&\left[\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_2^2 + K\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_1^2\right]B\\ &\quad + \left[\left(D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_1\lambda_2\right]C\\ &\quad = \rho_2\left[-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right],\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}&\left[\left(D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_1\lambda_2\right]B\\ &\quad + \left[\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1^2 + K\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_2^2\right]C\\ &\quad = \rho_2\left[-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right].\end{aligned} \tag{3.21}$$

Note that the system (3.20)–(3.21) can be rewritten as

$$\beta B + \gamma C = r, \tag{3.22}$$

$$\gamma B + \theta C = r, \tag{3.23}$$

such that its solutions is given by

$$B = \frac{r(\theta - \gamma)}{\beta\theta - \gamma^2}, \qquad C = \frac{r(\beta - \gamma)}{\beta\theta - \gamma^2}, \tag{3.24}$$

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where

$$\beta = \left[\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_2^2 + K\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_1^2\right],$$
$$\theta = \left[\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1^2 + K\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_2^2\right],$$
$$\gamma = \left[\left(D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_1\lambda_2\right],$$
$$r = \rho_2\left[-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right].$$

Thus, we have for B and C the following explicit expressions

$$B = \frac{\rho_2\left[\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1^2 + K\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_2^2\right]}{K\left(-\rho_1\lambda_n^2 + i\lambda_n d_1\right)\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K\right)} - \frac{\rho_2\left[\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_1\lambda_2\right]}{K\left(-\rho_1\lambda_n^2 + i\lambda_n d_1\right)\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K\right)}, \tag{3.25}$$

and

$$C = \frac{\rho_2\left[\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_2^2 + K\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_1^2\right]}{K\left(-\rho_1\lambda_n^2 + i\lambda_n d_1\right)\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K\right)} - \frac{\rho_2\left[\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2\right)\left(-\lambda_n^2\left(\rho_1 - K\frac{\rho_2}{D}\right) + i\lambda_n d_1\right) - K^2\delta^2\lambda_1\lambda_2\right]}{K\left(-\rho_1\lambda_n^2 + i\lambda_n d_1\right)\left(-D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K\right)}. \tag{3.26}$$

Substituting B and C given by (3.25) and (3.26) into (3.19), we get

$$A = -\frac{\delta\rho_2(\lambda_1 + \lambda_2)}{\left(-\rho_1\lambda_n^2 + i\lambda_n d_1\right)}. \tag{3.27}$$

From (3.25)–(3.27), we can conclude that

$$A \to 0, \tag{3.28}$$
$$B \to \left(\frac{L_1L_2 + L_2^2}{L_1^2 + L_2^2}\right)\frac{\rho_2}{\rho_1}\left(\frac{\rho_1}{K} - \frac{\rho_2}{D}\right), \tag{3.29}$$
$$C \to \left(\frac{L_1L_2 + L_1^2}{L_1^2 + L_2^2}\right)\frac{\rho_2}{\rho_1}\left(\frac{\rho_1}{K} - \frac{\rho_2}{D}\right), \tag{3.30}$$

when $n \to \infty$, where L_j is the length of interval $[0, L_j]$, for $j = 1, 2$. Since

$$\Psi_n = i\lambda_n\psi_n,$$

then using the definition of $||U_n||_{\mathscr{H}}$ and hypothesis (3.8), we have

$$\begin{aligned}
||U_n||^2_{\mathscr{H}} &\geqslant \rho_2 \int_\Omega |\Psi_n|^2 dxdy \\
&= \rho_2 \int_\Omega |\lambda_n \psi_n|^2 \, dxdy \\
&= \rho_2 \int_\Omega |\lambda_n B \cos(\delta\lambda_1 x)\sin(\delta\lambda_2 y)|^2 \, dxdy \\
&= \rho_2 \, |\lambda_n B|^2 \, \frac{L_1 L_2}{4} \to \infty \quad \text{as } n \to \infty. \qquad (3.31)
\end{aligned}$$

Therefore, applying the Theorem 1.1 we conclude that the semigroup $S(t)$ associated with the system (1.1)–(1.9) has lack of exponential decay. □

4 Asymptotic stability

In this section, we will show exponential decay as well as polynomial decay using the semigroup associated with the Reissner–Mindlin–Timoshenko system (1.1)–(1.9) according to a dependency between the speeds of wave propagation.

In order to show exponential decay from semigroup associated with the Reissner–Mindlin–Timoshenko system (1.1)–(1.9), first let us consider the product in $\mathscr{H}$ of $U = (\omega, W, \psi, \Psi, \varphi, \Phi)' \in \mathscr{D}(\mathscr{A})$ with the resolvent equation of $\mathscr{A}$, that is

$$i\lambda||U||^2_{\mathscr{H}} - (\mathscr{A}U, U)_{\mathscr{H}} = (F, U)_{\mathscr{H}}.$$

Then, taking the real part and using inequality (2.5), we obtain

$$d_1 \int_\Omega |W|^2 \, dxdy \leqslant ||U||_{\mathscr{H}} ||F||_{\mathscr{H}}. \qquad (4.1)$$

We will show that the resolvent is uniformly bounded over the imaginary axis. Thus, we state the following lemma.

Lemma 4.1 *With the above notation, we have*

$$i\mathbb{R} \subset \rho(\mathscr{A}).$$

Proof Since $(I - \mathscr{A})^{-1}$ is compact in $\mathscr{H}$, to check that $i\mathbb{R} \subset \rho(\mathscr{A})$ it is sufficient to check that $\mathscr{A}$ has no purely imaginary eigenvalue. Suppose that there exists $\lambda_0 \in \mathbb{R}^*$ such that $i\lambda_0$ is an eigenvalue and $U = (\omega, W, \psi, \Psi, \varphi, \Phi)'$ is a normalised eigenvector, that is

$$\mathscr{A}U = i\lambda_0 U.$$

Thus, we get

$$i\lambda_0\omega - W = 0, \qquad (4.2)$$

$$i\lambda_0\rho_1 W - K(\psi + \omega_x)_x - K(\varphi + \omega_y)_y + d_1 W = 0, \qquad (4.3)$$

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$$i\lambda_0\psi - \Psi = 0, \tag{4.4}$$

$$i\lambda_0\rho_2\Psi - D\left(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy} + \frac{1+\mu}{2}\varphi_{xy}\right) + K(\psi + \omega_x) = 0, \tag{4.5}$$

$$i\lambda_0\varphi - \Phi = 0, \tag{4.6}$$

$$i\lambda_0\rho_2\Phi - D\left(\frac{1-\mu}{2}\varphi_{xx} + \varphi_{yy} + \frac{1+\mu}{2}\psi_{xy}\right) + K(\varphi + \omega_y) = 0. \tag{4.7}$$

Therefore, from (4.1) with $F = 0$ we conclude that $W = 0$. Then, from (4.2) we get $\omega = 0$. Now, from (4.3), (4.5) and (4.7) and using Korn's and Poincaré's inequalities we can conclude that $\psi = \varphi = 0$. Finally, using (4.4) and (4.6) we have that $\Psi = \Phi = 0$. This implies that $U = 0$. But this is a contradiction, therefore there is no purely imaginary eigenvalues. □

In particular this result implies that the semigroup is strongly stable, that is

$$S(t)U_0 \to 0,$$

where $S(t) := e^{\mathscr{A}t}$ is the C_0-semigroup of contractions on Hilbert space $\mathscr{H}$ and U_0 is the initial data.

4.1 Exponential decay

Here, we will prove that the C_0-semigroup associated with the Reissner–Mindlin–Timoshenko system (1.1)–(1.9) is exponential stability if and only if

$$v_1^2 = v_2^2,$$

where v_1^2 and v_2^2 are given in (1.16). To do this, let us consider the resolvent equation

$$i\lambda U - \mathscr{A}U = F \quad \text{in } \mathscr{H}, \tag{4.8}$$

which can be rewritten in terms of its components as

$$i\lambda\omega - W = f^1, \tag{4.9}$$

$$i\lambda\rho_1 W - K(\psi + \omega_x)_x - K(\varphi + \omega_y)_y + d_1 W = f^2, \tag{4.10}$$

$$i\lambda\psi - \Psi = f^3, \tag{4.11}$$

$$i\lambda\rho_2\Psi - D\left(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy} + \frac{1+\mu}{2}\varphi_{xy}\right) + K(\psi + \omega_x) = f^4, \tag{4.12}$$

$$i\lambda\varphi - \Phi = f^5, \tag{4.13}$$

$$i\lambda\rho_2\Phi - D\left(\frac{1-\mu}{2}\varphi_{xx} + \varphi_{yy} + \frac{1+\mu}{2}\psi_{xy}\right) + K(\varphi + \omega_y) = f^6, \tag{4.14}$$

where $F = (f^1, f^2, f^3, f^4, f^5, f^6)' \in \mathscr{H}$ and $U = (\omega, W, \psi, \Psi, \varphi, \Phi)' \in \mathscr{D}(\mathscr{A})$. Note that for simplicity in our calculations we put $\rho_1 f^2 = f^2$, $\rho_2 f^4 = f^4$ and $\rho_2 f^6 = f^6$.

We will use a series of lemmas aiming to reached the exponential decay.

Lemma 4.2 *There exists a positive constant* M *such that any strong solution of system* (1.1)–(1.9) *satisfies*

$$\frac{K}{2}\int_\Omega |\psi+\omega_x|^2\,dxdy+\frac{K}{2}\int_\Omega |\varphi+\omega_y|^2\,dxdy \leqslant C_1\Bigg[\int_\Omega |\psi_x|^2\,dxdy + \int_\Omega |\varphi_y|^2\,dxdy+\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y+\varphi_x|^2\,dxdy + \int_\Omega \mu\varphi_y\overline{\psi}_x\,dxdy+\int_\Omega \mu\psi_x\overline{\varphi}_y\,dxdy\Bigg]+M\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}, \tag{4.15}$$

where C_1 *is a positive constant.*

Proof Multiplying equation (4.10) by $\overline{\varpi}$ and integrating on Ω, we get

$$\underbrace{i\lambda\rho_1\int_\Omega W\overline{\varpi}\,dxdy}_{:=I_1}-K\int_\Omega(\psi+\omega_x)_x\overline{\varpi}\,dxdy-K\int_\Omega(\varphi+\omega_y)_y\overline{\varpi}\,dxdy + d_1\int_\Omega W\overline{\varpi}\,dxdy=\int_\Omega f^2\overline{\varpi}\,dxdy.$$

Substituting ω given by (4.9) into I_1 and integrating by parts, we get

$$-\rho_1\int_\Omega W(\overline{f^1+W})\,dxdy+K\int_\Omega(\psi+\omega_x)\overline{\varpi}_x\,dxdy + K\int_\Omega(\varphi+\omega_y)\overline{\varpi}_y\,dxdy-K\int_\Gamma(\psi+\omega_x)\overline{\varpi}\nu_1\,d\Gamma - K\int_\Gamma(\varphi+\omega_y)\overline{\varpi}\nu_2\,d\Gamma+d_1\int_\Omega W\overline{\varpi}\,dxdy=\int_\Omega f^2\overline{\varpi}\,dxdy,$$

from where it follows using the boundary conditions (1.7)–(1.9) that

$$K\int_\Omega|\psi+\omega_x|^2\,dxdy+K\int_\Omega|\varphi+\omega_y|^2\,dxdy=\rho_1\int_\Omega|W|^2\,dxdy + \underbrace{K\int_\Omega(\psi+\omega_x)\overline{\psi}\,dxdy+K\int_\Omega(\varphi+\omega_y)\overline{\varphi}\,dxdy}_{:=I_2}+\rho_1\int_\Omega W\overline{f^1}\,dxdy \underbrace{-d_1\int_\Omega W\overline{\varpi}\,dxdy}_{:=I_3}+\int_\Omega f^2\overline{\varpi}\,dxdy. \tag{4.16}$$

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Using Lemma 2.1, we have that

$$\begin{aligned} I_2 &\leqslant \frac{K}{2}\int_\Omega |\psi+\omega_x|^2\,dxdy + \frac{K}{2}\int_\Omega |\psi|^2\,dxdy + \frac{K}{2}\int_\Omega |\varphi+\omega_y|^2\,dxdy + \frac{K}{2}\int_\Omega |\varphi|^2\,dxdy \\ &\leqslant \frac{K}{2}\int_\Omega |\psi+\omega_x|^2\,dxdy + \frac{K}{2}\int_\Omega |\varphi+\omega_y|^2\,dxdy \\ &\quad + C_1\left[\int_\Omega |\psi_x|^2+|\varphi_y|^2+\left(\frac{1-\mu}{2}\right)|\psi_y+\varphi_x|^2+\mu\varphi_y\overline{\psi}_x+\mu\psi_x\overline{\varphi}_y\,dxdy\right]. \end{aligned} \tag{4.17}$$

In addition, we have

$$\begin{aligned} I_3 &\leqslant \frac{d_1}{|\lambda|}\int_\Omega |W||W+f^1|\,dxdy \\ &\leqslant \frac{d_1}{|\lambda|}\int_\Omega |W|^2\,dxdy + \frac{d_1}{|\lambda|}\int_\Omega |W||f^1|\,dxdy. \end{aligned} \tag{4.18}$$

Substituting I_2 and I_3 into (4.16), we get

$$\begin{aligned} &\frac{K}{2}\int_\Omega |\psi+\omega_x|^2\,dxdy + \frac{K}{2}\int_\Omega |\varphi+\omega_y|^2\,dxdy \leqslant \left(\rho_1+\frac{d_1}{|\lambda|}\right)\int_\Omega |W|^2\,dxdy \\ &\quad + C_1\left[\int_\Omega |\psi_x|^2+|\varphi_y|^2+\left(\frac{1-\mu}{2}\right)|\psi_y+\varphi_x|^2+\mu\varphi_y\overline{\psi}_x+\mu\psi_x\overline{\varphi}_y\,dxdy\right] \\ &\quad + \frac{d_1}{|\lambda|}\int_\Omega |W||f^1|\,dxdy + \int_\Omega f^2\overline{\omega}\,dxdy. \end{aligned}$$

From the above inequality and from (4.1), we conclude that there exists a positive constant M such that

$$\begin{aligned} &\frac{K}{2}\int_\Omega |\psi+\omega_x|^2\,dxdy + \frac{K}{2}\int_\Omega |\varphi+\omega_y|^2\,dxdy \\ &\quad \leqslant C_1\left[\int_\Omega |\psi_x|^2+|\varphi_y|^2+\left(\frac{1-\mu}{2}\right)|\psi_y+\varphi_x|^2+\mu\varphi_y\overline{\psi}_x+\mu\psi_x\overline{\varphi}_y\,dxdy\right] \\ &\quad + M||U||_{\mathscr{H}}||F||_{\mathscr{H}}, \end{aligned}$$

from where our conclusion follows. The proof is now complete. □

Lemma 4.3 *There exists a positive constant* M *such that any strong solution of system* (1.1)–(1.9) *satisfies*

$$\begin{aligned} \rho_2\int_\Omega |\Psi|^2\,dxdy + \rho_2\int_\Omega |\Phi|^2\,dxdy &\leqslant C_2\left[\int_\Omega |\psi_x|^2\,dxdy\right. \\ &+ D\int_\Omega |\varphi_y|^2\,dxdy + D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y+\varphi_x|^2\,dxdy \end{aligned}$$

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$$+ D\mu \int_\Omega \varphi_y \overline{\psi}_x \, dxdy + D\mu \int_\Omega \psi_x \overline{\varphi}_y \, dxdy \Big] + \frac{K}{2} \int_\Omega |\psi + \omega_x|^2 \, dxdy$$
$$+ \frac{K}{2} \int_\Omega |\varphi + \omega_y|^2 \, dxdy + M||U||_{\mathscr{H}} ||F||_{\mathscr{H}}, \quad (4.19)$$

where C_2 is a positive constant.

Proof Multiplying equation (4.12) by $\overline{\psi}$ and integrating on Ω, we get

$$\underbrace{i\lambda\rho_2 \int_\Omega \Psi \overline{\psi} \, dxdy}_{:=I_4} - D \int_\Omega \psi_{xx} \overline{\psi} \, dxdy - D \left(\frac{1-\mu}{2}\right) \int_\Omega \psi_{yy} \overline{\psi} \, dxdy$$
$$- D \left(\frac{1-\mu}{2}\right) \int_\Omega \varphi_{xy} \overline{\psi} \, dxdy - D\mu \int_\Omega \varphi_{xy} \overline{\psi} \, dxdy$$
$$+ K \int_\Omega (\psi + \omega_x) \overline{\psi} \, dxdy = \int_\Omega f^4 \overline{\psi} \, dxdy.$$

Substituting ψ given by (4.11) into I_4 and integrating by parts, we get

$$- \rho_2 \int_\Omega \Psi (\overline{f^3 + \Psi}) \, dxdy + D \int_\Omega |\psi_x|^2 \, dxdy$$
$$+ D \left(\frac{1-\mu}{2}\right) \int_\Omega |\psi_y|^2 \, dxdy + D \left(\frac{1-\mu}{2}\right) \int_\Omega \varphi_x \overline{\psi}_y \, dxdy$$
$$+ D\mu \int_\Omega \varphi_y \overline{\psi}_x \, dxdy + K \int_\Omega (\psi + \omega_x) \overline{\psi} \, dxdy$$
$$- D \int_{\Gamma_2} \left(\psi_x + \mu \varphi_y, \frac{1-\mu}{2} (\varphi_x + \psi_y) \right) \cdot \nu \overline{\psi} \, d\Gamma_2$$
$$= \int_\Omega f^4 \overline{\psi} \, dxdy,$$

from where it follows using the boundary conditions (1.8)–(1.9) that

$$\rho_2 \int_\Omega |\Psi|^2 \, dxdy = D \int_\Omega |\psi_x|^2 \, dxdy$$
$$+ D \left(\frac{1-\mu}{2}\right) \int_\Omega |\psi_y|^2 \, dxdy + D \left(\frac{1-\mu}{2}\right) \int_\Omega \varphi_x \overline{\psi}_y \, dxdy$$
$$+ D\mu \int_\Omega \varphi_y \overline{\psi}_x \, dxdy + K \int_\Omega (\psi + \omega_x) \overline{\psi} \, dxdy$$
$$- \rho_2 \int_\Omega \Psi \overline{f^3} \, dxdy - \int_\Omega f^4 \overline{\psi} \, dxdy. \quad (4.20)$$

On the other hand, multiplying equation (4.14) by $\overline{\varphi}$, integrating by parts on Ω, using

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(4.13) and finally using the boundary conditions (1.8)–(1.9), we get

$$\begin{aligned}\rho_2 \int_\Omega |\Phi|^2\, dxdy &= D \int_\Omega |\varphi_y|^2\, dxdy \\ &+ D\left(\frac{1-\mu}{2}\right)\int_\Omega |\varphi_x|^2\, dxdy + D\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_y \overline{\varphi}_x\, dxdy \\ &+ D\mu \int_\Omega \psi_x \overline{\varphi}_y\, dxdy + K\int_\Omega (\varphi+\omega_y)\overline{\varphi}\, dxdy \\ &- \rho_2 \int_\Omega \Phi \overline{f^5}\, dxdy - \int_\Omega f^6 \overline{\varphi}\, dxdy. \end{aligned} \tag{4.21}$$

Summing the equalities (4.20) and (4.21), we arrive at

$$\begin{aligned}\rho_2 \int_\Omega |\Psi|^2\, dxdy + \rho_2 \int_\Omega |\Phi|^2\, dxdy &= D\int_\Omega |\psi_x|^2\, dxdy \\ &+ D\int_\Omega |\varphi_y|^2\, dxdy + D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y+\varphi_x|^2\, dxdy \\ &+ D\mu\int_\Omega \varphi_y\overline{\psi}_x\, dxdy + D\mu\int_\Omega \psi_x\overline{\varphi}_y\, dxdy + K\int_\Omega (\psi+\omega_x)\overline{\psi}\, dxdy \\ &+ K\int_\Omega(\varphi+\omega_y)\overline{\varphi}\, dxdy - \rho_2\int_\Omega \Psi\overline{f^3}\, dxdy - \int_\Omega f^4\overline{\psi}\, dxdy \\ &- \rho_2\int_\Omega \Phi\overline{f^5}\, dxdy - \int_\Omega f^6\overline{\varphi}\, dxdy. \end{aligned} \tag{4.22}$$

Substituting (4.17) into (4.22), we get

$$\begin{aligned}\rho_2 \int_\Omega |\Psi|^2\, dxdy + \rho_2 \int_\Omega |\Phi|^2\, dxdy &\leqslant C_2\Big[\int_\Omega |\psi_x|^2\, dxdy \\ &+ D\int_\Omega |\varphi_y|^2\, dxdy + D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y+\varphi_x|^2\, dxdy \\ &+ D\mu\int_\Omega \varphi_y\overline{\psi}_x\, dxdy + D\mu\int_\Omega \psi_x\overline{\varphi}_y\, dxdy\Big] + \frac{K}{2}\int_\Omega |\psi+\omega_x|^2\, dxdy \\ &+ \frac{K}{2}\int_\Omega|\varphi+\omega_y|^2\, dxdy - \rho_2\int_\Omega \Psi\overline{f^3}\, dxdy - \int_\Omega f^4\overline{\psi}\, dxdy \\ &- \rho_2\int_\Omega \Phi\overline{f^5}\, dxdy - \int_\Omega f^6\overline{\varphi}\, dxdy, \end{aligned}$$

from where we can conclude that

$$\begin{aligned}\rho_2 \int_\Omega |\Psi|^2\, dxdy + \rho_2 \int_\Omega |\Phi|^2\, dxdy &\leqslant C_2\Big[\int_\Omega |\psi_x|^2\, dxdy \\ &+ D\int_\Omega |\varphi_y|^2\, dxdy + D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y+\varphi_x|^2\, dxdy \end{aligned}$$

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$$+ D\mu \int_\Omega \varphi_y \overline{\psi}_x \, dxdy + D\mu \int_\Omega \psi_x \overline{\varphi}_y \, dxdy \Big] + \frac{K}{2} \int_\Omega |\psi + \omega_x|^2 \, dxdy$$
$$+ \frac{K}{2} \int_\Omega |\varphi + \omega_y|^2 \, dxdy + M||U||_{\mathscr{H}} ||F||_{\mathscr{H}}.$$

The proof is now complete. □

The next lemma gives the important relation between the coefficients of the Reissner–Mindlin–Timoshenko for obtaining the necessary and sufficient condition for exponential stability of system (1.1)–(1.9).

Lemma 4.4 *There exists a positive constant* M *such that any strong solution of system* (1.1)–(1.9) *satisfies*

$$\begin{aligned} D\Bigg[\int_\Omega |\psi_x|^2 \, dxdy + \int_\Omega |\varphi_y|^2 \, dxdy + \left(\frac{1-\mu}{2}\right) \int_\Omega |\psi_y + \varphi_x|^2 \, dxdy \\ + \mu \int_\Omega \varphi_y \overline{\psi}_x \, dxdy + \mu \int_\Omega \psi_x \overline{\varphi}_y \, dxdy \Bigg] \\ \leqslant |\lambda| \left| \frac{D\rho_1}{K} - \rho_2 \right| \int_\Omega \left| \overline{W} \right| |\psi_x + \varphi_y| \, dxdy + M||U||_{\mathscr{H}} ||F||_{\mathscr{H}}. \end{aligned} \tag{4.23}$$

Proof Multiplying equation (4.12) by $\overline{\omega}_x$, integrating by parts on Ω and using (4.11), we have

$$\begin{aligned} & i\lambda\rho_2 \int_\Omega \Psi \overline{\omega}_x \, dxdy + D \int_\Omega \psi_x \overline{\omega}_{xx} \, dxdy \\ & + D\left(\frac{1-\mu}{2}\right) \int_\Omega \psi_y \overline{\omega}_{xy} \, dxdy + D\left(\frac{1-\mu}{2}\right) \int_\Omega \varphi_x \overline{\omega}_{xy} \, dxdy \\ & + D\mu \int_\Omega \varphi_y \overline{\omega}_{xx} \, dxdy + K \int_\Omega (\psi + \omega_x) \overline{\omega}_x \, dxdy \\ & - D \int_\Gamma \left(\psi_x + \mu\varphi_y, \frac{1-\mu}{2} (\varphi_x + \psi_y) \right) \cdot \nu \overline{\omega}_x \, d\Gamma \\ & = \int_\Omega f^4 \overline{\omega}_x \, dxdy. \end{aligned} \tag{4.24}$$

On the other hand, multiplying equation (4.14) by $\overline{\omega}_y$, integrating by parts on Ω and using equation (4.13), we obtain

$$\begin{aligned} & i\lambda\rho_2 \int_\Omega \Phi \overline{\omega}_y \, dxdy + D\left(\frac{1-\mu}{2}\right) \int_\Omega \varphi_x \overline{\omega}_{xy} \, dxdy \\ & + D \int_\Omega \varphi_y \overline{\omega}_{yy} \, dxdy + D\left(\frac{1-\mu}{2}\right) \int_\Omega \psi_y \overline{\omega}_{xy} \, dxdy \\ & + D\mu \int_\Omega \psi_x \overline{\omega}_{yy} \, dxdy + K \int_\Omega (\varphi + \omega_y) \overline{\omega}_y \, dxdy \end{aligned}$$

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$$- D \int_{\Gamma} \left(\frac{1-\mu}{2}(\varphi_x + \psi_y), \varphi_y + \mu\psi_x \right) \cdot \nu \overline{\omega}_y \, d\Gamma$$
$$= \int_{\Omega} f^6 \overline{\omega}_y \, dxdy. \tag{4.25}$$

Summing up the results and taking into account the boundary conditions (1.7)–(1.9), it follows that

$$\begin{aligned} & i\lambda\rho_2 \int_{\Omega} \Psi \overline{\omega}_x \, dxdy + i\lambda\rho_2 \int_{\Omega} \Phi \overline{\omega}_y \, dxdy \\ & \underbrace{+D \int_{\Omega} \psi_x \overline{\omega}_{xx} \, dxdy + D \int_{\Omega} \varphi_y \overline{\omega}_{yy} \, dxdy}_{:=I_4} \\ & + 2D\left(\frac{1-\mu}{2}\right) \int_{\Omega} \psi_y \overline{\omega}_{xy} \, dxdy + 2D\left(\frac{1-\mu}{2}\right) \int_{\Omega} \varphi_x \overline{\omega}_{xy} \, dxdy \\ & + D\mu \int_{\Omega} \varphi_y \overline{\omega}_{xx} \, dxdy + D\mu \int_{\Omega} \psi_x \overline{\omega}_{yy} \, dxdy \\ & + K \int_{\Omega} (\psi + \omega_x)\overline{\omega}_x \, dxdy + K \int_{\Omega} (\varphi + \omega_y)\overline{\omega}_y \, dxdy \\ & = \int_{\Omega} f^4 \overline{\omega}_x \, dxdy + \int_{\Omega} f^6 \overline{\omega}_y \, dxdy. \end{aligned} \tag{4.26}$$

On the other hand, from (4.10) we have

$$\begin{aligned} & - i\lambda\rho_1 \int_{\Omega} \overline{W}(\psi_x + \varphi_y) \, dxdy - K \int_{\Omega} (\overline{\psi + \omega_x})_x \psi_x \, dxdy \\ & - K \int_{\Omega} (\overline{\psi + \omega_x})_x \varphi_y \, dxdy - K \int_{\Omega} (\overline{\varphi + \omega_y})_y \psi_x \, dxdy \\ & - K \int_{\Omega} (\overline{\varphi + \omega_y})_y \varphi_y \, dxdy + d_1 \int_{\Omega} \overline{W}(\psi_x + \varphi_y) \, dxdy \\ & = \int_{\Omega} \overline{f^2}(\psi_x + \varphi_y) \, dxdy, \end{aligned}$$

from where it follows that

$$\begin{aligned} & K \int_{\Omega} \psi_x \overline{\omega}_{xx} \, dxdy + K \int_{\Omega} \varphi_y \overline{\omega}_{yy} \, dxdy \\ & = -i\lambda\rho_1 \int_{\Omega} \overline{W}(\psi_x + \varphi_y) \, dxdy - K \int_{\Omega} |\psi_x|^2 \, dxdy \\ & - K \int_{\Omega} |\varphi_y|^2 \, dxdy - K \int_{\Omega} (\overline{\psi + \omega_x})_x \varphi_y \, dxdy \\ & - K \int_{\Omega} (\overline{\varphi + \omega_y})_y \psi_x \, dxdy + d_1 \int_{\Omega} \overline{W}(\psi_x + \varphi_y) \, dxdy \\ & - \int_{\Omega} \overline{f^2}(\psi_x + \varphi_y) \, dxdy. \end{aligned} \tag{4.27}$$

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Substituting (4.27) into I_4, we can rewrite (4.26) as follow

$$
\begin{aligned}
& i\lambda\rho_2 \int_\Omega \Psi \overline{\omega}_x \, dxdy + i\lambda\rho_2 \int_\Omega \Phi \overline{\omega}_y \, dxdy \\
& \quad - \frac{D}{K}\left[i\lambda\rho_1 \int_\Omega \overline{W}(\psi_x+\varphi_y)\, dxdy + K\int_\Omega |\psi_x|^2 \, dxdy \right.\\
& \quad + K\int_\Omega |\varphi_y|^2\, dxdy + K\int_\Omega (\overline{\psi+\omega_x})_x \varphi_y \, dxdy \\
& \quad + K\int_\Omega (\overline{\varphi+\omega_y})_y \psi_x \, dxdy - d_1 \int_\Omega \overline{W}(\psi_x+\varphi_y)\, dxdy \\
& \quad \left. + \int_\Omega \overline{f^2}(\psi_x+\varphi_y)\, dxdy \right] + 2D\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_y \overline{\omega}_{xy}\, dxdy \\
& \quad + 2D\left(\frac{1-\mu}{2}\right)\int_\Omega \varphi_x \overline{\omega}_{xy}\, dxdy + D\mu \int_\Omega \varphi_y \overline{\omega}_{xx}\, dxdy \\
& \quad + D\mu\int_\Omega \psi_x \overline{\omega}_{yy}\, dxdy + K\int_\Omega (\psi+\omega_x)\overline{\omega}_x\, dxdy \\
& \quad + K\int_\Omega (\varphi+\omega_y)\overline{\omega}_y \, dxdy = \int_\Omega f^4 \overline{\omega}_x\, dxdy + \int_\Omega f^6 \overline{\omega}_y \, dxdy,
\end{aligned}
$$

from where it follows that

$$
\begin{aligned}
& \underbrace{i\lambda\rho_2 \int_\Omega \Psi \overline{\omega}_x \, dxdy}_{:=I_5} + \underbrace{i\lambda\rho_2 \int_\Omega \Phi \overline{\omega}_y \, dxdy}_{:=I_6} \\
& \quad - i\lambda \frac{D\rho_1}{K}\int_\Omega \overline{W}(\psi_x+\varphi_y)\, dxdy - D\int_\Omega |\psi_x|^2\, dxdy \\
& \quad - D\int_\Omega |\varphi_y|^2 \, dxdy - D\int_\Omega (\overline{\psi+\omega_x})_x \varphi_y\, dxdy \\
& \quad - D\int_\Omega (\overline{\varphi+\omega_y})_y \psi_x \, dxdy + 2D\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_y\overline{\omega}_{xy}\, dxdy \\
& \quad + 2D\left(\frac{1-\mu}{2}\right)\int_\Omega \varphi_x \overline{\omega}_{xy}\, dxdy + D\mu\int_\Omega \varphi_y \overline{\omega}_{xx}\, dxdy \\
& \quad + D\mu \int_\Omega \psi_x \overline{\omega}_{yy}\, dxdy + K\int_\Omega (\psi+\omega_x)\overline{\omega}_x \, dxdy \\
& \quad + K\int_\Omega (\varphi+\omega_y)\overline{\omega}_y\, dxdy + \frac{d_1 D}{K}\int_\Omega \overline{W}(\psi_x+\varphi_y)\, dxdy \\
& \quad = \frac{D}{K}\int_\Omega \overline{f^2}(\psi_x+\varphi_y)\, dxdy + \int_\Omega f^4\overline{\omega}_x\, dxdy + \int_\Omega f^6 \overline{\omega}_y\, dxdy. \qquad (4.28)
\end{aligned}
$$

Substituting ω given by (4.9) into I_5 and I_6, we have

$$
\begin{aligned}
I_5 + I_6 = & \underbrace{\rho_2 \int_\Omega \overline{W}\Psi_x \, dxdy}_{:=I_7} + \underbrace{\rho_2\int_\Omega \overline{W}\Phi_y\, dxdy}_{:=I_8} \\
& + \rho_2 \int_\Omega \overline{f^1}\Psi_x \, dxdy + \rho_2\int_\Omega \overline{f^1}\Phi_y\, dxdy.
\end{aligned}
$$

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Substituting Ψ given by (4.11) and Φ given by (4.13) into I_7 and I_8, respectively, we get

$$I_5 + I_6 = i\lambda\rho_2 \int_\Omega \overline{W}(\psi_x + \varphi_y)\,dxdy - \rho_2 \int_\Omega \overline{W}(f_x^3 + f_y^5)\,dxdy \\ - \rho_2 \int_\Omega \overline{f_x^1}\Psi\,dxdy - \rho_2 \int_\Omega \overline{f_y^1}\Phi\,dxdy. \tag{4.29}$$

Substituting (4.29) into (4.28) and after simplifications, we get

$$D\int_\Omega |\psi_x|^2\,dxdy + D\int_\Omega |\varphi_y|^2\,dxdy + D\int_\Omega \overline{\psi}_y\varphi_x\,dxdy + D\int_\Omega \overline{\varphi}_x\psi_y\,dxdy \\ = i\lambda\left(\rho_2 - \frac{D\rho_1}{K}\right)\int_\Omega \overline{W}(\psi_x+\varphi_y)\,dxdy + K\int_\Omega(\psi+\omega_x)\overline{\varpi}_x\,dxdy \\ + K\int_\Omega(\varphi+\omega_y)\overline{\varpi}_y\,dxdy + \frac{d_1 D}{K}\int_\Omega \overline{W}(\psi_x+\varphi_y)\,dxdy \\ - \frac{D}{K}\int_\Omega \overline{f^2}(\psi_x+\varphi_y)\,dxdy - \int_\Omega f^4\overline{\varpi}_x\,dxdy - \int_\Omega f^6\overline{\varpi}_y\,dxdy \\ -\rho_2\int_\Omega \overline{W}(f_x^3+f_y^5)\,dxdy - \rho_2\int_\Omega \overline{f_x^1}\Psi\,dxdy - \rho_2\int_\Omega \overline{f_y^1}\Phi\,dxdy, \tag{4.30}$$

from where it follows that

$$D\int_\Omega |\psi_x|^2\,dxdy + D\int_\Omega |\varphi_y|^2\,dxdy + D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y+\varphi_x|^2\,dxdy \\ + D\mu\int_\Omega \varphi_y\overline{\psi}_x\,dxdy + D\mu\int_\Omega \psi_x\overline{\varphi}_y\,dxdy \\ = i\lambda\left(\rho_2 - \frac{D\rho_1}{K}\right)\int_\Omega \overline{W}(\psi_x+\varphi_y)\,dxdy + D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y-\varphi_x|^2\,dxdy \\ \underbrace{+K\int_\Omega(\psi+\omega_x)\overline{\varpi}_x\,dxdy + K\int_\Omega(\varphi+\omega_y)\overline{\varpi}_y\,dxdy}_{:=I_9} \\ + \frac{d_1 D}{K}\int_\Omega \overline{W}(\psi_x+\varphi_y)\,dxdy - \frac{D}{K}\int_\Omega \overline{f^2}(\psi_x+\varphi_y)\,dxdy \\ - \int_\Omega f^4\overline{\varpi}_x\,dxdy - \int_\Omega f^6\overline{\varpi}_y\,dxdy \\ -\rho_2\int_\Omega \overline{W}(f_x^3+f_y^5)\,dxdy - \rho_2\int_\Omega \overline{f_x^1}\Psi\,dxdy - \rho_2\int_\Omega \overline{f_y^1}\Phi\,dxdy. \tag{4.31}$$

Noting that

$$I_9 = \rho_1\int_\Omega |W|^2\,dxdy + d_1\int_\Omega W\overline{\varpi}\,dxdy + \rho_1\int_\Omega W\overline{f^1}\,dxdy + \int_\Omega f^2\overline{\varpi}\,dxdy \\ \leqslant \left(\rho_1 + \frac{d_1}{|\lambda|}\right)\int_\Omega |W|^2\,dxdy + \int_\Omega f^2\overline{\varpi}\,dxdy + \left(\rho_1 + \frac{d_1}{|\lambda|}\right)\int_\Omega |W||f^1|\,dxdy,$$

and using Lemma 2.1, our conclusion follows. Therefore, the proof is now complete. □

Now we are in the position to prove the main result of this paper.

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Theorem 4.5 *The semigroup associated with the Reissner–Mindlin–Timoshenko system* (1.1)–(1.9) *is exponentially stable if and only if* $v_1^2 = v_2^2$.

Proof From Lemmas 4.2–4.4, we can conclude that

$$||U||^2_{\mathscr{H}} \leqslant M||U||_{\mathscr{H}}||F||_{\mathscr{H}} \quad ,\forall U \in \mathscr{D}(\mathscr{A}),$$

from where we obtain

$$||U||_{\mathscr{H}} \leqslant M, \ \forall U \in \mathscr{D}(\mathscr{A}).$$

Using Prüss's result [20], the conclusion of the theorem follows. □

4.2 Polynomial decay

In this section, we will show that in general the Reissner–Mindlin–Timoshenko system (1.1)–(1.9) goes to zero polynomially as $1/\sqrt{t}$.

Theorem 4.6 *Let us suppose that* $v_1^2 - v_2^2 \neq 0$. *Then, the semigroup associated with the system*(1.1)–(1.9) *is polynomially stable and*

$$||S(t)U_0||_{\mathscr{H}} \leqslant \frac{1}{\sqrt{t}}||U_0||_{\mathscr{D}(\mathscr{A})}. \tag{4.32}$$

Moreover, this rate of decay is optimal, in the sense that decay must be slower than $t^{-\frac{1}{2-\epsilon}}$ *for any* $\epsilon > 0$.

Proof We note that, by using Young's inequality, the following inequality holds

$$\int_\Omega |W||\psi_x + \varphi_y|\, dxdy \leqslant \frac{|\lambda|}{2}\int_\Omega |W|^2\, dxdy + \frac{1}{2|\lambda|}\int_\Omega |\psi_x + \varphi_y|^2\, dxdy, \tag{4.33}$$

for $|\lambda| > 0$. On the other hand, from Lemma 4.4, we get

$$\begin{aligned}\int_\Omega |\psi_x + \varphi_y|^2\, dxdy &\leqslant |\lambda|\left|\frac{D\rho_1}{K} - \rho_2\right|\int_\Omega |W||\psi_x + \varphi_y|\, dxdy + C_1||U||_{\mathscr{H}}||F||_{\mathscr{H}}\\ &\leqslant \frac{|\lambda|^2}{2}\left|\frac{D\rho_1}{K} - \rho_2\right|^2\int_\Omega |W|^2\, dxdy + \frac{1}{2}\int_\Omega |\psi_x + \varphi_y|^2\, dxdy\\ &+ C_2||U||_{\mathscr{H}}||F||_{\mathscr{H}},\end{aligned}$$

which implies that

$$\frac{1}{2}\int_\Omega |\psi_x + \varphi_y|^2\, dxdy \leqslant C_1|\lambda|^2\int_\Omega |W|^2\, dxdy + C_2||U||_{\mathscr{H}}||F||_{\mathscr{H}}, \tag{4.34}$$

where C_1, C_2 are positive constants. Substituting (4.34) into (4.33), we obtain

$$\int_\Omega |W||\psi + \varphi_y|\, dxdy \leqslant \frac{|\lambda|}{2}\int_\Omega |W|^2\, dxdy + C_1|\lambda|\int_\Omega |W|^2\, dxdy + \frac{C_2}{|\lambda|}||U||_{\mathscr{H}}||F||_{\mathscr{H}}.$$

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Then, using (4.1) we have that

$$\int_\Omega |W||\psi + \varphi_y|\, dxdy \leqslant C_3|\lambda|||U||_{\mathscr{H}}||F||_{\mathscr{H}} + \frac{C_2}{|\lambda|}||U||_{\mathscr{H}}||F||_{\mathscr{H}}. \tag{4.35}$$

Combining the Lemmas 4.2–4.4, we obtain that there exists a positive constant M_1 such that

$$||U||^2_{\mathscr{H}} \leqslant M_1|\lambda|^2||U||_{\mathscr{H}}||F||_{\mathscr{H}},$$

for $|\lambda| > 0$ large enough. Therefore, we get

$$\frac{1}{|\lambda|^2}||U||_{\mathscr{H}} \leqslant M_1||F||_{\mathscr{H}},$$

which is equivalent to

$$||(\lambda I - \mathscr{A})^{-1}||_{\mathscr{L}(\mathscr{H})} \leqslant M_1|\lambda|^2.$$

Then using Theorem 2.4 in [5], we obtain

$$||S(t)\mathscr{A}^{-1}||_{\mathscr{L}(\mathscr{H})} = \mathscr{O}(t^{-1/2}) \Rightarrow ||S(t)\mathscr{A}^{-1}||_{\mathscr{L}(\mathscr{H})} \leqslant \frac{M_1}{\sqrt{t}}.$$

Since $0 \in \rho(\mathscr{A})$, it follows that $\mathscr{A}$ is onto $\mathscr{H}$, then taking $\mathscr{A}U_0 = F$, we get

$$||S(t)U_0||_{\mathscr{H}} \leqslant \frac{M_1}{\sqrt{t}}||U_0||_{\mathscr{D}(\mathscr{A})},$$

completing the first assertion of this theorem. To prove that the rate of decay is optimal, we will argue by contradiction. Suppose that the rate $t^{-1/2}$ can be improved. That is to say, that the rate is $t^{-\frac{1}{2-\epsilon}}$ for some $0 < \epsilon < 2$. From Theorem 5.3 in [7] the operator

$$|\lambda|^{-2+\frac{\epsilon}{2}}||(\lambda I - \mathscr{A})^{-1}||_{\mathscr{L}(\mathscr{H})},$$

should be limited, but this does not happen. For this, let us suppose that there exists a sequence $(\lambda_n) \subset \mathbb{R}$ with $\lim_{n\to\infty}|\lambda_n| = \infty$ and $(U_n) \subset \mathscr{D}(\mathscr{A})$ for $(F_n) \subset \mathscr{H}$ such that

$$(i\lambda_n I - \mathscr{A})U_n = F_n.$$

Then, we can consider

$$F_n = \left(0, F^2\sin(\delta\lambda_1 x)\sin(\delta\lambda_2 y), 0, F^4\cos(\delta\lambda_1 x)\sin(\delta\lambda_2 y), 0, F^6\sin(\delta\lambda_1 x)\cos(\delta\lambda_2 y)\right)',$$

for each $n \in \mathbb{N}$, with $F^2 \neq 0, F^4 \neq 0, F^6 \neq 0$ constants, where $\lambda_1 = \frac{n\pi}{\delta L_1}$, $\lambda_2 = \frac{n\pi}{\delta L_2}$, $\delta = \sqrt{\frac{\rho_2}{D}}$ and $U_n = (\omega_n, W_n, \psi_n, \Psi_n, \varphi_n, \Phi_n)'$. Moreover, we choose

$$\begin{aligned}\omega_n &= A\sin(\delta\lambda_1 x)\sin(\delta\lambda_2 y),\\ \psi_n &= B\cos(\delta\lambda_1 x)\sin(\delta\lambda_2 y),\\ \varphi_n &= C\sin(\delta\lambda_1 x)\cos(\delta\lambda_2 y).\end{aligned}$$

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So, choosing

$$\lambda = \lambda_n := \sqrt{\lambda_1^2 + \lambda_2^2} = \mathcal{O}(n), \quad \forall n \in \mathbb{N},$$

and proceeding as in the proof of Theorem 3.1, we can conclude that

$$|\lambda_n|^{-2+\frac{\epsilon}{2}} ||U_n||_{\mathcal{H}} \geqslant \mathcal{O}\left(n^{\frac{\epsilon}{2}}\right) \to \infty,$$

as $n \to \infty$. Therefore, the rate cannot be improved and the proof is now complete. □

5 Numerical approach

In this section, we consider a numerical scheme using finite difference and we reproduce numerically the analytical results established on exponential decay for the Reissner–Mindlin–Timoshenko system. We present a numerical method consistent of second order in all mesh parameters and ensuring naturally decay of energy like obtained in the previous sections.

We are concerned mainly with the lack of exponential decay according with the speeds of wave propagation v_1^2 and v_2^2. More precisely, if (3.8) holds, then the dissipative system of Reissner–Mindlin–Timoshenko treated here is not exponentially stable. Otherwise, we get the exponential decay of solutions.

5.1 Fully-discrete scheme in finite differences and properties

Given $I, J, N \in \mathbb{N}$ we set $\Delta x = \frac{L_1}{I+1}, \Delta y = \frac{L_2}{J+1}$ and $\Delta t = \frac{T}{N+1}$ and we introduce the nets

$$x_0 = 0 < x_1 = \Delta x < \cdots < x_I = I\Delta x < x_{I+1} = (I+1)\Delta x = L_1, \tag{5.1}$$

$$y_0 = 0 < y_1 = \Delta y < \cdots < y_J = J\Delta y < x_{J+1} = (J+1)\Delta y = L_2, \tag{5.2}$$

$$t_0 = 0 < t_1 = \Delta t < \cdots < t_N = N\Delta t < t_{N+1} = (N+1)\Delta t = T, \tag{5.3}$$

with $x_i = i\Delta x,\ y_j = j\Delta y$ and $t_n = n\Delta t$ for $i = 0, 1, 2, \ldots, I+1,\ j = 0, 1, 2, \ldots, J+1$ and $n = 0, 1, 2, \ldots, N+1$.

Taking an explicit scheme using finite differences, our problem consists of finding $(\omega_{i,j}^n, \psi_{i,j}^n, \varphi_{i,j}^n)$ satisfying the following numerical scheme:

$$\rho_1 \bar{\partial}_t \partial_t \omega_{i,j}^n = K \bar{\partial}_x \partial_x \omega_{i,j}^n + K \frac{\partial_x + \bar{\partial}_x}{2} \psi_{i,j}^n + K \bar{\partial}_y \partial_y \omega_{i,j}^n + K \frac{\partial_y + \bar{\partial}_y}{2} \varphi_{i,j}^n - d_1 \frac{\partial_t + \bar{\partial}_t}{2} \omega_{i,j}^n, \tag{5.4}$$

$$\begin{aligned} \rho_2 \bar{\partial}_t \partial_t \psi_{i,j}^n &= D \bar{\partial}_x \partial_x \psi_{i,j}^n + D \frac{1-\mu}{2} \bar{\partial}_y \partial_y \psi_{i,j}^n + D \frac{1+\mu}{2} \left(\frac{\partial_y + \bar{\partial}_y}{2} \frac{\partial_x + \bar{\partial}_x}{2} \right) \varphi_{i,j}^n \\ &\quad - \frac{K}{2} (\psi_{i+1/2,j}^n + \psi_{i-1/2,j}^n + \psi_{i,j+1/2}^n + \psi_{i,j-1/2}^n) - K \frac{\partial_x + \bar{\partial}_x}{2} \omega_{i,j}^n, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \rho_2 \bar{\partial}_t \partial_t \varphi_{i,j}^n &= D \bar{\partial}_y \partial_y \varphi_{i,j}^n + D \frac{1-\mu}{2} \bar{\partial}_x \partial_x \varphi_{i,j}^n + D \frac{1+\mu}{2} \left(\frac{\partial_x + \bar{\partial}_x}{2} \frac{\partial_y + \bar{\partial}_y}{2} \right) \psi_{i,j}^n \\ &\quad - \frac{K}{2} (\varphi_{i+1/2,j}^n + \varphi_{i-1/2,j}^n + \varphi_{i,j+1/2}^n + \varphi_{i,j-1/2}^n) - K \frac{\partial_y + \bar{\partial}_y}{2} \omega_{i,j}^n, \end{aligned} \tag{5.6}$$

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for all $i = 1, 2, \ldots, I$ $j = 1, 2, \ldots, J$ and $n = 1, 2, \ldots, N$. To simplicity our numerical calculations, we consider the homogeneous boundary conditions given by

$$\omega^n_{0,j} = \omega^n_{I+1,j} = u^n_{i,0} = \omega^n_{i,J+1} = 0, \quad \forall n = 1, 2, \ldots, N, \tag{5.7}$$
$$\psi^n_{0,j} = \psi^n_{I+1,j} = \psi^n_{i,0} = \psi^n_{i,J+1} = 0, \quad \forall n = 1, 2, \ldots, N, \tag{5.8}$$
$$\varphi^n_{0,j} = \varphi^n_{I+1,j} = \varphi^n_{i,0} = \varphi^n_{i,J+1} = 0, \quad \forall n = 1, 2, \ldots, N, \tag{5.9}$$

and initial conditions given by

$$\omega^0_{i,j} = \omega(x_i, y_j, 0),\ \omega^1_{i,j} = \omega^0_{i,j} + \Delta t \omega_t(x_i, y_j, 0), \quad \forall i = 1, \ldots, I, j = 1, \ldots, J, \tag{5.10}$$
$$\psi^0_{i,j} = \psi(x_i, y_j, 0),\ \psi^1_{i,j} = \psi^0_{i,j} + \Delta t \psi_t(x_i, y_j, 0), \quad \forall i = 1, \ldots, I, j = 1, \ldots, J, \tag{5.11}$$
$$\varphi^0_{i,j} = \varphi(x_i, y_j, 0),\ \varphi^1_{i,j} = \varphi^0_{i,j} + \Delta t \varphi_t(x_i, y_j, 0), \quad \forall i = 1, \ldots, I, j = 1, \ldots, J. \tag{5.12}$$

The numerical operators used in (5.4)–(5.6) are given by

$$\partial_x \omega^n_{i,j} := \frac{\omega^n_{i+1,j} - \omega^n_{i,j}}{\Delta x},\ \bar{\partial}_x \omega^n_{i,j} := \frac{\omega^n_{i,j} - \omega^n_{i-1,j}}{\Delta x},\ \partial_y \omega^n_{i,j} := \frac{\omega^n_{i,j+1} - \omega^n_{i,j}}{\Delta y},\ \bar{\partial}_y \omega^n_{i,j} := \frac{\omega^n_{i,j} - \omega^n_{i,j-1}}{\Delta y}, \partial_t \omega^n_{i,j} := \frac{\omega^{n+1}_{i,j} - \omega^n_{i,j}}{\Delta t},\quad \bar{\partial}_t \omega^n_{i,j} := \frac{\omega^n_{i,j} - \omega^{n-1}_{i,j}}{\Delta t},$$

$$\frac{\partial_x + \bar{\partial}_x}{2} \omega^n_{i,j} := \frac{\omega^n_{i+1,j} - \omega^n_{i-1,j}}{2\Delta x},\quad \frac{\partial_y + \bar{\partial}_y}{2} \omega^n_{i,j} := \frac{\omega^n_{i,j+1} - \omega^n_{i,j-1}}{2\Delta y},\quad \frac{\partial_t + \bar{\partial}_t}{2} \omega^n_{i,j} := \frac{\omega^{n+1}_{i,j} - \omega^{n-1}_{i,j}}{2\Delta t}, \bar{\partial}_x \partial_x \omega^n_{i,j} := \frac{\omega^n_{i+1,j} - 2\omega^n_{i,j} + \omega^n_{i-1,j}}{\Delta x^2},\ \bar{\partial}_y \partial_y \omega^n_{i,j} := \frac{\omega^n_{i,j+1} - 2\omega^n_{i,j} + \omega^n_{i,j-1}}{\Delta y^2}, \bar{\partial}_t \partial_t \omega^n_{i,j} := \frac{\omega^{n+1}_{i,j} - 2\omega^n_{i,j} + \omega^{n-1}_{i,j}}{\Delta t^2},$$

with the same approximations to the functions ψ and φ on the mesh. Here, we are denoting by $\omega^n_{i,j}, \varphi^n_{i,j}$ and $\psi^n_{i,j}$ the numerical approximations of the exact solutions ω, φ and ψ, respectively, evaluated on the mesh. More precisely, we have $\omega^n_{i,j} \approx \omega(x_i, y_j, t_n)$, $\psi^n_{i,j} \approx \psi(x_i, y_j, t_n)$ and $\varphi^n_{i,j} \approx \varphi(x_i, y_j, t_n)$. Also $\psi^n_{i-1/2,j}$ and $\psi^n_{i+1/2,j}$ denote the average of $\psi^n_{i,j}$ at the points $(x_{i-1}, y_j, t_n), (x_i, y_j, t_n)$ and $(x_{i+1}, y_j, t_n), (x_i, y_j, t_n)$, respectively. Similar meanings hold for $\psi^n_{i,j-1/2}$ and $\psi^n_{i,j+1/2}$.

The numerical scheme presented here is explicit and its computational implementation requires knowledge of the approximations at time level t_n and t_{n-1} in order to approximate the numerical solutions at time level t_{n+1}. We note that the proposed scheme (5.4)–(5.12) is consistent with the problem studied. In particular, the stability criterion in one-dimensional case obeys a relation between the time step Δt and the thickness h (see [24, 25]). It is expected that for the two-dimensional case a similar relation prevails but the proof is still to be done. However, for the purposes of numerical convergence, we fix the thickness h and we choose $\Delta t < \Delta x$ for $\Delta x = \Delta y$.

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5.2 Discrete energy

In this section, we prove that the numerical scheme (5.4)–(5.12) has a property of numerical consistency that preserves the instantaneous rate of change of energy according with Proposition 2.3. With this aim in mind, we present a first property concerning the total energy of discrete system (5.4)–(5.12).

The total energy to the numerical equations (5.4)–(5.12) at the time step t_n will be computed using the expression

$$\begin{aligned}
E^n := \frac{\Delta x \Delta y}{2} \sum_{i=0}^{I} \sum_{j=0}^{J} \Bigg[& \rho_1 \left(\frac{\omega_{i,j}^{n+1} - \omega_{i,j}^n}{\Delta t} \right)^2 + \rho_2 \left(\frac{\psi_{i,j}^{n+1} - \psi_{i,j}^n}{\Delta t} \right)^2 + \rho_2 \left(\frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^n}{\Delta t} \right)^2 \\
& + D \frac{\psi_{i+1,j}^{n+1} - \psi_{i,j}^{n+1}}{\Delta x} \frac{\psi_{i+1,j}^{n} - \psi_{i,j}^{n}}{\Delta x} + D \left(\frac{1-\mu}{2} \right) \frac{\psi_{i,j+1}^{n+1} - \psi_{i,j}^{n+1}}{\Delta y} \frac{\psi_{i,j+1}^{n} - \psi_{i,j}^{n}}{\Delta y} \\
& + D \left(\frac{1-\mu}{2} \right) \frac{\varphi_{i+1,j}^{n+1} - \varphi_{i,j}^{n+1}}{\Delta x} \frac{\varphi_{i+1,j}^{n} - \varphi_{i,j}^{n}}{\Delta x} + D \frac{\varphi_{i,j+1}^{n+1} - \varphi_{i,j}^{n+1}}{\Delta y} \frac{\varphi_{i,j+1}^{n} - \varphi_{i,j}^{n}}{\Delta y} \\
& + K \left(\frac{\omega_{i+1,j}^{n+1} - \omega_{i,j}^{n+1}}{\Delta x} + \frac{\psi_{i+1,j}^{n+1} + \psi_{i,j}^{n+1}}{2} \right) \left(\frac{\omega_{i+1,j}^{n} - \omega_{i,j}^{n}}{\Delta x} + \frac{\psi_{i+1,j}^{n} + \psi_{i,j}^{n}}{2} \right) \\
& + K \frac{\psi_{i,j+1}^{n+1} + \psi_{i,j}^{n+1}}{2} \frac{\psi_{i,j+1}^{n} + \psi_{i,j}^{n}}{2} \\
& + K \left(\frac{\omega_{i,j+1}^{n+1} - \omega_{i,j}^{n+1}}{\Delta y} + \frac{\varphi_{i,j+1}^{n+1} + \varphi_{i,j}^{n+1}}{2} \right) \left(\frac{\omega_{i,j+1}^{n} - \omega_{i,j}^{n}}{\Delta y} + \frac{\varphi_{i,j+1}^{n} + \varphi_{i,j}^{n}}{2} \right) \\
& + K \frac{\varphi_{i+1,j}^{n+1} + \varphi_{i,j}^{n+1}}{2} \frac{\varphi_{i+1,j}^{n} + \varphi_{i,j}^{n}}{2} \\
& + D \left(\frac{1+\mu}{2} \right) \Bigg(\frac{\psi_{i+1,j+1}^{n+1} - \psi_{i,j}^{n+1}}{2\Delta x} \frac{\varphi_{i+1,j+1}^{n} - \varphi_{i,j}^{n}}{2\Delta y} + \frac{\psi_{i,j+1}^{n+1} - \psi_{i+1,j}^{n+1}}{2\Delta x} \frac{\varphi_{i+1,j}^{n} - \varphi_{i,j+1}^{n}}{2\Delta y} \\
& + \frac{\varphi_{i+1,j+1}^{n+1} - \varphi_{i,j}^{n+1}}{2\Delta x} \frac{\psi_{i+1,j+1}^{n} - \psi_{i,j}^{n}}{2\Delta y} + \frac{\varphi_{i,j+1}^{n+1} - \varphi_{i+1,j}^{n+1}}{2\Delta x} \frac{\psi_{i+1,j}^{n} - \psi_{i,j+1}^{n}}{2\Delta y} \Bigg) \Bigg].
\end{aligned} \tag{5.13}$$

We note that E^n is the discrete version of the continuous energy (2.12). This total energy built from discrete system (5.4)–(5.12) is free from any over-estimation on the mesh size Δx and Δy. In that direction, our discrete system avoids a numerical anomaly known as locking phenomenon on shear force. To guidance of the reader, see Almeida Júnior [1] and references contained therein.

Moreover, one can show that E^n decreases for $d_1 > 0$ and that it is constant for $d_1 = 0$. Instead of computing the time derivative of the energy we can use the summation by parts.

Next, we establish the discrete counterpart of the Proposition 2.3.

Theorem 5.1 (Discrete energy) *Let* $(\omega_{i,j}^n, \varphi_{i,j}^n, \psi_{i,j}^n)$ *be a solution of the finite difference scheme* (5.4)–(5.12) *with* $d_1 > 0$. *Then for all* $\Delta t, \Delta x$ *and* Δy, *the discrete rate of change of energy of the numerical scheme* (5.4)–(5.12) *at the instant of time* t_n *is given by*

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$$\frac{E^n - E^{n-1}}{\Delta t} = -d_1 \sum_{i=1}^{I} \sum_{j=1}^{J} \left(\frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n-1}}{2\Delta t} \right)^2 \leqslant 0, \tag{5.14}$$

for all $n = 1, \ldots, N, N+1$.

Proof The proof is too long and we omit it here. Analogously to continuous case, we use the multipliers at discrete level given by $(\frac{\partial_t + \bar{\partial}_t}{2} \omega_{i,j}^n)$, $(\frac{\partial_t + \bar{\partial}_t}{2} \varphi_{i,j}^n)$ and $(\frac{\partial_t + \bar{\partial}_t}{2} \psi_{i,j}^n)$ and we organise the results in order to make up the difference $E^n - E^{n-1}$. □

5.3 Numerical simulations

In this section, we focus on the numerical scheme (5.4)–(5.12) and its energy E^n to illustrate by means of the numerical experiments the analytical results established in previous sections. We emphasise that we are not concerned with issues of numerical convergence between exact solution and numerical solution and the respective rate of convergence.

Taking into account several numerical experiments by using the discrete energy E^n, we get the exponential decay as well as the lack of exponential decay according with the relationship between speeds of wave propagation v_1^2 and v_2^2. In that direction, a measure of the numerical consistence of the numerical scheme (5.4)–(5.12) can be seen through the energy conservation law. Indeed, for $d_1 = 0$ in (5.14) we obtain that $E^n = E^0$, $n = 1, \ldots, N+1$.

In our numerical experiments, we use the following data: $L_1 = L_2 = 1$, $T = 4$ and thickness $h = 0.015$. In the initial conditions, we assume that

$$\omega(x_i, y_j, 0) = \psi(x_i, y_j, 0) = \varphi(x_i, y_j, 0) = 0, \tag{5.15}$$

$$\omega_t(x_i, y_j, 0) = 0, \quad \forall v \in \mathbb{N}, \tag{5.16}$$

$$\psi_t(x_i, y_j, 0) = \cos\left(v \frac{\pi x_i}{L_1}\right) \sin\left(v \frac{\pi y_j}{L_2}\right), \quad \forall v \in \mathbb{N}, \tag{5.17}$$

$$\varphi_t(x_i, y_j, 0) = \sin\left(v \frac{\pi x_i}{L_1}\right) \cos\left(v \frac{\pi y_j}{L_2}\right), \quad \forall v \in \mathbb{N}. \tag{5.18}$$

In the computational mesh, we use $\Delta x = \Delta y = 0.03125$ and $\Delta t = 0.00195$ such that $\Delta t / \Delta x = 0.0624$.

5.3.1 *Undamped and full damped cases*

Here, we consider the Reissner–Mindlin–Timoshenko system (1.1)–(1.9) in two physical situations: undamped and full damped cases. For the full damped case, we consider three frictional internal dissipations into the system, this is, we introduced the terms $d_1\omega_t$, $d_2\psi_t$ and $d_3\varphi_y$, for d_1, d_2 and d_3 to be positive constants.

In both cases, we have used different speeds of wave propagation.

Comments 1 *For a better comparison between the results of decay, we normalise the numerical energy by the initial energy, i.e., we define* $E^n := E^n / E^0$.

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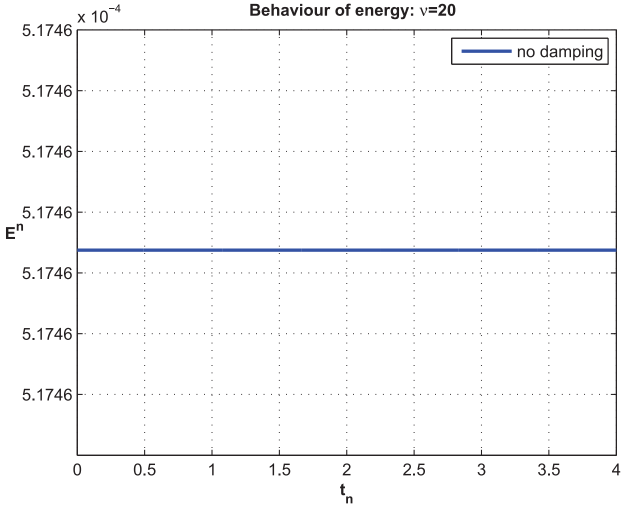

FIGURE 1. Undamped case.

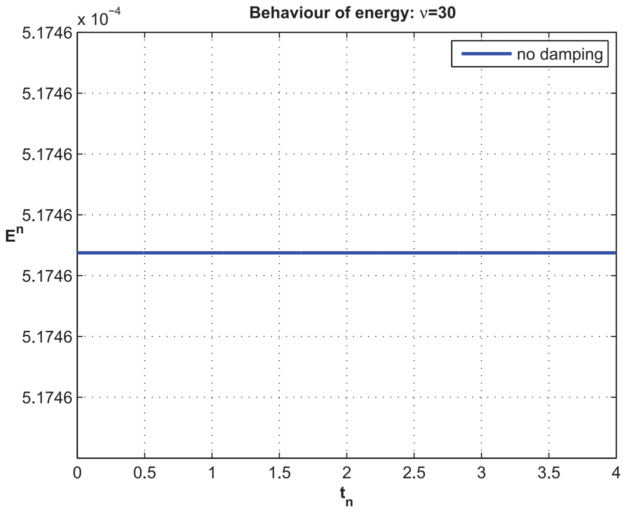

FIGURE 2. Undamped case.

We can see from Figures (1)–(2) that the discrete energy E^n *is constant for all discrete time* t_n *and this numerical behaviour is a measure of the precision of our numerical scheme* (5.4)–(5.12). *That is to say, the energy conservation law* (2.16) *and its discrete counterpart are qualitatively in agreement. On the other hand, Figures (3)–(4) show that the energy* E^n *is like an exponential function* $e^{-\omega t_n}$ *for* $\omega > 0$, *i.e., in the full damping case the discrete counterpart of the Reissner–Mindlin–Timoshenko system is exponentially stable independent from relationship between* v_1^2 *and* v_2^2. *The similar result in infinite dimensional was not included in our analysis, but it is not difficult to obtain this result by using for example the energy method.*

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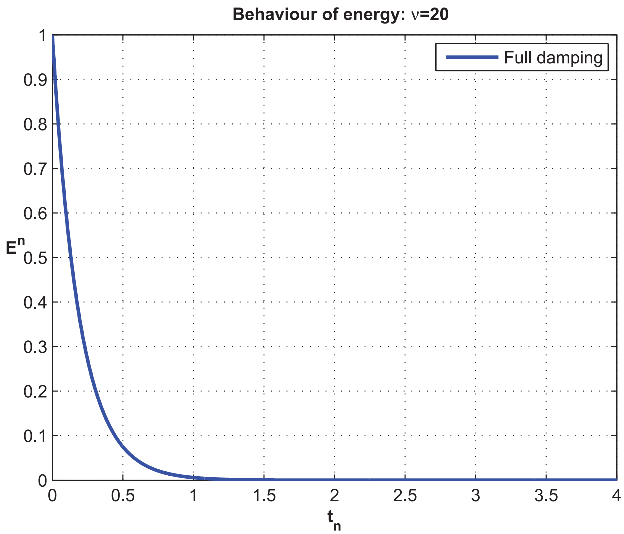

FIGURE 3. Full damped case.

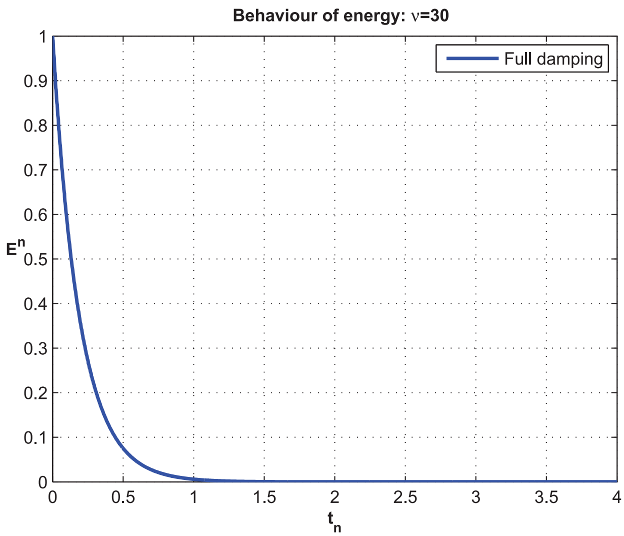

FIGURE 4. Full damped case.

5.3.2 *Damping only on transverse displacement*

Here, we show the numerical experiments concerning the main result of this work. In particular, we consider the initial data (5.16) equal to zero. The following are the results of our simulations.

Comments 2 *The Figures (5) and (7) represent a decay more slowly of the numerical solutions when speeds of wave propagation are different (see Theorem 3.1). This case is*

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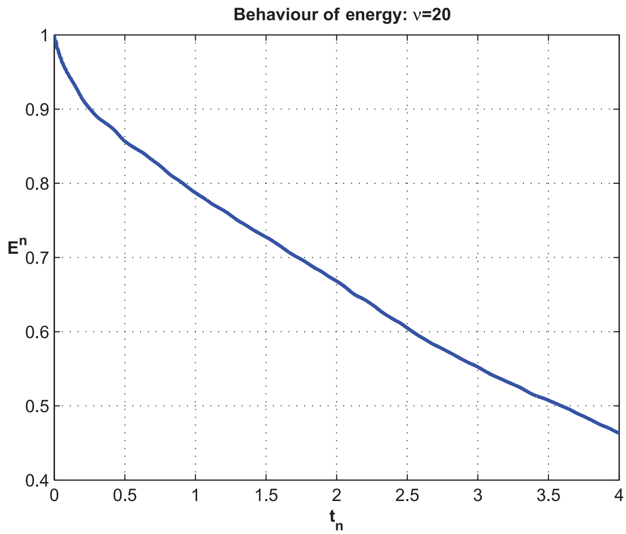

FIGURE 5. $v_1^2 \neq v_2^2$.

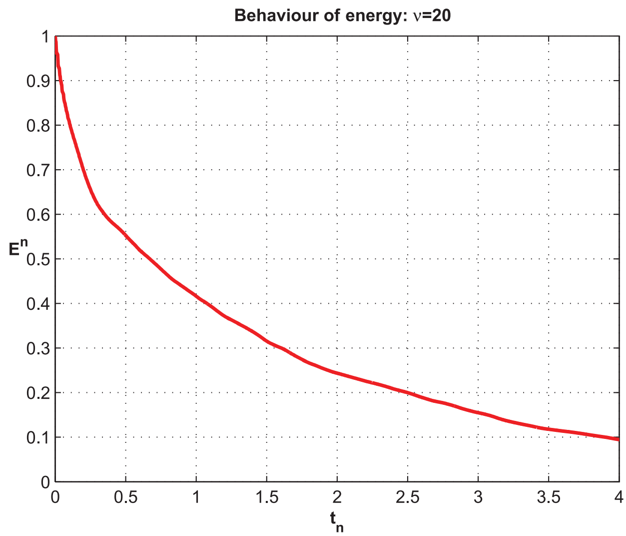

FIGURE 6. $v_1^2 = v_2^2$.

more realistic from physical point of view. In the right hand side, Figures (6) and (8), one has reproduced the exponential decay according with Theorem 4.5.

By comparison qualitative between Figures (5) and (7) and Figures (6) and (8), the lack of decay exponential can be seen as a typical behaviour of polynomial decay in accordance with the analytical results established in our mathematical analysis. For the same data of the simulations, the graphics have changed of exponential to a curve next of a straight line.

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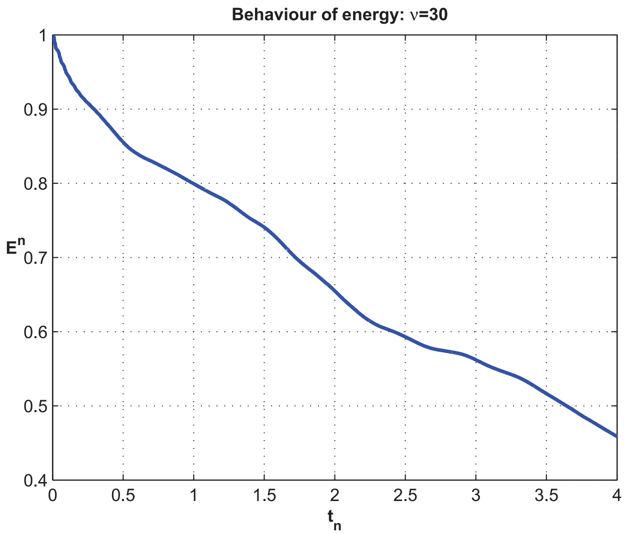

FIGURE 7. $v_1^2 \neq v_2^2$.

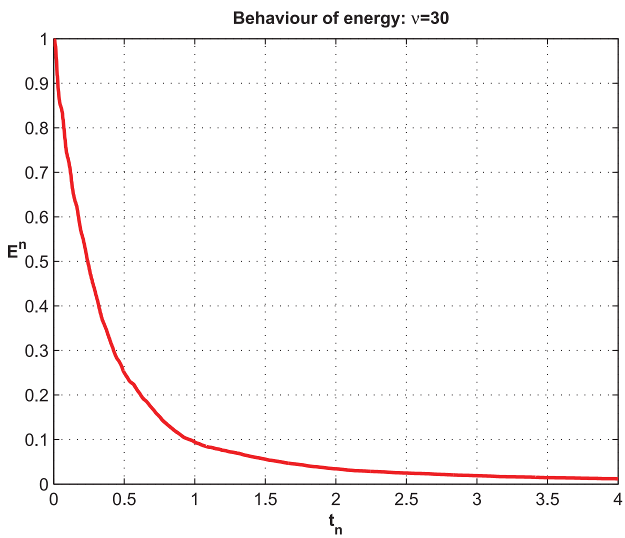

FIGURE 8. $v_1^2 = v_2^2$.

5.3.3 *Comparison between two damping cases*

Here, we present a comparison between two damping cases: the full damping and the damping on displacement function. In the figures below, we consider the relationship between the speeds of wave propagation only in the case of an only damping (on displacement function). On the other hand, in full damping case, we consider the speeds are different.

Comments 3 *These figures illustrate the important of the speeds of wave propagation* v_1^2 *and* v_2^2 *in order to obtain the exponential decay of the Reissner–Mindlin–Timoshenko by*

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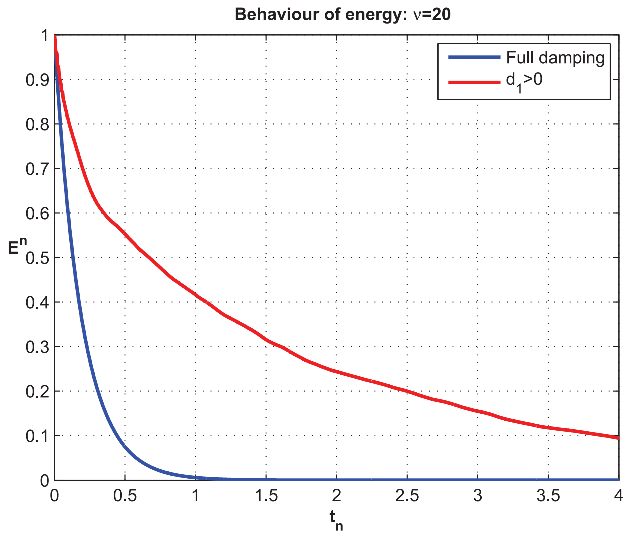

FIGURE 9. Exponential curves.

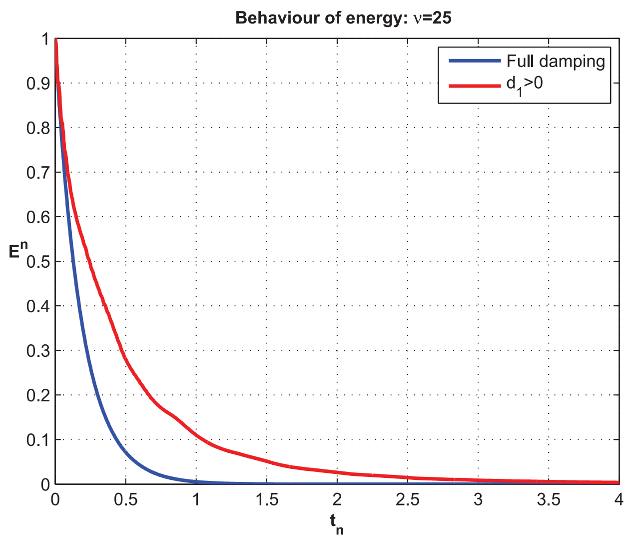

FIGURE 10. Exponential curves.

considering an only damping. The Figures (9)–(12) contain two exponential curves. The exponential curve in blue colour represents the full damping case where $v_1^2 \neq v_2^2$ *and the curve in red colour represents the exponential decay according with relationships* $v_1^2 = v_2^2$. *Therefore, looking for our analytical results established in previous sections and also for the several results studied on literature on stabilisation of the plates and beams, the same exponential decay (full damping case) can be obtained by taking into consideration an only damping and taking into account the equality* $v_1^2 = v_2^2$. *All numerical results presented here are qualitatively in agreement with the results established in infinite dimensional.*

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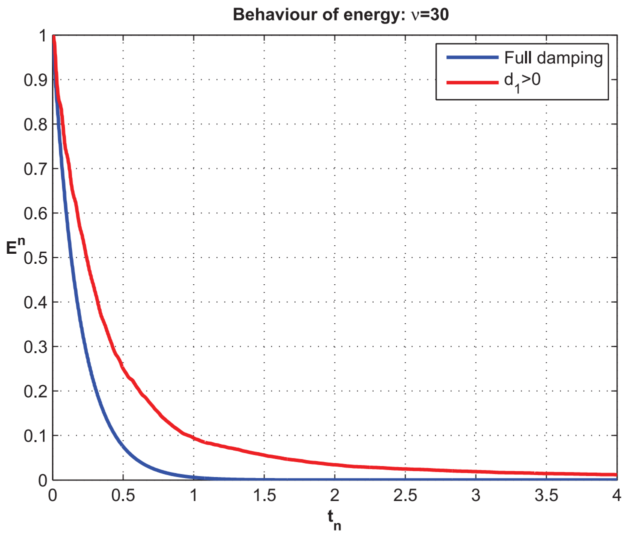

FIGURE 11. Exponential curves.

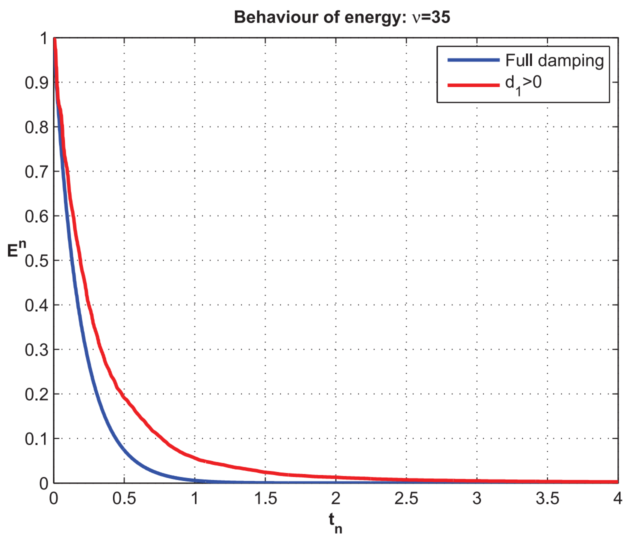

FIGURE 12. Exponential curves.

6 Conclusions

In this work, we have addressed an important problem in mathematical analysis of beams and plates models: the problem of determining the exponential decay by taking into account few dissipative mechanisms.

It is well known that the speeds of wave propagation play an important role to the dissipative Timoshenko systems in one-dimensional domain (see [2] and references contained therein). In that direction, in all literature concerning the stability of the Reissner–Mindlin–Timoshenko system we have not found any mention of the speeds of

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wave propagation. We have identified that the Reissner–Mindlin–Timoshenko system has two speeds of wave propagation and we have proved that is sufficient takes into account only one mechanism dissipation in order to obtain the exponential decay, for which a particular relationships between these speeds must hold.

Other dissipative cases can be considered. For example, looking at Reissner–Mindlin–Timoshenko systems in linear thermoelasticity, hypotheses such as Fourier's or Cattaneo's law as well as memory terms. These and many other constitute important models to be analysed at light of the relationships between v_1^2 and v_2^2.

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