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The unified transform for the heat equation: II. Non-separable boundary conditions in two dimensions

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We use the two-dimensional heat equation as an illustrative example to show that the unified transform is capable of constructing analytical solutions for linear evolution partial differential equations (PDEs) in two spatial dimensions involving non-separable boundary conditions. Such non-separable boundary value problems apparently cannot be solved by the usual transforms. We note that the unified transform always yields integral expressions which, in contrast to the expressions obtained by the usual transforms, have the advantage that are uniformly convergent at the boundary. Thus, even for the cases of separable boundary value problems where the usual transforms can be implemented, the unified transform provides alternative solution expressions which have advantages for both numerical and asymptotic considerations. The former advantage is illustrated by providing the numerical evaluation of a typical boundary value problem, by extending the approach of Flyer and Fokas (2008 *Proc. R. Soc.* **464**, 1823–1849). This work is the two-dimensional continuation of the heat equation with oblique Robin boundary conditions which was analysed in Mantzavinos and Fokas (2013 *Eur. J. Appl. Math.* **24**(6), 857–886.

Key words: initial-boundary value problems; two-dimensional heat equation; unified transform method; non-separable boundary conditions; oblique Neumann boundary conditions

1 Introduction

Self-adjoint boundary value problems for linear partial differential equations (PDEs) formulated in a separable domain and involving separable boundary conditions can be solved via the usual transforms. The unified transform introduced by one of the authors [1,2], although originally developed for integrable *nonlinear* evolution PDEs in one spatial dimension, has had important implications for *linear* PDEs. In particular, it is capable of constructing analytical solutions to certain linear boundary value problems which are either non-self-adjoint or involve non-separable boundary conditions; see [3] for a recent review comparing the unified transform with the classical approaches in two dimensions.

Most of the existing results using the unified transform have been restricted to two dimensions. Regarding linear PDEs, the only works in higher than two dimensions are the following: (i) the analytical solution of evolution PDEs in two spatial dimensions involving either second or third order spatial derivatives formulated in the quarter plane and satisfying separable boundary conditions; (ii) the analytical solution of the two-dimensional heat equation formulated in the interior of an equilateral triangle with Dirichlet boundary conditions [4].

Here we study the two-dimensional heat equation on the quarter plane $\{0 < x < \infty, 0 < y < \infty\}$ with given initial data $u_0(x, y)$ and oblique Neumann boundary conditions:

$$u_t = u_{xx} + u_{yy}, \qquad 0 < x < \infty,\ 0 < y < \infty,\ t > 0, \tag{1.1a}$$

$$u(x, y, 0) = u_0(x, y), \quad 0 < x < \infty,\ 0 < y < \infty, \tag{1.1b}$$

$$\cos\beta_1\, u_x(x, 0, t) - \sin\beta_1\, u_y(x, 0, t) = F_1(x, t), \quad 0 < x < \infty,\ t > 0, \tag{1.1c}$$

$$\cos\beta_2\, u_y(0, y, t) + \sin\beta_2\, u_x(0, y, t) = F_2(y, t), \quad 0 < y < \infty,\ t > 0, \tag{1.1d}$$

where $\beta_1, \beta_2 \in [0, 2\pi)$ are constants and $F_1(x, t), F_2(y, t)$ are given functions with appropriate smoothness and decay. We obtain an analytical solution in the case that β_1, β_2 satisfy any of the following conditions:

$$\beta_1 = \beta_2 + (2n + 1)\frac{\pi}{2}, \quad n = -2, -1, 0, 1, \tag{1.2a}$$

$$\beta_1 = (2m + 1)\frac{\pi}{2},\ m = 0, 1, \quad \beta_2 = (2n + 1)\frac{\pi}{2},\ n = 0, 1, \tag{1.2b}$$

$$\beta_1 = m\pi,\ m = 0, 1, \quad \beta_2 = n\pi,\ n = 0, 1. \tag{1.2c}$$

We note that equations (1.2*b*) and (1.2*c*) are the conditions that the boundary data vectors shown in Figure 1 are *orthogonal*, whereas equation (1.2*a*) is the condition that these vectors are *parallel*. It will be shown in Section 2 that the condition (1.2*a*) implies the following compatibility requirements for the functions $F_1(x, t)$ and $F_2(y, t)$:

$$n = -1, 1 :\ F_1(0, t) = F_2(0, t); \qquad n = -2, 0 :\ F_1(0, t) = -F_2(0, t). \tag{1.3}$$

It turns out that in the case of condition (1.2*a*) with $\beta_1 \in (0, \pi/2) \cup (\pi, 3\pi/2)$ well-posedness requires the additional boundary condition

$$u(0, 0, t) = h(t), \quad t > 0, \tag{1.4}$$

where $h(t)$ is a given function with appropriate smoothness.

The initial-boundary value problem (1.1) can be regarded as a two-dimensional analogue of the heat equation posed on the half-line $\{0 < x < \infty\}$ with a non-separable boundary condition of oblique Robin type:

$$u_t = u_{xx}, \qquad 0 < x < \infty,\ t > 0, \tag{1.5a}$$

$$u(x, 0) = u_0(x), \quad 0 < x < \infty, \tag{1.5b}$$

$$u_t(0, t) + \alpha u_x(0, t) + \beta u(0, t) = \gamma(t), \quad t > 0, \tag{1.5c}$$

where α, β are constants and $\gamma(t)$ is a known function. Problem (1.5) as well as the corresponding problem on the finite interval were analysed in [5], where in addition the authors considered the case of non-local integral constraints instead of boundary conditions of the form (1.5*c*).

We emphasise that boundary value problems with non-separable boundary conditions cannot in general be solved by the usual transforms. In this respect, we note that the application of the Laplace transform to the two-dimensional heat equation yields a Helmholtz type equation but with a "complex wave number"; the analysis of such an equation is rather complicated.

Boundary value problems for the modified Helmholtz equation on the quarter plane with oblique Neumann boundary conditions were investigated in [6]; the conditions (1.2) for β_1, β_2 are identical with the conditions for β_1, β_2 appearing in [6].

1.1 The unified transform

The implementation of this method to the heat equation in two dimensions requires the followings steps:

(1) Rewrite the heat equation

$$u_t = u_{xx} + u_{yy}, \tag{1.6}$$

in the divergence form

$$\left(e^{-ik_1x-ik_2y+(k_1^2+k_2^2)t}u\right)_t = \left[e^{-ik_1x-ik_2y+(k_1^2+k_2^2)t}\,(u_x + ik_1\,u)\right]_x + \left[e^{-ik_1x-ik_2y+(k_1^2+k_2^2)t}\left(u_y + ik_2\,u\right)\right]_y, \quad k_1, k_2 \in \mathbb{C}. \tag{1.7}$$

Use Gauss' theorem in the given domain to obtain the so-called *global relation*, namely an equation which couples the Fourier transform of $u(x, y, t)$, denoted by $\hat{u}(k_1, k_2, t)$, with appropriate transforms of the given data, as well as with transforms of certain unknown boundary values.

(2) Solve the global relation for $\hat{u}(k_1, k_2, t)$ and use the inverse Fourier transform to obtain $u(x, y, t)$ in terms of transforms of the initial and boundary data, as well as of transforms of certain unknown boundary values. Deform the contours of integration with respect to k_1 and k_2 from the real line to appropriate contours in the complex k_1 and k_2-planes to obtain an integral representation of $u(x, y, t)$. This representation is not yet effective because it involves transforms of certain unknown boundary values.

(3) Use the global relation, as well as three additional equations obtained from the global relation under the transformations $k_1 \mapsto -k_1$ and $k_2 \mapsto -k_2$, to eliminate the transforms of the unknown boundary values and thus obtain an effective solution expression.

2 The heat equation on the quarter-plane with oblique Neumann boundary conditions

Consider the two-dimensional heat equation

$$u_t = u_{xx} + u_{yy}, \quad u = u(x, y, t), \quad (x, y, t) \in \Omega, \tag{2.1}$$

where

$$\Omega = \{0 < x < \infty,\ 0 < y < \infty,\ t > 0\}, \tag{2.2}$$

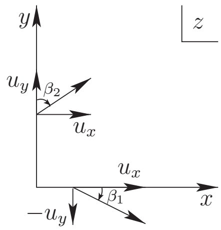

FIGURE 1. Oblique Neumann boundary conditions on the quarter-plane.

with the initial datum

$$u(x, y, 0) = u_0(x, y), \quad 0 < x < \infty, \quad 0 < y < \infty, \tag{2.3}$$

where $u_0(x, y)$ has appropriate smoothness and decay.

We pose an initial-boundary value problem by supplementing the above equations with the so-called *oblique Neumann* boundary conditions:

$$\cos\beta_1\, u_x(x, 0, t) - \sin\beta_1\, u_y(x, 0, t) = F_1(x, t), \quad 0 < x < \infty, \ t > 0, \tag{2.4a}$$
$$\cos\beta_2\, u_y(0, y, t) + \sin\beta_2\, u_x(0, y, t) = F_2(y, t), \quad 0 < y < \infty, \ t > 0, \tag{2.4b}$$

where $F_1(x, t)$ and $F_2(y, t)$ are known functions.

In what follows we apply steps 1–3 of the introduction to the heat equation on the quarter-plane with the oblique Neumann boundary conditions (2.4).

2.1 Apply Gauss' theorem on the divergence form

The formal adjoint of equation (2.1) is

$$-\tilde{u}_t = \tilde{u}_{xx} + \tilde{u}_{yy}, \quad \tilde{u} = \tilde{u}(x, y, t). \tag{2.5}$$

Thus,

$$(\tilde{u}u)_t = (\tilde{u}u_x - \tilde{u}_x u)_x + \left(\tilde{u}u_y - \tilde{u}_y u\right)_y. \tag{2.6}$$

Equation (2.5) admits the two-parameter family of solutions

$$\tilde{u}(x, y, t) = e^{-ik_1x - ik_2y + \omega(k_1,k_2)t}, \quad k_1 \in \mathbb{C}, \ k_2 \in \mathbb{C}, \tag{2.7}$$

where the function $\omega(k_1, k_2)$, hereafter denoted by ω, is defined by

$$\omega(k_1, k_2) = k_1^2 + k_2^2. \tag{2.8}$$

Inserting the particular solution (2.7) into equation (2.6) yields the following divergence form for the two-dimensional heat equation:

$$\begin{aligned}\left[e^{-ik_1x-ik_2y+\omega t}u(x, y, t)\right]_t = &\left\{e^{-ik_1x-ik_2y+\omega t}\left[u_x(x, y, t) + ik_1\, u(x, y, t)\right]\right\}_x \\ &+ \left\{e^{-ik_1x-ik_2y+\omega t}\left[u_y(x, y, t) + ik_2\, u(x, y, t)\right]\right\}_y.\end{aligned} \tag{2.9}$$

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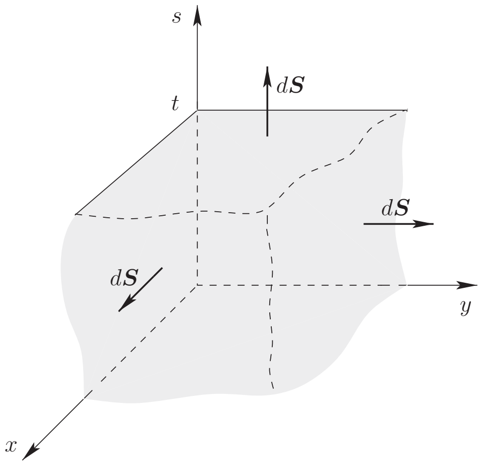

FIGURE 2. The domain V for Gauss' theorem.

Let $\hat{u}(k_1, k_2, t)$ denote the two-dimensional Fourier transform of $u(x, y, t)$ with respect to the spatial variables x and y:

$$\hat{u}(k_1, k_2, t) = \int_0^\infty dx \int_0^\infty dy\, e^{-ik_1 x - ik_2 y}\, u(x, y, t), \quad k_1 \in \mathbb{C}^-, \ k_2 \in \mathbb{C}^-. \tag{2.10a}$$

The inverse Fourier transforms yields

$$u(x, y, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_2\, e^{ik_1 x + ik_2 y}\, \hat{u}(k_1, k_2, t), \quad 0 < x < \infty, \ 0 < y < \infty. \tag{2.10b}$$

Note that we require k_1 and k_2 to have non-positive imaginary parts in order for the integrand in equation (2.10a) to be bounded as x and y tend to infinity.

Employing Gauss' theorem, equation (2.9) yields

$$\iint_{\partial V} e^{-ik_1 x - ik_2 y + \omega s} \begin{pmatrix} u_x(x, y, s) + ik_1\, u(x, y, s) \\ u_y(x, y, s) + ik_2\, u(x, y, s) \\ -u(x, y, s) \end{pmatrix} \cdot d\boldsymbol{S} = 0, \tag{2.11}$$

where $V = \{0 < x < \infty, \ 0 < y < \infty, \ 0 < s < t\}$ is depicted in Figure 2 and $d\boldsymbol{S}$ is the element of the surface of V pointing in the outward direction. Thus, we obtain

$$\begin{aligned} e^{\omega t} \hat{u}(k_1, k_2, t) = \hat{u}_0(k_1, k_2) &- \left[g_1^{(1)}(\omega, k_2, t) + ik_1\, g_0^{(1)}(\omega, k_2, t) \right] \\ &- \left[g_1^{(2)}(\omega, k_1, t) + ik_2\, g_0^{(2)}(\omega, k_1, t) \right], \quad k_1 \in \mathbb{C}^-, \ k_2 \in \mathbb{C}^-, \end{aligned} \tag{2.12}$$

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where

$$g_0^{(1)}(\omega,k_2,t)=\int_0^t ds\int_0^\infty dy\, e^{-ik_2y+\omega s}u(0,y,s),\quad k_1\in\mathbb{C},\ k_2\in\mathbb{C}^-, \tag{2.13a}$$

$$g_1^{(1)}(\omega,k_2,t)=\int_0^t ds\int_0^\infty dy\, e^{-ik_2y+\omega s}u_x(0,y,s),\quad k_1\in\mathbb{C},\ k_2\in\mathbb{C}^-, \tag{2.13b}$$

$$g_0^{(2)}(\omega,k_1,t)=\int_0^t ds\int_0^\infty dx\, e^{-ik_1x+\omega s}u(x,0,s),\quad k_1\in\mathbb{C}^-,\ k_2\in\mathbb{C}, \tag{2.13c}$$

$$g_1^{(2)}(\omega,k_1,t)=\int_0^t ds\int_0^\infty dx\, e^{-ik_1x+\omega s}u_y(x,0,s),\quad k_1\in\mathbb{C}^-,\ k_2\in\mathbb{C}. \tag{2.13d}$$

Let us hereafter write

$$\cos\beta_j=c_j,\quad \sin\beta_j=s_j,\quad j=1,2.$$

Then, the boundary conditions (2.4) become

$$c_1\,u_x(x,0,t)-s_1\,u_y(x,0,t)=F_1(x,t),\quad 0<x<\infty,\ t>0, \tag{2.14a}$$

$$c_2\,u_y(0,y,t)+s_2\,u_x(0,y,t)=F_2(y,t),\quad 0<y<\infty,\ t>0. \tag{2.14b}$$

We introduce the unknown functions $Q_1(x,t)$ and $Q_2(y,t)$ which satisfy the equations

$$s_1\,u_x(x,0,t)+c_1\,u_y(x,0,t)=Q_1(x,t),\quad 0<x<\infty,\ t>0, \tag{2.15a}$$

$$s_2\,u_y(0,y,t)-c_2\,u_x(0,y,t)=Q_2(y,t),\quad 0<y<\infty,\ t>0. \tag{2.15b}$$

Solving equations (2.14) and (2.15) for the boundary values of u, we find

$$u_x(x,0,t)=c_1F_1(x,t)+s_1Q_1(x,t), \tag{2.16a}$$

$$u_y(x,0,t)=c_1Q_1(x,t)-s_1F_1(x,t), \tag{2.16b}$$

$$u_y(0,y,t)=c_2F_2(y,t)+s_2Q_2(y,t), \tag{2.16c}$$

$$u_x(0,y,t)=s_2F_2(y,t)-c_2Q_2(y,t). \tag{2.16d}$$

Let $\{f_1,f_2,q_1,q_2\}$ denote the relevant transforms of $\{F_1,F_2,Q_1,Q_2\}$, that is

$$f_1(\omega,k_1,t)=\int_0^t ds\int_0^\infty dx\, e^{-ik_1x+\omega s}F_1(x,s), \tag{2.17a}$$

$$f_2(\omega,k_2,t)=\int_0^t ds\int_0^\infty dy\, e^{-ik_2y+\omega s}F_2(y,s), \tag{2.17b}$$

$$q_1(\omega,k_1,t)=\int_0^t ds\int_0^\infty dx\, e^{-ik_1x+\omega s}Q_1(x,s), \tag{2.17c}$$

$$q_2(\omega,k_2,t)=\int_0^t ds\int_0^\infty dy\, e^{-ik_2y+\omega s}Q_2(y,s). \tag{2.17d}$$

Multiplying equations (2.16a) and (2.16b) by $e^{-ik_1x+\omega s}$, taking the double integral over

$(x, s) \in (0, \infty) \times (0, t)$ and integrating by parts, we obtain

$$ik_1 g_0^{(2)}(\omega, k_1, t) = c_1 f_1(\omega, k_1, t) + s_1 q_1(\omega, k_1, t) + \tilde{u}(\omega, t), \tag{2.18a}$$

$$g_1^{(2)}(\omega, k_1, t) = c_1 q_1(\omega, k_1, t) - s_1 f_1(\omega, k_1, t), \tag{2.18b}$$

where the function $\tilde{u}$ is defined by

$$\tilde{u}(\omega, t) = \int_0^t ds\, e^{\omega s}\, u(0, 0, s). \tag{2.19}$$

Similarly, using equations (2.16*c*) and (2.16*d*) we find

$$ik_2 g_0^{(1)}(\omega, k_2, t) = c_2 f_2(\omega, k_2, t) + s_2 q_2(\omega, k_2, t) + \tilde{u}(\omega, t), \tag{2.20a}$$

$$g_1^{(1)}(\omega, k_2, t) = s_2 f_2(\omega, k_2, t) - c_2 q_2(\omega, k_2, t). \tag{2.20b}$$

Employing equations (2.18) and (2.20) into equation (2.12) we obtain the *global relation*

$$\begin{aligned} e^{\omega t}\hat{u}(k_1, k_2, t) = \hat{u}_0(k_1, k_2) &+ \left[\left(c_2 - s_2\frac{k_1}{k_2}\right) q_2(\omega, k_2, t) - \left(s_2 + c_2\frac{k_1}{k_2}\right) f_2(\omega, k_2, t)\right] \\ &- \left[\left(c_1 + s_1\frac{k_2}{k_1}\right) q_1(\omega, k_1, t) - \left(s_1 - c_1\frac{k_2}{k_1}\right) f_1(\omega, k_1, t)\right] \\ &- \left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)\tilde{u}(\omega, t), \quad k_1 \in \mathbb{C}^-,\ k_2 \in \mathbb{C}^-. \end{aligned} \tag{2.21}$$

This relation couples the known functions $\{\hat{u}_0, f_1, f_2\}$ with the unknown functions $\hat{u}$ and $\{q_1, q_2\}$.

2.2 Invert the global relation

Inserting equation (2.21) into the inverse Fourier transform (2.10*b*), we find

$$\begin{aligned} u(x, y, t) = &\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1 x + ik_2 y - \omega t}\, \hat{u}_0(k_1, k_2) \\ &- \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1 x + ik_2 y - \omega t}\left[\left(s_2 + c_2\frac{k_1}{k_2}\right) f_2(\omega, k_2, t) + \frac{k_1}{k_2}\tilde{u}(\omega, t)\right. \\ &\left. - \left(c_2 - s_2\frac{k_1}{k_2}\right) q_2(\omega, k_2, t)\right] \\ &- \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1 x + ik_2 y - \omega t}\left[-\left(s_1 - c_1\frac{k_2}{k_1}\right) f_1(\omega, k_1, t) + \frac{k_2}{k_1}\tilde{u}(\omega, t)\right. \\ &\left. + \left(c_1 + s_1\frac{k_2}{k_1}\right) q_1(\omega, k_1, t)\right]. \end{aligned} \tag{2.22}$$

Define the regions D^+ and D^- of the complex k_j-planes, $j = 1, 2$, by

$$D^+ = \{k_j \in \mathbb{C}^+ : \mathrm{Re}\left(k_j^2\right) < 0\}, \quad D^- = \{k_j \in \mathbb{C}^- : \mathrm{Re}\left(k_j^2\right) < 0\}, \quad j = 1, 2. \tag{2.23}$$

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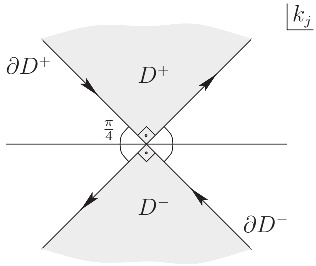

FIGURE 3. The regions D^+ and D^- of the k_j-planes, $j = 1, 2$.

The second term on the right-hand side of equation (2.22) involves the exponential

$$e^{ik_1x+ik_2(y-\eta)-\omega(t-s)}, \quad x \geqslant 0, \ t \geqslant s, \tag{2.24}$$

where in the definitions (2.13) we have used η instead of y. The definitions (2.8) and (2.23) imply that this exponential is bounded for $k_1 \in \mathbb{C}^+ \setminus D^+$, thus we can deform the contour with respect to k_1 from the real axis to the contour ∂D^+ shown in Figure 3. Similarly, in the third term on the right-hand side of equation (2.22) we can deform the contour with respect to k_2 from the real axis to the contour ∂D^+. Hence, the following proposition is established.

Proposition 1 (Integral representation) *The two-dimensional heat equation* (2.1) *posed on the quarter-plane with the initial datum* (2.3) *admits the integral representation*

$$\begin{aligned} u(x,y,t) = &\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\ \hat{u}_0(k_1,k_2) \\ &- \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\left(s_2 + c_2\frac{k_1}{k_2}\right) f_2(\omega,k_2,t) + \frac{k_1}{k_2}\tilde{u}(\omega,t)\right. \\ &\left.-\left(c_2 - s_2\frac{k_1}{k_2}\right) q_2(\omega,k_2,t)\right] \\ &+ \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\left(s_1 - c_1\frac{k_2}{k_1}\right) f_1(\omega,k_1,t) - \frac{k_2}{k_1}\tilde{u}(\omega,t)\right. \\ &\left.-\left(c_1 + s_1\frac{k_2}{k_1}\right) q_1(\omega,k_1,t)\right], (x,y,t) \in \Omega, \end{aligned} \tag{2.25}$$

where $\hat{u}_0(k_1,k_2)$ *is the Fourier transform* (2.10*a*) *of the initial datum* $u_0(x,y)$, *the functions* $f_j(\omega,k_j,t), q_j(\omega,k_j,t), j = 1,2$, *are defined by equations* (2.17) *and the contour* ∂D^+ *is depicted in Figure 3.*

2.3 Elimination of the unknown transforms

The integral representation equation (2.25) involves the unknown functions $\{q_1(\omega,k_1,t), q_2(\omega,k_2,t)\}$ defined by equations (2.17*c*) and (2.17*d*) as the relevant transforms of $\{Q_1(x,t), Q_2(y,t)\}$, as well as the unknown function $\tilde{u}(\omega,t)$ defined by equation (2.19).

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In what follows, we will use the global relation (2.21) to eliminate $q_1(\omega,k_1,t)$ and $q_2(\omega,k_2,t)$ from equation (2.25). In this respect, we note that the function $\omega(k_1,k_2)$ is invariant under the transformations $k_j \mapsto -k_j$, $j=1,2$. Hence, we supplement the global relation (2.21) with the following additional identities obtained from equation (2.21) via the aforementioned transformations:

$$\begin{aligned} e^{\omega t}\hat{u}(-k_1,k_2,t) = \hat{u}_0(-k_1,k_2) &+ \left[\left(c_2+s_2\frac{k_1}{k_2}\right)q_2(\omega,k_2,t)-\left(s_2-c_2\frac{k_1}{k_2}\right)f_2(\omega,k_2,t)\right] \\ &- \left[\left(c_1-s_1\frac{k_2}{k_1}\right)q_1(\omega,-k_1,t)-\left(s_1+c_1\frac{k_2}{k_1}\right)f_1(\omega,-k_1,t)\right] \\ &+ \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t), \quad k_1\in\mathbb{C}^+,\ k_2\in\mathbb{C}^-, \end{aligned} \tag{2.26a}$$

$$\begin{aligned} e^{\omega t}\hat{u}(k_1,-k_2,t) = \hat{u}_0(k_1,-k_2) &+ \left[\left(c_2+s_2\frac{k_1}{k_2}\right)q_2(\omega,-k_2,t)-\left(s_2-c_2\frac{k_1}{k_2}\right)f_2(\omega,-k_2,t)\right] \\ &- \left[\left(c_1-s_1\frac{k_2}{k_1}\right)q_1(\omega,k_1,t)-\left(s_1+c_1\frac{k_2}{k_1}\right)f_1(\omega,k_1,t)\right] \\ &+ \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t), \quad k_1\in\mathbb{C}^-,\ k_2\in\mathbb{C}^+, \end{aligned} \tag{2.26b}$$

and

$$\begin{aligned} e^{\omega t}\hat{u}(-k_1,-k_2,t) = \hat{u}_0(-k_1,-k_2) &+ \left[\left(c_2-s_2\frac{k_1}{k_2}\right)q_2(\omega,-k_2,t)-\left(s_2+c_2\frac{k_1}{k_2}\right)f_2(\omega,-k_2,t)\right] \\ &- \left[\left(c_1+s_1\frac{k_2}{k_1}\right)q_1(\omega,-k_1,t)-\left(s_1-c_1\frac{k_2}{k_1}\right)f_1(\omega,-k_1,t)\right] \\ &- \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t), \quad k_1\in\mathbb{C}^+,\ k_2\in\mathbb{C}^+. \end{aligned} \tag{2.26c}$$

Solving equation (2.26*a*) for $q_2(\omega,k_2,t)$ and equation (2.26*b*) for $q_1(\omega,k_1,t)$, we find respectively the following equations:

$$\begin{aligned} \left(c_2+s_2\frac{k_1}{k_2}\right)q_2(\omega,k_2,t) = e^{\omega t}\hat{u}(-k_1,k_2,t)-\hat{u}_0(-k_1,k_2) &+ \left(s_2-c_2\frac{k_1}{k_2}\right)f_2(\omega,k_2,t) \\ &+ \left(c_1-s_1\frac{k_2}{k_1}\right)q_1(\omega,-k_1,t)-\left(s_1+c_1\frac{k_2}{k_1}\right)f_1(\omega,-k_1,t) \\ &- \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t), \qquad k_1\in\mathbb{C}^+,\ k_2\in\mathbb{C}^-, \end{aligned} \tag{2.27a}$$

and

$$\begin{aligned} \left(c_1-s_1\frac{k_2}{k_1}\right)q_1(\omega,k_1,t) = -e^{\omega t}\hat{u}(k_1,-k_2,t)+\hat{u}_0(k_1,-k_2) &- \left(s_2-c_2\frac{k_1}{k_2}\right)f_2(\omega,-k_2,t) \\ &+ \left(c_2+s_2\frac{k_1}{k_2}\right)q_2(\omega,-k_2,t)+\left(s_1+c_1\frac{k_2}{k_1}\right)f_1(\omega,k_1,t) \\ &+ \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t), \qquad k_1\in\mathbb{C}^-,\ k_2\in\mathbb{C}^+. \end{aligned} \tag{2.27b}$$

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Inserting equations (2.27) in the integral representation (2.25), we obtain

$$\begin{aligned}
u(x,y,t) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\hat{u}_0(k_1,k_2) \\
&- \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\Bigg\{\left(s_2+c_2\frac{k_1}{k_2}\right)f_2(\omega,k_2,t)+\frac{k_1}{k_2}\tilde{u}(\omega,t) \\
&- \frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\Bigg[e^{\omega t}\hat{u}(-k_1,k_2,t)-\hat{u}_0(-k_1,k_2) \\
&+ \left(s_2-c_2\frac{k_1}{k_2}\right)f_2(\omega,k_2,t)+\left(c_1-s_1\frac{k_2}{k_1}\right)q_1(\omega,-k_1,t) \\
&- \left(s_1+c_1\frac{k_2}{k_1}\right)f_1(\omega,-k_1,t)-\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t)\Bigg]\Bigg\} \\
&+ \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\Bigg\{\left(s_1-c_1\frac{k_2}{k_1}\right)f_1(\omega,k_1,t)-\frac{k_2}{k_1}\tilde{u}(\omega,t) \\
&+ \frac{c_1k_1+s_1k_2}{c_1k_1-s_1k_2}\Bigg[e^{\omega t}\hat{u}(k_1,-k_2,t)-\hat{u}_0(k_1,-k_2) \\
&+ \left(s_2-c_2\frac{k_1}{k_2}\right)f_2(\omega,-k_2,t)-\left(c_2+s_2\frac{k_1}{k_2}\right)q_2(\omega,-k_2,t) \\
&- \left(s_1+c_1\frac{k_2}{k_1}\right)f_1(\omega,k_1,t)-\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t)\Bigg]\Bigg\}. \qquad (2.28)
\end{aligned}$$

The contributions of the terms involving $\hat{u}(-k_1,k_2,t)$ and $\hat{u}(k_1,-k_2,t)$ are zero. Indeed, the relevant integrands are analytic and, furthermore, since $x+\xi \geqslant 0$ and $y+\eta \geqslant 0$,

$$\lim_{|k_1|\to\infty} e^{ik_1(x+\xi)+ik_2(y-\eta)} = 0, \quad k_1 \in D^+,\ k_2 \in \mathbb{R}, \qquad (2.29a)$$

and

$$\lim_{|k_2|\to\infty} e^{ik_1(x-\xi)+ik_2(y+\eta)} = 0, \quad k_1 \in \mathbb{R},\ k_2 \in D^+. \qquad (2.29b)$$

Thus, these integrands have exponential decay at infinity and hence, by Cauchy's theorem and Jordan's lemma the corresponding integrals vanish.

2.3.1 *The relation between β_1 and β_2*

The representation (2.28) still involves the unknown functions $q_1(\omega,-k_1,t)$ and $q_2(\omega,-k_2,t)$. It turns out that we can eliminate these functions by using equation (2.26*c*). However, we have been able to achieve this only if the angles β_1 and β_2 are such that the combination of $q_1(\omega,-k_1,t)$ and $q_2(\omega,-k_2,t)$ appearing in equation (2.28) is proportional to the combination of these functions appearing in equation (2.26*c*), *for all* complex values

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of the spectral variables k_1 and k_2, i.e.

$$\frac{c_2k_2 - s_2k_1}{c_2k_2 + s_2k_1}\left(c_1 - s_1\frac{k_2}{k_1}\right)q_1(\omega, -k_1, t) - \frac{c_1k_1 + s_1k_2}{c_1k_1 - s_1k_2}\left(c_2 + s_2\frac{k_1}{k_2}\right)q_2(\omega, -k_2, t)$$
$$= \lambda\left[\left(c_1 + s_1\frac{k_2}{k_1}\right)q_1(\omega, -k_1, t) - \left(c_2 - s_2\frac{k_1}{k_2}\right)q_2(\omega, -k_2, t)\right], \tag{2.30}$$

for some constant λ which does not depend on k_1 and k_2. By equating the coefficients of powers of k_1 and k_2, we find that equation (2.30) is satisfied *for all* k_1 and k_2 if the following three equations are satisfied:

$$(c_1c_2 + s_1s_2)(\lambda - 1) = 0, \quad c_1s_2(\lambda + 1) = 0, \quad c_2s_1(\lambda + 1) = 0. \tag{2.31}$$

In what follows, we analyse these equations.

(1) $\lambda = -1$. Then $c_1c_2 + s_1s_2 = \cos(\beta_1 - \beta_2) = 0$ or, equivalently,

$$\beta_1 = \beta_2 + (2n+1)\frac{\pi}{2}, \quad n = -2, -1, 0, 1. \tag{2.32}$$

(2) $\lambda = 1$. Then $c_1s_2 = c_2s_1 = 0$.

(a) If $c_1 = c_2 = 0$, then

$$\beta_1 = (2m+1)\frac{\pi}{2},\ m = 0, 1, \quad \beta_2 = (2n+1)\frac{\pi}{2},\ n = 0, 1. \tag{2.33a}$$

(b) If $s_2 = s_1 = 0$, then

$$\beta_1 = m\pi,\ m = 0, 1, \quad \beta_2 = n\pi,\ n = 0, 1. \tag{2.33b}$$

(3) If $|\lambda| \neq 1$ then equation (2.31) cannot be satisfied for any choice of the angles β_1, β_2.

Equations (2.33) are actually the conditions that the boundary data vectors shown in Figure 1 are *orthogonal*, whereas condition (2.32) forces these vectors to be parallel. Equation (2.32) implies the compatibility requirements (1.3). Indeed, according to the condition (2.32) for $n = -1, 1$, we have $c_1 = s_2$ and $s_1 = -c_2$. Thus, evaluating the boundary conditions (2.14) at $x = y = 0$, we find

$$\left.\begin{aligned} s_2u_x(0,0,t) + c_2u_y(0,0,t) &= F_1(0,t) \\ c_2u_y(0,0,t) + s_2u_x(0,0,t) &= F_2(0,t) \end{aligned}\right\} \Rightarrow F_1(0,t) = F_2(0,t).$$

On the other hand, for $n = -2, 0$ we have $c_1 = -s_2$ and $s_1 = c_2$. Thus, evaluating the boundary conditions (2.14) at $x = y = 0$, we find

$$\left.\begin{aligned} -s_2u_x(0,0,t) - c_2u_y(0,0,t) &= F_1(0,t) \\ c_2u_y(0,0,t) + s_2u_x(0,0,t) &= F_2(0,t) \end{aligned}\right\} \Rightarrow F_1(0,t) = -F_2(0,t).$$

The integrals on the right-hand side of the representation (2.28) involving the functions $q_1(\omega,-k_1,t)$ and $q_2(\omega,-k_2,t)$ are given by

$$\frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\ \frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\left(c_1-s_1\frac{k_2}{k_1}\right)q_1(\omega,-k_1,t),$$

and

$$-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\ \frac{c_1k_1+s_1k_2}{c_1k_1-s_1k_2}\left(c_2+s_2\frac{k_1}{k_2}\right)q_2(\omega,-k_2,t).$$

The integrands of the above integrals are analytic for all $k_j \in \mathbb{C}^+ \setminus D^+$, $j=1,2$, hence, by Cauchy's theorem and Jordan's lemma, the contours with respect to k_2 and k_1 can be deformed from the real axis to the contour ∂D^+, so that

$$\frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\ \frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\left(c_1-s_1\frac{k_2}{k_1}\right)q_1(\omega,-k_1,t)$$
$$=\frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\ \frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\left(c_1-s_1\frac{k_2}{k_1}\right)q_1(\omega,-k_1,t), \qquad (2.34a)$$

and

$$-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\ \frac{c_1k_1+s_1k_2}{c_1k_1-s_1k_2}\left(c_2+s_2\frac{k_1}{k_2}\right)q_2(\omega,-k_2,t)$$
$$=-\frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\ \frac{c_1k_1+s_1k_2}{c_1k_1-s_1k_2}\left(c_2+s_2\frac{k_1}{k_2}\right)q_2(\omega,-k_2,t). \qquad (2.34b)$$

Under condition (2.30), we can combine equations (2.34) with equation (2.26c). Neglecting the function $\hat{u}(-k_1,-k_2,t)$ (using the earlier arguments, see equations (2.29)), we find that the terms of equation (2.28) that involve $q_1(\omega,-k_1,t)$ and $q_2(\omega,-k_2,t)$ are equal to

$$\frac{\lambda}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\Big[\hat{u}_0(-k_1,-k_2)-\left(s_2+c_2\frac{k_1}{k_2}\right)f_2(\omega,-k_2,t)$$
$$+\left(s_1-c_1\frac{k_2}{k_1}\right)f_1(\omega,-k_1,t)-\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t)\Big]. \qquad (2.35)$$

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Inserting equation (2.35) in the integral representation (2.28) we obtain the following representation, which does not involve any of the unknown functions q_1 and q_2:

$$
\begin{aligned}
u(x,y,t) = {} & \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\hat{u}_0(k_1,k_2) \\
& - \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\left\{\frac{2k_1}{c_2k_2+s_2k_1} f_2(\omega,k_2,t)\right. \\
& + \frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\left[\hat{u}_0(-k_1,k_2) + \left(s_1 + c_1\frac{k_2}{k_1}\right) f_1(\omega,-k_1,t)\right. \\
& \left.\left. + \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t)\right] + \frac{k_1}{k_2}\tilde{u}(\omega,t)\right\} \\
& - \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left\{\frac{2k_2}{c_1k_1-s_1k_2} f_1(\omega,k_1,t)\right. \\
& + \frac{c_1k_1+s_1k_2}{c_1k_1-s_1k_2}\left[\hat{u}_0(k_1,-k_2) - \left(s_2 - c_2\frac{k_1}{k_2}\right) f_2(\omega,-k_2,t)\right. \\
& \left.\left. + \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t)\right] + \frac{k_2}{k_1}\tilde{u}(\omega,t)\right\} \\
& + \frac{\lambda}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\hat{u}_0(-k_1,-k_2) - \left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t)\right. \\
& \left. + \left(s_1 - c_1\frac{k_2}{k_1}\right) f_1(\omega,-k_1,t) - \left(s_2 + c_2\frac{k_1}{k_2}\right) f_2(\omega,-k_2,t)\right].
\end{aligned} \tag{2.36}
$$

The above representation contains the unknown function $\tilde{u}(\omega,t)$, thus we rewrite it in the form

$$
\begin{aligned}
u(x,y,t) = {} & \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\hat{u}_0(k_1,k_2) \\
& - \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\left\{\frac{2k_1}{c_2k_2+s_2k_1} f_2(\omega,k_2,t)\right. \\
& \left. + \frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\left[\hat{u}_0(-k_1,k_2) + \left(s_1 + c_1\frac{k_2}{k_1}\right) f_1(\omega,-k_1,t)\right]\right\} \\
& - \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left\{\frac{2k_2}{c_1k_1-s_1k_2} f_1(\omega,k_1,t)\right. \\
& \left. + \frac{c_1k_1+s_1k_2}{c_1k_1-s_1k_2}\left[\hat{u}_0(k_1,-k_2) - \left(s_2 - c_2\frac{k_1}{k_2}\right) f_2(\omega,-k_2,t)\right]\right\} \\
& + \frac{\lambda}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\hat{u}_0(-k_1,-k_2)\right. \\
& \left. + \left(s_1 - c_1\frac{k_2}{k_1}\right) f_1(\omega,-k_1,t) - \left(s_2 + c_2\frac{k_1}{k_2}\right) f_2(\omega,-k_2,t)\right] \\
& + u^*(x,y,t),
\end{aligned} \tag{2.37}
$$

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where $u^*(x,y,t)$ is defined by

$$u^*(x,y,t) = -\frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\frac{k_1}{k_2} + \frac{c_2k_2 - s_2k_1}{c_2k_2+s_2k_1}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\right]\tilde{u}(\omega,t)$$
$$- \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\frac{k_2}{k_1} + \frac{c_1k_1 + s_1k_2}{c_1k_1-s_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\right]\tilde{u}(\omega,t)$$
$$- \frac{\lambda}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t) \qquad (2.38)$$

with $\tilde{u}(\omega,t)$ defined by equation (2.19).

Case 1 $\lambda = 1$**.** We have two subcases: $c_1 = c_2 = 0$ or $s_1 = s_2 = 0$. (a) $c_1 = c_2 = 0$. Then, formula (2.38) becomes

$$u^*(x,y,t) = \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\,\frac{k_2}{k_1}\,\tilde{u}(\omega,t)$$
$$+ \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\,\frac{k_1}{k_2}\,\tilde{u}(\omega,t)$$
$$- \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t) \equiv 0, \qquad (2.39)$$

after deforming all the contours of integration to ∂D^+.

(b) $s_1 = s_2 = 0$. Then, after deforming all the contours of integration to ∂D^+, formula (2.38) becomes

$$u^*(x,y,t) = -\frac{1}{\pi^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t). \qquad (2.40)$$

Using equations (2.4), however, it is possible to compute $\tilde{u}(\omega,t)$ explicitly. Indeed, we have

$$c_1u_x(x,0,t) = F_1(x,t), \quad c_2u_y(0,y,t) = F_2(y,t), \qquad (2.41)$$

hence, differentiating with respect to x and y respectively, we find

$$c_1u_{xx}(x,0,t) = \partial_x F_1(x,t), \quad c_2u_{yy}(0,y,t) = \partial_y F_2(y,t). \qquad (2.42)$$

Evaluating the above expressions at $x = y = 0$, we obtain

$$u_t(0,0,t) = u_{xx}(0,0,t) + u_{yy}(0,0,t) = \frac{1}{c_1}\,\partial_x F_1(0,t) + \frac{1}{c_2}\,\partial_y F_2(0,t), \qquad (2.43)$$

which in turn yields the formula

$$u(0,0,t) = \int_0^t ds\left[\frac{1}{c_1}\,\partial_x F_1(0,s) + \frac{1}{c_2}\,\partial_y F_2(0,s)\right] + u_0(0,0), \qquad (2.44)$$

where we have used the fact that $u(0,0,0) = u_0(0,0)$. Inserting formula (2.44) in the

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definition (2.19) of $\tilde{u}(\omega, t)$ we then find

$$\begin{aligned}\tilde{u}(\omega,t) &= \int_0^t ds\, e^{\omega s}\left\{\int_0^s dt'\left[\frac{1}{c_1}\,\partial_x F_1(0,t') + \frac{1}{c_2}\,\partial_y F_2(0,t')\right] + u_0(0,0)\right\}\\ &= \int_0^t ds\, e^{\omega s}\int_0^s dt'\left[\frac{1}{c_1}\,\partial_x F_1(0,t') + \frac{1}{c_2}\,\partial_y F_2(0,t')\right] + \frac{1}{\omega}\left(e^{\omega t}-1\right)u_0(0,0). \qquad (2.45)\end{aligned}$$

Formula (2.45) is an explicit formula for $\tilde{u}(\omega, t)$ in terms of the initial and boundary conditions and, therefore, when inserted in equation (2.40) yields an explicit formula for $u^*(x, y, t)$.

Case 2 $\lambda = -1$. Then, the function $u^*(x, y, t)$ is equal to

$$\begin{aligned}u^*(x,y,t) = &-\frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\,\frac{c_2k_2 - s_2k_1}{c_2k_2 + s_2k_1}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t)\\ &-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\,\frac{c_1k_1 + s_1k_2}{c_1k_1 - s_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t). \qquad (2.46)\end{aligned}$$

We have the following two subcases:

(a) $c_1s_1 \leqslant 0$. Then $c_2s_2 \geqslant 0$ and hence $(c_2k_2 + s_2k_1)(c_1k_1 - s_1k_2) \neq 0$ for all $k_1, k_2 \in \mathbb{C}^+$. Thus, both integrands on the right-hand side of equation (2.38) are analytic for $k_1, k_2 \in \mathbb{C}^+$. Hence, we can deform the contour of integration with respect to k_2 in the first double integral, as well as the contour of integration with respect to k_1 in the second double integral, to the contour ∂D^+. Then, due to condition (2.32)

$$\frac{c_2k_2 - s_2k_1}{c_2k_2 + s_2k_1} + \frac{c_1k_1 + s_1k_2}{c_1k_1 - s_1k_2} = \frac{2(k_1^2 + k_2^2)(c_1c_2 + s_1s_2)}{(c_1k_1 - k_2s_1)(c_2k_2 + k_1s_2)} \equiv 0,$$

therefore, $u^*(x, y, t)$ vanishes.

(b) $c_1s_1 > 0$. Then condition (2.32) implies $c_2s_2 < 0$, hence the integrands in equation (2.46) have poles at $k_2 = k_1 \cot\beta_1$ for $k_1, k_2 \in \mathbb{C}^+$. The contours of integration with respect to k_1 can be deformed to the boundary ∂E^+, shown in Figure 4. Thus, noting also that condition (2.32) is now equivalent to $\tan\beta_1 \tan\beta_2 = -1$, we find

$$\begin{aligned}u^*(x,y,t) = &-\frac{1}{(2\pi)^2}\int_{\partial E^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\,\frac{c_2k_2 - s_2k_1}{c_2k_2 + s_2k_1}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t)\\ &+\frac{1}{(2\pi)^2}\int_{\partial E^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\,\frac{c_2k_2 - s_2k_1}{c_2k_2 + s_2k_1}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t).\end{aligned} \qquad (2.47)$$

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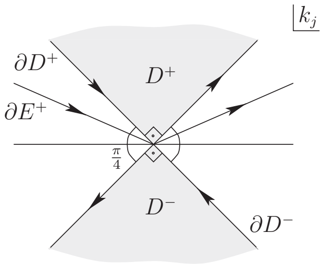

FIGURE 4. The contours ∂D^+, ∂D^- and ∂E^+.

Cauchy's theorem in the region $\mathbb{C}^+ \setminus D^+$ of the k_2-plane for the first term of equation (2.47) yields

$$\begin{aligned} u^*(x,y,t) = &-\frac{1}{(2\pi)^2}\int_{\partial E^+} dk_1 \int_{\partial D^+} dk_2\; e^{ik_1x+ik_2y-\omega t}\, \frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t) \\ &+\frac{1}{i\pi}\csc^2\beta_1 \int_{\partial E^+} dk_1\; e^{ik_1(x+y\cot\beta_1)-\omega^* t}\, k_1\tilde{u}(\omega^*,t) \\ &+\frac{1}{(2\pi)^2}\int_{\partial E^+} dk_1 \int_{\partial D^+} dk_2\; e^{ik_1x+ik_2y-\omega t}\, \frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\tilde{u}(\omega,t), \end{aligned}$$

where the second term above corresponds to the residues from the poles at $k_2 = k_1\cot\beta_1$ and $\omega^* = k_1^2\csc^2\beta_1$. Hence, for $c_1s_1 > 0$ we have

$$u^*(x,y,t) = \frac{1}{i\pi}\csc^2\beta_1 \int_{\partial E^+} dk_1\; e^{ik_1(x+y\cot\beta_1)-\omega^* t}\, k_1\tilde{u}(\omega^*,t). \tag{2.48}$$

In order to determine $\tilde{u}(\omega^*,t)$, we need to employ appropriate version(s) of the global relations (2.26) evaluated at $k_2 = k_1\cot\beta_1$. Recalling that q_1 and q_2 are *unknown* and yield a non-zero contribution due to the exponential $e^{\omega^*(t-s)}$, we see that the only versions that may be used are equations (2.26*a*) and (2.26*b*), since in these identities the coefficients of q_1 and q_2 vanish at $k_2 = k_1\cot\beta_1$. However, equations (2.26*a*) and (2.26*b*) are valid for $\{k_1 \in \mathbb{C}^+,\ k_2 \in \mathbb{C}^-\}$ and $\{k_1 \in \mathbb{C}^-,\ k_2 \in \mathbb{C}^+\}$ respectively, thus they cannot be employed since $k_2 = k_1\cot\beta_1$ and $\cot\beta_1 > 0$.

Consequently, for $c_1s_1 > 0$ the representation (2.37) contains the function $u^*(x,y,t)$ defined by equation (2.38). Equation (2.48) expresses $u^*(x,y,t)$ in terms of the unknown function $\tilde{u}(\omega^*,t)$, i.e. the function $\tilde{u}(\omega,t)$ evaluated at $k_2 = k_1\cot\beta_1$. On the other hand, for $c_1s_1 \leqslant 0$ the function $u^*(x,y,t)$ vanishes and the representation (2.37) depends entirely on given functions.

The above are summarised by the following proposition.

Proposition 2 (Solution) *Let $u(x,y,t)$ satisfy the heat equation* (2.1) *with the initial datum* (2.3) *and the oblique Neumann boundary conditions* (2.4).

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If the constants $\beta_1, \beta_2, \lambda$ *satisfy condition* (2.33*a*), *or condition* (2.32) *with the restriction* $\cos\beta_1 \sin\beta_1 \leqslant 0$, *then the solution is given by*

$$\begin{aligned} u(x,y,t) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\hat{u}_0(k_1,k_2) \\ &- \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\left\{\frac{2k_1}{c_2k_2+s_2k_1}f_2(\omega,k_2,t)\right. \\ &\left. + \frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\left[\hat{u}_0(-k_1,k_2)+\left(s_1+c_1\frac{k_2}{k_1}\right)f_1(\omega,-k_1,t)\right]\right\} \\ &- \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left\{\frac{2k_2}{c_1k_1-s_1k_2}f_1(\omega,k_1,t)\right. \\ &\left. + \frac{c_1k_1+s_1k_2}{c_1k_1-s_1k_2}\left[\hat{u}_0(k_1,-k_2)-\left(s_2-c_2\frac{k_1}{k_2}\right)f_2(\omega,-k_2,t)\right]\right\} \\ &+ \frac{\lambda}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\hat{u}_0(-k_1,-k_2)\right. \\ &\left. + \left(s_1-c_1\frac{k_2}{k_1}\right)f_1(\omega,-k_1,t)-\left(s_2+c_2\frac{k_1}{k_2}\right)f_2(\omega,-k_2,t)\right], \quad (x,y,t)\in\Omega, \end{aligned} \tag{2.49}$$

where $\hat{u}_0(k_1,k_2)$ *is the Fourier transform* (2.10*a*) *of the initial datum, the functions* $f_1(\omega,k_1,t)$, $f_2(\omega,k_2,t)$ *are defined by equations* (2.17) *and the contour* ∂D^+ *is shown in Figure 4.*

If the constants $\beta_1, \beta_2, \lambda$ *satisfy condition* (2.33*b*) *then the solution* (2.49) *involves the additional term* $u^*(x,y,t)$, *which is given explicitly in terms of the initial and boundary conditions via equations* (2.40) *and* (2.45).

If the constants β_1, β_2 *satisfy condition* (2.32) *with the restriction* $\cos\beta_1 \sin\beta_1 > 0$ *then we assume the additional boundary condition*

$$u(0,0,t) = h(t), \quad t > 0. \tag{2.50}$$

In this case, the solution (2.49) *involves the additional term* $u^*(x,y,t)$,

$$u^*(x,y,t) = \frac{1}{i\pi}\csc^2\beta_1 \int_{\partial D^+} dk_1\; e^{ik_1(x+y\cot\beta_1)-\omega^* t}\; k_1\tilde{u}(\omega^*,t), \quad t>0, \tag{2.51}$$

where the function $\tilde{u}(\omega^*,t)$ *is defined by*

$$\tilde{u}(\omega^*,t) = \int_0^t ds\, e^{\omega^* s}h(s), \quad \omega^* = k_1^2\csc^2\beta_1, \quad t>0. \tag{2.52}$$

3 Numerical evaluations

The solution formulae obtained through the unified transform involve integrals along contours in the complex k_1 and k_2-planes. The analytical advantage of this feature was already exploited in Section 2, where we employed Jordan's lemma in the process of eliminating the transforms of the unknown boundary values from the integral representation.

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We will now utilise this feature for a different purpose, namely for evaluating numerically the solution formula (2.49) for various initial and boundary conditions.

(1) Suppose that

$$u_0(x,y) = A\left(x^2 e^{-a_1^2 x}\right)\left(y^2 e^{-a_2^2 y}\right), \quad A, a_1, a_2 \in \mathbb{R}, \tag{3.1}$$

and

$$F_1(x,t) = x^2 e^{-b_1 x}\sin(b_2 t), \quad b_1 > 0,\ b_2 \in \mathbb{R}, \quad F_2(y,t) \equiv 0. \tag{3.2}$$

Using the transforms (2.10a) and (2.17), straightforward computations yield

$$\hat{u}_0(k_1,k_2) = \frac{4A}{(a_1^2+ik_1)^3(a_2^2+ik_2)^3}, \tag{3.3}$$

and

$$f_1(\omega,k_1,t) = \frac{2(e^{\omega t}-1)\sin(b_2 t)}{\omega(b_1+ik_1)^3}, \quad f_2(\omega,k_2,t) \equiv 0. \tag{3.4}$$

Suppose further that we have

$$\beta_1 = \frac{2\pi}{3}, \quad \beta_2 = \frac{\pi}{6},$$

so that

$$c_1 = -\frac{1}{2},\ s_1 = \frac{\sqrt{3}}{2}, \quad c_2 = \frac{\sqrt{3}}{2},\ s_2 = \frac{1}{2},$$

and hence

$$\cos\beta_1 \sin\beta_1 \leqslant 0, \quad \beta_1 = \beta_2 + \frac{\pi}{2}.$$

Then, condition (2.32) holds with $n = 0$ and $\lambda = -1$. Moreover, note that the compatibility requirement (1.3) also holds for our choice of $F_1(x,t)$ and $F_2(y,t)$.

Inserting the above in the solution formula (2.49), we find

$$\begin{aligned}
u(x,y,t) = {} & \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\, \frac{4A}{(a_1^2+ik_1)^3(a_2^2+ik_2)^3} \\
& - \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\, \frac{\sqrt{3}k_2-k_1}{\sqrt{3}k_2+k_1} \\
& \times \left[\frac{4A}{(a_1^2-ik_1)^3(a_2^2+ik_2)^3} + \left(\sqrt{3}-\frac{k_2}{k_1}\right)\frac{(e^{\omega t}-1)\sin(b_2t)}{\omega(b_1-ik_1)^3}\right] \\
& - \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[-\frac{8k_2}{k_1+\sqrt{3}k_2}\cdot\frac{(e^{\omega t}-1)\sin(b_2t)}{\omega(b_1+ik_1)^3}\right. \\
& \left. + \frac{k_1-\sqrt{3}k_2}{k_1+\sqrt{3}k_2}\cdot\frac{4A}{(a_1^2+ik_1)^3(a_2^2-ik_2)^3}\right] \\
& - \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\frac{4A}{(a_1^2-ik_1)^3(a_2^2-ik_2)^3}\right. \\
& \left. + \left(\sqrt{3}+\frac{k_2}{k_1}\right)\frac{2(e^{\omega t}-1)\sin(b_2t)}{\omega(b_1-ik_1)^3}\right].
\end{aligned} \tag{3.5}$$

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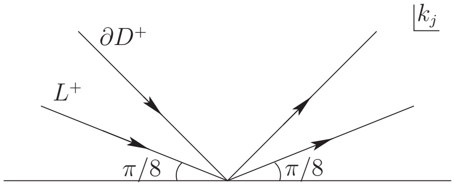

FIGURE 5. The steepest descent path L^+.

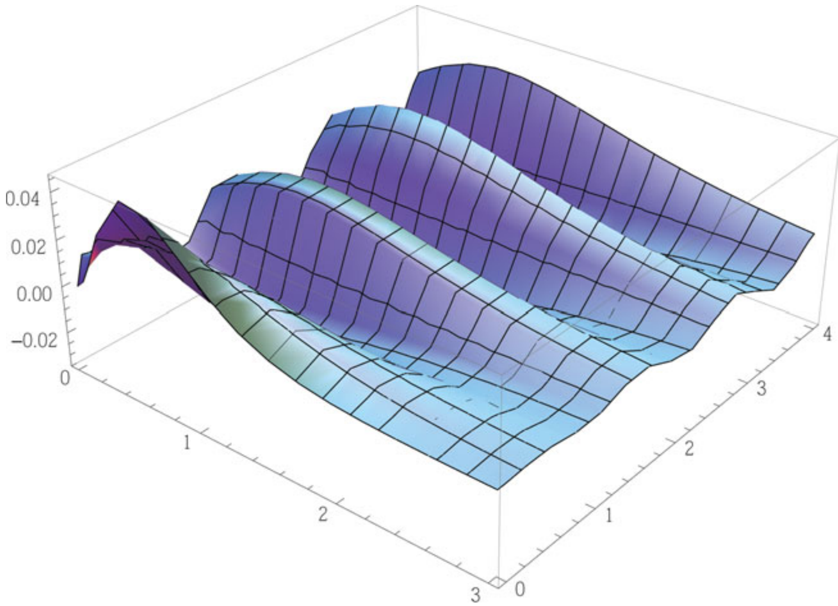

FIGURE 6. The solution (3.5) depicted over $\{(x,t) \in [0,3] \times [0,4], y = 0.3\}$ for $A = 100$, $a_1 = 2$, $a_2 = 2.5$, $b_1 = 3$, $b_2 = 5$.

Following the hybrid analytical-numerical method of Flyer and Fokas [7], we deform the contours of integration to the steepest descent path L^+ (see Figure 5), which forms an angle of $\pi/8$ with the real k_j-axis, $j = 1, 2$.

This deformation is very helpful for numerical computations, since now both the x-part and the t-part of the relevant exponentials decay as $|k_j| \to \infty$. Then, using the change of variables $k_1 = i\sin(\pi/8 - i\theta), k_2 = i\sin(\pi/8 - i\phi)$, a few lines of code in Mathematica produce the graphical representations shown in Figures 6–8.

(2) Suppose that

$$u_0(x,y) = A\left(xe^{-a^2x}\right)\left(ye^{-a^2y}\right), \quad A,\, a \in \mathbb{R}, \tag{3.6}$$

and

$$F_1(x,t) = x^3 e^{-b_1 x}\sin(b_2 t), \quad b_1 > 0,\, b_2 \in \mathbb{R}, \tag{3.7a}$$

$$F_2(y,t) = (y^2 e^{-b_3 y})(te^{-b_4 t}), \quad b_3 > 0,\, b_4 > 0. \tag{3.7b}$$

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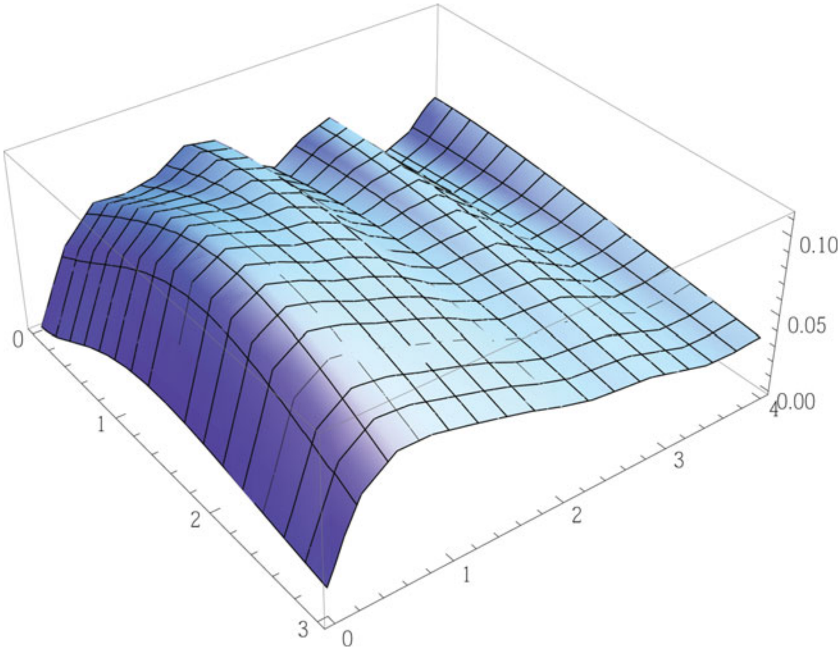

FIGURE 7. The solution (3.5) depicted over $\{x = 2, (y, t) \in [0, 3] \times [0, 4]\}$ for $A = 100$, $a_1 = 2$, $a_2 = 1.2$, $b_1 = 3$, $b_2 = 5$.

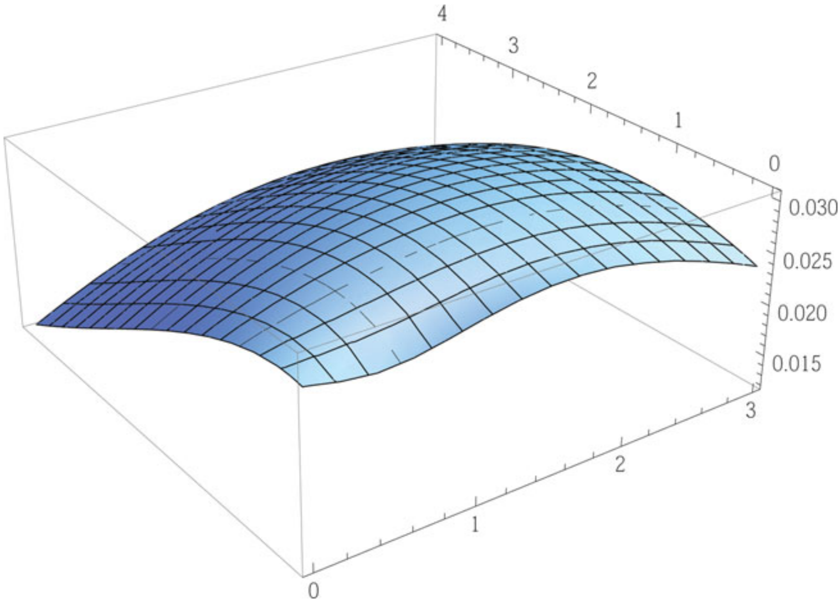

FIGURE 8. The solution (3.5) depicted over $\{(x, y) \in [0, 3] \times [0, 4], t = 5\}$ for $A = 100$, $a_1 = 2$, $a_2 = 1.2$, $b_1 = 3$, $b_2 = 5$.

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Straightforward computations yield

$$\hat{u}_0(k_1,k_2) = \frac{A}{(a_1^2+ik_1)^2(a_2^2+ik_2)^2}, \tag{3.8}$$

and

$$f_1(\omega,k_1,t) = \frac{6(e^{\omega t}-1)\sin(b_2 t)}{\omega(b_1+ik_1)^4}, \tag{3.9a}$$

$$f_2(\omega,k_2,t) = \frac{2\left(1+e^{(\omega-b_4)t}\left[(\omega-b_4)t-1\right]\right)}{(b_3+ik_2)^3(\omega-b_4)^2}. \tag{3.9b}$$

Suppose further that we have

$$\beta_1 = \frac{\pi}{2}, \quad \beta_2 = \frac{3\pi}{2},$$

so that

$$c_1 = 0,\ s_1 = 1, \quad c_2 = 0,\ s_2 = -1.$$

Then, the orthogonality condition (2.33*a*) with $m = 0$ and $n = 1$ is satisfied and we have $\lambda = 1$. Also, note that the compatibility requirement $\partial_x F_1(0,t) = \partial_y F_2(0,t)$ is satisfied.

The solution formula (2.49) now becomes

$$\begin{aligned}
u(x,y,t) = {} & \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\, \frac{A}{(a_1^2+ik_1)^2(a_2^2+ik_2)^2} \\
& + \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\frac{4\left(1+e^{(\omega-b_4)t}\left[(\omega-b_4)t-1\right]\right)}{(b_3+ik_2)^3(\omega-b_4)^2}\right. \\
& \left. + \frac{A}{(a_1^2-ik_1)^2(a_2^2+ik_2)^2} + \frac{6(e^{\omega t}-1)\sin(b_2t)}{\omega(b_1-ik_1)^4}\right] \\
& + \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\frac{12(e^{\omega t}-1)\sin(b_2t)}{\omega(b_1+ik_1)^4}\right. \\
& \left. + \frac{A}{(a_1^2+ik_1)^2(a_2^2-ik_2)^2} + \frac{2\left(1+e^{(\omega-b_4)t}\left[(\omega-b_4)t-1\right]\right)}{(b_3-ik_2)^3(\omega-b_4)^2}\right] \\
& + \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y-\omega t}\left[\frac{A}{(a_1^2-ik_1)^2(a_2^2-ik_2)^2}\right. \\
& \left. + \frac{6(e^{\omega t}-1)\sin(b_2t)}{\omega(b_1-ik_1)^4} + \frac{2\left(1+e^{(\omega-b_4)t}\left[(\omega-b_4)t-1\right]\right)}{(b_3-ik_2)^3(\omega-b_4)^2}\right].
\end{aligned} \tag{3.10}$$

The same technique as in the previous example yields the graphical representations of Figures 9–11.

4 Conclusion

We have used the particular example of the heat equation (1.6) formulated in the domain Ω specified by equation (2.2), with the oblique type boundary conditions (1.1*c*), (1.1*d*) and with the constants $\{\beta_1, \beta_2\}$ satisfying equation (1.2), to illustrate the fact that the unified

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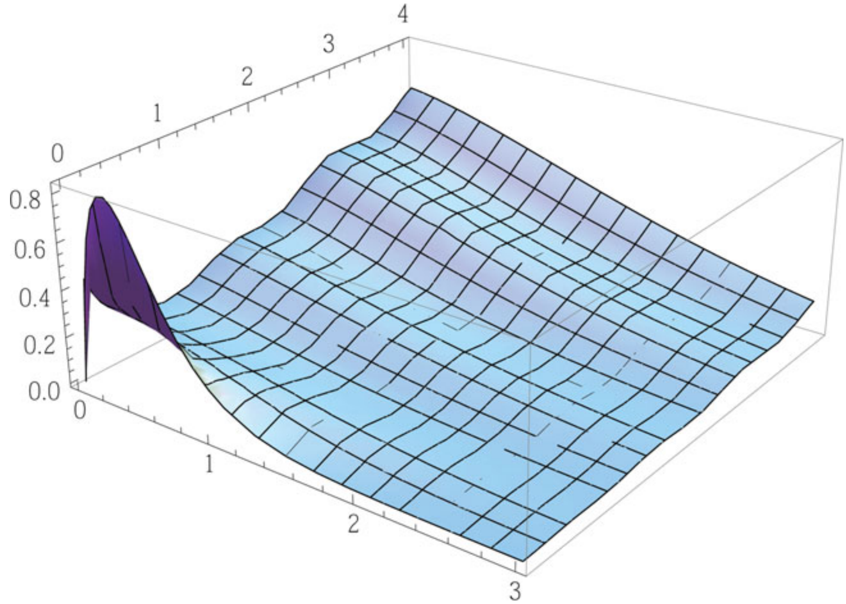

FIGURE 9. The solution (3.10) depicted over $\{(x, t) \in [0, 3] \times [0, 4], y = 0.3\}$ for $A = 100$, $a = 2$, $b_1 = 3$, $b_2 = 5$, $b_3 = 1$, $b_4 = 1/3$.

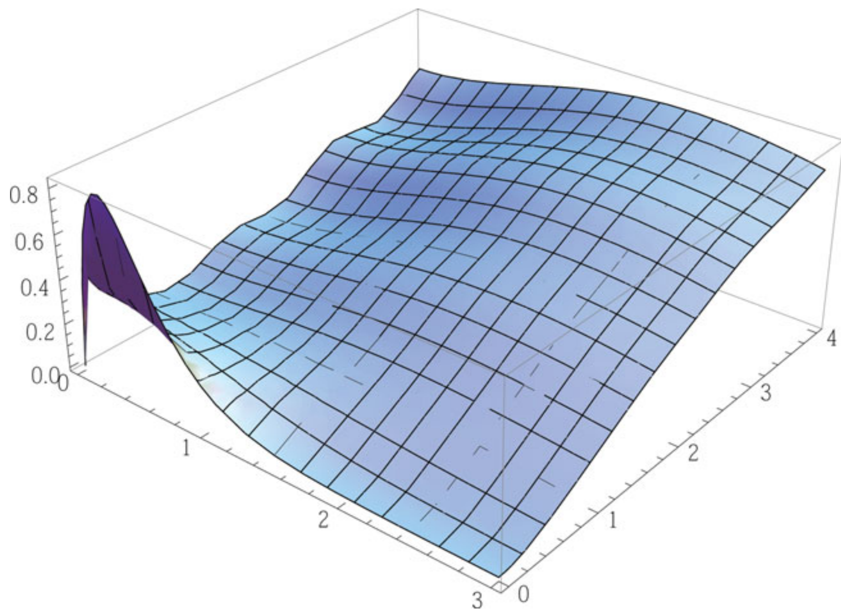

FIGURE 10. The solution (3.10) depicted over $\{x = 0.3, (y, t) \in [0, 3] \times [0, 4]\}$ for $A = 100$, $a = 2$, $b_1 = 3$, $b_2 = 5$, $b_3 = 1$, $b_4 = 1/3$.

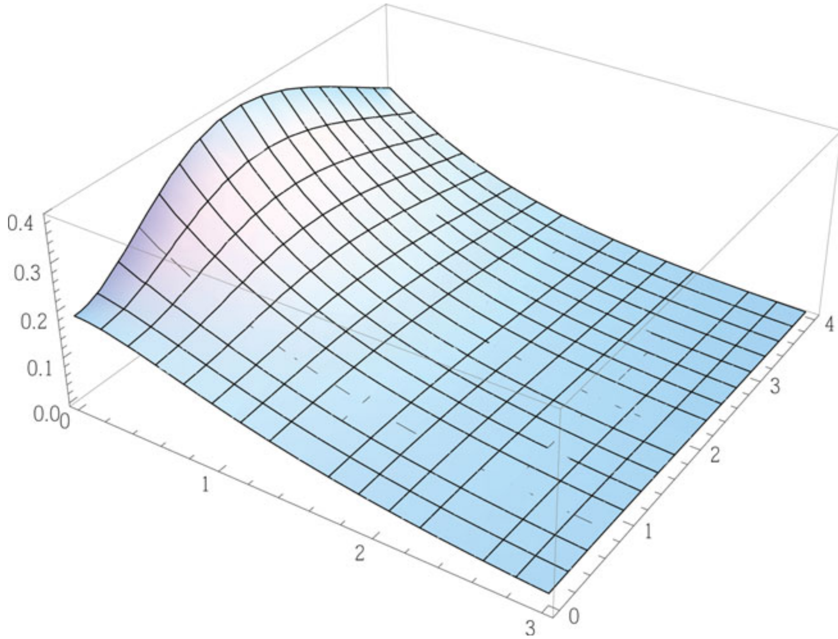

FIGURE 11. The solution (3.10) depicted over $\{(x, y) \in [0, 3] \times [0, 4], t = 1.2\}$ for $A = 100$, $a = 2$, $b_1 = 3$, $b_2 = 5$, $b_3 = 1$, $b_4 = 1/3$.

transform yields the solution of two-dimensional evolution PDEs with a large class of *non-separable* boundary conditions.

The solution of the above initial-boundary value problem is expressed in Proposition 2, in terms of certain integrals in the complex k_1 and the complex k_2-planes.

If the given initial and boundary data are such that their associated transforms can be computed analytically, then the solution representation of Proposition 2 yields an efficient way for the numerical evaluation of the solution. This is illustrated in Section 3 for the particular case that the initial and boundary data are given by either equations (3.1) and (3.2) with $\{\beta_1 = 2\pi/3, \beta_2 = \pi/6\}$, or by equations (3.6) and (3.7) with $\{\beta_1 = \pi/2, \beta_2 = 3\pi/2\}$.

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Appendix A Verification of the solution

In the applied mathematics literature, a PDE is solved under the *a priori assumption* of existence. Indeed, since the solution is obtained by applying an appropriate transform, this procedure makes sense only if one assumes *a priori* that the solution exists and has certain decay and smoothness properties. In order to eliminate this assumption one must prove *a posteriori* that the expression obtained by this approach satisfies the PDE and the given initial and boundary conditions (one then has to address independently the question of uniqueness). However, this verification is almost never carried out in the applied literature. Actually, this is *not* a straightforward task because *any* representation obtained via the usual transforms is *not uniformly convergent* at the boundary. A major advantage of the unified transform is that it constructs a solution which is uniformly convergent at the boundary. Thus, it is straightforward, at least *formally*, to verify that the solution satisfies the given PDE and the given data. The *rigorous* implementation of this verification for a large class of PDEs is discussed in (see note below).[1]

In what follows, by utilising the advantage of the unified transform to yield representations for the solution which are uniformly convergent at the boundaries, we will verify that the expression (2.49) does indeed satisfy the initial-boundary value problem posed by equations (2.1), (2.3) and (2.4).

A.1 The heat equation

By employing Cauchy's theorem and Jordan's lemma, we can show that the integrals involving the functions f_1, f_2 and $\tilde{u}$ depend on the temporal variable t only through the exponential $e^{-\omega t}$. Thus, the physical variables (x, y, t) enter equation (2.49) exclusively through the exponentials $e^{ik_1x+ik_2y-\omega t}$ and $e^{ik_1(x+y\cot\beta_1)-\omega^* t}$, which obviously satisfy equation (2.1).

A.2 Initial condition

Note that $u^*(x, y, 0) \equiv 0$. Moreover, evaluating equation (2.49) at $t = 0$ gives

$$\begin{aligned} u(x,y,0) = &\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y}\hat{u}_0(k_1,k_2) \\ &- \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y}\, \frac{c_2k_2 - s_2k_1}{c_2k_2 + s_2k_1}\, \hat{u}_0(-k_1,k_2) \\ &- \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y}\, \frac{c_1k_1 + s_1k_2}{c_1k_1 - s_1k_2}\, \hat{u}_0(k_1,-k_2) \\ &+ \frac{\lambda}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x+ik_2y}\, \hat{u}_0(-k_1,-k_2). \end{aligned}$$

[1] Fokas, A. S. & Sung, L. Y. (1999) Initial-boundary value problems for linear dispersive evolution equations on the half-line (unpublished).

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The last three terms above vanish by Cauchy's theorem and Jordan's lemma, hence

$$u(x,y,0) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x+ik_2y}\hat{u}_0(k_1,k_2) = u_0(x,y),$$

with the last equality due to the inverse Fourier transform formula (2.10*b*).

A.3 Boundary conditions

First, recall that the only case in which the function $u^*(x,y,t)$, given by equation (2.48), is not trivially zero is when condition (2.32) with the restriction $\cos\beta_1 \sin\beta_1 > 0$ holds. But then,

$$\begin{aligned}&\cos\beta_1 u_x^*(x,y,t) - \sin\beta_1 u_y^*(x,y,t)\\ &\quad= \frac{1}{\pi}\csc^2\beta_1 \int_{\partial E^+} dk_1\; e^{ik_1(x+y\cot\beta_1)-\omega^* t}\; k_1\,(\cos\beta_1 - \sin\beta_1\cot\beta_1)\,k_1\tilde{u}(\omega^*,t) \equiv 0\end{aligned}$$

and, due to condition (2.32),

$$\begin{aligned}&\cos\beta_2 u_y^*(x,y,t) + \sin\beta_2 u_x^*(x,y,t)\\ &\quad= \frac{1}{\pi}\csc^2\beta_1 \int_{\partial E^+} dk_1\; e^{ik_1(x+y\cot\beta_1)-\omega^* t}\; k_1\,(\cos\beta_2\cot\beta_1 + \sin\beta_2)\,k_1\tilde{u}(\omega^*,t) \equiv 0.\end{aligned}$$

Therefore, in all cases the function $u^*(x,y,t)$ satisfies the homogeneous version of the boundary conditions (2.4).

The same is true for the terms of equation (2.49) involving $u_0(x,y)$. Indeed, by inserting these terms in equation (2.4*a*) we find

$$\begin{aligned}&\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x-\omega t}\,(c_1k_1 - s_1k_2)\,\hat{u}_0(k_1,k_2)\\ &\quad - \frac{i}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x-\omega t}\,\frac{c_2k_2 - s_2k_1}{c_2k_2 + s_2k_1}\,(c_1k_1 - s_1k_2)\,\hat{u}_0(-k_1,k_2)\\ &\quad - \frac{i}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x-\omega t}\,(c_1k_1 + s_1k_2)\,\hat{u}_0(k_1,-k_2)\\ &\quad + \frac{i\lambda}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x-\omega t}\,(c_1k_1 - s_1k_2)\,\hat{u}_0(-k_1,-k_2).\end{aligned} \tag{A 1}$$

We let $k_2 \mapsto -k_2$ in the first term and then deform the k_2-contour from the real line to ∂D^+. Similarly, we deform the k_2-contour in the fourth term to the real line and then we

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let $k_2 \mapsto -k_2$. In this case, equation (A 1) becomes

$$\begin{aligned}
&\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x-\omega t}\,(c_1k_1+s_1k_2)\,\hat{u}_0(k_1,-k_2)\\
&\quad -\frac{i}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x-\omega t}\,\frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\,(c_1k_1-s_1k_2)\,\hat{u}_0(-k_1,k_2)\\
&\quad -\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x-\omega t}\,(c_1k_1+s_1k_2)\,\hat{u}_0(k_1,-k_2)\\
&\quad +\frac{i\lambda}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x-\omega t}\,(c_1k_1+s_1k_2)\,\hat{u}_0(-k_1,k_2). \qquad \text{(A 2)}
\end{aligned}$$

In the case of $\lambda=-1$, where condition (2.32) is valid, this expression equals

$$-\frac{i}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x-\omega t}\,\frac{2(c_1c_2+s_1s_2)k_1k_2}{c_2k_2+s_2k_1}\hat{u}_0(-k_1,k_2)\equiv 0.$$

On the other hand, if $\lambda=1$ then equation (A 2) becomes

$$\begin{aligned}
&\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x-\omega t}\,(c_1k_1+s_1k_2)\,\hat{u}_0(k_1,-k_2)\\
&\quad -\frac{i}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x-\omega t}\,\frac{c_2k_2-s_2k_1}{c_2k_2+s_2k_1}\,(c_1k_1-s_1k_2)\,\hat{u}_0(-k_1,k_2)\\
&\quad -\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x-\omega t}\,(c_1k_1+s_1k_2)\,\hat{u}_0(k_1,-k_2)\\
&\quad +\frac{i}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x-\omega t}\,(c_1k_1+s_1k_2)\,\hat{u}_0(-k_1,k_2).
\end{aligned}$$

Hence, the above expression vanishes under each one of the conditions (2.33).

The above analysis regarding $u^*(x,y,t)$ and the terms involving $u_0(x,y)$ implies

$$\begin{aligned}
&\cos\beta_1\ u_x(x,0,t)-\sin\beta_1\ u_y(x,0,t)\\
&\quad =-\frac{i}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{ik_1x-\omega t}\,\frac{c_1k_1-s_1k_2}{c_2k_2+s_2k_1}\Big[2k_1f_2(\omega,k_2,t)\\
&\qquad +(c_2k_2-s_2k_1)\Big(s_1+c_1\frac{k_2}{k_1}\Big)f_1(\omega,-k_1,t)\Big]\\
&\qquad -\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x-\omega t}\Big[2k_2f_1(\omega,k_1,t)-(c_1k_1+s_1k_2)\Big(s_2-c_2\frac{k_1}{k_2}\Big)f_2(\omega,-k_2,t)\Big]\\
&\qquad +\frac{i\lambda}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{ik_1x-\omega t}(c_1k_1-s_1k_2)\Big[\Big(s_1-c_1\frac{k_2}{k_1}\Big)f_1(\omega,-k_1,t)\\
&\qquad -\Big(s_2+c_2\frac{k_1}{k_2}\Big)f_2(\omega,-k_2,t)\Big]. \qquad \text{(A 3)}
\end{aligned}$$

Using Cauchy's theorem and the change of variables $k_2 \mapsto -k_2$ in the first integral, the

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terms involving f_2 in the above expression may be written in the form

$$\frac{i}{(2\pi)^2}\int_{\partial D^+}dk_1\int_{-\infty}^{\infty}dk_2\,e^{ik_1x-\omega t}\Bigg[2k_1\,\frac{c_1k_1+s_1k_2}{c_2k_2-s_2k_1}-(c_1k_1+s_1k_2)\Big(c_2\frac{k_1}{k_2}-s_2\Big)$$
$$-\lambda(c_1k_1-s_1k_2)\Big(s_2+c_2\frac{k_1}{k_2}\Big)\Bigg]f_2(\omega,-k_2,t).$$

If $\lambda=-1$, condition (2.32) is valid and then, the coefficient of f_2 simplifies to

$$2k_1\left[\frac{s_1}{c_2}+(c_1s_2-s_1c_2)\right]=2k_1\,\frac{s_1+c_1c_2s_2-s_1c_2^2}{c_2}=2k_1\,\frac{s_2(s_1s_2+c_1c_2)}{c_2}\equiv 0.$$

If $\lambda=1$, then according to conditions (2.33) either $c_1=c_2=0$ or $s_1=s_2=0$. Thus, the coefficient of f_2 again vanishes. Consequently, equation (A 3) becomes

$$\begin{aligned}&\cos\beta_1\;u_x(x,0,t)-\sin\beta_1\;u_y(x,0,t)\\&\quad=\frac{i}{(2\pi)^2}\int_{\partial D^+}dk_1\int_{-\infty}^{\infty}dk_2\,e^{ik_1x-\omega t}\,\frac{s_1k_2-c_1k_1}{c_2k_2+s_2k_1}\,(c_2k_2-s_2k_1)\Big(s_1+c_1\frac{k_2}{k_1}\Big)f_1(\omega,-k_1,t)\\&\qquad-\frac{i}{(2\pi)^2}\int_{-\infty}^{\infty}dk_1\int_{\partial D^+}dk_2\,e^{ik_1x-\omega t}\,2k_2f_1(\omega,k_1,t)\\&\qquad+\frac{i\lambda}{(2\pi)^2}\int_{\partial D^+}dk_1\int_{\partial D^+}dk_2\,e^{ik_1x-\omega t}\,(c_1k_1-s_1k_2)\Big(s_1-c_1\frac{k_2}{k_1}\Big)f_1(\omega,-k_1,t).\end{aligned}$$

Deforming the contours of integration, we find

$$\begin{aligned}&\cos\beta_1\;u_x(x,0,t)-\sin\beta_1\;u_y(x,0,t)\\&\quad=\frac{i}{(2\pi)^2}\int_{\partial D^+}dk_1\int_{-\infty}^{\infty}dk_2\,e^{ik_1x-\omega t}\,\frac{s_1k_2-c_1k_1}{c_2k_2+s_2k_1}\,(c_2k_2-s_2k_1)\Big(s_1+c_1\frac{k_2}{k_1}\Big)f_1(\omega,-k_1,t)\\&\qquad-\frac{i}{(2\pi)^2}\int_{\partial D^+}dk_1\int_{-\infty}^{\infty}dk_2\,e^{ik_1x-\omega t}\,2k_2f_1(\omega,k_1,t)\\&\qquad+\frac{i\lambda}{(2\pi)^2}\int_{\partial D^+}dk_1\int_{-\infty}^{\infty}dk_2\,e^{ik_1x-\omega t}\,(c_1k_1-s_1k_2)\Big(s_1-c_1\frac{k_2}{k_1}\Big)f_1(\omega,-k_1,t).\end{aligned}$$

In both cases (2.32) and (2.33), the above expression reduces to

$$\begin{aligned}&\cos\beta_1\;u_x(x,0,t)-\sin\beta_1\;u_y(x,0,t)\\&\quad=\frac{i}{(2\pi)^2}\int_{\partial D^+}dk_1\int_{-\infty}^{\infty}dk_2\,e^{ik_1x-\omega t}\,2k_2\,[f_1(\omega,-k_1,t)-f_1(\omega,k_1,t)]\\&\quad=\frac{1}{(2\pi)^2}\int_0^{\infty}d\xi\int_{-\infty}^{\infty}dk_1\int_0^t ds\,e^{-k_1^2(t-s)}\,F_1(\xi,s)\left[e^{ik_1(x+\xi)}-e^{ik_1(x-\xi)}\right]\int_{\partial D^+}dk_2\,e^{-k_2^2(t-s)}\,2ik_2.\end{aligned}$$

The change of variables $l=-ik_2^2$ maps ∂D^+ to $(-\infty,\infty)$ and

$$\int_{\partial D^+}dk_2\,e^{-k_2^2(t-s)}\,2ik_2=-\int_{-\infty}^{\infty}dl\,e^{il(s-t)}=-2\pi\delta(s-t), \qquad \text{(A 4)}$$

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therefore, we conclude that

$$\cos\beta_1 u_x(x,0,t) - \sin\beta_1 u_y(x,0,t) = \frac{1}{2\pi}\int_0^\infty d\xi \int_{-\infty}^\infty dk_1 \left[e^{ik_1(x-\xi)} - e^{ik_1(x+\xi)}\right] F_1(\xi,t)$$
$$= \int_0^\infty d\xi\, \delta(\xi - x) F_1(\xi,t) = F_1(x,t) \qquad \text{(A 5)}$$

and hence the boundary condition (2.4a) has been verified. The verification of the second boundary condition (2.4b) can be achieved in a similar fashion.

As noted in Proposition 2, in the case of condition (2.32) with the restriction $(\cos\beta_1 \sin\beta_1) > 0$ the expression (2.49) for the solution $u(x,y,t)$ involves the unknown boundary value $u(0,0,t)$ through the quantity u^* defined by equation (2.51). However, we have just verified that equation (2.49) solves the particular initial-boundary value problem specified by equations (2.1), (2.3) and (2.4) regardless of the sign of $(\cos\beta_1 \sin\beta_1)$.

Therefore, if condition (2.32) with $(\cos\beta_1 \sin\beta_1) > 0$ is satisfied then the expression (2.49) provides the solution up to the function u^* which satisfies the homogeneous version of the boundary conditions, thus not affecting the value of $u(x,y,t)$ at the boundary. Under condition (2.32) with $(\cos\beta_1 \sin\beta_1) \leqslant 0$, as well as under conditions (2.33), no such unknown function enters the solution.

We will now show that the presence of u^* on the right-hand side of the solution (2.49) in the case of $(\cos\beta_1 \sin\beta_1) > 0$ is necessary for consistency at $x = y = 0$. Indeed, evaluating equation (2.49) at $x = y = 0$ for $\lambda = -1$ and $\beta_1 \in [0, 2\pi)$ we have

$$\begin{aligned}
u(0,0,t) = {} & \frac{1}{(2\pi)^2}\int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_2\, e^{-\omega t}\hat{u}_0(k_1,k_2) \\
& - \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^\infty dk_2\, e^{ik_1x+ik_2y-\omega t}\left\{\frac{2k_1}{c_2k_2+s_2k_1} f_2(\omega,k_2,t)\right. \\
& \left. + \frac{c_2k_2 - s_2k_1}{c_2k_2+s_2k_1}\left[\hat{u}_0(-k_1,k_2) + \left(s_1 + c_1\frac{k_2}{k_1}\right) f_1(\omega,-k_1,t)\right]\right\} \\
& - \frac{1}{(2\pi)^2}\int_{-\infty}^\infty dk_1 \int_{\partial D^+} dk_2\, e^{-\omega t}\left\{\frac{2k_2}{c_1k_1 - s_1k_2} f_1(\omega,k_1,t)\right. \\
& \left. + \frac{c_1k_1+s_1k_2}{c_1k_1-s_1k_2}\left[\hat{u}_0(k_1,-k_2) - \left(s_2 - c_2\frac{k_1}{k_2}\right) f_2(\omega,-k_2,t)\right]\right\} \\
& - \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{\partial D^+} dk_2\, e^{-\omega t}\left[\hat{u}_0(-k_1,-k_2)\right. \\
& \left. + \left(s_1 - c_1\frac{k_2}{k_1}\right) f_1(\omega,-k_1,t) - \left(s_2 + c_2\frac{k_1}{k_2}\right) f_2(\omega,-k_2,t)\right] \\
& + u^*(0,0,t).
\end{aligned}$$

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Rearranging and using condition (2.32), we find

$$
\begin{aligned}
u(0,0,t) = \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{-\omega t}\, \frac{k_2 + k_1 \cot\beta_1}{k_2 - k_1\cot\beta_1}\Big[-\hat{u}_0(-k_1,k_2) \\
+ \csc\beta_2\, f_2(\omega,k_2,t) - \csc\beta_1\, f_1(\omega,-k_1,t)\Big] \\
- \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^+} dk_2\, e^{-\omega t}\, \frac{k_2 + k_1 \cot\beta_1}{k_2 - k_1\cot\beta_1}\Big[-\hat{u}_0(k_1,-k_2) \\
+ \csc\beta_2\, f_2(\omega,-k_2,t) - \csc\beta_1\, f_1(\omega,k_1,t)\Big] \\
+ u^*(0,0,t).
\end{aligned}
$$

Letting $k_1 \mapsto -k_1$ and $k_2 \mapsto -k_2$ in the second term above, we obtain

$$
\begin{aligned}
u(0,0,t) = \frac{1}{(2\pi)^2}\int_{\partial D^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{-\omega t}\, \frac{k_2 + k_1 \cot\beta_1}{k_2 - k_1\cot\beta_1}\Big[-\hat{u}_0(-k_1,k_2) \\
+ \csc\beta_2\, f_2(\omega,k_2,t) - \csc\beta_1\, f_1(\omega,-k_1,t)\Big] \\
+ \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D^-} dk_2\, e^{-\omega t}\, \frac{k_2 + k_1 \cot\beta_1}{k_2 - k_1\cot\beta_1}\Big[-\hat{u}_0(-k_1,k_2) \\
+ \csc\beta_2\, f_2(\omega,k_2,t) - \csc\beta_1\, f_1(\omega,-k_1,t)\Big] \\
+ u^*(0,0,t),
\end{aligned}
$$

where the contour ∂D^- is shown in Figure 4. Moreover, deforming the contours with respect to k_1 to the contour ∂E^+, also shown in Figure 4, we find

$$
\begin{aligned}
u(0,0,t) = \frac{1}{(2\pi)^2}\int_{\partial E^+} dk_1 \int_{-\infty}^{\infty} dk_2\, e^{-\omega t}\, \frac{k_2 + k_1 \cot\beta_1}{k_2 - k_1\cot\beta_1}\Big[-\hat{u}_0(-k_1,k_2) \\
+ \csc\beta_2\, f_2(\omega,k_2,t) - \csc\beta_1\, f_1(\omega,-k_1,t)\Big] \\
+ \frac{1}{(2\pi)^2}\int_{\partial E^+} dk_1 \int_{\partial D^-} dk_2\, e^{-\omega t}\, \frac{k_2 + k_1 \cot\beta_1}{k_2 - k_1\cot\beta_1}\Big[-\hat{u}_0(-k_1,k_2) \\
+ \csc\beta_2\, f_2(\omega,k_2,t) - \csc\beta_1\, f_1(\omega,-k_1,t)\Big] \\
+ u^*(0,0,t). \qquad \text{(A 6)}
\end{aligned}
$$

Hence, deforming the contour with respect to k_2 from the real axis to ∂D^- in the first term and computing the contribution from the residue at $k_2 = k_1 \cot\beta_1$, we obtain

$$
\begin{aligned}
u(0,0,t) = u^*(0,0,t) + \frac{i}{2\pi}\,\mathscr{H}(-\cot\beta_1)\int_{\partial E^+} dk_1\, e^{-\omega^* t}\, 2k_1\cot\beta_1\Big[\hat{u}_0(-k_1,k_1\cot\beta_1) \\
- \csc\beta_2\, f_2(\omega^*,k_1\cot\beta_1) + \csc\beta_1\, f_1(\omega^*,-k_1)\Big], \qquad \text{(A 7)}
\end{aligned}
$$

where $\mathscr{H}$ denotes the Heaviside function.

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If $\cot\beta_1 \leqslant 0$ then by evaluating the global relation (2.26*a*) at $k_2 = k_1\cot\beta_1$, we obtain

$$e^{\omega^* t}\hat{u}(-k_1, k_1\cot\beta_1, t) = \hat{u}_0(-k_1, k_1\cot\beta_1) - \csc\beta_2\ f_2(\omega^*, k_1\cot\beta_1) \\ + \csc\beta_1\ f_1(\omega^*, -k_1) + (\cot\beta_1 - \cot\beta_2)\,\tilde{u}(\omega^*, t), \quad k_1 \in \mathbb{C}^+.$$

Using this identity in equation (A 7), we find

$$u(0,0,t) = u^*(0,0,t) - \frac{i}{2\pi}\ \mathscr{H}(-\cot\beta_1)\,(\cot\beta_1 - \cot\beta_2)\int_{\partial E^+} dk_1\, e^{-\omega^* t}\ 2k_1\cot\beta_1\ \tilde{u}(\omega^*, t) \\ + \frac{i}{2\pi}\ \mathscr{H}(-\cot\beta_1)\int_{\partial E^+} dk_1\, 2k_1\cot\beta_1\ \hat{u}(-k_1, k_1\cot\beta_1, t).$$

Furthermore, analyticity in the upper half complex k_1-plane and Jordan's lemma imply that the integral involving $\hat{u}(-k_1, k_1\cot\beta_1, t)$ vanishes for $\cot\beta_1 \leqslant 0$, hence

$$u(0,0,t) = u^*(0,0,t) + \frac{i}{2\pi}\ \mathscr{H}(-\cot\beta_1)\,(\cot\beta_2 - \cot\beta_1)\int_{\partial E^+} dk_1\, e^{-\omega^* t}\ 2k_1\cot\beta_1\ \tilde{u}(\omega^*, t). \tag{A 8}$$

On the other hand, by the definition of $\tilde{u}$ and the change of variables $l = -ik_1^2$ it follows that

$$u(0,0,t) = \frac{i}{2\pi}\,(\cot\beta_2 - \cot\beta_1)\int_{\partial E^+} dk_1\ e^{-\omega^* t}\ 2k_1\cot\beta_1\ \tilde{u}(\omega^*, t). \tag{A 9}$$

Therefore, combining equations (A 8) and (A 9) we conclude that

$$u^*(0,0,t) = \frac{i}{2\pi}\ \mathscr{H}(\cot\beta_1)\,(\cot\beta_1 - \cot\beta_2)\int_{\partial E^+} dk_1\, e^{-\omega^* t}\ 2k_1\cot\beta_1\ \tilde{u}(\omega^*, t),$$

which is consistent with the expression (2.51) for u^*.