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AN ELEMENTARY PROOF OF JOHNSON–DULMAGE–MENDELSOHN'S REFINEMENT OF BIRKHOFF'S THEOREM ON DOUBLY STOCHASTIC MATRICES

BY

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SUMMARY. A purely combinatorial and elementary proof of Johnson–Dulmage–Mendelsohn's theorem, which gives a quite sharp upper bound on the number of permutation matrices needed for representing a doubly stochastic matrix by their convex combination, is given.

1. **Introduction.** Since Birkhoff [1] proved that the set of all doubly stochastic (d.s.) matrices of order n coincides with the convex hull of all permutation matrices of the same order, several authors have been seeking the least upper bound on the number of permutation matrices needed for representing a d.s. matrix by a convex combination of these matrices ([4], 323–325). Among them, Johnson *et al.* [2] gave the sharpest bound, which reflects the structure of the d.s. matrix, using the Birkhoff algorithm ([2], 240–241). In their paper the Lemma 2 and its corollary ([2], 238–240) play an essential role. Their proofs of these propositions require some graph theoretical prerequisites. In this note we will present elementary combinatorial proofs to these propositions.

2. **Fully indecomposable decomposition.** A square matrix A is called *reducible* if

$$PAP' = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B and D are non-empty square matrices, for some permutation matrix P and *irreducible* otherwise. Similarly A is called *fully indecomposable* (*f.i.*) if PAQ is irreducible for any permutation matrices P and Q. When A is a d.s. matrix, it is easily seen that any A, not f.i., can be decomposed into a direct sum of f.i. and d.s. submatrices. We call such a decomposition a *fully indecomposable decomposition* (*f.i.d.*) of A. Note that such a decomposition of A is concordant with the canonical decomposition of the bipartite graph introduced by A ([2] and [3]). Since the canonical decomposition is uniquely determined [3], the f.i.d. of A is also unique up to permutations on rows and columns of A. We will give, in the following lemma, a combinatorial proof to this fundamental result. The proof of this lemma occurred with the author while he was refining the proof of Birkhoff's theorem based on the celebrated P. Hall's marriage theorem ([6], 553–554).

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We denote $A \sim B$ if $B = PAQ$ for some permutation matrices P and Q.

LEMMA. *Let $A = (a_{ij})$ be a $n \times n$ d.s. matrix. If A has two f.i.d.s: $A \sim A_1 \dotplus \cdots \dotplus A_s$ and $A \sim B_1 \dotplus \cdots \dotplus B_t$, where A_i and B_j are f.i. matrices of order m_i and n_j respectively $(1 \leq i \leq s, 1 \leq j \leq t)$, then $t = s$ and there exists an appropriate permutation σ on $\{1, 2, \ldots, s\}$ such that $n_i = m_{\sigma(i)}$ and $B_i \sim A_{\sigma(i)}$ for all i.*

Proof. Let $\Gamma_i \equiv \{j \mid 1 \leq j \leq n, a_{ij} > 0\}$ and $L_i \equiv \{l_1, \ldots, l_{m_i}\}$ be the set of suffixes of rows in A corresponding to $A_i (1 \leq i \leq s)$. Since A_i is a f.i. and d.s. matrix, the following property is easily verified.

$$(*) \qquad \left|\bigcup_{\nu \in L_i} \Gamma_\nu\right| = m_i, \qquad \left|\bigcup_{\nu \in K} \Gamma_\nu\right| > |K| \quad \text{for} \quad \phi \subsetneqq {}^\forall K \subsetneqq L_i,$$

where $|S|$ denotes the cardinality of the set S. The following property also immediately follows from $(*)$ and the fact that A is a d.s. matrix.

$$(**) \qquad a_{kj} = 0 \quad \text{for} \quad k \in L_i \quad \text{and} \quad j \notin \bigcup_{\nu \in L_i} \Gamma_\nu;\ k \notin L_i \quad \text{and} \quad j \in \bigcup_{\nu \in L_i} \Gamma_\nu.$$

It is clear that there exists an obvious one-one correspondence in the sense that to a f.i.d. of A there corresponds a partition of $\{1, 2, \ldots, n\}$ with the property $(*)$. Let $\{L_1, \ldots, L_s\}$ and $\{M_1, \ldots, M_t\}$ be two partitions as described above. We will show below that if $L_i \cap M_j \neq \phi$, then $L_i = M_j$. Thus the equality $t = s$ and the existence of an appropriate permutation σ on $\{1, 2, \ldots, s\}$ such that $M_i = L_{\sigma(i)}$ is guaranteed. Therefore we have $n_i = m_{\sigma(i)}$ and $B_i \sim A_{\sigma(i)}$ for all i.

Now suppose $L_i \neq M_j$ in despite of $L_i \cap M_j \neq \phi$. Note that none of L_is contain any M_j, and vice versa. From $(*)$ and $(**)$, we have easily $a_{kl} = 0$ for $k \in L_i \cap M_j$ and $l \in (\bigcup_{\nu \in L_i} \Gamma_\nu - \bigcup_{\nu \in L_i \cap M_j} \Gamma_\nu) \cup (\bigcup_{\nu \in M_j} \Gamma_\nu - \bigcup_{\nu \in L_i \cap M_j} \Gamma_\nu)$; $k \in L_i - (L_i \cap M_j)$ and $l \in \bigcup_{\nu \in L_i - (L_i \cap M_j)} \Gamma_\nu$; $k \in M_j - (L_i \cap M_j)$ and $l \in \bigcup_{\nu \in M_j - (L_i \cap M_j)} \Gamma_\nu$. Thus the equality $|\bigcup_{\nu \in L_i \cap M_j} \Gamma_\nu| = |L_i \cap M_j|$ can be proved considering A being a d.s. matrix. This contradicts the property $(*)$ of L_i, because $\phi \subsetneqq L_i \cap M_j \subsetneqq L_i$.

The following corollary is a part of the Lemma 2 in [7]. We give, however, a simple proof different from the one therein.

COROLLARY. *Let A be a $n \times n$ square matrix of the form*

$$A = \begin{bmatrix} A_1 & B_1 & & & & \\ & A_2 & B_2 & & 0 & \\ & & \cdot & & & \\ & & & \cdot & & \\ & 0 & & & \cdot & \\ & & & & A_{s-1} & B_{s-1} \\ B_s & & & & & A_s \end{bmatrix}, \text{ where each } A_i \text{ is f.i.}$$

and B_i contains at least one non zero entry. Then A is f.i.

Proof. It is easily seen by our assumptions that $|\bigcup_{\nu\in L}\Gamma_\nu|>|L|$ for $\phi \subsetneqq {}^\forall L \subsetneqq \{1,2,\ldots,n\}$ and $|\bigcup_{\nu=1}^n \Gamma_\nu|=n$. Hence A can not contain any $t\times(n-t)$ zero submatrices, $1\le t\le n-1$.

EXAMPLE.

$$A=\begin{bmatrix} p_1 & q_1 & & & & \\ & p_2 & q_2 & & 0 & \\ & & \cdot & & & \\ & & & \cdot & & \\ & 0 & & & \cdot & \\ & & & & p_{n-1} & q_{n-1} \\ q_n & & & & & p_n \end{bmatrix}, \text{ where } p_i\neq 0 \text{ and } q_i\neq 0$$

$(1\le i\le n)$, is a particular case of the corollary described above, thus A is f.i. On the other hand

$$B=\begin{bmatrix} 0 & 1 & 0 \\ p & 0 & q \\ q & 0 & p \end{bmatrix},$$

where $p>0$, $q>0$ and $p+q=1$, is regular (i.e. irreducible and aperiodic in the sense of finite Markov chains) but is not f.i., since obviously

$$B\sim\begin{bmatrix} 1 & 0 & 0 \\ 0 & p & q \\ 0 & q & p \end{bmatrix}.$$

3. **Proof of Johnson–Dulmage–Mendelsohn's theorem.** Let A be a d.s. matrix of order n. Let $A^{(i)}$ $(i=0,1,2,\ldots, A^{(0)}\equiv A)$ be the matrix which comes out after the i-times consecutive performances of the Birkhoff algorithm starting at A. Let $\rho(A^{(i)})$ and $d^{(i)}$ $(i=0,1,2,\ldots)$ be the number of positive entries in $A^{(i)}$ and the number of f.i. components of $A^{(i)}$, respectively. Let $\nu(A)$ be the number of performances of the Birkhoff algorithm which are needful until the algorithm terminates, i.e. $A^{(\nu(A))}=0$. Then the theorem of Johnson *et al.* [2] may be stated as follows

(1) $$\nu(A)\le\rho(A)-2n+d+1, \quad \text{where} \quad d\equiv d^{(0)}.$$

It is worth while to note that $\nu(A)=\nu(B)$ provided $A\sim B$.

In the following we will confine i within $1\le i\le \nu(A)-1$. In case there are just l $(l\ge 1)$ f.i. components B_α $(\alpha=1,\ldots,l)$ of $A^{(i-1)}$ which, after the ith performance, come out as the direct sum of f.i. components $B_\alpha^{(1)},\ldots,B_\alpha^{(k_\alpha)}$

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$(k_\alpha \geq 2)$ of $A^{(i)}$ respectively, i.e.

$$B_\alpha \sim \begin{bmatrix} B_\alpha^{(1)} & & & \\ & \cdot & & * \\ & & \cdot & \\ * & & & \cdot \\ & & & B_\alpha^{(k_\alpha)} \end{bmatrix} \text{and every } B_\alpha \text{ is f.i.}$$

If $[* \cdots B_\alpha^{(j)} \cdots *] = [0 \cdots B_\alpha^{(j)} \cdots 0]$ for some α and j, then

$$\begin{bmatrix} * \\ \cdot \\ \cdot \\ \cdot \\ B_\alpha^{(j)} \\ \cdot \\ \cdot \\ \cdot \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ B_\alpha^{(j)} \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

because all row sums and all column sums of B_α are of the same value. This contradicts the fact that B_α is f.i. Thus there exist at least k_α positive entries in B_α which vanish at the ith performance. Since $d^{(i)} = \sum_{\alpha=1}^{l} k_\alpha + (d^{(i-1)} - l)$, we have the inequality

(2) $$\rho(A^{(i-1)}) - \rho(A^{(i)}) \geq \sum_{\alpha=1}^{l} k_\alpha = d^{(i)} - d^{(i-1)} + l = d^{(i)} - (d^{(i-1)} - l),$$

which is identical with the Lemma 2 of Johnson *et al.* [2]. In case every f.i. component of $A^{(i-1)}$ remains f.i. after the ith performance, $d^{(i)} = d^{(i-1)}$, and on the other hand there always exist at least one positive entry in $A^{(i-1)}$ which vanish at the ith performance by the definition of the Birkhoff algorithm. Thus we have the inequality

(3) $$\rho(A^{(i-1)}) - \rho(A^{(i)}) \geq 1 = d^{(i)} - d^{(i-1)} + 1.$$

In any case, considering the inequalities (2) and (3), we have the following inequality

(4) $$\rho(A^{(i-1)}) - \rho(A^{(i)}) \geq d^{(i)} - d^{(i-1)} + 1,$$

which is identical with the corollary to the Lemma 2 of Johnson *et al.* [2].

We rewrite here the proof of (1) ([2], 240–241) for convenience. Using the inequality (4), we have

$$
\begin{aligned}
\rho(A)-\rho(A^{(1)}) &\geq d^{(1)}-d+1, \\
\rho(A^{(1)})-\rho(A^{(2)}) &\geq d^{(2)}-d^{(1)}+1, \\
&\ \,\vdots \\
\rho(A^{(\nu(A)-2)})-\rho(A^{(\nu(A)-1)}) &\geq d^{(\nu(A)-1)}-d^{(\nu(A)-2)}+1.
\end{aligned} \tag{5}
$$

Adding the above inequalities term by term, we have $\rho(A)-\rho(A^{(\nu(A)-1)})\geq d^{(\nu(A)-1)}-d+(\nu(A)-1)$. Since $\rho(A^{(\nu(A)-1)})=d^{(\nu(A)-1)}=n$, we obtain $\nu(A)\leq\rho(A)-2n+d+1$.

Remarks. (i) The equality in (1) holds iff the inequality (4) becomes the equality for any i $(1\leq i\leq \nu(A)-1)$: i.e. in the case of the inequality (2) $l=1$ and the equality holds, moreover in the case of the inequality (3) the equality holds.

(ii) Let $\gamma(A)$ be the $\max\{r_i(A), c_j(A)\mid 1\leq i, j\leq n\}$, where $r_i(A)$ (resp. $c_j(A)$) is the number of positive entries in ith row (resp. jth column). We clearly have the inequality

$$\nu(A)\geq\gamma(A). \tag{6}$$

This inequality gives a lower bound on $\nu(A)$. For example, $\gamma(A)=3$, $n=3$, $d=1$ and $\rho(A)=7$ for

$$A=\begin{bmatrix} 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}.$$

We have $3\leq\nu(A)\leq 7-2\times 3+1+1=3$, thus $\nu(A)=3$.

Finally it will be worthwhile to cite, concerning d.s. matrices and the extention of P. Hall's theorem, again the expository papers of Mirsky [4], Mirsky and Perfect [6] and the book of Mirsky [5].

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