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Spectral analogues of the law of the wall, the defect law and the log law

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Unlike the classical scaling relations for the mean-velocity profiles of wall-bounded uniform turbulent flows (the law of the wall, the defect law and the log law), which are predicated solely on dimensional analysis and similarity assumptions, scaling relations for the turbulent-energy spectra have been informed by specific models of wall turbulence, notably the attached-eddy hypothesis. In this paper, we use dimensional analysis and similarity assumptions to derive three scaling relations for the turbulent-energy spectra, namely the spectral analogues of the law of the wall, the defect law and the log law. By design, each spectral analogue applies in the same spatial domain as the attendant scaling relation for the mean-velocity profiles: the spectral analogue of the law of the wall in the inner layer, the spectral analogue of the defect law in the outer layer and the spectral analogue of the log law in the overlap layer. In addition, as we are able to show without invoking any model of wall turbulence, each spectral analogue applies in a specific spectral domain (the spectral analogue of the law of the wall in the high-wavenumber spectral domain, where viscosity is active, the spectral analogue of the defect law in the low-wavenumber spectral domain, where viscosity is negligible, and the spectral analogue of the log law in a transitional intermediate-wavenumber spectral domain, which may become sizable only at ultra-high Re_τ), with the implication that there exist model-independent one-to-one links between the spatial domains and the spectral domains. We test the spectral analogues using experimental and computational data on pipe flow and channel flow.

Key words: turbulent boundary layers, turbulent flows

1. Introduction

In the 1920s Prandtl and Kármán argued that the mean-velocity profiles (MVPs) of wall-bounded uniform turbulent flows should satisfy three complementary model-independent scaling relations, each in a specific spatial domain of application: the law of the wall in the inner layer, the defect law in the outer layer and the log law

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in the overlap layer (Tennekes & Lumley 1972). The law of the wall, the defect law and the log law promptly came to be regarded as the classical scaling relations for the MVPs, yet their theoretical underpinnings would not be fully clarified until the 1990s, when Barenblatt *et al.* introduced to fluid mechanics the distinction between complete and incomplete similarity. Then the log law was shown to rest on the assumption of complete similarity in two dimensionless variables, and it was recognized that the similarity in one of these dimensionless variables could be incomplete instead of complete, resulting in an alternative scaling relation that is just as model independent as the log law but might be in better keeping (as it was asserted) with experimental data than the log law (Barenblatt 1993, 2003). Despite this major challenge to the log law, the classical scaling relations have remained in textbooks all along, and it appears to be broadly agreed that they express the most salient properties of the MVPs (Smits, McKeon & Marusic 2011).

Meanwhile, a number of scaling relations have been put forward for the turbulent-energy spectra of wall-bounded uniform turbulent flows. Best known are Perry *et al.*'s 'inner scaling', 'outer scaling' and 'overlap scaling' (Perry & Chong 1982; Perry, Henbest & Chong 1986). The overlap scaling involves turbulent-energy spectra that are inversely proportional to the wavenumber, and has received the most attention; whether it is valid or not, however, remains an unresolved question (Smits *et al.* 2011). Regardless of their names (which might suggest otherwise), the scaling relations of Perry *et al.* apply in the overlap layer (that is, the spatial domain of application of the log law). Perry *et al.* have also put forward a scaling relation that applies in the outer layer (Perry *et al.* 1986), but, curiously, no one appears to have proposed a scaling relation that applies in the inner layer. In any event, from the outlook of this paper, the existing scaling relations for the turbulent-energy spectra share a cardinal trait: they are all model dependent in that their respective spectral domains of application have been ascertained via the attached-eddy hypothesis, a model of wall turbulence that entails the existence of specific links between spatial domains and spectral domains (Perry & Chong 1982; Perry *et al.* 1986; Perry & Marusic 1995; Smits *et al.* 2011). Thus, for example, Perry *et al.*'s inner scaling, outer scaling and overlap scaling apply within the intermediate range of wavenumbers that the attached-eddy hypothesis pairs with the overlap layer.

Here, we seek to formulate a set of model-independent scaling relations for the turbulent-energy spectra of wall-bounded uniform turbulent flows. For guidance, we carry out a step-by-step derivation of the law of the wall, the defect law and the log law using dimensional analysis and suitable assumptions of complete similarity. We then follow the same steps and make the same assumptions to derive analogous scaling relations for the turbulent-energy spectra. We start with a review of the concept of complete similarity.

2. Complete similarity

Consider a dimensionless function $f(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are dimensionless variables. Function f is said to be completely similar in x_i for $x_i \rightarrow 0$ (or $x_i \rightarrow \infty$) if there exists a finite (that is, bounded and non-zero) function $f_1(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ such that $\lim_{x_i \rightarrow 0} f(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ (or $\lim_{x_i \rightarrow \infty} f(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$) (Barenblatt 2003). Suppose now that f is completely similar in x_i for $x_i \rightarrow 0$ (or $x_i \rightarrow \infty$); in this case, we can substitute $f_1(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ for $f(x_1, \ldots, x_n)$ and expect the substitution to be valid if $|x_i| \ll 1$ (or $x_i \gg 1$). (Note that '$|x_i| \ll 1$' (or '$x_i \gg 1$') is but a sufficient condition for the substitution to be valid. Thus, if f is independent of x_i, for example, we can substitute $f_1 \equiv \lim_{x_i\to 0} f$ for f and the substitution will be valid for all x_i, not just for $|x_i| \ll 1$.) The law of the wall, the defect law and the log law can be predicated on suitable assumptions of complete similarity, as reviewed in the section that follows, which will serve as a roadmap for the formulation of analogous spectral scaling relations in § 4.

3. Scaling relations for the turbulent mean-velocity profiles

We focus for concreteness on a turbulent channel flow. Here 2δ is the distance between the walls of the channel, τ_w is the shear stress at the walls, ρ is the density of the fluid and ν is the kinematic viscosity of the fluid. The coordinates are x (streamwise), y (wall-normal) and z (spanwise); and u, v and w are the corresponding instantaneous velocities. At any given distance y from a wall, we can compute the mean velocity U (that is, the time-averaged value of u) and the spectrum of the turbulent energy, $E(k)$. (Here, E stands for a generic one-dimensional turbulent-energy spectrum at the distance y from the wall, and k stands for a generic wavenumber. The specific realizations of $E(k)$ are $E_{uu}(k_x)$, $E_{uu}(k_z)$, $E_{vv}(k_x)$, $E_{vv}(k_z)$, $E_{ww}(k_x)$ and $E_{ww}(k_z)$.) In this section we concern ourselves with U.

Consider the six dimensional variables $U' \equiv \partial U/\partial y$, y, δ, τ_w, ρ and ν. (Note that U is not Galilean-invariant, thus the choice of U'.) From Buckingham's Π-theorem and the dimensional equations $[U'] = [y]^{-1}[\tau_w]^{1/2}[\rho]^{-1/2}$, $[\delta] = [y][\tau_w]^0[\rho]^0$ and $[\nu] = [y]^1[\tau_w]^{1/2}[\rho]^{-1/2}$, we conclude that the functional relation among the six dimensional variables can be expressed as an equivalent functional relation among three dimensionless variables (yU'/u_τ, y/δ and y/δ_v, where u_τ is the frictional velocity, $u_\tau \equiv (\tau_w/\rho)^{1/2}$ and δ_v is the viscous length scale, $\delta_v \equiv \nu/u_\tau$), in the form

$$\frac{yU'}{u_\tau} = F(y/\delta, y/\delta_v). \tag{3.1}$$

We can readily derive a version of (3.1) suitable for application in the viscous layer. Note that $\tau_w = \rho\nu U'(0)$ or, equivalently, $U'(0) = u_\tau/\delta_v$. For $U'(y)$ to be continuous at the wall ($y = 0$), it must be that $U'(y) \sim u_\tau/\delta_v$ as $y \to 0$ (where u_τ/δ_v is the first term of the Taylor expansion of $U'(y)$ about $y = 0$) and, therefore, that $F(y/\delta, y/\delta_v) \sim y/\delta_v$ as $y/\delta \to 0$ and $y/\delta_v \to 0$ (the set of limits associated with the viscous layer). Thus, for the viscous layer, a version of (3.1) can be written in the form $yU'/u_\tau = y/\delta_v$ or, after integration with boundary condition $U(0) = 0$, in the form $U/u_\tau = y/\delta_v$. We shall presently derive versions of (3.1) suitable for application in three other spatial domains: the inner layer $y \ll \delta$, the outer layer $y \gg \delta_v$ and the overlap layer $\delta_v \ll y \ll \delta$.

We start with the inner layer. We assume that F in (3.1) is completely similar in y/δ for $y/\delta \to 0$, and write $yU'/u_\tau = F_1(y/\delta_v)$. It follows that

$$\frac{U}{u_\tau} = I(y/\delta_v), \tag{3.2}$$

where $I(x) \equiv \int_0^x \xi^{-1} F_1(\xi)\,\mathrm{d}\xi$. (Note that, from our discussion of the viscous layer, $\xi^{-1}F_1(\xi) \sim 1$ as $\xi \to 0$.) Equation (3.2) is the law of the wall, a version of (3.1) suitable for application in the inner layer $y \ll \delta$. (Here '$y \ll \delta$' is a sufficient condition.) The velocity profile of creeping channel flow, $U/u_\tau = (y/\delta_v)(2 - y/\delta)/2$, simplifies in the limit $y/\delta \to 0$ to $U/u_\tau = y/\delta_v$, which is in keeping with (3.2), as we expect it to be when the sufficient condition $y \ll \delta$ is satisfied, regardless of the value of the frictional

Reynolds number $Re_\tau \equiv \delta/\delta_v$. Nevertheless, from the identity $y/\delta_v = (y/\delta)Re_\tau$, y/δ_v can remain finite as $y/\delta \to 0$ if and only if $Re_\tau \to \infty$. (In creeping channel flow, where Re_τ remains small, $y/\delta \to 0$ necessitates $y/\delta_v \to 0$, and the inner layer is coextensive with the viscous layer.) Thus, the law of the wall can be said to signify that, for any finite y/δ_v, the dimensionless velocity U/u_τ becomes independent of Re_τ (and therefore of y/δ) at large Re_τ. (To show formally that $Re_\tau \gg 1$ is another sufficient condition for the law of the wall to apply, we express (3.1) in the alternative form $yU'/u_\tau = F_2(Re_\tau, y/\delta_v)$, assume that F_2 is completely similar in Re_τ for $Re_\tau \to \infty$, and write $yU'/u_\tau = F_3(y/\delta_v)$ or, after integration, $U/u_\tau = F_4(y/\delta_v)$, which can also be written as $U/u_\tau = I(y/\delta_v)$, valid for $Re_\tau \gg 1$.)

Consider next the outer layer. We assume that F in (3.1) is completely similar in y/δ_v for $y/\delta_v \to \infty$, and write $yU'/u_\tau = F_5(y/\delta)$. It follows that

$$\frac{U_\delta - U}{u_\tau} = O(y/\delta), \tag{3.3}$$

where U_δ is the mean velocity at the midplane of the flow ($y = \delta$) and $O(x) \equiv \int_1^x F_5(\xi)\xi^{-1}\,\mathrm{d}\xi$. Equation (3.3) is the defect law, a version of (3.1) suitable for application in the outer layer $y \gg \delta_v$. (Here '$y \gg \delta_v$' is a sufficient condition.) Now, from the identity $y/\delta = (y/\delta_v)Re_\tau^{-1}$, y/δ can remain finite as $y/\delta_v \to \infty$ if and only if $Re_\tau \to \infty$. Thus, the defect law can be said to signify that, for any finite y/δ, the dimensionless defect velocity $(U_\delta - U)/u_\tau$ becomes independent of Re_τ (and therefore of y/δ_v) at large Re_τ. (To show formally that '$Re_\tau \gg 1$' is another sufficient condition for the defect law to apply, we express (3.1) in the alternative form $yU'/u_\tau = F_6(y/\delta, Re_\tau)$, assume that F_6 is completely similar in Re_τ for $Re_\tau \to \infty$, and write $yU'/u_\tau = F_7(y/\delta)$ or, after integration, $(U_\delta - U)/u_\tau = F_8(y/\delta)$, which can also be written as $(U_\delta - U)/u_\tau = O(y/\delta)$, valid for $Re_\tau \gg 1$.)

Consider last the overlap layer, where $y/\delta \to 0$ and $y/\delta_v \to \infty$. Under the assumption that F in (3.1) is completely similar in y/δ and y/δ_v for $y/\delta \to 0$ and $y/\delta_v \to \infty$, we can write $yU'/u_\tau = (\kappa)^{-1}$, where κ is a dimensionless constant known as the Kármán constant. It follows that

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\frac{y}{\delta_v} + B, \tag{3.4}$$

where B is another dimensionless constant. Equation (3.4) is the log law, a version of (3.1) suitable for application in the overlap layer $\delta_v \ll y \ll \delta$. (Here '$\delta_v \ll y \ll \delta$' is a sufficient condition.)

It bears emphasis that it is possible to have a well-developed turbulent inner layer and a well-developed outer layer and no overlap layer. Suppose, for example, that we write the inner-layer condition $y/\delta \ll 1$ and the outer-layer condition $y/\delta_v \gg 1$ in the widely used nominal form $y/\delta < 0.1$ and $y/\delta_v > 50$, respectively. In this case, the distal edge of the inner layer, y_I^d, can be computed as $y_I^d = 0.1\delta$, and the proximate edge of the outer layer, y_O^p, can be computed as $y_O^p = 50\delta_v$. By setting $y_I^d = y_O^p$, we conclude that for $Re_\tau = 500$ the overlap layer consists of one point – the only point that is shared in common by the inner layer and the outer layer. For the overlap layer to extend over one decade, so that $y_I^d = 10y_O^p$, it must be that $Re_\tau = 5000$.

4. Scaling relations for the turbulent-energy spectra

Consider the seven dimensional variables $E, k, y, \delta, \tau_w, \rho$ and ν. From Buckingham's Π-theorem and the dimensional equations $[E] = [k]^{-1}[\tau_w]^1[\rho]^{-1}$, $[y] = [k]^{-1}[\tau_w]^0[\rho]^0$,

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$[\delta] = [k]^{-1}[\tau_w]^0[\rho]^0$ and $[\nu] = [k]^{-1}[\tau_w]^{1/2}[\rho]^{-1/2}$, we conclude that the functional relation among the seven dimensional variables can be expressed as an equivalent functional relation among four dimensionless variables (kE/u_τ^2, ky, $k\delta$ and $k\delta_v$), in the form $kE/u_\tau^2 = \mathscr{F}_0(ky, k\delta, k\delta_v)$, which can also be written as

$$\frac{kE}{u_\tau^2} = \mathscr{F}(ky, y/\delta, y/\delta_v). \tag{4.1}$$

4.1. *Inner layer*

For the inner layer, we assume that $\mathscr{F}$ is completely similar in y/δ for $y/\delta \to 0$, and write

$$\frac{kE}{u_\tau^2} = \mathscr{I}(ky, y/\delta_v). \tag{4.2}$$

This is the spectral analogue of the law of the wall, a version of (4.1) suitable for application in the inner layer $y \ll \delta$. (Here '$y \ll \delta$' is a sufficient condition.) Now, we have seen that y/δ_v can remain finite as $y/\delta \to 0$ if and only if $Re_\tau \to \infty$. (To show formally that '$Re_\tau \gg 1$' is another sufficient condition for the spectral analogue of the law of the wall to apply, we can proceed as we did to show the equivalent result for the law of the wall.) Further, from the identity $ky = (k\delta)(y/\delta)$, ky can remain finite as $y/\delta \to 0$ if and only if $k\delta \to \infty$, with the implication that the inner layer $y \ll \delta$ is linked to the high-wavenumber domain $k \gg 1/\delta$ (or $ky \gg y/\delta$). Thus, the spectral analogue of the law of the wall can be said to signify that, for any finite y/δ_v, the dimensionless spectrum kE/u_τ^2 becomes independent of Re_τ for $ky \gg (y/\delta_v)Re_\tau^{-1}$ (a sufficient condition). (To show formally that '$k\delta \gg 1$' is yet another sufficient condition for the spectral analogue of the law of wall to apply, we express (4.1) in the alternative form $kE/u_\tau^2 = \mathscr{F}_1(ky, k\delta, y/\delta_v)$, assume that $\mathscr{F}_1$ is completely similar in $k\delta$ for $k\delta \to \infty$, and write $kE/u_\tau^2 = \mathscr{F}_2(ky, y/\delta_v)$, which can also be written as $kE/u_\tau^2 = \mathscr{I}(ky, y/\delta_v)$, valid for $k\delta \gg 1$.)

To test the spectral analogue of the law of the wall, we start by selecting a few values of y/δ_v. In a separate panel for each value of y/δ_v, we plot a few curves k_xE_{uu}/u_τ^2 versus k_xy, corresponding to different values of Re_τ, and verify that these curves collapse onto a master curve at high k_xy. (Note that function $\mathscr{I}$ of (4.2) depends on both y/δ_v and ky, and the master curve may therefore be different for different values of y/δ_v.) For figure 1(*a*), we use experimental data on pipe flow and channel flow at high Re_τ (Ng *et al.* 2011). (We make no distinction between channel flow and pipe flow because we expect turbulence to be independent of the type of flow in the inner layer.) For figure 1(*b*), we use computational direct numerical simulation (DNS) data on channel flow at moderate Re_τ (del Álamo *et al.* 2004; Hoyas & Jiménez 2006).

A quick glance through the individual panels of figure 1 indicates that the curves in any given panel tend to collapse onto a master curve at high k_xy, and thus provides a cursory verification of the spectral analogue of the law of the wall. (An exception must be made for the two panels at the bottom of figure 1, where the curves do not collapse onto a master curve except perhaps at the very highest values of k_xy for which data are available. Note, however, that those panels correspond to $y/\delta_v \approx 570$, a value of y/δ_v that falls outside the inner layer, even at $Re_\tau = 3000$.) For a more detailed verification, we concentrate on an individual panel and check that the first curve to peel off from the master curve, as k_xy is lessened, is the curve for the lowest value of Re_τ, which is followed in turn by the curve for the second-lowest value of Re_τ, and so forth.

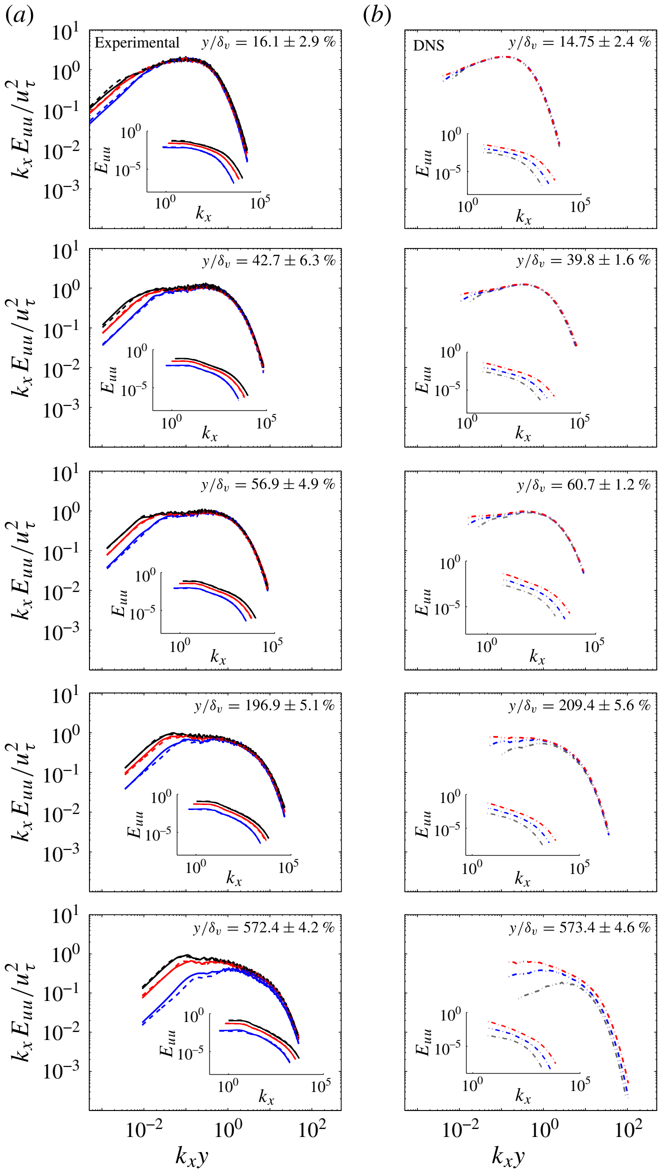

FIGURE 1. Test of the spectral analogue of the law of the wall at moderate and high Re_τ. The unscaled dimensional spectra in SI units (insets) are scaled (and rendered dimensionless) in accord with (4.2). (*a*) Experimental data on channel flow (—) and pipe flow (– –) for $Re_\tau = 1000$ (blue), 2000 (red) and 3000 (black) (from Ng *et al.* 2011). (*b*) Computational data on channel flow (–·–) for $Re_\tau = 550$ (grey), 934 (blue) and 2003 (red) (from del Álamo *et al.* 2004; Hoyas & Jiménez 2006, available at http://torroja.dmt.upm.es/channels/data/). Values of y/δ_v as indicated. Note that in this and all other figures we use a logarithmic scale for the coordinate (y-axis) of our plots; thus the gap, as seen in a plot, between a curve Y and a curve $Y + \delta Y$ (where δY is the absolute discrepancy between the curves) represents the relative discrepancy between the curves, $\delta Y/Y$ (that is to say, $\ln(Y + \delta Y) - \ln Y \approx \delta Y/Y$) – see supplementary material available at http://dx.doi.org/10.1017/jfm.2014.497.

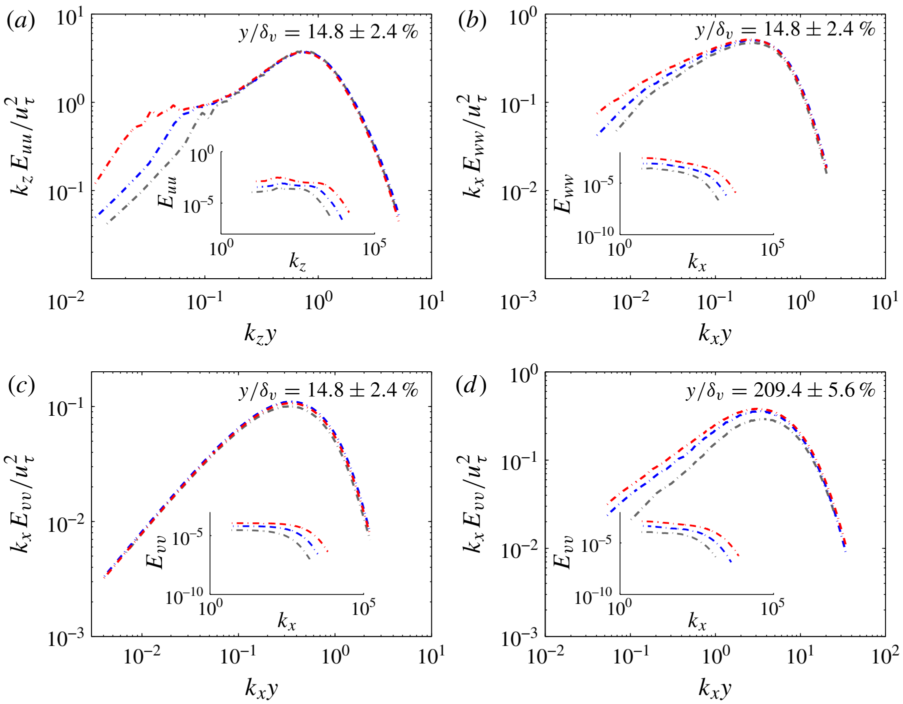

FIGURE 2. Further test of the spectral analogue of the law of the wall at moderate Re_τ. The unscaled dimensional spectra in SI units (insets) are scaled (and rendered dimensionless) in accord with (4.2). Computational data on channel flow for $Re_\tau = 550$ (grey), 934 (blue) and 2003 (red) (from del Álamo *et al.* 2004; Hoyas & Jiménez 2006, available at http://torroja.dmt.upm.es/channels/data/). Values of y/δ_v as indicated.

In other words, for a given y/δ_v, a higher Re_τ corresponds to a lower $k_x y$ at peel-off. This trend is consistent with the condition $k_x y \gg (y/\delta_v) Re_\tau^{-1}$, whereby the high-wavenumber domain of application of the spectral analogue of the law of the wall is expected to broaden as Re_τ is increased.

A second trend, also consistent with the condition $k_x y \gg (y/\delta_v)\, Re_\tau^{-1}$, can be discerned in either of the two columns of figure 1. For a given Re_τ, an increase in y/δ_v corresponds to an increase in $k_x y$ at peel-off, and to a concomitant narrowing of the high-wavenumber domain of application of the spectral analogue of law of the wall.

On the basis of figure 1, we propose '$k_x y \gtrsim 10 (y/\delta_v) Re_\tau^{-1}$' as a nominal form of '$k_x y \gg (y/\delta_v) Re_\tau^{-1}$'. This nominal form appears to be applicable, at least tentatively, to $E_{uu}(k_x)$.

From the computational data, it is possible to compute any realization of $E(k)$, not just $E_{uu}(k_x)$. Encouraged by the second column of figure 1, which shows that the spectral analogue of the law of the wall holds even at moderate values of Re_τ, at least for $E_{uu}(k_x)$, we use the same computational data (del Álamo *et al.* 2004; Hoyas & Jiménez 2006) to prepare figure 2. (Figures including all possible realizations of $E(k)$ can be seen in the supplementary material.)

In figure 2(*c*), a phenomenon is apparent that might seem impossible to reconcile with the condition $k_x y \gg (y/\delta_v)\, Re_\tau^{-1}$: the curves for $Re_\tau = 550$, 934 and 2003 collapse onto a master curve not only at high $k_x y$, where that condition is satisfied, but also at low $k_x y$. Note, however, that '$k_x y \gg (y/\delta_v)\,Re_\tau^{-1}$' is merely a sufficient condition for the spectral analogue of the law of the wall to apply. It is not a necessary condition. Thus, the phenomenon remains consistent with the spectral analogue of the law of the wall. We shall encounter a similar phenomenon later on, in a discussion of the spectral analogue of the defect law.

An alternative form of the spectral analogue of the law of the wall (4.2) may be written as

$$\frac{kE}{u_\tau^2} = \mathscr{I}^a(k\delta_v,\, y/\delta_v), \tag{4.3}$$

where

$$\mathscr{I}^a(k\delta_v,\, y/\delta_v) \equiv \mathscr{I}(k\delta_v \times y/\delta_v,\, y/\delta_v). \tag{4.4}$$

4.2. *Outer layer*

For the outer layer we assume that $\mathscr{F}$ in (4.1) is completely similar in y/δ_v for $y/\delta_v \to \infty$, and write

$$\frac{kE}{u_\tau^2} = \mathscr{O}(ky,\, y/\delta). \tag{4.5}$$

This is the spectral analogue of the defect law, a version of (4.1) suitable for application in the outer layer $y \gg \delta_v$. (Here '$y \gg \delta_v$' is a sufficient condition.) Now, y/δ can remain finite as $y/\delta_v \to \infty$ if and only if $Re_\tau \to \infty$. (To show formally that '$Re_\tau \gg 1$' is another sufficient condition for the spectral analogue of the defect law to apply, we can proceed as we did to show the equivalent result for the defect law.) Further, from the identity $ky = (k\delta_v)(y/\delta_v)$, ky can remain finite as $y/\delta_v \to \infty$ if and only if $k\delta_v \to 0$, with the implication that the outer layer $y \gg \delta_v$ is linked to the low-wavenumber domain $k \ll 1/\delta_v$ (or $ky \ll y/\delta_v$). Thus, the analogue of the defect law can be said to signify that, for any given y/δ, the dimensionless spectrum kE/u_τ^2 becomes independent of Re_τ for $ky \ll (y/\delta)Re_\tau$ (a sufficient condition). (To show formally that '$k\delta_v \ll 1$' is yet another sufficient condition for the spectral analogue of the defect law to apply, we express (4.1) in the alternative form $kE/u_\tau^2 = \mathscr{F}_3(ky,\, y/\delta,\, k\delta_v)$, assume that $\mathscr{F}_3$ is completely similar in $k\delta_v$ for $k\delta_v \to 0$, and write $kE/u_\tau^2 = \mathscr{F}_4(ky,\, y/\delta)$, which can also be written as $kE/u_\tau^2 = \mathscr{O}(ky,\, y/\delta)$, valid for $k\delta_v \ll 1$.)

To test the spectral analogue of the defect law, we start by selecting a few values of y/δ. In a separate panel for each value of y/δ, we plot a few curves $k_x E_{uu}/u_\tau^2$ versus $k_x y$, corresponding to different values of Re_τ, and verify that these curves collapse onto a master curve at low $k_x y$. (Note that function $\mathscr{O}$ of (4.5) depends on both y/δ and ky, and the master curve may therefore be different for different values of y/δ.) For figures 3 and 4, which correspond to $y/\delta \approx 0.1$ and $y/\delta \approx 1$, respectively, we use experimental data on pipe flow at high and ultra-high Re_τ (McKeon & Morrison 2007; Rosenberg *et al.* 2013). For figure 5(*a*), we use experimental data on pipe flow at high Re_τ (Ng *et al.* 2011). For figure 5(*b*), we use experimental data on channel flow at high Re_τ (Ng *et al.* 2011).

A quick glance through the individual panels of figures 3–5 indicates that the curves in any given panel tend to collapse onto a master curve at low $k_x y$, and thus provides a cursory verification of the spectral analogue of the defect law. (An exception must be made for the two panels at the top of figure 5. Note, however, that those panels correspond to $y/\delta \approx 0.01$, a value of y/δ that falls outside the outer layer.) For a more

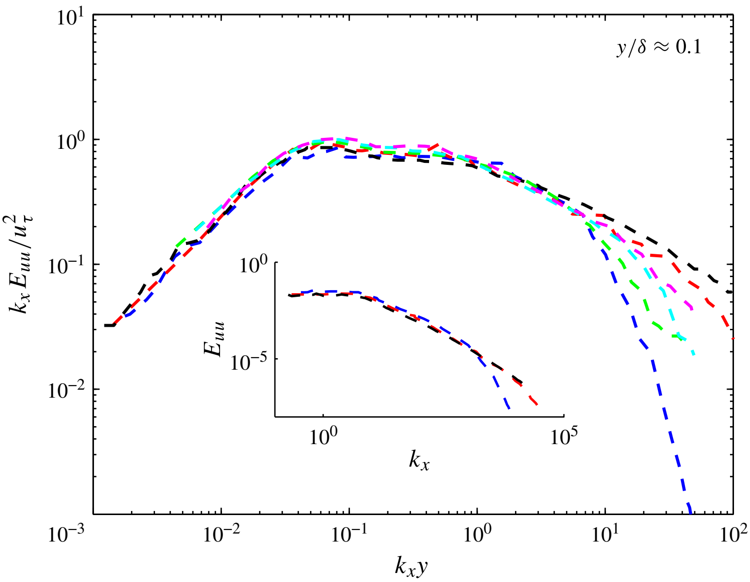

FIGURE 3. Test of the spectral analogue of the defect law at high and ultra-high Re_τ. The unscaled dimensional spectra in SI units (insets) are scaled (and rendered dimensionless) in accord with (4.5). Experimental data on pipe flow for Re_τ = 1985 (blue), 3350 (green), 8560 (cyan), 19 700 (magenta), 37 690 (red) and 98 190 (black) (from McKeon & Morrison 2007; Rosenberg *et al.* 2013). Value of y/δ as indicated.

detailed verification, we start by focusing on the experimental data at high and ultra-high Re_τ (figures 3 and 4). In figure 3, we see that the first curve to peel off from the master curve, as $k_x y$ is increased, is the curve for the lowest value of Re_τ, which is followed in turn by the curve for the second-lowest value of Re_τ, and so forth. In other words, for a given y/δ, an increase in Re_τ corresponds to an increase in $k_x y$ at peel-off. This trend is consistent with the condition $k_x y \ll (y/\delta)Re_\tau$, whereby the low-wavenumber domain of application of the spectral analogue of the defect law is expected to broaden as Re_τ is increased.

A second trend, also consistent with the condition $ky \ll (y/\delta)Re_\tau$, can be discerned from a comparison of figure 3 with figure 4. For a given Re_τ ($Re_\tau = 1985$), an increase in y/δ corresponds to an increase in $k_x y$ at peel-off, and to a concomitant broadening of the low-wavenumber domain of application of the spectral analogue of the defect law.

On the basis of figures 3 and 4, we propose '$k_x y \lesssim 0.02(y/\delta)Re_\tau$' as a nominal form of '$k_x y \ll (y/\delta)Re_\tau$'. This nominal form appears to be applicable, at least tentatively, to $E_{uu}(k_x)$.

Next, we turn our attention to the experimental data at high Re_τ (figure 5). In each panel of figure 5 there are three curves, corresponding to $Re_\tau = 1000$, 2000 and 3000. As might be expected from the condition $k_x y \ll (y/\delta)Re_\tau$, at low $k_x y$ the curves for $Re_\tau = 2000$ and 3000 collapse onto a master curve whereas the curve for $Re_\tau = 1000$ remains distinct. Now, if we scan the curves from left to right, in the direction of increasing $k_x y$, we can verify that the curve for Re_τ = 1000 eventually converges onto the master curve, starting from a value of $k_x y$ that increases systematically with y/δ. Close to the proximal edge of the outer layer, where $y/\delta \approx 0.1$, the curve

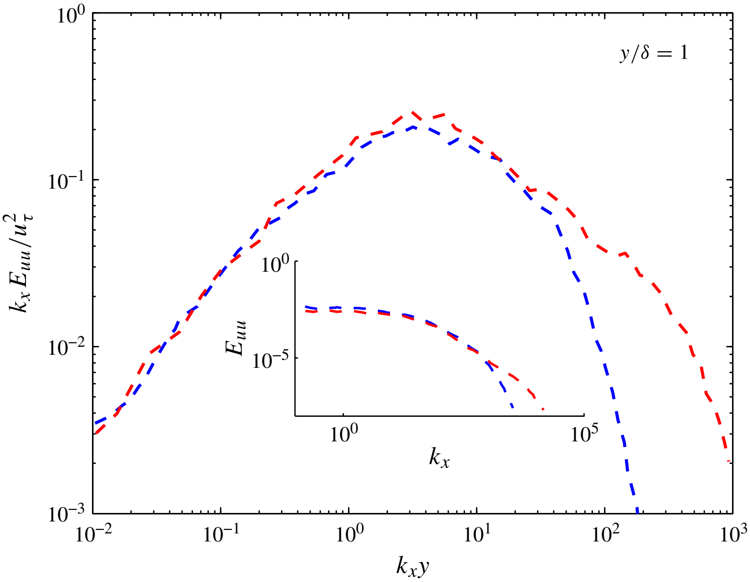

FIGURE 4. Further test of the spectral analogue of the defect law at high and ultra-high Re_τ. The unscaled dimensional spectra in SI units (insets) are scaled (and rendered dimensionless) in accord with (4.5). Experimental data on pipe flow for $Re_\tau = 1985$ (blue) and 37 690 (red) (from Rosenberg *et al.* 2013). Value of y/δ as indicated.

for $Re_\tau = 1000$ fans out again immediately beyond the point of convergence. Deep in the outer layer, where y/δ is relatively large, the curve for $Re_\tau = 1000$ fans out again only after remaining on the master curve over a sizable range of values of $k_x y$.

This phenomenon, whereby the curve for $Re_\tau = 1000$ satisfies the spectral analogue of the defect law only at intermediate values of $k_x y$, does not contradict the sufficient condition $k_x y \ll (y/\delta) Re_\tau$. Thus, the phenomenon remains consistent with the spectral analogue of the defect law. Yet the spectral analogue of the defect law is not rich enough to explain the phenomenon. To explain the phenomenon, we must introduce additional assumptions (and might have to commit to a particular model of wall turbulence).

Let us assume, as a plausible example, that the turbulent power per unit mass of fluid, ε, is independent of the viscosity and of the wavenumber over an intermediate range of values of k, as in the phenomenological theory of turbulence (Tennekes & Lumley 1972). To carry out a dimensional analysis of ε, we exclude the variables ν and k and consider the five dimensional variables ε, y, δ, τ_w and ρ. From Buckingham's Π-theorem and the dimensional equations $[\varepsilon] = [y]^{-1}[\tau_w]^{3/2}[\rho]^{-3/2}$ and $[\delta] = [y]^1[\tau_w]^0[\rho]^0$, we conclude that the functional relation among the five dimensional variables can be expressed as an equivalent functional relation among two dimensionless variables ($\varepsilon y/u_\tau^3$ and y/δ), in the form $\varepsilon = (u_\tau^3/y)H(y/\delta)$. In this case, Kolmogorov's turbulent-energy spectrum (which is usually written as $E = C\varepsilon^{2/3}k^{-5/3}$, valid for $1/\delta \ll k \ll 1/\eta$, where C is Kolmogorov's dimensionless constant and η is Kolmogorov's length scale, $\eta = \nu^{3/4}\varepsilon^{-1/4}$) can be expressed in the form

$$\frac{kE}{u_\tau^2} = C(ky)^{-2/3}[H(y/\delta)]^{2/3}, \tag{4.6}$$

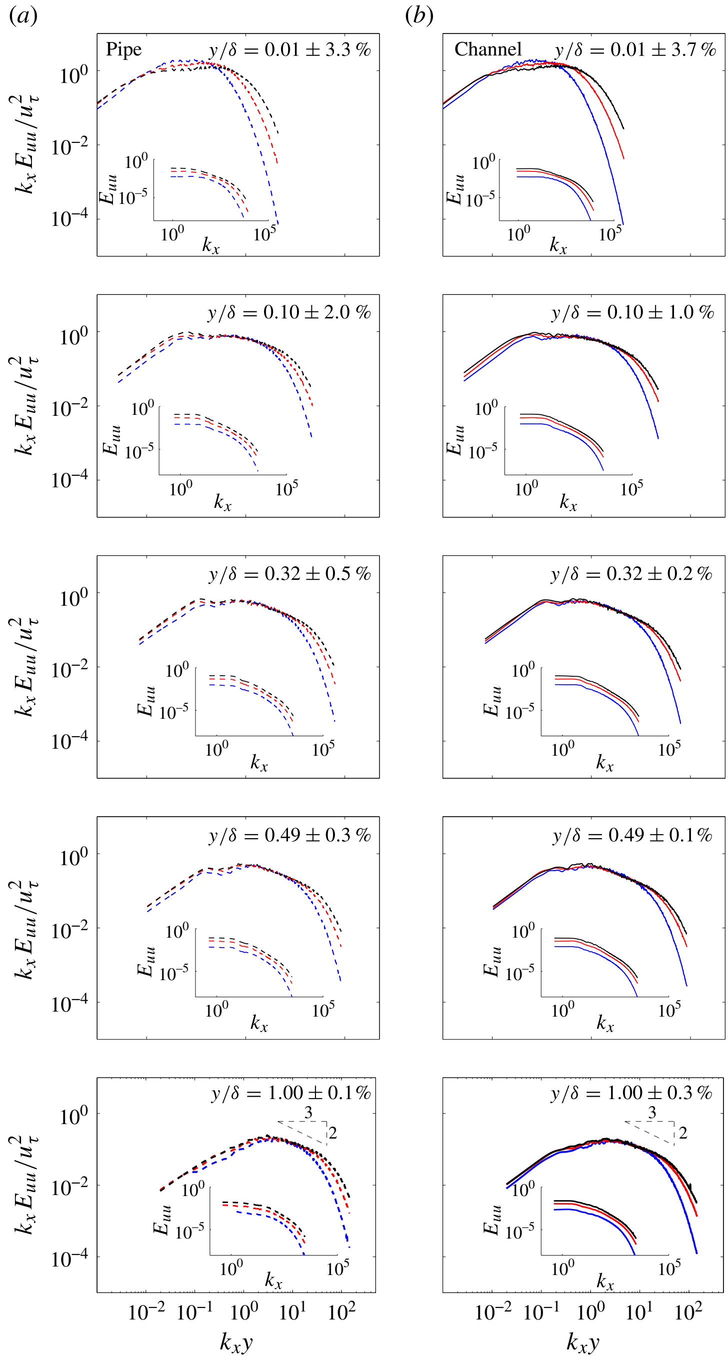

FIGURE 5. Further test of the spectral analogue of the defect law at high Re_τ. The unscaled dimensional spectra in SI units (insets) are scaled (and rendered dimensionless) in accord with (4.5). Experimental data on (*a*) pipe flow (– –) and (*b*) channel flow (—) for $Re_\tau = 1000$ (blue), 2000 (red) and 3000 (black) (from Ng *et al.* 2011). Values of y/δ as indicated.

which is valid over the 'Kolmogorov spectral domain', namely

$$\frac{y}{\delta} \ll ky \ll \frac{y}{\eta} = \left(\frac{y}{\delta}Re_\tau\right)^{3/4} [H(y/\delta)]^{1/4}. \tag{4.7}$$

Note that the scaling relation (4.6) is a special case of (4.5) (the spectral analogue of the defect law), and that, for any given y/δ and at sufficiently high Re_τ, the entire Kolmogorov spectral domain is subsumed under the sufficient condition $ky \ll (y/\delta)Re_\tau$. Further, the lower limit of the Kolmogorov spectral domain, y/δ, is consistent with the experimental data of figure 5, where we have seen that the curve for $Re_\tau = 1000$ converges onto the master curve starting from a value of $k_x y$ that increases systematically with y/δ. As we do not have an expression for $H(y/\delta)$, it is hard to predict the manner in which the Kolmogorov spectral domain broadens or shrinks as a function of y/δ. Nevertheless, for $y/\delta = 0.1$ we might assume that the mean velocity profile will not differ much from the log law, set $H(y/\delta) \propto 1 - y/\delta$ in (4.7), and draw the conclusion that the Kolmogorov spectral domain shrinks rapidly as the value of y/δ approaches 0.1 from above. This conclusion is consistent with the experimental data of figure 5, where we have seen that, close to the proximal edge of the outer layer, where $y/\delta \approx 0.1$, the curve for $Re_\tau = 1000$ converges onto the master curve only to fan out again almost immediately beyond the point of convergence. For larger values of y/δ, the curve for $Re_\tau = 1000$ remains on the master curve over a sizable range of values of $k_x y$, and the slope of the master curve is in rough accord with (4.6) (figure 5).

An alternative form of the spectral analogue of the defect law (4.5) may be written as

$$\frac{kE}{u_\tau^2} = \mathscr{O}^a(k\delta, y/\delta), \tag{4.8}$$

where

$$\mathscr{O}^a(k\delta, y/\delta) \equiv \mathscr{O}(k\delta \times y/\delta, y/\delta). \tag{4.9}$$

4.3. *Overlap layer*

For the overlap layer we assume that $\mathscr{F}$ in (4.1) is completely similar in y/δ and y/δ_v for $y/\delta \to 0$ and $y/\delta_v \to \infty$, and write

$$\frac{kE}{u_\tau^2} = \mathscr{V}(ky). \tag{4.10}$$

This is the spectral analogue of the log law, a version of (4.1) suitable for application in the overlap layer $\delta_v \ll y \ll \delta$ and the intermediate-wavenumber spectral domain $1/\delta \ll k \ll 1/\delta_v$ (or $y/\delta \ll ky \ll y/\delta_v$). It coincides with the 'inner scaling' of Perry *et al.* and can be said to signify that, for any given point in the overlap layer, the dimensionless spectrum kE/u_τ^2 becomes independent of Re_τ for $y/\delta \ll ky \ll y/\delta_v$ at large Re_τ.

To test the spectral analogue of the log law, we use experimental data on pipe flow at ultra-high Re_τ (Rosenberg *et al.* 2013) to plot curves kE_{uu}/u_τ^2 versus $k_x y$ for a large number of points in the overlap layer (figure 6). The spectral analogue of the log law appears to hold over a narrow range of intermediate values of $k_x y$ close to $k_x y \approx 1$, a conclusion that is in accord with the test of Perry *et al.*'s inner scaling carried out by Morrison *et al.* (2004). (An ancillary discussion of the spectral analogue of the log law can be found in appendix A.)

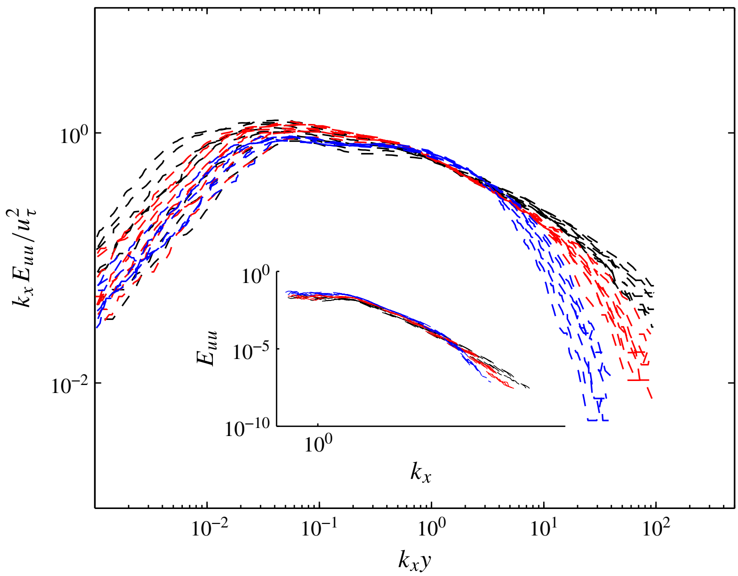

FIGURE 6. Test of the spectral analogue of the log law at ultra-high Re_τ. The unscaled dimensional spectra in SI units (insets) are scaled (and rendered dimensionless) in accord with (4.10). Experimental data on pipe flow for $Re_\tau = 3334$ (blue), 20 250 (red) and 37 690 (black) (from Rosenberg *et al.* 2013). All data from the overlap layer ($50\delta_\upsilon \leqslant y \leqslant 0.1\delta$).

5. Discussion

We have derived three scaling relations for the turbulent-energy spectra of wall-bounded uniform turbulent flows. Each of these scaling relations is the spectral analogue of one the classical scaling relations for the MVPs (the law of the wall, the defect law and the log law). The spectral analogues and their respective spatial and spectral domains of application are summarized in table 1.

The spectral analogues of the law of the wall, the defect law and the log law might be fittingly termed the 'inner scaling', the 'outer scaling' and the 'overlap scaling', respectively, on the basis of their respective spatial domains of application. Nevertheless, the spectral analogues should be distinguished from Perry *et al.*'s inner scaling, outer scaling and overlap scaling, all three of which, their prevalent names notwithstanding, apply in the overlap layer.

The spectral analogue of the log law is the same as Perry *et al.*'s inner scaling, which has been widely used to display spectra from the overlap layer. Such spectra corresponding to several different values of Re_τ may be plotted in the form of curves kE/u_τ^2 versus ky (one curve for each value of Re_τ). Then, as per the spectral analogue of the log law, the curves should fall onto a master curve at intermediate values of ky.

Unlike the inner and outer scalings of Perry *et al.*, the spectral analogues of the law of the wall and the defect law can be used to display spectra from the inner and outer layers, respectively. Thus, for example, spectra corresponding to a fixed value of y/δ_υ and several values of Re_τ (where $y/\delta_\upsilon \lesssim 0.1Re_\tau$ so as to satisfy the inner-layer condition) may be plotted in the form of curves kE/u_τ^2 versus ky (one curve for each value of Re_τ). Then, as per the spectral analogue of the law of the wall, the curves should fall on a master curve at high ky; in particular, the curve corresponding to a specific value of Re_τ should coincide with the master curve for $ky \gtrsim 10(y/\delta_\upsilon)Re_\tau^{-1}$

Spectral analogue	Expression	Spatial domain	Spectral domain
Of the law of the wall	$kE/u_\tau^2 = \mathscr{I}(ky, y/\delta_\nu)$	Inner layer	High ky
Of the defect law	$kE/u_\tau^2 = \mathscr{O}(ky, y/\delta)$	Outer layer	Low ky
Of the log law	$kE/u_\tau^2 = \mathscr{V}(ky)$	Overlap layer	Intermediate ky

TABLE 1. Spectral analogues. Here, 'inner layer' corresponds to the sufficient condition $y/\delta_\nu \ll Re_\tau$ (nominally $y/\delta_\nu \lesssim 0.1Re_\tau$), 'outer layer' to $y/\delta \gg Re_\tau^{-1}$ (nominally $y/\delta \gtrsim 50Re_\tau^{-1}$), 'overlap layer' to $\delta_\nu \ll y \ll \delta$ (nominally $50\delta_\nu \lesssim y \lesssim 0.1\delta$), 'high ky' to $ky \gg (y/\delta_\nu)Re_\tau^{-1}$ (nominally $ky \gtrsim 10(y/\delta_\nu)Re_\tau^{-1}$), 'low ky' to $ky \ll (y/\delta)Re_\tau$ (nominally $ky \lesssim 0.02(y/\delta)Re_\tau$), and 'intermediate ky' to $y/\delta \ll ky \ll y/\delta_\nu$ (nominally $10y/\delta \lesssim ky \lesssim 0.02y/\delta_\nu$). The nominal forms of the sufficient conditions on ky are tentative, and might apply only to $E_{uu}(k_x y)$.

(table 1). The same procedure may be repeated for other values of y/δ_ν, but the master curve might differ for different values of y/δ_ν (because function $\mathscr{I}$ depends on y/δ_ν). Similarly, spectra corresponding to a fixed value of y/δ and several values of Re_τ (where $y/\delta \gtrsim 50Re_\tau^{-1}$ so as to satisfy the outer-layer condition) may be plotted in the form of curves kE/u_τ^2 versus ky (one curve for each value of Re_τ). Then, as per the spectral analogue of the defect law, the curves should fall on a master curve at low ky; in particular, the curve corresponding to a specific value of Re_τ should coincide with the master curve for $ky \lesssim 0.02(y/\delta)Re_\tau$ (table 1). The same procedure may be repeated for other values of y/δ, but the master curve might differ for different values of y/δ (because function $\mathscr{O}$ depends on y/δ).

Tennekes & Lumley (1972) have conjectured that there is a 'close analogy between the spatial structure of turbulent boundary layers and the spectral structure of turbulence'. The spectral analogues entail the existence of specific links between spatial domains and spectral domains (table 1) and can thus be interpreted as concrete realizations of Tennekes & Lumley's 'close analogy'. Further, we have derived the spectral analogues and ascertained their respective domains of application, spatial as well as spectral, by using only dimensional analysis and similarity assumptions, without having recourse to any model of wall turbulence. Thus the spectral analogues, unlike all existing scaling relations for the energy spectra, are model independent. The attendant specific links between spatial domains and spectral domains are model independent, too, and as such they can always be counted on to act as an internal constraint in any model of wall turbulence that incorporates both the turbulent MVP and the turbulent-energy spectra. Yet, with few exceptions (e.g. Gioia *et al.* 2010), models of wall turbulence (most notably the classical model of the turbulent mean-velocity profile) ignore the turbulent-energy spectra altogether, even where they account in detail for the spatial structure of turbulent boundary layers. If the turbulent-energy spectra, on which much is known, are included as an integral part of future models of wall turbulence, prospects may open up for gaining new insights into an old but hardly exhausted problem. We hope that our findings will help spur such models.

Acknowledgements

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Supplementary material

Supplementary material is available at http://dx.doi.org/10.1017/jfm.2014.497.

Appendix A. An ancillary discussion of the spectral analogue of the log law

It might be tempting to try to simplify the spectral analogue of the log law, $kE/u_\tau^2 = \mathscr{V}(ky)$ (4.10), to $kE/u_\tau^2 = \text{const.}$, valid in the overlap layer (and in the attendant spectral domain), by arguing, for example, that function $\mathscr{I}^a$ of the alternative scaling relation (4.3) may be assumed to be completely similar in y/δ_v for $y/\delta_v \to \infty$. Under this new assumption, it would be possible to substitute a function $\mathscr{I}^V(k\delta_v)$ for $\mathscr{I}^a(k\delta_v, y/\delta_v)$ and to conclude that $\mathscr{V}(ky) = \mathscr{I}^V(k\delta_v)$, a condition that would immediately lead to '$kE/u_\tau^2 = \text{const.}$'. Note, however, that even though both sides of the equals sign in '$\mathscr{V}(ky) = \mathscr{I}^V(k\delta_v)$' correspond ostensibly to the same limits, $y/\delta \to 0$ and $y/\delta_v \to \infty$, in the case of the left-hand side $y/\delta_v \to \infty$ while ky remains finite (resulting in a function of ky, namely $\mathscr{V}$) whereas in the case of the right-hand side $y/\delta_v \to \infty$ while $k\delta_v$ remains finite (resulting in a function of $k\delta_v$, namely $\mathscr{I}^V$), and it is not possible for both ky and $k\delta_v$ to remain finite where $y/\delta_v \to \infty$. In other words, the assumption of complete similarity in y/δ_v that underlies the scaling relation $kE/u_\tau^2 = \mathscr{V}(ky)$ is inconsistent with the assumption of complete similarity in y/δ_v that underlies the scaling relation $kE/u_\tau^2 = \mathscr{I}^V(k\delta_v)$, and '$kE/u_\tau^2 = \text{const.}$' was predicated on two mutually exclusive assumptions. In keeping with this conclusion, if we proceed from (4.4) and compute $\lim_{y/\delta_v \to \infty} \mathscr{I}^a(k\delta_v, y/\delta_v)$ as $\lim_{y/\delta_v \to \infty} \mathscr{I}(k\delta_v \times y/\delta_v, y/\delta_v)$, then, under the prevailing assumption that $\mathscr{I}$ is completely similar in y/δ_v for $y/\delta_v \to \infty$, the result will not be a constant but a function of ky (regardless of the behaviour of $k\delta_v$ as $y/\delta_v \to \infty$). It bears emphasis, however, that kE/u_τ^2 might still be constant on a portion of the overlap layer and of the attendant spectral domain. Interestingly, '$kE/u_\tau^2 = \text{const.}$' is Perry *et al.*'s 'overlap scaling', which according to Perry *et al.* applies in a small segment of the spectral domain of the spectral analogue of the log law (that is, a small segment of the low-wavenumber spectral domain where viscosity is negligible).

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