Simplexes and other configurations upon a rational normal curve.

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[*Received and read* 25 July 1927.]

The theorem that if two triangles be inscribed in a conic their six sides touch another conic is, of course, to be found in all the text-books; it is apparently due in the first place to Brianchon*. The further remark, that if three triangles be inscribed in a conic the three conics obtained from them in pairs have a common tangent, is to be found in Taylor's *Ancient and Modern Geometry of Conics*†; it was made independently by Wakeford.

The first generalization to the space cubic is due, so far as I have been able to discover, to von Staudt‡, who shewed that if two tetrads of points be taken on a space cubic, the eight faces are planes of a cubic developable. This theorem, again, was discovered independently by Hurwitz§ in 1875; he also pointed out that there is then on the cubic a single infinity of tetrads whose faces touch the same developable, the tetrads forming, in fact, an involution of sets of four points on the cubic curve. Meanwhile Cremona‖ had considered three n-ads on the cubic curve, and the doubly-infinite involution of sets of n points determined therefrom, and had shewn that their faces, $\binom{n}{3}$ from each n-ad, touch a surface of class $n-2$. Independently, again, Pasch¶ had considered the particular case $n=4$, in which the twelve faces of three tetrads touch a quadric.

In 1882 Emil Weyr**, who was acquainted with the work of Cremona, considered his theorem in more detail, shewing that the surface of class $n-2$ passes through $\frac{1}{2}(n-1)(n-2)$ chords of the cubic curve. For $n=5$ we get a class-cubic surface and the six chords of the cubic curve form half of a double six thereon. Weyr was, however, chiefly concerned with involutions on rational curves in the plane, and his numerous other papers†† are not much to the point.

* Brianchon, *Mémoire sur les lignes du second ordre*, Paris, 1817, p. 35 (reference taken from *Encykl. Math. Wiss.*, III, C. 1, p. 35, fn. 100).

† Cambridge, 1881, p. 360. Taylor refers to H. Picquet, *Étude géométrique des systèmes ponctuels*, Paris, 1872, but this author only gives the converse.

‡ Von Staudt, *Beitrage zur Geometrie der Lage*, 1860, p. 378.

§ Announced in *Math. Ann.*, 15, 1879, p. 14; proof *ibid.*, 20, 1882, p. 135.

‖ Cremona, *Rendiconti Lombardo* (2), 12, 1879, pp. 347–52; *Opera*, t. III, p. 441.

¶ Pasch, *Journal für Math.*, 89, 1880, p. 256.

** Weyr, *Bulletin de l'acad. roy. de Belgique* (3), 3, pp. 472–85.

†† See list in the *Royal Society Catalogue of Scientific Papers*.

Finally W. F. Meyer* gave the general theorem: The faces of k n-ads (and of the ∞^{k-1} n-ads of the involution determined by them) upon a rational normal curve of order m in space of m dimensions touch a variety of primes of dimension $k-1$ and class $\binom{n-k+1}{m-k+1}$, $(m \geqslant k \geqslant 2)$. This result is also given in Segre's article on *Mehrdimensionale Räume*†; with the extension made by F. Deruyts‡ to rational curves of order m in space of less than m dimensions we are not here concerned.

Recently Professor Kubota§ has returned to the matter, apparently without being aware of the work of Cremona, Weyr and Meyer; he has considered in particular the case of pentads upon a space cubic curve and has made a notable extension, as follows:

Taking three pentads upon the curve we get a class-cubic surface touched by the 30 faces; four pentads give four such surfaces which have the planes of a cubic developable in common; five pentads give five cubic developables which have one plane in common.

Kubota's work, like that of all writers on this matter except von Staudt, Hurwitz and Wakeford, is algebraical; and no author treats n-ads upon a rational normal curve in what would appear to be the natural manner, namely as arising from the prime sections of a curve of order n in space of n dimensions, of which the given curve is a projection. It therefore seems worth while to point out that a large number of the results hitherto obtained, including those of Kubota, with generalizations, may be very readily seen without any calculation by the help of the generation of the rational normal curve of order n from projective systems, which is explained in Veronese's paper "Princip des Projicirens und Schneidens‖."

For the sake of clearness, a few particular cases will be considered, and generalizations will be merely indicated¶.

1. *Pentads upon a cubic curve.*

Consider a rational normal quintic curve in space of five dimensions; and take any three chords l, l', l''. Then it is clear that the trisecant planes of the curve are obtained as the intersections of corresponding primes of three triply infinite projectively related systems of primes through l, l', l'' respectively.

* Meyer, *Apolarität und rationale Curven*, 1883, p. 387.

† *Encykl. Math. Wiss.*, III, C 7, p. 896. ‡ See reference in Segre, *loc. cit.*

§ Kubota, *Science Reports, Tohoku Imperial University*, 15, 1926, pp. 39–44; *Math. Zeits.*, 26, 1927, pp. 450–6.

‖ Veronese, *Math. Ann.*, 19, 1881, pp. 161–234, especially pp. 219–20.

¶ The proof of the theorem on the twelve faces of three tetrads on a space cubic curve from four dimensions is indicated in Baker's *Principles of Geometry*, vol. IV, 1925, p. 147.

Now take two arbitrary primes meeting in a solid S; a trisecant plane lying in any prime through S meets S in a line. We consider then the trisecant planes which meet S in lines.

Let P be a point of S; join it to l, l', l'' by planes ϖ_1, ϖ_2', ϖ_3'', and let ϖ_2, ϖ_3 be the planes through l corresponding to ϖ_2' and to ϖ_3'' in the projectivity mentioned above. As P varies in S we thus get three projectively related triply infinite systems of planes through l, and taking the section by an arbitrary solid Σ we get three projectivities of the space Σ. Three corresponding planes ϖ_1, ϖ_2, ϖ_3, passing through l, lie in a prime Π through l which, with its corresponding primes through l', l'', gives the trisecant plane of the quintic curve which passes through P. This trisecant plane thus projects from l on to Σ into the plane joining three corresponding points in the projectivities of Σ. If, however, the trisecant plane meets S in a line m, the solid joining it to l lies with the solids corresponding to it, in the projectively related systems through l, in the prime which projects the trisecant plane, and hence in this case the projection is a plane which contains three corresponding lines in the projectivities of Σ. This is the dual of the point of concurrence of three corresponding lines in three projectivities between the planes of a space of three dimensions and the locus of such points is known to be a sextic curve of genus three, the residual intersection of two cubic surfaces with a space cubic curve in common*.

Hence we have the result that the trisecant planes of a quintic curve in five dimensions which meet a given solid in lines project, from a chord of the curve, into the planes of a sextic developable of genus three, obtained from two class-cubic surfaces with a cubic developable in common.

On projection the quintic curve gives rise to a cubic curve in Σ, the intersections of primes through S with the quintic give an involution of sets of five points upon the cubic, determined by any two of the sets, and the ten trisecant planes lying in any such prime give the ten faces of the corresponding pentad on the cubic.

Hence the 20 *faces of any two pentads on a space cubic curve touch a sextic developable, which is touched by the faces of all the pentads of the involution which they determine.*

It is easy to see that the ∞^1 lines in which S is met by trisecant planes are the trisecant chords of the sextic curve (of genus three) in which S meets the sextic three dimensional variety of chords of the quintic.

The case of three pentads upon the cubic curve is rather simpler. These clearly arise in a similar way from the intersections

* Schur, *Math. Ann.*, 18, 1881, pp. 1–32.

https://doi.org/10.1017/S0305004100013748 Published online by Cambridge University Press

of the quintic curve with three primes, which meet in a plane ϖ. The faces of the pentads of the doubly infinite involution determined by the three arise from the trisecant planes which meet ϖ, each in a point. Joining ϖ to l, l', l'' by primes Π_1, Π_2', Π_3'' and taking Π_2, Π_3, through l, corresponding to Π_2', Π_3'' respectively, we have three primes through l containing respectively the three planes ϖ_1, ϖ_2, ϖ_3 arising from any point P in ϖ. On projection we get from Π_1, Π_2, Π_3 three planes in Σ, projectively related, and the join of corresponding points in these planes is the projection of a trisecant plane of the quintic curve which meets ϖ.

Again we have the dual of a case considered by Schur and others, who obtain a cubic surface as the locus of intersections of corresponding planes of three projectively related bundles (or stars), each with a base point. Moreover, as is well known, there are six sets of corresponding planes which meet in lines, giving six lines of the cubic surface, the half of a double six. In our case we get a class-cubic surface and six lines of it; clearly these six lines arise from the chords of the quintic curve which meet ϖ, in the six intersections of ϖ with the variety of chords.

Hence we have the result:

Three pentads of points on a space cubic curve determine a doubly infinite involution of pentads, and the ten faces of each pentad touch the same class-cubic surface, which has six chords of the curve as half of a double six.

The configuration for four pentads is similarly obtained from four primes in the space of five dimensions, which have in common a line m. The threes of the four pentads determine four planes through this line and the planes common to the four class-cubic surfaces are obtained from the trisecant planes of the quintic curve which meet m. In this case we get three solids through l and sets of three planes through l, lying in them and projectively related. On projection we get, from the trisecant planes meeting m, the planes joining corresponding points of three projectively related lines, that is, the planes of a cubic developable. Hence, *the four class-cubic surfaces touching the faces of threes of four pentads upon the cubic curve have in common a cubic developable.*

Lastly, take five pentads upon the cubic curve. These arise from primes through a point P in the five-dimensional space. Through this point just one trisecant plane of the quintic can be drawn. We have thus the final result:

The five cubic developables arising from the sets of four pentads taken from the five have one plane in common.

https://doi.org/10.1017/S0305004100013748 Published online by Cambridge University Press

2. $(2r-1)$-*ads upon a rational normal curve of order r.*

Precisely the same method applies to sets of $2r-1$ points upon a curve of order r in space of r dimensions. We need only state the results.

A set of $2r-1$ points in r dimensions gives, by the joins of r points, $\binom{2r-1}{r}$ primes, which we shall call the faces of the $(2r-1)$-ad.

The faces of r such $(2r-1)$-ads upon the curve of order r touch a variety of primes of dimension $r-1$ and class r, obtained from r projectively related primes, say a V^r_{r-1}.

The $r+1$ such V^r_{r-1} arising from the sets of r of $r+1$ $(2r-1)$-ads have in common a $V^{\frac{1}{2}r(r-1)}_{r-2}$.

The $r+2$ such $V^{\frac{1}{2}r(r-1)}_{r-1}$ arising from the sets of $r+1$ of $r+2$ $(2r-1)$-ads have in common a $V^{\frac{1}{6}r(r-1)(r-2)}_{r-3}$.

....

The $r+k$ such $V^{\binom{r}{k}}_{r-k}$ arising from the sets of $r+k-1$ of $r+k$ $(2r-1)$-ads have in common a $V^{\binom{r}{k+1}}_{r-k-1}$.

....

The $2r-1$ such $V_1{}^r$ (developables of class r) arising from the sets of $2r-2$ of $2r-1$ $(2r-1)$-ads have in common a single prime.

This general statement includes the case of § 1 for $r=3$, and for $r=2$ gives the theorem in conics mentioned in the introduction.

3. *Hexads upon a cubic curve.*

As another example, take the case of three hexads upon a space cubic curve.

These should be considered as arising from three primes and a sextic curve in space of six dimensions. The three primes intersect in a solid, which meets a plane in a prime through it in a point. We have thus to consider the trisecant planes of the sextic curve which meet a given solid S in points.

Now the trisecant planes of a rational normal sextic curve arise as the intersections of corresponding primes of four projectively related triply infinite systems of primes, the base of each being a trisecant plane of the curve. Let α, β, γ, δ be the bases. Then joining a point P to α, β, γ, δ by solids and proceeding as before, we get four solids through α. If these lie in a prime this prime projects from α a trisecant plane which then passes through P. Taking all points of S and projecting from α upon a solid Σ we thus get four projectivities of Σ, and the tri-

https://doi.org/10.1017/S0305004100013748 Published online by Cambridge University Press

secant planes which meet S project into planes which join four corresponding and coplanar points in these projectivities. The locus of planes in Σ is thus* a surface of class 4, whose tangential equation is given by the vanishing of a determinant of four rows and columns—a surface which, following Jessop†, we may call a determinant class-quartic surface.

This surface is, however, not the most general determinant class-quartic; it contains 10 lines which are chords of the cubic curve, these arising from the 10 chords of the sextic curve which meet the solid S. (The locus of chords of the sextic is a variety of dimension 3 and of order 10.) Hence the result:

The 45 faces of three hexads upon a space cubic curve touch a surface of class 4 which is the Jacobian of four quadrics, given in plane coordinates.

It is not necessary to do more than state the results for four and for five hexads upon the cubic curve.

Taking four hexads, the four class-quartic surfaces arising from the threes thereof have in common the planes of a developable of class 6 and of genus 3; taking five hexads, the five sextic developables have in common four planes‡.

The extension to the rational normal curve of order r is at once clear. We take r sets of $2r$ points upon the curve; the $r\binom{2r}{r}$ faces touch a variety of class $r+1$; $r+1$ sets give $r+1$ such varieties, with the primes of a $V_{r-2}^{\frac{1}{2}r(r+1)}$ in common, and so on, until finally we arrive at $2r-1$ sets of $2r$ points and $r+1$ common primes.

The elementary case in which $r=2$ is interesting; the twelve sides of two quadrangles inscribed in a conic touch a curve of class 3; the three class-cubics arising from three quadrangles have three common tangents.

4. $(n+1)$-*ads upon a rational normal curve of order* n.

Another generalization of the theorem concerning tetrads upon a space cubic curve is worth special investigation, particularly as Segre§ calls attention to it in a foot-note. This is the case of simplexes upon a rational normal curve of any order.

* Schur, *loc. cit.*

† Jessop, *Quartic Surfaces*, Cambridge, 1916, Chap. IX.

‡ The points P in S through which pass trisecant planes of the sextic curve describe a symmetroid, the 10 nodes of which are the points of intersection of S with the variety of chords of the sextic. For the four hexads we get a plane in the six-dimensional space and on it a quartic curve, through the points of which pass trisecant planes; with five hexads we get a straight line and four points of it. The symmetroid, the quartic curve and the four points are the intersections of S, the plane and the line respectively with the variety of trisecant planes of the sextic, which is of dimension 5 and order 4.

§ Segre, *loc. cit.*, p. 896, fn. 375.

For this we consider the curve as the projection of a curve of order $n+1$ from a point of itself. As before, a simplex arises by projection from the intersections of this curve by a prime, and the faces of the simplex from the n-secant spaces of dimension $n-1$. Consider two such simplexes and the corresponding primes, which meet in a space S of dimension $n-1$. In a prime through S a space of dimension $n-1$ meets S in a space of dimension $n-2$. We have thus to consider the n-secant spaces of the curve of order $n+1$ which meet S in spaces of dimension $n-2$.

The n-secant spaces arise as the intersections of corresponding primes of two related ∞^n systems whose bases are respectively two points A, B of the curve. Taking any point P and joining to A, B, and taking the line through A which corresponds to the line BP through B, we have two lines through A, and any prime through the plane determined by these lines meets the corresponding prime through B in an n-secant space which passes through P.

Varying P in S and projecting from A upon a prime Σ we get two projectively related primes ϖ, ρ of Σ, corresponding points arising from the pairs of lines through A, and an n-secant space through P projects into a prime in Σ which contains two corresponding points of the primes ϖ, ρ. An n-secant space which meets S in a space of dimension $n-2$ thus gives rise in Σ to two spaces of dimension $n-2$ in ϖ, ρ which correspond in the projectivity and which lie in a prime of Σ.

The dual of all this is, first, a projectivity of the primes through two points in a space of n-dimensions, and then two corresponding lines in the projectivity which intersect; the locus of such intersections, as is well known, is a rational normal curve of order n.

Hence, in our case, *the faces of the two simplexes and of the simplexes in the involution determined by them are the osculating primes of a rational normal curve of order n.*

Three simplexes on the curve of order n arise in a similar way from three primes in the space of $n+1$ dimensions, which have in common a space of dimension $n-2$. We have to consider n-secant spaces of the curve of order $n+1$ which meet this in a space of dimension $n-3$. Projecting from A upon Σ we get two spaces α, β of dimension $n-2$, projectively related, and the faces of the simplexes arise from corresponding spaces of dimension $n-3$ in α, β which lie in a prime of Σ. We thus get a double infinity of primes of class $n-1$*.

Skipping intermediate cases, which may easily be worked out if desired, let us go on to the case of n simplexes.

* The dual is a surface of order $n-1$ arising as the locus of intersections of corresponding planes of two projectively related systems of primes through two lines.

In the space of $n+1$ dimensions we have n primes, which meet in a line l. A space of $n-1$ dimensions in a prime through l meets l in a point. We have thus to consider the n-secant spaces of the curve of order $n+1$ which pass through the points of l. We clearly get, on projection from A, two lines m, n in Σ, projectively related, two corresponding points M, N upon them arising from the same point P of l. Any prime through MN is the projection from A of an n-secant space of the curve of order $n+1$ which passes through P. The primes through MN are the tangent primes of a quadric, which is moreover degenerate tangentially into a quadric surface in the three-dimensional space of the lines m, n.

The dual theorem is perhaps easier to state:

*Taking n simplexes, each formed by n osculating primes of a rational normal curve of order n, the $n(n+1)$ vertices lie upon a quadric cone of which the "vertex" is a space of dimension $n-4$**.

For $n+1$ simplexes (going back to the original form of the theorems), we have to consider the n-secant spaces through a point P, and thus on projection the primes through a line MN. Hence *the $n+1$ quadrics arising from the simplexes taken n at a time have in common all primes through a line*—the dual of the statement that quadric cones of the $(n-4)$-th kind have a generating space of dimension $n-2$ in common†.

* Cf. Segre, *loc. cit.* He does not remark that the quadric is degenerate if $n>3$. For $n=3$, 2 the cone becomes an ordinary quadric and a conic, respectively.

† The result, included in this for $n=3$, that, for four tetrads on a cubic curve, the four quadrics touching the faces of threes have a common generator, was remarked by Mr J. H. Grace.