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THE AKIYAMA MEAN-MEDIAN MAP HAS UNBOUNDED TRANSIT TIME AND DISCONTINUOUS LIMIT

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Abstract

Open conjectures state that, for every $x \in [0,1]$, the orbit $(x_n)_{n=1}^{\infty}$ of the mean-median recursion

$$x_{n+1} = (n+1) \cdot \operatorname{median}(x_1, \ldots, x_n) - (x_1 + \cdots + x_n), \quad n \geqslant 3,$$

with initial data $(x_1, x_2, x_3) = (0, x, 1)$, is eventually constant, and that its transit time and limit functions (of x) are unbounded and continuous, respectively. In this paper, we prove that for the slightly modified recursion

$$x_{n+1} = n \cdot \operatorname{median}(x_1, \ldots, x_n) - (x_1 + \cdots + x_n), \quad n \geqslant 3,$$

first suggested by Akiyama, the transit time function is unbounded but the limit function is discontinuous.

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1. Introduction

The *mean-median map* (MMM) enlarges a finite nonempty real multiset $[x_1, \ldots, x_n]$ to $[x_1, \ldots, x_n, x_{n+1}]$, where x_{n+1} is the unique real number which equates the (*arithmetic*) *mean* of the latter multiset and the *median* of the former multiset:

$$\langle x_1, \ldots, x_n, x_{n+1} \rangle = \mathcal{M}_n, \quad \text{that is,} \quad x_{n+1} = (n+1)\mathcal{M}_n - \mathcal{S}_n, \tag{1.1}$$

where $\langle x_1, \ldots, x_n \rangle$, $\mathcal{M}_n$ and $\mathcal{S}_n$ denote the mean, the median and the sum of the elements of $[x_1, \ldots, x_n]$, respectively. The median $\mathcal{M}_n$ of the multiset $[x_1, \ldots, x_n]$ is, as ordinarily known, the middle number in the sorted multiset if n is odd, and the mean of the middle pair otherwise.

Given an initial multiset $[x_1, \ldots, x_{n_0}]$, $n_0 \in \mathbb{N}$, iterating the map generates an *orbit* $(x_n)_{n=1}^{\infty}$ which is conjectured to *stabilise*, that is, to be eventually constant.

STRONG TERMINATING CONJECTURE [9]. The MMM orbit of every initial multiset stabilises.

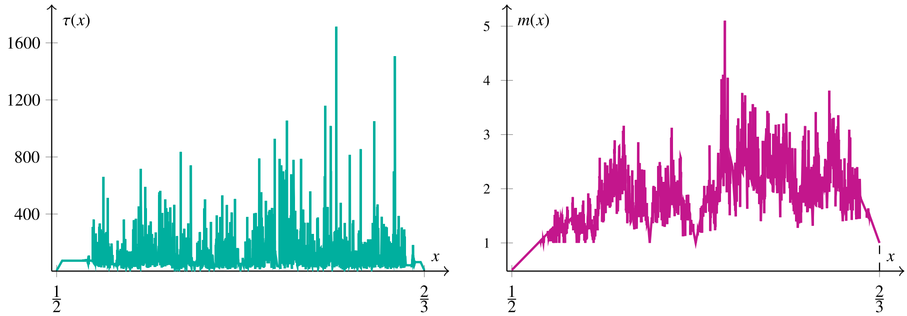

FIGURE 1. Graphs of τ (left) and m (right).

It is known that the *median sequence* $(\mathcal{M}_n)_{n=n_0}^{\infty}$ associated to the orbit is monotonic [4, Theorem 2.1], and converges once a repeated orbit point appears above (below) a median in the nondecreasing (nonincreasing) case [4, Theorem 2.4]. Such repeated points are observed to be ubiquitous [4, paragraph preceding Section 3], suggesting the following weaker version of the above conjecture.

WEAK TERMINATING CONJECTURE [4]. The median sequence of every initial multiset converges.

Despite intensive research effort [3–10], these terminating conjectures, as well as two additional conjectures to follow, are still open even in the case of the smallest nontrivial initial multisets: those of size three. The fact that the MMM commutes with elementwise affine transformations [4, Section 3] makes the orbit of every such multiset affine-equivalent to that of a univariate initial multiset $[0, x, 1]$, for some real number $x \in [\frac{1}{2}, \frac{2}{3}]$ which we call the *initial condition*. We associate to this multiset the *transit time* $\tau(x) \in \mathbb{N}_{>3} \cup \{\infty\}$ of its MMM orbit—the time step at which the orbit stabilises—and the *limit* $m(x) \in \mathbb{R}$ of its median sequence. These functions, sketched in Figure 1, are conjectured to possess the following properties.

UNBOUNDEDNESS CONJECTURE [5]. The function τ is unbounded.

CONTINUITY CONJECTURE [4]. The function m is continuous.

A sufficient condition for the appearance of a repeated point—which guarantees convergence of the median sequence—is available for bounded rational orbits. Such an orbit is forced to repeat if its time-dependent *effective exponent*—the largest exponent of 2 in the denominators of existing points—grows sublogarithmically over time [6, (2.2)]. From (1.1), it is apparent that after each iteration, this exponent either stays unchanged or increases by 1. Thus, for a sublogarithmic growth, the increments must occur sufficiently infrequently. This infrequency of increments, although well supported by computational evidence, seems to originate from an arithmetical phenomenon which is very difficult to elaborate rigorously.

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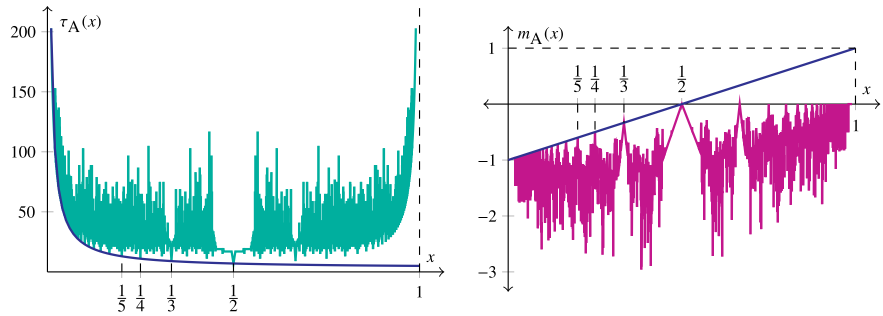

FIGURE 2. Graphs of τ_A (left) and m_A (right) in $(0, 1)$ with bounds given in the main theorem.

To eliminate this difficulty, Akiyama [1] suggested modifying the recursion (1.1) into

$$x_{n+1} = n\mathcal{M}_n - \mathcal{S}_n, \tag{1.2}$$

thereby introducing a new variant of the MMM, which we call the *Akiyama* MMM, whose rational orbits have a *constant* effective exponent. Naturally, for the Akiyama MMM, there are analogous terminating conjectures; these are also open. However, for this map, clearly, every bounded rational orbit stabilises.

As we shall see, the Akiyama MMM has the same smallest nontrivial form of initial multisets, namely $[0, x, 1]$, whose transit time $\tau_A(x) \in \mathbb{N}_{>3} \cup \{\infty\}$ and limit $m_A(x) \in \mathbb{R}$ are defined analogously for $x \in (-\infty, 1)$, and are sketched in Figure 2. For these functions, one naturally questions the analogous unboundedness and continuity conjectures. The main purpose of this paper is to prove analytically that the former holds, whereas the latter fails. More precisely, we will prove the following theorem.

THEOREM. *If $x \in (0, 1)$, then*

$$\tau_A(x) \geqslant \frac{2}{x} + 3 \quad \text{and} \quad m_A(x) \leqslant 2x - 1,$$

where equality holds if and only if x is a unit fraction (that is, a positive fraction with unit numerator).

The first inequality clearly implies the unboundedness of τ_A. Since $m_A(0) = 0$, the second inequality implies that m_A is discontinuous at $x = 0$.

Our proof of this theorem is methodologically similar to that of the bounds for the transit time and limit of the so-called *normal form* of the original MMM [7, Theorem 6.2]; we first show that every orbit begins with a *predictable phase* whose length depends on an arithmetical property of the initial condition. The bounds for τ_A and m_A in the theorem can then be inferred, respectively, from the number of existing points and the location of the median at the end of the phase.

The simultaneous occurrence of the unboundedness of the transit time and the discontinuity of the limit function is unsurprising. Indeed, in the original MMM, we have pointed out that these will be two interrelated consequences if a local functional

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orbit is found to be divergent [7, Theorems 5.4 and 5.6]. While such divergence has not been found in the original MMM, we find it near $x = 0$ in the Akiyama MMM.

Let us now describe the structure of this paper. In the next section, we define the Akiyama MMM more formally and discuss its basic properties. There are properties which are the same as those of the original MMM (the proofs of which are thus omitted): the median sequence is monotonic (Proposition 2.2), a repeated orbit point guarantees convergence and two equal consecutive medians cause stabilisation (Proposition 2.3). There is also a different property: the map commutes with scalar multiplications, but not with nonidentity translations (Proposition 2.1). In Section 3, we present our main result, namely an explicit description of the predictable phase for every initial condition (Lemma 3.1) from which the above theorem then follows. Finally, the graphs in Figure 2 suggest the presence of symmetry around $x = \frac{1}{2}$; a brief discussion on this in Section 4 concludes the paper.

2. Preliminaries

The Akiyama MMM is a self-map on the space of finite nonempty real multisets. The image $\mathbf{M}_{\mathrm{A}}(\xi)$ of such a multiset ξ is obtained by increasing the multiplicity of the real number

$$\mathrm{M}_{\mathrm{A}}(\xi) := |\xi|\mathcal{M}(\xi) - \mathcal{S}(\xi)$$

in ξ by one, where $|\xi|$, $\mathcal{M}(\xi)$ and $\mathcal{S}(\xi)$ denote the cardinality, median and sum of elements of ξ, respectively. Employing the additive union notation [2, page 50], we write

$$\mathbf{M}_{\mathrm{A}}(\xi) := \xi \uplus [\mathrm{M}_{\mathrm{A}}(\xi)].$$

Generally, the map $\mathbf{M}_{\mathrm{A}}$ does *not* commute with elementwise affine transformations (see [4, Theorem 2.2]). However, it commutes with elementwise scalar multiplications.

Proposition 2.1. *For every $a, b \in \mathbb{R}$ with $a \neq 0$, we have*

$$\mathbf{M}_{\mathrm{A}}(a\xi + b) = (a\xi + b) \uplus [a\mathrm{M}_{\mathrm{A}}(\xi)],$$

and, in particular,

$$\mathbf{M}_{\mathrm{A}}(a\xi) = a\mathbf{M}_{\mathrm{A}}(\xi), \tag{2.1}$$

that is, $\mathbf{M}_{\mathrm{A}}$ commutes with elementwise scalar multiplications.

Proof. Since $\mathcal{M}(a\xi + b) = a\mathcal{M}(\xi) + b$ and $\mathcal{S}(a\xi + b) = a\mathcal{S}(\xi) + |\xi|b$, the map $\mathbf{M}_{\mathrm{A}}$ increases in the multiset $a\xi + b$ the multiplicity of the number

$$\begin{aligned} |a\xi + b|\mathcal{M}(a\xi + b) - \mathcal{S}(a\xi + b) &= |\xi|[a\mathcal{M}(\xi) + b] - [a\mathcal{S}(\xi) + |\xi|b] \\ &= a[|\xi|\mathcal{M}(\xi) - \mathcal{S}(\xi)] \\ &= a\mathrm{M}_{\mathrm{A}}(\xi), \end{aligned}$$

proving the first identity. Setting $b = 0$ gives the second identity. □

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Under iterations of $\mathbf{M}_{\mathrm{A}}$, every initial multiset $\xi_{n_0} = [x_1, \ldots, x_{n_0}]$, $n_0 \in \mathbb{N}$, is associated to a sequence of multisets $(\xi_n)_{n=n_0}^{\infty}$, an orbit $(x_n)_{n=1}^{\infty}$ and a median sequence $(\mathcal{M}_n)_{n=n_0}^{\infty}$, where

$$\xi_{n+1} = \mathbf{M}_{\mathrm{A}}(\xi_n), \quad x_{n+1} = \mathrm{M}_{\mathrm{A}}(\xi_n) \quad \text{and} \quad \mathcal{M}_n := \mathcal{M}(\xi_n) \quad \text{for every } n \geqslant n_0.$$

Moreover,

$$x_{n+2} = (n+1)\mathcal{M}_{n+1} - n\mathcal{M}_n \quad \text{for every } n \geqslant n_0, \tag{2.2}$$

an expression of an orbit point as an affine combination of the last two medians. Exactly as in the original MMM [4, Theorem 2.1], we deduce from (2.2) that the median sequence is monotonic.

PROPOSITION 2.2. *The median sequence* $(\mathcal{M}_n)_{n=n_0}^{\infty}$ *is monotonic.*

Loosely speaking, an Akiyama MMM orbit reaches stabilisation in a similar way as an original MMM orbit: the orbit first generates a repeated point which guarantees the convergence of the median sequence [4, Theorem 2.4]. (In the case of $x_1, \ldots, x_{n_0} \in \mathbb{Q}$, since the effective exponent is constant, convergence implies stabilisation.) Once one of these repeated points is reached by the median sequence, two equal consecutive medians are created; as is apparent from (2.2), this causes stabilisation. Formally, we have the following result.

PROPOSITION 2.3.

(i) *If* $n \geqslant n_0$ *is such that* $\mathcal{M}_n = \mathcal{M}_{n+1}$, *then* $x_j = \mathcal{M}_{n+1}$ *for every* $j \geqslant n+2$.
(ii) *The nondecreasing (nonincreasing) median sequence converges if there exist* $i, j, s \in \mathbb{N}$ *with* $i \neq j$ *and* $s \geqslant n_0$ *such that* $\mathcal{M}_s \leqslant x_i = x_j$ ($\mathcal{M}_s \geqslant x_i = x_j$).

The orbits of a singleton multiset $[x]$, a two-element multiset containing a zero $[0, x]$, and a multiset of two equal elements $[x, x]$, where $x \in \mathbb{R}$, are straightforward to compute; these are $(x, 0, 0, -x, \overline{0})$, $(0, x, 0, -x, \overline{0})$ and $(x, x, 0, \overline{x})$, respectively, where the overlines denote infinite repetitions. The smallest nontrivial initial multisets are those of the form $[x, y]$, where x, y are nonzero and $x < y$. By (2.1), these are represented by multisets of the form $[x, 1]$, $x < 1$, whose limit $m_{\mathrm{A}}(x)$ and transit time $\tau_{\mathrm{A}}(x)$ are plotted in Figure 2. For these multisets, the median sequence is nonincreasing. It is straightforward to show that $\mathbf{M}_{\mathrm{A}}([x, 1]) = [0, x, 1]$; in this sense, the smallest nontrivial initial multisets of the original and Akiyama MMMs have the same form.

3. Main result

We are now ready to present our main result. For $x \in (0, 1)$, we show that the orbit of the smallest nontrivial initial multiset $[x, 1]$ begins with a *predictable phase*: an initial segment of length $2\ell + 2$, where $\ell := \lceil 1/x \rceil \geqslant 2$, in which every term has an explicit formula. (Here, $\lceil 1/x \rceil$ denotes the smallest integer not less than $1/x$.) In this phase, the first four terms are given by $(x_n)_{n=1}^{4} = (x, 1, 0, 2x - 1)$, as easily verified, and the rest by the following lemma. Moreover, the phase is immediately followed by

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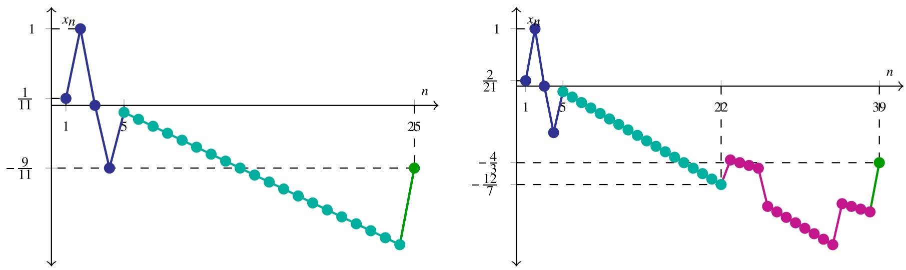

FIGURE 3. The orbit of $[x, 1]$ for $x = 1/11$ (left) and for $x = 2/21 \in (1/11, 1/10)$ (right). The first four terms are shown in dark blue, the terms prescribed by Lemma 3.1 in light blue, the unprescribed terms in purple and the term from which the orbit stabilises in green. (Colour available online.)

stabilisation—hence the available formulae describe the entire orbit—if and only if x is a unit fraction, that is, the reciprocal of ℓ. See Figure 3.

LEMMA 3.1. *Let x_n be the nth term of the orbit of the multiset $[x, 1]$, where $x \in (0, 1)$.*

(i) *If $x = 1/\ell$ for some integer $\ell \geqslant 2$, then $x_n = -(n-4)x$ for every $n \in \{5, \ldots, 2\ell+2\}$ and $x_n = 2x - 1$ for every $n \geqslant 2\ell + 3$. Thus, $m_A(x) = 2x - 1$ and $\tau_A(x) = 2\ell + 3$.*

(ii) *If $x \in (1/\ell, 1/(\ell-1))$ for some integer $\ell \geqslant 2$, then $x_n = -(n-4)x$ for every $n \in \{5, \ldots, 2\ell\}$,*

$$x_{2\ell+1} = (\ell^2 - 2\ell + 3)x - \ell \quad \text{and} \quad x_{2\ell+2} = (\ell^2 - \ell + 2)x - \ell - 1. \tag{3.1}$$

Moreover, $m_A(x) < 2x - 1$ and $\tau_A(x) > 2\ell + 3$.

PROOF. Let $x \in [1/\ell, 1/(\ell-1))$ for some integer $\ell \geqslant 2$. First, suppose $\ell = 2$. Then $x \in [\frac{1}{2}, 1)$. If $x = \frac{1}{2}$, then $(x_n)_{n=1}^{\infty} = (\frac{1}{2}, 1, 0, 0, -\frac{1}{2}, -1, \overline{0})$, satisfying property (i). Otherwise, $(x_n)_{n=1}^{6} = (x, 1, 0, 2x-1, 3x-2, 4x-3)$, satisfying property (ii).

Therefore, it remains to prove the lemma for $\ell \geqslant 3$. In this case, we have $x \in (0, \frac{1}{2})$. We divide the proof into two parts.

Part I: Formulae for $x_5, \ldots, x_{2\ell}$. Let us prove that for every $n \in \{5, \ldots, 2\ell\}$, we have

$$x_n = -(n-4)x \tag{3.2}$$

by strong induction on n. First, since $x \in (0, \frac{1}{2})$, then $x_4 < x_3 < x_1 < x_2$, so $\mathcal{M}_4 = \langle x_3, x_1 \rangle = x/2$ and

$$x_5 = 4\mathcal{M}_4 - \mathcal{S}_4 = 4 \cdot \frac{x}{2} - 3x = -x,$$

proving that the statement holds for $n = 5$.

Next, let $r \in \{5, \ldots, 2\ell - 1\}$ be such that $x_n = -(n-4)x$ for every $n \in \{5, \ldots, r\}$. We shall prove that $x_{r+1} = -(r-3)x$, dividing the proof into two cases.

Case I: $r \in \{5, \ldots, \ell + 1\}$. Since $x < 1/(\ell - 1)$, then

$$x_4 - x_r = (2x - 1) + (r - 4)x \leqslant (2x - 1) + [(\ell + 1) - 4]x < 0, \quad \text{that is,} \quad x_4 < x_r,$$

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so

$$x_4 < x_r < x_{r-1} < \cdots < x_5 < x_3 < x_1 < x_2,$$

from which we can see that if r is odd,

$$\mathcal{M}_{r-1} = \begin{cases} \langle x_{r-2}, x_{r-4} \rangle & \text{if } r \in \{5,7\}, \\ \langle x_{(r+3)/2}, x_{(r+1)/2} \rangle & \text{if } r \geqslant 9, \end{cases} \quad \text{and} \quad \mathcal{M}_r = \begin{cases} x_3 & \text{if } r = 5, \\ x_{(r+3)/2} & \text{if } r \geqslant 7, \end{cases}$$

and otherwise

$$\mathcal{M}_{r-1} = \begin{cases} x_3 & \text{if } r = 6, \\ x_{(r+2)/2} & \text{if } r \geqslant 8, \end{cases} \quad \text{and} \quad \mathcal{M}_r = \begin{cases} \langle x_3, x_5 \rangle & \text{if } r = 6, \\ \langle x_{(r+4)/2}, x_{(r+2)/2} \rangle & \text{if } r \geqslant 8. \end{cases}$$

Case II: $r \in \{\ell+2, \ldots, 2\ell-1\}$. Since $1/\ell \leqslant x < 1/(\ell-1)$, then

$$x_4 - x_{\ell+1} = (2x-1) + [(\ell+1)-4]x < 0, \quad \text{that is,} \quad x_4 < x_{\ell+1}$$

and

$$x_{\ell+2} - x_4 = [(\ell+2)-4]x - (2x-1) \leqslant 0, \quad \text{that is,} \quad x_{\ell+2} \leqslant x_4,$$

so

$$x_r < \cdots < x_{\ell+2} \leqslant x_4 < x_{\ell+1} < x_\ell < \cdots < x_5 < x_3 < x_1 < x_2,$$

from which we can see that

$$\mathcal{M}_{r-1} = \begin{cases} \langle x_{(r+3)/2}, x_{(r+1)/2} \rangle & \text{if } r \text{ is odd,} \\ x_{(r+2)/2} & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathcal{M}_r = \begin{cases} x_{(r+3)/2} & \text{if } r \text{ is odd,} \\ \langle x_{(r+4)/2}, x_{(r+2)/2} \rangle & \text{otherwise.} \end{cases}$$

In both cases, $\mathcal{M}_{r-1} = -(r-6)x/2$ and $\mathcal{M}_r = -(r-5)x/2$, so

$$x_{r+1} = r\mathcal{M}_r - (r-1)\mathcal{M}_{r-1} = r\Big(-\frac{r-5}{2}x\Big) - (r-1)\Big(-\frac{r-6}{2}x\Big) = -(r-3)x,$$

as desired.

Part II: Formulae for $x_{2\ell+1}$ and $x_{2\ell+2}$. From the previous part, $\mathcal{M}_{2\ell-1} = x_{\ell+1}$. Moreover, since

$$x_{2\ell} < \cdots < x_{\ell+2} \leqslant x_4 < x_{\ell+1} < x_\ell < \cdots < x_5 < x_3 < x_1 < x_2,$$

then $\mathcal{M}_{2\ell} = \langle x_4, x_{\ell+1} \rangle$. Therefore,

$$x_{2\ell+1} = 2\ell\mathcal{M}_{2\ell} - (2\ell-1)\mathcal{M}_{2\ell-1} = \ell x_4 - (\ell-1)x_{\ell+1} = x_4 - (\ell-1)(x_{\ell+1} - x_4) < x_4, \tag{3.3}$$

so that $\mathcal{M}_{2\ell+1} = x_4$, implying

$$x_{2\ell+2} = (2\ell+1)\mathcal{M}_{2\ell+1} - 2\ell\mathcal{M}_{2\ell} = (\ell+1)x_4 - \ell x_{\ell+1}. \tag{3.4}$$

Next, we split into two cases.

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Case I: $x = 1/\ell$. In this case, $x_4 = x_{\ell+2} = -(\ell-2)x$. Substituting this and $x_{\ell+1} = -(\ell-3)x$ into (3.3) and (3.4) gives $x_{2\ell+1} = -(2\ell-3)x$ and $x_{2\ell+2} = -(2\ell-2)x$, extending (3.2). Moreover, since $x_{2\ell+2} < x_{2\ell+1} < x_4$, then $\mathcal{M}_{2\ell+2} = \langle x_{\ell+2}, x_4\rangle = x_4 = \mathcal{M}_{2\ell+1}$, so, by part (ii) of Proposition 2.3, we have $x_n = x_4 = 2x-1$ for every $n \geqslant 2\ell+3$. This means $m_{\mathrm{A}}(x) = 2x-1$ and $\tau_{\mathrm{A}}(x) = 2\ell+3$, completing the proof.

Case II: $x \in (1/\ell, 1/(\ell-1))$. Substituting $x_4 = 2x-1$ and $x_{\ell+1} = -(\ell-3)x$ into (3.3) and (3.4) gives (3.1). Moreover,

$$x_{2\ell+2} = (\ell^2-\ell+2)x-\ell-1 = 2x-1+\ell(\ell-1)x-\ell < 2x-1 = x_4,$$

because $\ell(\ell-1)x-\ell < 0$ as $x < 1/(\ell-1)$. Consequently, $\mathcal{M}_{2\ell+2} < \mathcal{M}_{2\ell+1}$, so $m_{\mathrm{A}}(x) < \mathcal{M}_{2\ell+1} = 2x-1$ and $\tau_{\mathrm{A}}(x) > 2\ell+3$, completing the proof. □

To show how our main theorem follows from Lemma 3.1, let $x \in (0,1)$. If $x = 1/\ell$ for some integer $\ell \geqslant 2$, then, by Lemma 3.1, we have $m_{\mathrm{A}}(x) = 2x-1$ and $\tau_{\mathrm{A}}(x) = 2\ell+3 = 2/x+3$. Otherwise, $x \in (1/\ell, 1/(\ell-1))$ for some integer $\ell \geqslant 2$, so by Lemma 3.1, $m_{\mathrm{A}}(x) < 2x-1$ and $\tau_{\mathrm{A}}(x) \geqslant 2\ell+3 = 2/(1/\ell)+3 > 2/x+3$.

4. Remarks on symmetries

One of the most striking features of Figure 2 is the presence of symmetries, particularly around $x = \frac{1}{2}$. In this closing section, we briefly explain the symmetry near $x = \frac{1}{2}$ in the light of what has been done for the original MMM [7].

As in [7], we now regard $[x, 1]$, $x \in (0,1)$, as a multiset of univariate piecewise-affine continuous real functions [in this case, $Y_1(x) = x$ and $Y_2(x) = 1$]; we refer to such a multiset as a *bundle* [7, Section 2.2]. Observing that

$$\mathbf{M}_{\mathrm{A}}([x,1]) = [x,1,0] \quad \text{and} \quad \mathbf{M}_{\mathrm{A}}([x,1,0]) = [x,1,0,2x-1],$$

it is natural to regard $\mathbf{M}_{\mathrm{A}}$ as a self-map on the space of nonempty bundles with pointwise action.

The point $\frac{1}{2}$ is an *X-point* [7, Section 2.2]: a transversal intersection of two bundle functions, namely $Y_3(x) = 0$ and $Y_4(x) = 2x-1$ (see Figure 4). Let

$$\Omega := [Y_3, Y_4, Y_1]$$

be the subbundle containing these two functions and the function Y_1 immediately above the X-point. Notice that for

$$f(z) = z - 2x + 1 \quad \text{and} \quad \mu(x) = 1 - x, \tag{4.1}$$

the subbundle Ω satisfies

$$\Omega(\mu(x)) = [\mu(x), 2\mu(x)-1, 0] = [1-x, -2x+1, 0] = f([0, 2x-1, x]) = f(\Omega(x)).$$

Moreover, it is possible to show that the multiset of all functions Y satisfying the same identity, $Y(\mu(x)) = f(Y(x))$, is precisely

$$\Psi := \{\alpha \min\{Y_3, Y_4\} + \beta \max\{Y_3, Y_4\} + \gamma Y_1 : \alpha+\beta+\gamma = 1\},$$

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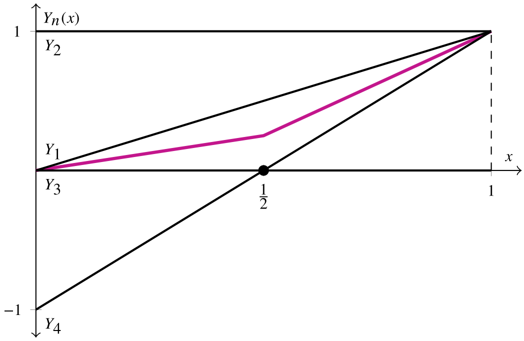

FIGURE 4. The bundle $[x, 1, 0, 2x-1]$ and its median $\mathcal{M}_4$ in purple. (Colour available online.)

that is, the multiset of all affine combinations of the functions $\min\{Y_3, Y_4\}$, $\max\{Y_3, Y_4\}$ and Y_1, the minimum and maximum being defined pointwise [7, Lemma 5.1]. One shows that

$$Y_5 = 4\mathcal{M}_4 - 3\mathcal{M}_3 = 0 \cdot \min\{Y_3, Y_4\} + 2 \cdot \max\{Y_3, Y_4\} + (-1) \cdot Y_1 \in \Psi.$$

Moreover, for every $n \geqslant 5$, the fact that $Y_5, \ldots, Y_n \in \Psi$ implies $Y_{n+1} \in \Psi$, since

$$Y_{n+1} = n\mathcal{M}_n - (n-1)\mathcal{M}_{n-1}$$

is an affine combination of $\mathcal{M}_n$ and $\mathcal{M}_{n-1}$, each of which is either a function in the multiset $[Y_5, \ldots, Y_n] \uplus [\min\{Y_3, Y_4\}, \max\{Y_3, Y_4\}, Y_1]$ or the mean of two such functions. This inductively proves that $Y_n \in \Psi$ for every $n \geqslant 5$ (see [7, Lemma 5.2]).

In other words,

$$Y_n(\mu(x)) = f(Y_n(x))$$

for every $n \geqslant 5$, where f and μ are given by (4.1). Since $\mu : (0, \frac{1}{2}] \to [\frac{1}{2}, 1)$ is a bijection, the transformation f connects the dynamics at every initial condition $x \in (0, \frac{1}{2}]$ to that at a unique initial condition $\mu(x) \in [\frac{1}{2}, 1)$. In particular, for every $x \in (0, \frac{1}{2}]$,

$$\tau_{\mathrm{A}}(\mu(x)) = \tau_{\mathrm{A}}(x) \quad \text{and} \quad m_{\mathrm{A}}(\mu(x)) = f(m_{\mathrm{A}}(x)),$$

that is,

$$\tau_{\mathrm{A}}(1-x) = \tau_{\mathrm{A}}(x) \quad \text{and} \quad m_{\mathrm{A}}(1-x) = m_{\mathrm{A}}(x) - 2x + 1,$$

explaining the symmetry seen in Figure 2.

The symmetry also means that the bounds in our main theorem—although already sufficient to achieve the goal of this paper—can be improved to

$$\tau_{\mathrm{A}}(x) \geqslant \begin{cases} 2/x + 3, & \text{if } x \in (0, \frac{1}{2}] \\ 2/(1-x) + 3, & \text{if } x \in (\frac{1}{2}, 1) \end{cases} \quad \text{and} \quad m_{\mathrm{A}}(x) \leqslant \begin{cases} 2x - 1 & \text{if } x \in (0, \frac{1}{2}], \\ 0 & \text{if } x \in (\frac{1}{2}, 1), \end{cases}$$

where equalities in $(0, \frac{1}{2}]$ occur at unit fractions, whereas those in $[\frac{1}{2}, 1)$ occur at fractions whose numerator and denominator differ by 1. These two families of fractions form two sequences, converging to the points 0 and 1 where m_A is discontinuous, along which τ_A becomes arbitrarily large.

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