Proceedings of the Edinburgh Mathematical Society (2017) **60**, 1–16
DOI:10.1017/S0013091516000122

CIRCLE OF SARKISOV LINKS ON A FANO 3-FOLD

HAMID AHMADINEZHAD[1] AND FRANCESCO ZUCCONI[2]

[1]*School of Mathematics, University of Bristol, Bristol BS8 1TW, UK*
(h.ahmadinezhad@bristol.ac.uk)
[2]*Dipartimento di Matematica e Informatica, Università degli Studî Udine, Via delle Scienze, 206, 33100 Udine, Italy*
(zucconi@dimi.uniud.it)

(Received 10 February 2014)

Abstract For a general Fano 3-fold of index 1 in the weighted projective space $\mathbb{P}(1,1,1,1,2,2,3)$ we construct two new birational models that are Mori fibre spaces in the framework of the so-called Sarkisov program. We highlight a relation between the corresponding birational maps, as a circle of Sarkisov links, visualizing the notion of relations in the Sarkisov program.

Keywords: birational automorphism; Fano varieties; Sarkisov program; variation of geometric invariant theory

2010 *Mathematics subject classification:* Primary 14E05; 14E30; 14E07; 14E08

1. Introduction

All varieties in this paper are projective over the field of complex numbers.

The question of rationality in algebraic geometry is classic. It asks whether a given algebraic variety X of dimension n is birational to $\mathbb{P}^n$, or, equivalently, whether the field of rational functions $\mathbb{C}(X)$ is isomorphic to $\mathbb{C}(x_1,\dots,x_n)$. In dimension 1 the only rational variety is the projective line. In dimension 2 we have a classification of rational surfaces, thanks to the Italian school of Castelnuovo, Cremona, Enriques and others.

In dimension 3 this question boils down to the rationality question for uni-ruled varieties, as rationality implies uni-ruledness. The minimal model program (MMP for short) in birational geometry (see [**26**]) produces a so-called Mori fibre space (MFS) for a smooth uni-ruled variety (see Definition 2.1). For example, Fano varieties with Picard number 1 form a class of MFS. The MMP provides a framework for classification of algebraic varieties, and in particular of MFSs.

The next step is to study relations between the MFSs. This, in particular, includes the rationality question since $\mathbb{P}^n$ is itself an MFS. In fact, the notion of pliability for an MFS, introduced by Corti [**15**], is an invariant that formalizes the classification of MFSs up to a natural equivalence relation $\sim$, the square birationality (see Definition 2.2).

Definition 1.1 (Corti). The *pliability* of an MFS $X \to S$ is the set

$$\mathcal{P}(X/S) = \{\text{MFS } Y \to T \mid X \text{ is birational to } Y\}/\sim.$$

We sometimes use the term pliability to mean the cardinality of this set. When the base S is a point we use the notation $\mathcal{P}(X)$.

For instance, if $\mathcal{P}(X/S)$ is finite, then X is not rational, as $|\mathcal{P}(\mathbb{P}^3)| = \infty$. Varieties with $\mathcal{P}(X/S) = 1$, the so-called *birationally rigid varieties*, have attracted particular attention in recent decades. The first example of such a variety is a smooth quartic in $\mathbb{P}^4$, discovered by Iskovskikh and Manin [**22**].

1.1. The graph structure of $\mathcal{P}(X/S)$

The theory of the Sarkisov program, developed by Corti [**13**] and generalized to higher dimensions by Hacon and McKernan [**20**], states that any birational map between MFSs decomposes as a finite sequence of *elementary Sarkisov links* (ESLs). Each ESL is produced, as explained in § 2.2, by a chain of forced moves called a *2-ray game*.

Definition 1.2. For an MFS X/S, define its *Mori graph* to be the graph $\mathrm{MG}(X/S) = (V, E)$, where the vertices of $\mathrm{MG}(X/S)$ are $V = \{v_\alpha\}$ with each v_α being an MFS birational to X, and edges are $E = \{e_{\alpha\beta}\}$, where v_α is connected to v_β if and only if $e_{\alpha\beta}$ is an ESL.

The square relation is defined to be $v_\alpha \sim v_\beta$ if $e_{\alpha\beta} \in E$ can be made square birational after a self-map of v_α (see Definition 2.2).

Define the *pliability graph* of an MFS X/S to be $\mathrm{PG}(X/S)$, that is, the quotient of the graph MG with *square relation*.

A Sarkisov relation, introduced by Kaloghiros [**24**], is a loop in G that contains at least three vertices. And an elementary Sarkisov relation is a triangle (in G).

The following question gives an insight into the structure of $\mathcal{P}(X/S)$.

Question 1.3. *Given an MFS X/S, if $|V| > 2$, does $\mathrm{PG}(X/S)$ contain a tree in the sense that there are two vertices in the graph that are connected by at least two paths that intersect each other only at these two vertices?*

Question 1.4. *Given an MFS X/S with$|V| > 2$, when is $\mathrm{PG}(X/S)$ complete?*

It follows from the main result of [**24**] that a loop in $\mathrm{MG}(X/S)$ is triangulizable. Hence, if the answer to Question 1.3 is 'no', then $\mathcal{P}(X/S)$ is naturally constructed by ESLs on X/S only.

In this paper we construct such a triangle, in $\mathrm{PG}(X/S)$, on a non-complete interaction Fano 3-fold. The expectation is that if Question 1.4 has a negative answer (it has a chain with 3 vertices and they do not form a triangle), it will be for a variety with the properties similar to the one we study: we start with a Fano 3-fold with two distinct singular points and construct two new models from the blow up of these points. Then, using some delicate machinery, we show that these three models make a loop (triangle). Namely, we prove the following theorem.

Theorem 1.5. *Let $X \subset \mathbb{P}(1,1,1,1,2,2,3)$ be a codimension 3 Fano 3-fold. If X is general, then it is an MFS, i.e. X is $\mathbb{Q}$-factorial with Picard number 1, and it is quasi-smooth with only two singular points, of type $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$. Moreover, X is birational to an MFS over $\mathbb{P}^1$, with cubic surface fibres. The birational map factors in two different ways through Sarkisov links, one of which includes two links, through a codimension 2 Fano 3-fold $Y_{3,3} \subset \mathbb{P}(1,1,1,1,1,2)$. In particular, the pliability of X is at least 3 and there is a circle of Sarkisov links between the models.*

In particular, this theorem also provides a family of examples for [**24**] in higher codimension with pliability bigger than 2.

2. MFSs and relations among them

In this section we recall some definitions and known facts about MFSs. The formal definition of an MFS, in any dimension, is as follows.

Definition 2.1. An MFS is a variety X together with a morphism $\varphi\colon X \to S$ such that

(i) X is $\mathbb{Q}$-factorial and has at worst terminal singularities;

(ii) $-K_X$, the anti-canonical class of X, is φ-ample; and

(iii) X/S has relative Picard number 1.

It is natural to not differentiate between two MFSs if they have the same structure up to a fibre-wise transform. Formally speaking this leads to the following definition.

Definition 2.2. Let $\varphi\colon X \to S$ and $\varphi'\colon X' \to S'$ be MFSs such that there is a birational map $f\colon X \dashrightarrow X'$. The map f is said to be *square* if there is a birational map $g\colon S \dashrightarrow S'$ that makes the diagram

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ S & \overset{g}{\dashrightarrow} & S' \end{array}$$

commute and, in addition, the induced birational map $f_L\colon X_L \to X'_L$ between the generic fibres is biregular. In this situation, we say that the two MFSs $X \to S$ and $X' \to S'$ are *birational square*, and denote this by $(X/S) \sim (X'/S')$.

MFSs in dimension 3 form three classes, decided by the dimension of S:

(1) Fano varieties, when $\dim S = 0$;

(2) del Pezzo fibrations, when $\dim S = 1$; and

(3) conic bundles, when $\dim S = 2$.

2.1. The graded ring approach

Concerning the comparatively less studied Fano case, there are two natural problems to tackle: construct all possible Fano 3-folds (classification), and find the relations between these models. The classical approach to the classification of Fano 3-folds can be found in [**23**]. The modern approach, however, is to view these objects as varieties embedded in weighted projective spaces via study of the graded ring $R(X, -K_X)$. There are 95 families of Fano 3-folds embedded in a weighted projective space as hypersurfaces (see [**21**]). Similarly, there are 85 families in codimension 2, 70 candidates in codimension 3 and 145 candidates in codimension 4; see [**3**, **4**, **7**, **8**, **10**] for explicit construction and description of the models, or [**6**] for the database. The next step, as we discussed, is to study birational relations between these models. As a generalization of the work of Iskovskikh and Manin [**22**], it was shown by Corti *et al*. [**16**] that a general member in the 95 families has pliability 1. This was recently generalized for quasi-smooth models by Cheltsov and Park [**12**]. The codimension 2 case was recently studied by Okada [**27**, **28**], and it was shown in [**9**] that codimension 3 models have pliability bigger than 1. Our model of study can also be considered as a first step in the analysis of the relations in the pliability set of codimension 3 Fano 3-folds.

2.2. The 2-ray game on a Fano 3-fold

We start with a Fano 3-fold X with $\mathrm{Pic}(X) = \mathbb{Z}$. According to [**14**, § 2.2], a 2-ray game, as the building block of an ESL, starts by blowing up a centre (a point or a curve C in X) $(Y, E) \to (X, C)$ such that Y is still $\mathbb{Q}$-factorial and terminal. By assumption, $\mathrm{Pic}(Y) = \mathbb{Z}^2$, and hence $\overline{\mathrm{NE}}(Y)$ is a convex cone in $\mathbb{Q}^2$. Therefore, there are at most two extremal projective morphisms from Y corresponding to the two boundaries of $\overline{\mathrm{NE}}(Y)$, and we know one of them! If the other map exists with positive-dimensional fibres and it does not contract a codimension 2 locus, then it is either a divisorial contraction or a fibration, and the game stops. If the contracted locus is one dimensional, we check whether the flip exists, and if so we replace Y by the new variety Y_1, which has Picard number 2. One boundary of $\overline{\mathrm{NE}}(Y_1)$ corresponds to the map that goes back to the base of the flip and we seek the other boundary and continue the game. If the game terminates and all flips and divisorial contractions and fibrations are in the Mori category (cf. [**26**]), then we have an ESL. The generalization for MFSs is natural and we refer the reader to [**14**] for this.

Throughout this paper we play the 2-ray game by means of Cox rings, as in [**9**] and [**2**].

3. The initial model and its singularities

The variety under consideration is a 3-fold X embedded in the weighted projective space

$$\mathbb{P}(1, 1, 1, 1, 2, 2, 3);$$

for brevity we denote this weighted projective space by $\mathbb{P}$. Let the coordinates of $\mathbb{P}$ be x, x_1, x_2, x_3, y, y_1, z. The 3-fold X is defined by the vanishing of the Pfaffians of a 5×5

skew-symmetric matrix with upper triangle block given by

$$\begin{pmatrix} y & A_3 & y_1 + C_2 & -x_1 \\ & B_3 & D_2 & x \\ & & z & -y_1 \\ & & & x_3 \end{pmatrix},$$

where A and B are general cubic forms, and C and D are general quadratic forms in variables x, x_1, x_2, x_3; see [**3**] for general construction. In other words, X is the vanishing of

$$\mathrm{Pf}_{1234}: \quad yz = AD - (y_1 + C)B, \tag{3.1 a}$$

$$\mathrm{Pf}_{1235}: \quad yy_1 = -xA - x_1 B, \tag{3.1 b}$$

$$\mathrm{Pf}_{1245}: \quad yx_3 = x(y_1 + C) + x_1 D, \tag{3.1 c}$$

$$\mathrm{Pf}_{1345}: \quad x_3 A = -y_1(y_1 + C) + x_1 z, \tag{3.1 d}$$

$$\mathrm{Pf}_{2345}: \quad x_3 B = -y_1 D - xz. \tag{3.1 e}$$

3.1. Singular locus of the 3-fold

Note that A, B, C, D are general so that X is quasi-smooth. In particular, the singular locus of X is $\mathrm{Sing}(X) = \{p_y, p_z\}$, where, for example, p_y is the point $(0:0:0:0:1:0:0) \in X \subset \mathbb{P}$. It is a standard verification, using (3.1), to see that the germ $p_y \in X$ is isomorphic to the terminal singularity of type $\frac{1}{2}(1,1,1)$. Here we explain how this isomorphism is obtained. This notation will be repeatedly used throughout the paper without further explanation.

Define the Zariski open subset $U_y \subset \mathbb{P}$ as the complement of the locus $(y = 0)$. This subset is defined by

$$U_y = \mathrm{Spec}\left[x, x_1, x_2, x_3, y, y_1, z, \frac{1}{y}\right]^{\mathbb{C}^*},$$

which is isomorphic to the quotient space

$$\mathrm{Spec}[x, x_1, x_2, x_3, y_1, z]/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts, on coordinates, by

$$\varepsilon \cdot (x, x_1, x_2, x_3, y_1, z) \mapsto (\varepsilon x, \varepsilon x_1, \varepsilon x_2, \varepsilon x_3, y_1, \varepsilon z),$$

where ε is a second root of unity. In other words, $U_y \cong \mathbb{C}^6/\mathbb{Z}_2$; a typical case of a quotient singularity.

Note that the point p_y can be viewed as the origin in this quotient. The singular locus of this quotient is a line passing through the origin. We use the notation $\frac{1}{2}(1,1,1,1,0,1)$ for this point, on the 6-fold. There are three tangent variables near this point, z, y_1 and x_3, by (3.1 a), (3.1 b) and (3.1 c). Hence, the germ $p_y \in X$ is isomorphic to the origin in the quotient space $\frac{1}{2}(1,1,1)$. In particular, p_y is an isolated singularity on the 3-fold. Similarly, one can check that p_z is of type $\frac{1}{3}(1,1,2)$, and X has no other singularities.

In §§ 4.1 and 4.2 we construct two birational maps, starting by Sarkisov links initiated by blowing up these two points. Recall that the *index* of a terminal point of type $1/n(a,b,c)$ is defined to be the number n.

At the heart of our calculations lies the theorem of Kawamata that asserts which divisorial contractions are centred at quotient singularities.

Theorem 3.1 (Kawamata [25]). *Let $(p \in X) \cong 1/r(a, r-a, 1)$ be the germ of a 3-fold terminal quotient singularity. In particular, a and r are coprime and $r \geqslant 2$. Suppose that $\varphi\colon (E \subset Y) \to (\Gamma \subset X)$ is a divisorial contraction such that Y is terminal and $p \in \Gamma$. Then $\Gamma = \{p\}$ and φ is the weighted blow up with weights $(a, r-a, 1)$.*

We refer to such an operation as a 'Kawamata blow up'.

3.2. Building Sarkisov links

The variety X is embedded in $\mathbb{P}$. We aim to find a toric variety with a divisorial contraction to $\mathbb{P}$ such that the restriction of this map to the birational transform of X is a divisorial contraction to a point X and is (locally) the Kawamata blow up we are after. The toric variety will have rank 2, the rank of its Picard group. Then we run the 2-ray game on it, following [**1**] and [**9**]. Next we check whether this 2-ray game restricts to a 2-ray game on the 3-fold under study, and if so, we check to see if it is a Sarkisov link. In similar situations, we blow up other MFS 3-folds embedded in weighted projective spaces or rank 2 toric varieties, and follow these instructions. When the rank of the ambient toric variety is 2 we use techniques of [**2**] in order to realize the rank 3 toric variety after the blow up and the 2-ray game played on it.

3.3. $\mathbb{Q}$-factoriality and Picard number

The fact that $\operatorname{rank}\operatorname{Pic}(X) = 1$ is an immediate consequence of the Lefschetz hyperplane section theorem. For factoriality of X, as $\operatorname{Pic}(X) \cong \mathrm{H}^2(X,\mathbb{Z})$ and $\operatorname{Cl}(X) \cong \mathrm{H}_4(X,\mathbb{Z})$, it would be enough to show that $\mathrm{H}^2(X,\mathbb{Q}) \cong \mathrm{H}_4(X,\mathbb{Q})$. But this follows from Poincaré duality for orbifolds over the rationals as X is quasi-smooth.

4. The birational maps

4.1. Blowing up the index 2 point

The fan of $\mathbb{P}$, as a toric variety, consists of seven one-dimensional rays $\{\rho_i\}$ in $\mathbb{Z}^6$, forming a complete fan with six (six-dimensional) cones $\sigma_i = \langle \rho_1, \dots, \widehat{\rho_i}, \dots, \rho_7 \rangle$, with a single relation between the rays

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 + 2\rho_5 + 2\rho_6 + 3\rho_7 = 0,$$

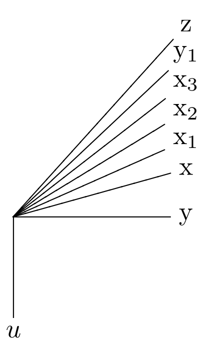
where the coefficients are indicated by the weights that define $\mathbb{P}$. Adding a new ray and performing the consequent subdivision results in a blow up of this variety. We aim to blow up the point p_y, and hence the new ray should be in the cone σ_5 (see [**19**, § 2.6]). Let the new ray be ρ_0 and let the blow up variety be denoted by $\mathfrak{X}$. By the earlier discussion, some multiple of ρ_0 can be written as the positive sum of the rays other than ρ_5. Now we explain how to decide the coefficients in this relation. The Cox ring (set of relations) of $\mathfrak{X}$ is the polynomial ring with eight variables u, y, x, x_1, x_2, x_3, y_1, x associated with the matrix

$$\begin{pmatrix} \rho_0 & \rho_5 & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_6 & \rho_7 \\ 0 & 2 & 1 & 1 & 1 & 1 & 2 & 3 \\ -\omega & 0 & \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_6 & \omega_7 \end{pmatrix},$$

where ω and the ω_i are positive integers to be determined.

Each column of this matrix indicates a ray in the fan, or represents a variable in the Cox ring. The numerical rows of the matrix represent the relations between the rays, or, equivalently, the numerical columns represent the bi-degree of the variables in the homogenous coordinate ring (Cox ring) of $\mathfrak{X}$. See [**17**] for the basic theory and explanation.

Let the new variable, associated with the ray ρ, be u. The geometric invariant theory (GIT) chambers of this toric variety has the following shape:



Note that we do not claim that x, x_1, x_2, x_3, y_1, z are in that order. What is clear is that they all fall in that side of y in comparison with u, because ω and the ω_is are all strictly positive. The blow up $\varphi\colon \mathfrak{X} \to \mathbb{P}$ is equivalent to taking the birational map defined by the linear system $|\mathcal{O}(1,0)|$; in other words,

$$(u, \mathrm{y}, \mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{z}) \in \mathfrak{X} \\ \mapsto (u^{\omega_1/\omega}\mathrm{x}, u^{\omega_2/\omega}\mathrm{x}_1, u^{\omega_3/\omega}\mathrm{x}_2, u^{\omega_4/\omega}\mathrm{x}_3, \mathrm{y}, u^{\omega_6/\omega}\mathrm{y}_1, u^{\omega_7/\omega}\mathrm{z}) \in \mathbb{P}.$$

In order to determine the values of ω and ω_i we use Theorem 3.1. As mentioned before, this map in a local neighbourhood of the point p_y on X is sought to be $u^{1/2}\mathrm{x}$, $u^{1/2}\mathrm{x}_1$, $u^{1/2}\mathrm{x}_2$. Hence, $\omega = 2$ and $\omega_1 = \omega_2 = \omega_3 = 1$. On the other hand, replacing these in (3.1), we aim to cancel the highest possible power of u in each equation. This indicates that $\omega_4 = 3$, $\omega_6 = 4$ and $\omega_7 = 5$. Note that this changes the Cox ring of $\mathfrak{X}$ to

$$\begin{pmatrix} u & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{z} & \mathrm{y}_1 & \mathrm{x}_3 \\ 0 & 2 & 1 & 1 & 1 & 3 & 2 & 1 \\ -2 & 0 & 1 & 1 & 1 & 5 & 4 & 3 \end{pmatrix}.$$

https://doi.org/10.1017/S0013091516000122 Published online by Cambridge University Press

However, this matrix defines a stacky fan and not a toric fan (see [**2**,**5**,**18**]). However, the suitable toric variety can be obtained by well-forming this matrix (see [**2**]) by subtracting the first row from the second and then dividing by 2, which essentially removes a factor of 2 from the determinant of all 2×2 minors of the matrix above. Hence, the Cox ring of the toric variety $\mathfrak{X}$ is

$$\begin{pmatrix} u & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{z} & \mathrm{y}_1 & \mathrm{x}_3 \\ 0 & 2 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

with irrelevant ideal $I = (u, \mathrm{y}) \cap (\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, z, \mathrm{y}_1, \mathrm{x}_3)$. Note that the map $\mathfrak{X} \to \mathbb{P}$ has not changed.

The birational transform of X under this map defines a 3-fold X_1' that is the Kawamata blow up of X at the point p_y. It is a codimension 3 subvariety of $\mathfrak{X}$ defined by the vanishing of the five equations

$$\mathrm{Pf}_{1234}: \quad \mathrm{yz} = AD - (u\mathrm{y}_1 + C)B, \tag{4.1 a}$$

$$\mathrm{Pf}_{1235}: \quad \mathrm{yy}_1 = -\mathrm{x}A - \mathrm{x}_1 B, \tag{4.1 b}$$

$$\mathrm{Pf}_{1245}: \quad \mathrm{yx}_3 = \mathrm{x}(u\mathrm{y}_1 + C) + \mathrm{x}_1 D, \tag{4.1 c}$$

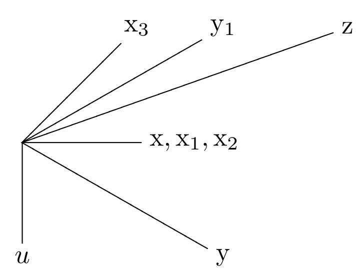
$$\mathrm{Pf}_{1345}: \quad \mathrm{x}_3 A = -\mathrm{y}_1(u\mathrm{y}_1 + C) + \mathrm{x}_1\mathrm{z}, \tag{4.1 d}$$

$$\mathrm{Pf}_{2345}: \quad \mathrm{x}_3 B = -\mathrm{y}_1 D - \mathrm{xz}, \tag{4.1 e}$$

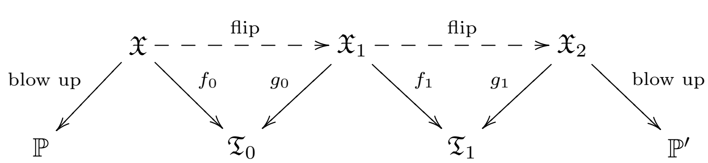
where A, B, C, D are the same as before with the following replacements:

$$x \mapsto \mathrm{x}, \qquad x_1 \mapsto \mathrm{x}_1, \qquad x_2 \mapsto \mathrm{x}_2, \qquad x_3 \mapsto u\mathrm{x}_3.$$

Running the 2-ray game on $\mathfrak{X}$ is essentially the variation of geometric invariant theory (vGIT) in the Mori chambers of $\mathfrak{X}$, given by



The 2-ray game follows the diagram



https://doi.org/10.1017/S0013091516000122 Published online by Cambridge University Press

The map f_0 contracts the locus $(\mathrm{z}=\mathrm{y}_1=\mathrm{x}_3=0)\subset\mathfrak{X}$, which is isomorphic to the three-dimensional scroll

$$\begin{pmatrix} u & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}_2 \\ 0 & 2 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix},$$

to the $\mathbb{P}^2_{\mathrm{x}:\mathrm{x}_1:\mathrm{x}_2}\subset\mathfrak{T}_0$. The map g_0, on the other hand, contracts the four-dimensional scroll defined by $(u=\mathrm{y}=0)\subset\mathfrak{X}_1$ to the same $\mathbb{P}^2$. Therefore, the birational map $\mathfrak{X}\dashrightarrow\mathfrak{X}_1$ is a $(-1,-1,1,1,1)$ flip above a $\mathbb{P}^2$. Similarly, the birational map $\mathfrak{X}_1\dashrightarrow\mathfrak{X}_2$ can be seen as the flip $(3,5,1,1,1,-1,-2)$ that is the contraction of a $\mathbb{P}(1,1,1,3,5)\subset\mathfrak{X}_1$ to a point in $\mathfrak{T}_1$ by f_1 and extraction of a $\mathbb{P}(1,2)\subset\mathfrak{X}_2$ from that point by g_1. The last map is simply the contraction of the divisor $(\mathrm{x}_3=0)\subset\mathfrak{X}_2$. It can be written explicitly via the linear system $|\mathcal{O}(2,1)|$, that is,

$$(u,\mathrm{y},\mathrm{x},\mathrm{x}_1,\mathrm{x}_2,\mathrm{z},\mathrm{y}_1,x_3)\in\mathfrak{X}_2\mapsto(\mathrm{y}_1,\mathrm{x}_3\mathrm{x},x_3\mathrm{x}_1,\mathrm{x}_3\mathrm{x}_2,x_3^2u,\mathrm{x}_3\mathrm{z},x_3^4\mathrm{y})\in\mathbb{P}(1,1,1,1,1,2,3).$$

Theorem 4.1. *The 2-ray game on $\mathfrak{X}$ restricts to a 2-ray game on X_1'. In particular, X is birational to Y, a complete intersection of two cubics in $\mathbb{P}(1,1,1,1,1,2)$, and the birational map between them consists of 11 flops followed by a $(3,1,-1,-2)$ flip.*

Proof. Substituting $u=\mathrm{y}=\mathrm{z}=\mathrm{y}-1=\mathrm{x}_3=0$ in (4.1) and then solving in $\mathbb{P}^2_{\mathrm{x}:\mathrm{x}_1:\mathrm{x}_2}$, we obtain 11 points, by the Hilbert–Burch theorem. Note that setting $\mathrm{z}=\mathrm{y}-1=\mathrm{x}_3=0$ gives the same equations, and hence f_0 restricted to Y contracts 11 lines to 11 points (on $\mathbb{P}^2$). On the other hand, near any of these points the variable z is a tangent, by (4.1 d) and (4.1 e). Therefore, this variable can be eliminated in a neighbourhood of any point in $\mathfrak{X}_1$ that maps to one of those 11 points. Hence, g_0 restricted to the 3-fold (defined by (4.1)) is the contraction of a $\mathbb{P}^1_{\mathrm{y}_1:\mathrm{x}_3}$. Therefore, the first toric flip restricts to 11 flops $(1,1,-1,-1)$ on X_1' to a 3-fold X_2'. At the next step, the tangency of variables y, x and x_1 near $p_z\in\mathfrak{T}_1$ (the image of contractions from f_1 and g_1) allows one to eliminate these variables locally, and hence the toric flip restricts to a 3-fold flip of type $(3,1,-1,-2)$ to a 3-fold X_3'. In particular, the singular point p_{uz} of type $\frac{1}{3}(1,1,2)$ is replaced by the $\frac{1}{2}(1,1,1)$ point $p_{\mathrm{x}_3\mathrm{z}}$ after the flip. The last step of the game contracts the divisor $(\mathrm{x}_3=0)$ to the point $p_{\mathrm{y}_1}\in\mathbb{P}'$. Restricting this map to (4.1) shows that the variable y can be globally eliminated on this 3-fold. After this elimination one can see that the transforms of (4.1 a) and (4.1 b) are in the ideal generated by (4.1 d) and (4.1 e).

Hence, we have a divisorial contraction from X_3' to a 3-fold Y defined by the vanishing of

$$A+y_1(uy_1+C)-x_1z=B+xz+y_1D=0$$

in the weighted projective space $\mathbb{P}(1,1,1,1,1,2)$, with variables u, x, x_1, x_2, y_1, z, where A and B are general cubics and C and D are general quadrics in variables u, x_1, x_2. In particular, Y has two singular points: p_z of quotient type $\frac{1}{2}(1,1,1)$, and p_{y_1}, a cA_1 with local analytic isomorphism

$$(p_{y_1}\in Y)\cong(0\in(\mathrm{xz}+\mathrm{x}_1^2+\mathrm{x}_2^2=0)\subset\mathbb{C}^4).$$

□

https://doi.org/10.1017/S0013091516000122 Published online by Cambridge University Press

4.2. Blowing up the index 3 point

In this section we perform a similar construction for blowing up the point p_z, which has singularity of type $\frac{1}{3}(1,1,2)$. Recall that X is defined by the vanishing of

$$yz = AD - (y_1 + C)B, \tag{4.2a}$$
$$x_1 z = x_3 A + y_1(y_1 + C), \tag{4.2b}$$
$$xz = -y_1 D - x_3 B, \tag{4.2c}$$
$$yy_1 = -xA - x_1 B, \tag{4.2d}$$
$$yx_3 = x(y_1 + C) + x_1 D. \tag{4.2e}$$

Similarly to the previous section for the Cox ring of the 6-fold $\mathfrak{X}'$, the toric blow up is

$$\begin{pmatrix} w & \mathrm{z} & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 \\ 0 & 3 & 1 & 1 & 2 & 2 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The 3-fold X_1'', the Kawamata blow up of X at the point p_z, is defined by

$$\mathrm{yz} = AD - (\mathrm{y}_1 + C)B, \tag{4.3a}$$
$$\mathrm{x}_1\mathrm{z} = \mathrm{x}_3 A + \mathrm{y}_1(\mathrm{y}_1 + C), \tag{4.3b}$$
$$\mathrm{xz} = -\mathrm{y}_1 D - \mathrm{x}_3 B, \tag{4.3c}$$
$$\mathrm{yy}_1 = -\mathrm{x}A - \mathrm{x}_1 B, \tag{4.3d}$$
$$\mathrm{yx}_3 = \mathrm{x}(\mathrm{y}_1 + C) + \mathrm{x}_1 D, \tag{4.3e}$$

where A, B, C, D are as before with replacements

$$x \mapsto w\mathrm{x}, \qquad x_1 \mapsto w\mathrm{x}_1, \qquad x_2 \mapsto \mathrm{x}_2, \qquad x_3 \mapsto \mathrm{x}_3.$$

The first step of the ambient 2-ray game restricts to 7 flops, and maps to X_2'', and is followed at the next step by a Francia flip $(2,1,-1,-1)$, mapping to $\tilde{Z}$. At the end, we have a fibration over $\mathbb{P}^1_{\mathrm{x}:\mathrm{x}_1}$ from $\tilde{Z}$. Using (4.3 b) and (4.3 c) we can eliminate the variable z above all fibres, and therefore the new model is the complete intersection of

$$\mathrm{yy}_1 = -\mathrm{x}A - \mathrm{x}_1 B \quad \text{and} \quad \mathrm{yx}_3 = \mathrm{x}(\mathrm{y}_1 + C) + \mathrm{x}_1 D$$

as two hypersurfaces in the toric variety $\mathfrak{X}''$,

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & -1 & -1 & -2 & 0 & 1 & 1 \end{pmatrix}.$$

Note that this matrix is obtained by removing the variable z and re-scaling using $\mathrm{GL}(2,\mathbb{Z})$.

Each fibre is a complete intersection of a quadric with a cubic in $\mathbb{P}(1,1,1,1,2)$; in particular, the generic fibre is a cubic surface. Furthermore, this model has only a singular point of type $\frac{1}{2}(1,1,1)$, at the point $p_{y_1 x_1}$.

The next theorem follows from the construction above.

https://doi.org/10.1017/S0013091516000122 Published online by Cambridge University Press

Theorem 4.2. *The 2-ray game on $\mathfrak{X}'$ restricts to a 2-rag game on X''. In particular, X is birational to $\tilde{Z} \subset \mathfrak{X}''$, a fibration over $\mathbb{P}^1$ with fibres being the complete intersection of a quadric and a cubic in $\mathbb{P}(1,1,1,1,2)$, and the birational map between them consists of seven flops followed by a $(2,1,-1,-1)$ Francia flip.*

5. The relation among the models

In this section we obtain two other models, both fibred over $\mathbb{P}^1$ with cubic surface fibres, from Y and $\tilde{Z}$ and show that they are isomorphic. It turns out that these models are square birational to $\tilde{Z}/\mathbb{P}^1$.

5.1. Strict Mori fibrations and square birationality

Let us first construct a cubic surface fibration by an ESL from Y.

Lemma 5.1. *The blow up of the $\frac{1}{2}(1,1,1)$ singular point in $Y \subset \mathbb{P}(1,1,1,1,1,2)$ has a Sarkisov link of type I to a cubic surface fibration over $\mathbb{P}^1$.*

Proof. Consider the toric variety with Cox ring

$$\begin{pmatrix} v & \mathrm{z} & u & \mathrm{x}_2 & \mathrm{y}_1 & \mathrm{x} & \mathrm{x}_1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

with irrelevant ideal $I = (v, \mathrm{z}) \cap (u, \mathrm{x}_2, \mathrm{y}_1, \mathrm{x}, \mathrm{x}_1)$.

Similar to constructions in the previous section, it follows that the 3-fold Y_1' defined by the vanishing of

$$A + y_1(uy_1 + C) - x_1 z = B + xz + y_1 D = 0$$

in this toric variety is the Kawamata blow up of Y at the point p_z, where A, B, C, D are two cubics and two quadrics in $v\mathrm{x}$, $v\mathrm{x}_1$, x_2, u. Furthermore, the 2-ray game of the ambient space restricts to a 2-ray game on Y_1', which consists of nine flops followed by a fibration to $\mathbb{P}^1_{\mathrm{x}:\mathrm{x}_1}$. Note that above every point in the base of this fibration the variable z can be eliminated using the ratio

$$\mathrm{z} = \frac{A + \mathrm{y}_1(u\mathrm{y}_1 + C)}{-\mathrm{x}} = \frac{B + \mathrm{y}_1 D}{\mathrm{x}_1}.$$

Hence, the new variety Z can be viewed as the hypersurface defined by

$$\mathrm{x}_1(A + \mathrm{y}_1(u\mathrm{y}_1 + C)) + \mathrm{x}(B + \mathrm{y}_1 D) = 0$$

in a toric variety with Cox ring

$$\begin{pmatrix} v & u & \mathrm{x}_2 & \mathrm{y}_1 & \mathrm{x} & \mathrm{x}_1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and irrelevant ideal $I = (\mathrm{x}, \mathrm{x}_1) \cap (v, u, \mathrm{x}_2, \mathrm{y}_1)$. The 3-fold Z is a fibration of cubic surfaces over $\mathbb{P}^1_{\mathrm{x}:\mathrm{x}_1}$ with a singular point $p_{\mathrm{y}_1\mathrm{x}}$, which is a cA_1 singularity. □

https://doi.org/10.1017/S0013091516000122 Published online by Cambridge University Press

Lemma 5.2. *The blow up of the $\frac{1}{2}(1,1,1)$ singular point in $\tilde{Z}$ has a Sarkisov link of type II to another cubic surface fibration over $\mathbb{P}^1$. Furthermore, these two models are square birational.*

Proof. Let us recall that $\tilde{Z}$ is the complete intersection of

$$\mathrm{yy}_1 = -\mathrm{x}A - \mathrm{x}_1 B \quad \text{and} \quad \mathrm{yx}_3 = \mathrm{x}(\mathrm{y}_1 + C) + \mathrm{x}_1 D$$

in the toric variety

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & -1 & -1 & -2 & 0 & 1 & 1 \end{pmatrix}$$

with irrelevant ideal $I = (\mathrm{x}, \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y})$. It has a singular point of type $\frac{1}{2}(1,1,1)$ at $p_{\mathrm{x}_1\mathrm{y}_1}$. We aim to blow up this point. Using the techniques introduced in [**2**], we consider the toric variety T of rank 3 with Cox ring

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 3 & 2 & 0 & -2 \end{pmatrix}$$

with irrelevant ideal $J = (\mathrm{x}, \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y}) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}, \mathrm{x}) \cap (\mathrm{x}', \mathrm{x}_1) \cap (\mathrm{x}', \mathrm{y}_1)$. To recover the map (the blow up) $T \to \mathfrak{X}''$ from the Cox ring (i.e. a map from rank 3 to rank 2) we only need to write down the graded ring orthogonal to the exceptional divisor $(x' = 0)$, that is, with respect to the GIT (Mori) chambers of T, the morphism that maps coordinates of T to monomials that span the linear subspace $\langle e_1, e_2 \rangle = \langle (1,0,0), (0,1,0) \rangle = \{(a,b,0) \mid (a,b) \neq 0\} \subset \mathbb{Z}^3$. In other words, this map is

$$(w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y}, \mathrm{x}, \mathrm{x}_1, \mathrm{x}') \mapsto (w{\mathrm{x}'}^{1/2}, \mathrm{x}_2{\mathrm{x}'}^{1/2}, \mathrm{x}_3{\mathrm{x}'}^{1/2}, \mathrm{y}_1, \mathrm{y}{\mathrm{x}'}^{3/2}, \mathrm{xx}', \mathrm{x}_1),$$

which is, when restricted to $\tilde{Z}$, the Kawamata blow up of the singular point. However, note that the well-formed model of the toric ambient space is

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \end{pmatrix}.$$

We aim to play the 2-ray game on this toric variety over the base $\mathbb{P}^1_{\mathrm{x}:\mathrm{x}_1}$, that is, the base of the Mori fibration. The game corresponds to $\mathrm{Pic}(T/\mathbb{P}^1)$. In order to visualize this we need to get rid of the contribution of the base ($\mathbb{P}^1$ here) in the Picard group. As T is obtained by blowing up a point in the fibre above the point $p_{x_1} = (0:1) \in \mathbb{P}^1_{x:x1}$, we can consider the open subset $(x_1 \neq 0) \subset \mathbb{P}^1$ to realize the relative Picard group and play the relative 2-ray game. That is to say, fixing the action of $\mathbb{C}^*$ (with respect to the last row of the matrix above) we have a 2×7 matrix that represents the relative Picard group and the corresponding grading, namely,

$$\begin{pmatrix} \mathrm{x} & \mathrm{y} & w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{x}' \\ 0 & 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

with the ideal $J' = (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}, \mathrm{x}) \cap (\mathrm{x}', \mathrm{y}_1)$. The 2-ray game of this toric variety, over the base, restricts to six flops, corresponding to the six-point solutions of the general quadric D and general cubic B in $\mathbb{P}^2_{w\,:\,\mathrm{x}_2\,:\,\mathrm{x}_3}$. The next step is a divisorial contraction. It is more useful to see this contraction on the global model, i.e. the rank 3 variety, preserving the base $\mathbb{P}^1_{x\,:\,x_1}$. Note that after flops the toric variety has the same Cox ring with irrelevant ideal $J = (\mathrm{x}, \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y}) \cap (\mathrm{y}, \mathrm{x}) \cap (\mathrm{x}', \mathrm{x}_1) \cap (\mathrm{x}', \mathrm{y}_1, w, \mathrm{x}_2, \mathrm{x}_3)$. Rewrite the matrix, using some $\mathrm{GL}(3, \mathbb{Z})$ action, in the form

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & -3 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Note that the map corresponding to the wall $\langle e_1, e_2 \rangle$ in the GIT cone is a morphism that contracts the divisor $(\mathrm{x} = 0)$ to the point $p_{\mathrm{x}'\mathrm{y}}$ in the toric variety with Cox ring

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

with irrelevant ideal $I' = (\mathrm{x}', \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y})$. The blow up (contraction) map is

$$(w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y}, \mathrm{x}, \mathrm{x}_1, \mathrm{x}') \mapsto (w\mathrm{x}, \mathrm{x}_2\mathrm{x}, \mathrm{x}_3\mathrm{x}, \mathrm{y}_1\mathrm{x}^3, \mathrm{y}, \mathrm{x}_1, \mathrm{x}'\mathrm{x})$$

Carrying all these maps on the equation of $\tilde{Z}$ we end up with a new 3-fold $\bar{Z}$, defined as the complete intersection of

$$\mathrm{y}\mathrm{y}_1 = -\mathrm{x}'A - \mathrm{x}_1 B \quad \text{and} \quad \mathrm{y}\mathrm{x}_3\mathrm{x}' = \mathrm{y}_1 + \mathrm{x}'C + \mathrm{x}_1 D$$

in the latter rank 2 toric variety. Clearly, the variable y_1 can be eliminated. Hence, $\bar{Z}$ is the hypersurface

$$\mathrm{y}^2\mathrm{x}_3\mathrm{x}' - \mathrm{x}'\mathrm{y}C - \mathrm{x}_1\mathrm{y}D = -\mathrm{x}'A - \mathrm{x}_1 B$$

in the toric variety with Cox ring

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and irrelevant ideal $\bar{I} = (\mathrm{x}', \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y})$. It is easy to see that $\bar{Z}$ is a cubic surface fibration over $\mathbb{P}^1_{\mathrm{x}'\,:\,\mathrm{x}_1}$ and has a cA_1 singularity at the point $p_{\mathrm{y}\mathrm{x}_1}$. □

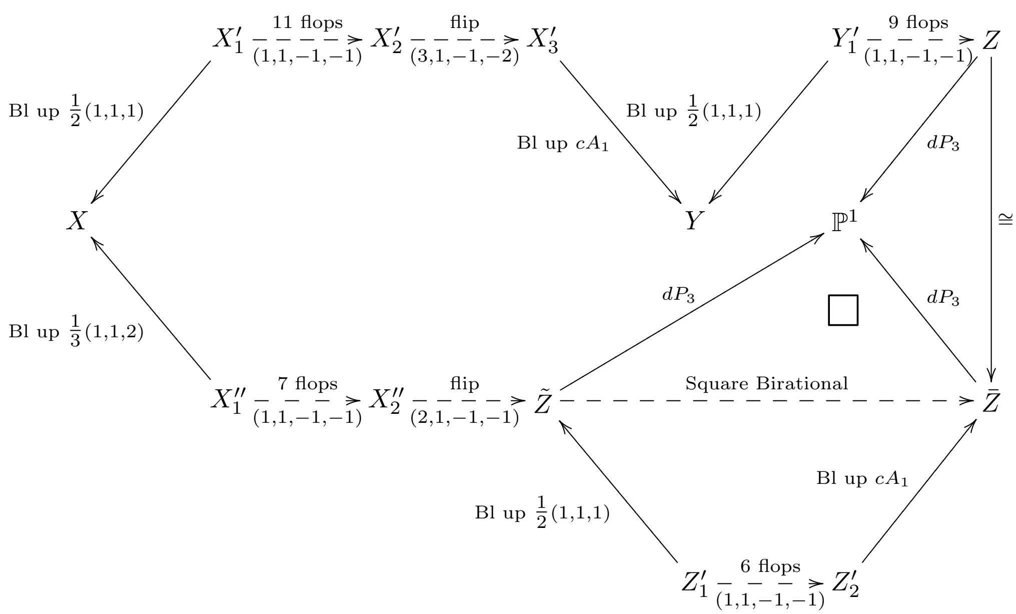
Proposition 5.3. *The two models obtained in Lemmas 5.1 and 5.2 are isomorphic.*

Proof. This is clear from Lemmas 5.1 and 5.2. □

5.2. Pliability of the models and the circle

It follows from Theorem 4.1, Lemmas 5.1 and 5.2, and Corollary 5.3 that the Fano 3-fold $X \subset \mathbb{P}(1,1,1,1,2,2,3)$ is birational to a Fano complete intersection $Y_{3,3} \subset \mathbb{P}(1,1,1,1,1,2)$ and a fibration of cubic surfaces over $\mathbb{P}^1$, up to square birational equivalence. Hence, the pliability of this 3-fold is at least 3.

The following diagram visualizes the relation between all models discussed in this paper:



Remark 5.4. The embedding of the model Z into a scroll indicates that perhaps the result of [**11**] can be used to predict irrationality of X. However, this result does not immediately apply to this situation as Z is not smooth. However, the very mild singularity of Z, which is an ordinary double point, suggests that the irrationality criterion of Cheltsov [**11**] could potentially be generalized to this class of varieties.

The computations that we carried out in this paper, together with our experience working with Fano 3-folds (and especially in codimension 3), suggest the following question, which relates the number of singular points on a general Fano 3-fold in codimension 3 to its pliability.

Question 5.5. *Let X be a Fano 3-fold embedded in a weighted projective space in codimension 3 and suppose that X is quasi-smooth. Let n be the number of different analytic types of singularities that appear in X. Is it true that if X is general, then $\mathcal{P}(X) = n + 1$?*

https://doi.org/10.1017/S0013091516000122 Published online by Cambridge University Press

Acknowledgements. H.A. was partly supported by the Austrian Science Fund (FWF) (Grant P22766-N18).

References

1. H. Ahmadinezhad, On del Pezzo fibrations that are not birationally rigid, *J. Lond. Math. Soc. (2)* **86**(1) (2012), 36–62.
2. H. Ahmadinezhad, On pliability of del Pezzo fibrations and Cox rings, *J. Reine Angew. Math.* (2014), DOI:10.1515/crelle-2014–0095.
3. S. Altinok, Constructing new $K3$ surfaces, *Turkish J. Math.* **29**(2) (2005), 175–192.
4. S. Altinok, G. Brown and M. Reid, Fano 3-folds, K3 surfaces and graded rings, in *Topology and geometry: commemorating SISTAG*, Contemporary Mathematics, Volume 314, pp. 25–61 (American Mathematical Society, Providence, RI, 2002).
5. L. A. Borisov, L. Chen and G. G. Smith, The orbifold Chow ring of toric Deligne–Mumford stacks, *J. Am. Math. Soc.* **18**(1) (2005), 193–215.
6. G. Brown, Graded ring database (available at www.grdb.co.uk).
7. G. Brown and K. Suzuki, Computing certain Fano 3-folds, *Jpn. J. Ind. Appl. Math.* **24**(3) (2007), 241–250.
8. G. Brown and K. Suzuki, Fano 3-folds with divisible anticanonical class, *Manuscr. Math.* **123**(1) (2007), 37–51.
9. G. Brown and F. Zucconi, Graded rings of rank 2 Sarkisov links, *Nagoya Math. J.* **197** (2010), 1–44.
10. G. Brown, M. Kerber and M. Reid, Fano 3-folds in codimension 4, Tom and Jerry, Part I, *Compositio Math.* **148**(4) (2012), 1171–1194.
11. I. Cheltsov, Nonrational del Pezzo fibrations, *Adv. Geom.* **8**(3) (2008), 441–450.
12. I. Cheltsov and J. Park, Birationally rigid Fano threefold hypersurfaces, *Mem. Am. Math. Soc.* (2014), in press.
13. A. Corti, Factoring birational maps of threefolds after Sarkisov, *J. Alg. Geom.* **4**(2) (1995), 223–254.
14. A. Corti, Singularities of linear systems and 3-fold birational geometry, in *Explicit birational geometry of 3-folds*, London Mathematical Society Lecture Notes Series, Volume 281, pp. 259–312 (Cambridge University Press, 2000).
15. A. Corti and M. Mella, Birational geometry of terminal quartic 3-folds, I, *Am. J. Math.* **126**(4) (2004), 739–761.
16. A. Corti, A. Pukhlikov and M. Reid, Fano 3-fold hypersurfaces, in *Explicit birational geometry of 3-folds*, London Mathematical Society Lecture Note Series, Volume 281, pp. 175–258 (Cambridge University Press, 2000).
17. D. A. Cox, The homogeneous coordinate ring of a toric variety, *J. Alg. Geom.* **4**(1) (1995), 17–50.
18. B. Fantechi, E. Mann and F. Nironi, Smooth toric Deligne–Mumford stacks, *J. Reine Angew. Math.* **648** (2010), 201–244.
19. W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, Volume 131 (Princeton University Press, 1993).
20. C. D. Hacon and J. McKernan, The Sarkisov program, *J. Alg. Geom.* **22**(2) (2013), 389–405.
21. A. R. Iano-Fletcher, Working with weighted complete intersections, in *Explicit birational geometry of 3-folds*, London Mathematical Society Lecture Note Series, Volume 281, pp. 101–173 (Cambridge University Press, 2000).
22. V. A. Iskovskih and Y. I. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem, *Math. USSR Sb.* **15** (1971), 141–166.

23. V. A. Iskovskikh and Yu. G. Prokhorov, *Algebraic geometry V: Fano varieties*, Encyclopaedia of Mathematical Sciences, Volume 47 (Springer, 1999).
24. A.-S. Kaloghiros, Relations in the Sarkisov program, *Compositio Math.* **149**(10) (2013), 1685–1709.
25. Y. Kawamata, Divisorial contractions to 3-dimensional terminal quotient singularities, in *Higher dimensional complex varieties*, De Gruyter Proceedings in Mathematics, pp. 241–246 (De Gruyter, Berlin, 1996).
26. J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, Volume 134 (Cambridge University Press, 1998).
27. T. Okada, Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections, *Proc. Lond. Math. Soc.* **109**(6) (2014), 1549–1600.
28. T. Okada, Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections, II, *J. Reine Angew. Math.* (2015), DOI:10.1515/crelle-2015-0054.