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Symmetry via the moving plane method for a class of quasilinear elliptic problems involving the Hardy potential

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We consider positive solutions to a class of quasilinear elliptic problems involving the Hardy potential under zero Dirichlet boundary condition. Via moving plane method, proving a weak comparison principle, we prove symmetry and monotonicity properties for the solutions defined on strictly convex symmetric domains.

Keywords: Quasilinear elliptic equations; Hardy potential; Supercritical problems; Simmetry and Monotonicity

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1. Introduction

The aim of this paper is to study qualitative properties such as symmetry and monotonicity of positive weak solutions to the quasilinear elliptic problems involving Hardy potential:

$$\begin{cases} -\mathrm{div}\,(A(|\nabla u|)\nabla u) = \vartheta \frac{u^q}{|x|^p} + g(x,u) & \text{in } \ \Omega, \\ u > 0 & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a C^2-bounded domain in $\mathbb{R}^N$, $0 \in \Omega$, $\vartheta \geqslant 0$, $p \in (1, N)$, $q \in (0, p)$.

The interest in this class of problems in recent years is related to the possibility of modelling wide classes of elliptic problems. For instance, taking into account the results contained in [4–6] we know that if the real function $A : \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be of class $C^1(\mathbb{R}^+)$ and fulfills the following

$$\bullet\ -1 < \inf_{t>0} \frac{tA'(t)}{A(t)} =: m_A \leqslant M_A := \sup_{t>0} \frac{tA'(t)}{A(t)} < +\infty, \tag{1.2}$$

we can consider problems in which the differential operator satisfies ellipticity and monotonicity conditions, not necessarily of power type. On the other hand, the choice $A(t) = t^{p-2} + bt^{q-2}$, $b \geqslant 0$ leads to the well-known classical semilinear elliptic problem [**14**, **15**, **17**] for $b = 0$, or to double-phase type problems (see [**8**–**12**, **19**, **24**]). Such problems are therefore particular cases contained in the more general setting that we consider here. We shall borrow some ideas from [**13**, **15**] but, contrary to the first feeling, the adaptation of such techniques is a quite delicate issue, once we want to deal with a real general class of operators. In the following we will also suppose that

- $\exists c_1 > 0$ and $\exists K \geqslant 1 : \forall \eta \in \mathbb{R}^N$ with $|\eta| \geqslant K,\ |A(|\eta|)\eta| \leqslant c_1|\eta|^{p-1}$; (1.3)
- $\exists c_2 > 0 : \forall \eta \in \mathbb{R}^N, A(|\eta|)|\eta|^2 \geqslant c_2|\eta|^p$; (1.4)
- If $\liminf_{t\to 0^+} A(t) = 0,\ \exists \delta > 0 : A(t)$ is non-decreasing on $I_0 := (0, \delta)$. (1.5)

We remark that hypothesis (1.3), that we prefer to state in this way, is equivalent to request that

$$\exists A, B \in \mathbb{R}^+ : \forall \eta \in \mathbb{R}^N, |A(|\eta|)\eta| \leqslant A + B|\eta|^{p-1}.$$

The position of the origin with respect to the domain is connected to the presence of the Hardy potential and it is crucial already to prove existence results (see [**7**] and the references therein). Regarding the nonlinear term $g \in C^1(\bar{\Omega} \times \mathbb{R})$ we will suppose that:

(g1) $g(x, \cdot\,)$ is a locally Lipschitz continuous, uniformly with respect to x, i.e. for every $\Omega' \subset \Omega$ and for every $M > 0$, there is a positive constant $L = (M, \Omega')$ such that for every $x \in \Omega'$ and every $u, v \in [0, M]$, we have that

$$|g(x, u) - g(x, v)| \leqslant L|u - v|;$$

(g2) $g(\cdot\,, u)$ is locally Lipschitz continuous, uniformly with respect to u.

(g3) $g(\cdot\,, u)$ is non-decreasing w.r.t. the x_1-direction in the set $\Omega_\lambda := \{x = (x_1, \ldots, x_N) \in \Omega : x_1 < \lambda \text{with } \lambda < 0\}$;

(g4) $g(x, \cdot\,) > 0$ is positive and, more precisely, $g(x, u) > 0$ in Ω' for every $\Omega' \subset \Omega$ and for every $u > 0$.

The monotonicity assumption on g, with respect to the first variable, is necessary for the applicability of the moving plane method; this is well-known already in the case of non singular source terms. We refer to [**3**, **25**, **26**] for a discussion about such a condition. In this setting the notion of solution has to be understood in the weak sense as in the next definition

Definition 1.1. *We say that u is a weak solution to problem (1.1) if $u \in W_0^{1,p}(\Omega)$, $g(\cdot, u(\cdot)) \in L^1(\Omega)$ and*

$$\int_\Omega A(|\nabla u|)(\nabla u, \nabla \varphi)\,\mathrm{d}x = \vartheta \int_\Omega \frac{u^q}{|x|^p}\varphi\,\mathrm{d}x + \int_\Omega g(x, u)\varphi\,\mathrm{d}x, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

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Remark 1.2. We do not prefer to discuss here the wide literature regarding the L^∞-boundedness of the solutions, that is a well-known issue e.g. in the critical or sub-critical case. Once we reduce to deal with bounded far from the origin $W_0^{1,p}(\Omega)$ solutions, by [16, 29], we get that $u \in C^{1,\alpha}_{loc}(\Omega\backslash\{0\})$; furthermore, supposing that Ω is smooth, the $C^{1,\alpha}_{loc}(\bar{\Omega}\backslash\{0\})$ regularity follows by [23] while, from [22], it also follows that $u \in C^2(\Omega\backslash(Z_u \cup \{0\}))$, where

$$Z_u = \{x \in \Omega : \nabla u(x) = 0\}.$$

Moreover, arguing as in [7], we have that $u_i \in W^{1,2}_{loc}(\Omega\backslash\{0\})$ for $p \in (1, 3)$ and $u_i \in W^{1,q}_{loc}(\Omega\backslash\{0\})$ for every $q < \frac{p-1}{p-2}$ and $p \geqslant 3$.

Our result will be obtained by means of the moving plane method; this technique is mostly used in this topic and it goes back to the seminal papers of Alexandrov [1] and Serrin [28] and the celebrated papers [2, 21].

This tool was adapted to the case of the p-Laplacian operator in *bounded* domains firstly in [14] for the case $1 < p < 2$ and, later on, in [15] for the case of positive nonlinearities and $p > 2$. In this paper we will apply the moving plane technique by means of the achievement of a weak comparison principle *in small domains* and the strong comparison principle; this approach is more close to those introduced in [14, 15] and goes back to the illuminating papers [2, 21].

Such a technique can be performed in general convex domains providing partial monotonicity results near the boundary. For simplicity of exposition and without loss of generality, we assume directly in all the paper that Ω is a strictly convex domain in the x_1-direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. In this setting, our main result is the following:

Theorem 1.3. *Let $u \in C^1(\bar{\Omega}\backslash\{0\})$ be a weak solution to (1.1). Let Ω be strictly convex with respect to x_1-direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Then it follows that u is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and non-decreasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$.*

Moreover, if Ω is a ball centred at the origin, then u is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $r \neq 0$.

To prove theorem (1.3) we have to face some difficulties, mainly related to the nonlinear degenerate nature of the operator. In particular, in order to carry on the procedure, we need to study the asymptotic behaviour of the solution near to zero. This task was also faced in [25]. However, the presence of the distorsion $A(\cdot)$, causes that we can not repeat the same argument of [25]. Furthermore, when proving the weak comparison principle in small domains, we have to overcome the difficulty arising from many homogeneity problems, that the reader will appreciate during the reading. Finally, note that, taking into account the literature regarding the existence of the solutions, we choose a setting of assumptions that involves also cases when the nonlinearity is not Lipschitz continuous at zero.

The paper is organized as follows: in next section we introduce some preliminary tools and results such as summability properties of the second derivatives of the solutions. Furthermore we prove that the solution u blow-up near the origin and we develop the proof of the weak comparison principle for small domains. The main result, namely theorem (1.3), is proved in the last section. Finally, for the readers convenience, we add the proofs of some standard results in the appendix.

2. Preliminaries and useful results

In this section, briefly, we enclose some definitions, results and remarks that it will be useful in the rest of the paper. From now on, in order to get a readable notation, generic numerical constant will be denoted by c and will be allowed to vary within a single line or formula. Moreover we denote with $f^+ := \max\{f, 0\}$.

REMARK 2.1. With respect to the setting (1.2)–(1.4), we observe that:

- From [**5**, lemma 4.2], calling $\bar{c} := (1 + \min\{0, m_A\}) > 0$, one has that

$$[A(|\xi|)\xi - A(|\eta|)\eta] \cdot [\xi - \eta] \geqslant \bar{c}|\xi - \eta|^2 \int_0^1 A(|\eta + s(\xi - \eta)|)\,\mathrm{d}s, \tag{2.1}$$

  for every $\xi, \eta \in \mathbb{R}^N$. Using (1.4), we get

$$[A(|\xi|)\xi - A(|\eta|)\eta] \cdot [\xi - \eta] \geqslant c_2\bar{c}|\xi - \eta|^2 \int_0^1 |\eta + s(\xi - \eta)|^{p-2}\,\mathrm{d}s$$

  and, arguing as in [**13**, lemma 2.1], one has that for every $\xi, \eta \in \mathbb{R}^N$

$$[A(|\xi|)\xi - A(|\eta|)\eta] \cdot [\xi - \eta] \geqslant c_2\bar{c}(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2. \tag{2.2}$$

- In [**5**, proposition 4.1], it is already proved that

$$A(1)\min\{t^{m_A}, t^{M_A}\} \leqslant A(t) \leqslant A(1)\max\{t^{m_A}, t^{M_A}\}. \tag{2.3}$$

  Since $m_A > -1$, by (2.3) there exists $\eta \in [0, 1)$ such that $m_A + \eta > 0$; hence

$$\lim_{t\to 0} t^\eta A(t) = 0 \tag{2.4}$$

  and, in particular $\lim_{t\to 0} tA(t) = 0$.

  Then, if $0 < t < K$ there exists a constant $C_K := C(K) > 0$ such that

$$|tA(t)| \leqslant C_K. \tag{2.5}$$

  Moreover, $j(t) := tA(t)$ is a non-decreasing function on $[0, +\infty)$ since

$$j'(t) = A(t) + tA'(t) = A(t)\left[1 + \frac{tA'(t)}{A(t)}\right] \geqslant A(t)(1 + m_A) \geqslant 0.$$

- If $m_A \geqslant 0$, by (1.2), one has

$$\frac{tA'(t)}{A(t)} \geqslant m_A \Rightarrow A'(t) \geqslant \frac{m_A A(t)}{t} \geqslant 0.$$

  Then,

$$\text{if } m_A \geqslant 0,\ A(t) \text{ is a non-decreasing function on } (0, +\infty). \tag{2.6}$$

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- For $t \in \mathbb{R}^+$ and $t \geqslant K$, using (1.3) and (2.3), we get

$$A(1)t^{m_A} \leqslant A(t) \leqslant c_1 t^{p-2} \Rightarrow t^{m_A-p+2} \leqslant \frac{c_1}{A(1)}.$$

Hence, we note that $(m_A - p + 2)$ has to be necessarily negative, i.e.

$$m_A \leqslant p-2. \tag{2.7}$$

Analogously, for $t \in \mathbb{R}^+$ and $t \geqslant K$, using (1.4) and (2.3), we get

$$c_2 t^{p-2} \leqslant A(t) \leqslant A(1)t^{M_A} \Rightarrow t^{p-2-M_A} \leqslant \frac{A(1)}{c_2}.$$

Hence, we note that $(p - 2 - M_A)$ has to be necessarily negative, i.e.

$$M_A \geqslant p-2. \tag{2.8}$$

The following theorems are devoted to obtain some summability properties of the second derivatives and the gradient of solutions to (1.1).

Theorem 2.2. *Assume that Ω is a bounded smooth domain and $1 < p < N$. Consider $u \in C^{1,\alpha}_{loc}(\bar\Omega\backslash\{0\}) \cap C^2(\Omega\backslash(Z_u \cup \{0\}))$ a solution to (1.1), where $g(\cdot, u) \in W^{1,\infty}(\Omega)$ and $g(x, \cdot) \in W^{1,\infty}(\mathbb{R})$. We have*

$$\int_E \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta}\,\mathrm{d}x \leqslant \mathcal{C} \quad \forall i = 1, \dots, N$$

for any $E \Subset \Omega\backslash\{0\}$ and uniformly for any $y \in E$, with

$$\mathcal{C} := \mathcal{C}(\gamma, m_A, M_A, \beta, g, \|\nabla u\|_\infty, E)$$

for $0 \leqslant \beta < 1$ and $\gamma < (N-2)$ if $N \geqslant 3$ ($\gamma = 0$ if $N = 2$).

Moreover, if we also assume that g is positive in Ω in the sense of (g4), we have that

$$\int_{\tilde\Omega} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta}\,\mathrm{d}x \leqslant \mathcal{C} \quad \forall i = 1, \dots, N.$$

where $\tilde\Omega$ is a compact such that $\tilde\Omega \subset \bar\Omega\backslash\{0\}$.

Remark 2.3. As showed in [7, remark 4.4], if g is a positive function, the set Z_u is such that $|Z_u| = 0$.

Theorem 2.4. *Let $u \in C^{1,\alpha}(\Omega\backslash\{0\}) \cap C^2(\Omega\backslash(Z_u \cup \{0\}))$ be a solution to (1.1) with $g(\cdot, u) \in W^{1,\infty}(\Omega)$ and $g(x, \cdot) \in W^{1,\infty}(\mathbb{R})$ and $g(x, u)$ positive in the sense of (g4)*

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in $B_{2\rho}(x_0) \subset \Omega\backslash\{0\}$*. Then*

$$\int_{B_\rho(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma}\,\mathrm{d}x \leqslant \mathcal{C}$$

for any $y \in B_\rho(x_0)$*, with* $\alpha := \frac{p-1}{p-2}$ *if* $p > 2$ *and* $\alpha := \frac{m_A+1}{m_A}$ *if* $p \in (1,2)$*,* $r \in (0,1)$*,* $\gamma < N-2$ *if* $N \geqslant 3$*,* $\gamma = 0$ *if* $N = 2$ *and*

$$\mathcal{C} = \mathcal{C}(\gamma, \eta, g, ||\nabla u||_\infty, \rho, x_0, \alpha, M_A, c_2).$$

If Ω *is a smooth domain and* g *is nonnegative in* Ω

$$\int_{\tilde\Omega} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma}\,\mathrm{d}x \leqslant \mathcal{C}.$$

where $\tilde\Omega$ *is a compact such that* $\tilde\Omega \subset \bar\Omega\backslash\{0\}$ *and* $y \in \tilde\Omega$*.*

Remark 2.5. Even if we consider a solution to the problem (1.1), the proof of theorems 2.2 and 2.4 repeats verbatim the arguments exploited in [**7**, theorems 4.2, 4.6, 4.8].

Assume that $\tilde\Omega \Subset \Omega\backslash\{0\}$. We recall the definition of weighted Sobolev space $H^{1,2}_\rho(\tilde\Omega)$.

Definition 2.6. *Let* $\rho, \rho^{-1} \in L^1(\Omega)$*. The space* $H^{1,2}_\rho(\tilde\Omega)$ *is defined as the completion of* $C^1(\tilde\Omega)$ *(or* $C^\infty(\tilde\Omega)$*) with the norm*

$$\|v\|_{H^{1,2}_\rho} = \|v\|_{L^2(\tilde\Omega)} + \|\nabla v\|_{L^2(\tilde\Omega,\rho)}\,, \tag{2.9}$$

where

$$\|\nabla v\|^2_{L^2(\tilde\Omega,\rho)} := \int_{\tilde\Omega} \rho(x)|\nabla v(x)|^2\,\mathrm{d}x\,.$$

*For a discussion on this topic we refer the reader to [**31**].*

We also recall that $H^{1,2}_{0,\rho}(\tilde\Omega)$ is defined as the completion of $C^1_c(\tilde\Omega)$ (or $C^\infty_c(\tilde\Omega)$) under the norm

$$\|v\|_{H^{1,2}_{0,\rho}(\tilde\Omega)} = \|\nabla v\|_{L^2(\tilde\Omega,\rho)}\,.$$

Theorem 2.7 Weighted Sobolev inequality, [**20**]. *Let* ρ *be a weight function such that*

$$\int_{\tilde\Omega} \frac{1}{\rho^\sigma |x-y|^\gamma}\,\mathrm{d}x \leqslant \mathcal{C},$$

with $1 < \sigma < \frac{p-1}{p-2}$ *if* $p > 2$*,* $\gamma < N-2$ *if* $N \geqslant 3$*,* $\gamma = 0$ *if* $N = 2$*. Assume, in the case* $N \geqslant 3$*, without no loss generality that* $\gamma > N - 2\sigma$*, which implies* $N\sigma - 2N +$

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$2\sigma + \gamma > 0$. *Then, for any* $w \in H^{1,2}_{0,\rho}(\tilde{\Omega})$, *there exists a constant* C_s *such that*

$$\|w\|_{L^q(\tilde{\Omega})} \leqslant C_s \|\nabla w\|_{L^2(\tilde{\Omega},\rho)},$$

for any $1 \leqslant q < 2^*(\sigma)$ *where*

$$\frac{1}{2^*(\sigma)} = \frac{1}{2} - \frac{1}{N} + \frac{1}{\sigma}\left(\frac{1}{2} - \frac{\gamma}{2N}\right).$$

In particular we prove the following Poincaré's inequality that we will use later considering the weight $\rho = A(|\nabla u|)$. Note that this choice is possible thanks to theorem 2.4.

Corollary 2.8. *Let us consider* $w \in H^{1,2}_{0,\rho}(\tilde{\Omega})$. *Then*

$$\int_{\tilde{\Omega}} w^2 \, \mathrm{d}x \leqslant C_{pp}(\tilde{\Omega}) \int_{\tilde{\Omega}} \rho |\nabla w|^2 \, \mathrm{d}x, \tag{2.10}$$

where $C_{pp}(\tilde{\Omega}) \to 0$ *when* $|\tilde{\Omega}| \to 0$.

Proof. Choose $2 < q < 2^*(\sigma)$. By Lebesgue's spaces embedding

$$\int_{\tilde{\Omega}} w^2 \, \mathrm{d}x \leqslant |\tilde{\Omega}|^{\frac{q-2}{q}} \left(\int_{\tilde{\Omega}} w^q \, \mathrm{d}x\right)^{\frac{2}{q}},$$

then using theorem 2.7 we get

$$\int_{\tilde{\Omega}} w^2 \, \mathrm{d}x \leqslant |\tilde{\Omega}|^{\frac{q-2}{q}} \left(\int_{\tilde{\Omega}} w^q \, \mathrm{d}x\right)^{\frac{2}{q}} \leqslant |\tilde{\Omega}|^{\frac{q-2}{q}} C_s^2 \int_{\tilde{\Omega}} \rho |\nabla w|^2 \, \mathrm{d}x = C_{pp}(\tilde{\Omega}) \int_{\tilde{\Omega}} \rho |\nabla w|^2 \, \mathrm{d}x.$$

where $C_{pp}(\tilde{\Omega}) := |\tilde{\Omega}|^{\frac{q-2}{q}} C_s^2$. □

To state the next results we need some notations.
For a real number λ we set

$$\Omega_\lambda = \{x \in \Omega : x_1 < \lambda\},$$

$$x_\lambda = R_\lambda(x) = (2\lambda - x_1, x_2, \ldots, x_N),$$

$$u_\lambda(x) = u(x_\lambda)$$

where x_λ is the reflection through the hyperplane $T_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\}$.

Also let us define $a := \inf_{x \in \Omega} x_1$.

Let us denote with B_ρ the open ball with centre 0 and radius $\rho > 0$. Define $\phi_\rho(x) \in C^1_c(\Omega)$, $\phi_\rho \geqslant 0$ such that

$$\phi_\rho = \begin{cases} 1 & \text{in } \Omega \backslash B_{2\rho}, \\ 0 & \text{in } B_\rho \end{cases} \quad \text{and} \quad |\nabla \phi_\rho| \leqslant \frac{k}{\rho} \text{ in } B_{2\rho} \backslash B_\rho, \tag{2.11}$$

where k is a positive constant.

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LEMMA 2.9. *Let $u \in C^1(\bar{\Omega}\setminus\{0\})$ a solution to (1.1). Then, the set $\Omega\setminus Z_u$ does not contain any connected component $\mathcal{C}$ such that $\bar{\mathcal{C}} \subset \Omega$.*

Moreover, if we assume that Ω is a smooth bounded domain with connected boundary, it follows that $\Omega\setminus Z_u$ is connected.

Proof. We proceed by contradiction. Let us assume that such component exists, namely

$$\mathcal{C} \subset \Omega\setminus Z_u \text{ such that } \partial\mathcal{C} \subset Z_u.$$

By remark 2.3, we get that $|Z_u| = 0$.

For all $\varepsilon > 0$, let us define $J_\varepsilon : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ by setting

$$J_\varepsilon(t) = \begin{cases} t & \text{if } t \geqslant 2\varepsilon, \\ 2t - 2\varepsilon & \text{if } \varepsilon \leqslant t \leqslant 2\varepsilon, \\ 0 & \text{if } 0 \leqslant t \leqslant \varepsilon, \end{cases}$$

We consider the function

$$\psi_\varepsilon = \phi_\rho(x)\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\chi_{\mathcal{C}}.$$

We point out that supp $\psi_\varepsilon \subset \mathcal{C}$ and $\psi_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. By using ψ_ε as test function in problem (1.1), since

$$\nabla\psi_\varepsilon = \nabla\phi_\rho(x)\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\chi_{\mathcal{C}} + \phi_\rho(x)\nabla\left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\right)\chi_{\mathcal{C}}$$

we obtain

$$\begin{aligned} &\int_{\mathcal{C}} A(|\nabla u|)\left(\nabla u, \nabla\left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\right)\right)\phi_\rho(x)\,\mathrm{d}x + \int_{\mathcal{C}} A(|\nabla u|)\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}(\nabla u, \nabla\phi_\rho(x))\,\mathrm{d}x \\ &= \vartheta\int_{\mathcal{C}} \frac{u^q}{|x|^p}\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\phi_\rho(x)\,\mathrm{d}x + \int_{\mathcal{C}} g(x,u)\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\phi_\rho(x)\,\mathrm{d}x. \end{aligned} \tag{2.12}$$

Denoting $h_\varepsilon(t) = \frac{J_\varepsilon(t)}{t}$, we get

$$\nabla\left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\right) = \nabla(h_\varepsilon(|\nabla u|)) = h'_\varepsilon(|\nabla u|)\nabla(|\nabla u|)$$

and, by straightforward calculation, we see that $|\nabla(|\nabla u|)| \leqslant \|D^2u\|$. Therefore, the first term on the left-hand side of (2.12) can be estimated as

$$\left|\int_{\mathcal{C}} A(|\nabla u|)\left(\nabla u, \nabla\left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\right)\right)\phi_\rho\,\mathrm{d}x\right| \leqslant \int_{\mathcal{C}} A(|\nabla u|)\,\|D^2u\|\,\phi_\rho|\nabla u||h'_\varepsilon(|\nabla u|)|\,\mathrm{d}x. \tag{2.13}$$

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Choosing η as in (2.4) and using theorem 2.2, one has

$$\int_{\mathcal{C}} A^2(|\nabla u|)\left\|D^2 u\right\|^2 \phi_\rho^2 \,\mathrm{d}x = \int_{\mathcal{C}\setminus B_\rho} A(|\nabla u|)|\nabla u|^\eta \frac{A(|\nabla u|)\left\|D^2 u\right\|^2}{|\nabla u|^\eta}\phi_\rho^2 \,\mathrm{d}x$$
$$\leqslant \sup_{\mathcal{C}\setminus B_\rho}\left(A(|\nabla u|)|\nabla u|^\eta\right)\int_{\mathcal{C}\setminus B_\rho} \frac{A(|\nabla u|)\left\|D^2 u\right\|^2}{|\nabla u|^\eta}\,\mathrm{d}x \leqslant c,$$

hence, by Lebesgue embedding space, $A(|\nabla u|)\left\|D^2 u\right\|\phi_\rho \in L^1(\mathcal{C})$, for all $\rho > 0$. Since

$$h_\varepsilon(t) = \begin{cases} 1 & \text{if } t \geqslant 2\varepsilon, \\ 2 - \frac{2\varepsilon}{t} & \text{if } \varepsilon \leqslant t \leqslant 2\varepsilon, \\ 0 & \text{if } 0 \leqslant t \leqslant \varepsilon \end{cases} \quad h'_\varepsilon(t) = \begin{cases} 0 & \text{if } t > 2\varepsilon, \\ \frac{2\varepsilon}{t^2} & \text{if } \varepsilon < t < 2\varepsilon, \\ 0 & \text{if } 0 < t < \varepsilon, \end{cases} \tag{2.14}$$

we get that $\lim_{\varepsilon\to 0}|\nabla u||h'_\varepsilon(|\nabla u|)| = 0$ a.e. in $\mathcal{C}$ and $|\nabla u||h'_\varepsilon(|\nabla u|)| \leqslant |\nabla u|\dfrac{2\varepsilon}{|\nabla u|^2} \leqslant 2$.

Then, since $A(|\nabla u|)|\nabla u||h'_\varepsilon(|\nabla u|)|\left\|D^2 u\right\|\phi_\rho \leqslant 2A(|\nabla u|)\left\|D^2 u\right\|\phi_\rho \in L^1(\mathcal{C})$, using Dominated convergence theorem in (2.13), we obtain

$$\lim_{\varepsilon\to 0}\left|\int_{\mathcal{C}} A(|\nabla u|)\left(\nabla u, \nabla\left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\right)\right)\phi_\rho \,\mathrm{d}x\right| = 0, \quad \forall \rho > 0. \tag{2.15}$$

From (2.15), passing to the limit in (2.12), we get

$$\lim_{\varepsilon\to 0}\int_{\mathcal{C}} A(|\nabla u|)\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}(\nabla u, \nabla\phi_\rho)\,\mathrm{d}x$$
$$= \lim_{\varepsilon\to 0}\left(\vartheta\int_{\mathcal{C}} \frac{u^q}{|x|^p}\phi_\rho\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\,\mathrm{d}x + \int_{\mathcal{C}} g(x,u)\phi_\rho\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\,\mathrm{d}x\right). \tag{2.16}$$

Since

$$\lim_{\varepsilon\to 0}\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} = 1,$$

using Dominated convergence theorem, we obtain

$$\int_{\mathcal{C}} A(|\nabla u|)(\nabla u, \nabla\phi_\rho)\,\mathrm{d}x = \vartheta\int_{\mathcal{C}}\frac{u^q}{|x|^p}\phi_\rho\,\mathrm{d}x + \int_{\mathcal{C}} g(x,u)\phi_\rho\,\mathrm{d}x$$

and in particular, by (2.11), one has

$$\int_{\mathcal{C}\cap(B_{2\rho}\setminus B_\rho)} A(|\nabla u|)(\nabla u, \nabla\phi_\rho)\,\mathrm{d}x = \vartheta\int_{\mathcal{C}}\frac{u^q}{|x|^p}\phi_\rho\,\mathrm{d}x + \int_{\mathcal{C}} g(x,u)\phi_\rho\,\mathrm{d}x. \tag{2.17}$$

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Using Hölder inequality and (2.5), we estimate the left-hand side of (2.17) as

$$
\begin{aligned}
&\left|\int_{\mathcal{C}\cap(B_{2\rho}\setminus B_\rho)} A(|\nabla u|)(\nabla u,\nabla\phi_\rho)\,\mathrm{d}x\right| \\
&\leqslant \int_{\mathcal{C}\cap(B_{2\rho}\setminus B_\rho)\cap\{|\nabla u|<K\}} |A(|\nabla u|)\nabla u||\nabla\phi_\rho|\,\mathrm{d}x \\
&\quad + \int_{\mathcal{C}\cap(B_{2\rho}\setminus B_\rho)\cap\{|\nabla u|\geqslant K\}} |A(|\nabla u|)\nabla u||\nabla\phi_\rho|\,\mathrm{d}x \\
&\leqslant kC_K \int_{\mathcal{C}\cap(B_{2\rho}\setminus B_\rho)\cap\{|\nabla u|<K\}} \frac{1}{\rho}\,dx + c_1\int_{\mathcal{C}\cap(B_{2\rho}\setminus B_\rho)\cap\{|\nabla u|\geqslant K\}} |\nabla u|^{p-1}|\nabla\phi_\rho|\,\mathrm{d}x \\
&\leqslant ckC_K\left(\frac{1}{\rho}\int_\rho^{2\rho} r^{N-1}\,\mathrm{d}r\right) \\
&\quad + c_1\left(\int_{\mathcal{C}\cap(B_{2\rho}\setminus B_\rho)\cap\{|\nabla u|\geqslant K\}} |\nabla u|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{\mathcal{C}\cap(B_{2\rho}\setminus B_\rho)\cap\{|\nabla u|\geqslant K\}} |\nabla\phi_\rho|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leqslant c\left[kC_K\rho^{N-1} + \left(\int_{B_{2\rho}\setminus B_\rho}\frac{1}{\rho^p}\,\mathrm{d}x\right)^{\frac{1}{p}}\right] = c\left[kC_K\rho^{N-1} + \left(\frac{1}{\rho^p}\int_\rho^{2\rho} r^{N-1}\,\mathrm{d}r\right)^{\frac{1}{p}}\right] \\
&= c\left[kC_K\rho^{N-1} + \left(\frac{\rho^N}{\rho^p}\right)^{\frac{1}{p}}\right] \xrightarrow{\rho\to 0} 0.
\end{aligned}
$$

Therefore, passing to the limit in (2.17) for $\rho \to 0$, we get

$$
0 = \lim_{\rho\to 0}\left(\vartheta\int_{\mathcal{C}}\frac{u^q}{|x|^p}\phi_\rho\,\mathrm{d}x + \int_{\mathcal{C}} g(x,u)\phi_\rho\,\mathrm{d}x\right). \tag{2.18}
$$

On the other hand, since $\phi_\rho(x) \to 1$ for $\rho \to 0$ and $g(x,u) > 0$, using again dominated convergence theorem, we get

$$
\lim_{\rho\to 0}\vartheta\int_{\mathcal{C}}\frac{u^q}{|x|^p}\phi_\rho\,\mathrm{d}x + \lim_{\rho\to 0}\int_{\mathcal{C}} g(x,u)\phi_\rho\,\mathrm{d}x = \vartheta\int_{\mathcal{C}}\frac{u^q}{|x|^p}\,\mathrm{d}x + \int_{\mathcal{C}} g(x,u)\,\mathrm{d}x > 0,
$$

which contradicts (2.18). If Ω is smooth, since the right-hand side of (1.1) is positive, by Höpf's Lemma (see [**27**, **30**]), a neighbourhood of the boundary belongs to a component $\mathcal{C}$ of $\Omega\setminus Z_u$. The case $\Omega\setminus Z_u$ not connected would imply the existence of a second connected component $\mathcal{C}'$ with $\partial\mathcal{C}' \subset Z_u$, but this would provide a contradiction with the first part of the proof. Thus $\Omega\setminus Z_u$ is connected. □

Remark 2.10. We remark that the previous theorem still holds if we replace the assumption $g(x,u) > 0$ by the assumption $\frac{u^q}{|x|^p} + g(x,u) > 0$ for every $x \in \Omega$.

Lemma 2.11. *Let $u \in W_0^{1,p}(\Omega)$ be a nonnegative weak solution to the problem* (1.1). *Then*

$$
\lim_{|x|\to 0} u(x) = +\infty. \tag{2.19}
$$

Proof. Since $g(x,u)>0$, u is a supersolution to $-\mathrm{div}(A(|\nabla u|)\nabla u)=\vartheta\frac{u^q}{|x|^p}$. By strong maximum principle (see [**27**, theorem 7.1.2]), if we set $\tilde{C}:=\inf_{B_R} u>0$, we get

$$-\operatorname{div}\left(A(|\nabla u|)\nabla u\right)\geqslant\vartheta\frac{\tilde{C}^q}{|x|^p}:=\frac{C}{|x|^p}\ \text{in } B_R, \tag{2.20}$$

If we consider

$$\begin{cases}-\mathrm{div}\left(A(|\nabla w|)\nabla w\right)=\frac{C}{|x|^p} & \text{in } B_R,\\ w>0 & \text{in } B_R,\\ w=0 & \text{on } \partial B_R,\end{cases} \tag{2.21}$$

we have that (2.21) admits an unique radial non-increasing solution $w\in W_0^{1,p}(B_R)$ such that $w(r)\to+\infty$ for $r\to 0^+$ (see the appendix for details).

Since $u\geqslant w$ on ∂B_R, using $(w-u)^+\in W_0^{1,p}(B_R)$ as function test in (2.20) and (2.21), we get

$$\int_{B_R}(|\nabla w|^{p-2}\nabla w-|\nabla u|^{p-2}\nabla u,\nabla(w-u)^+)\,\mathrm{d}x\leqslant 0.$$

Using (2.2), we have that

$$\begin{aligned}c_2\bar{c}&\int_{B_R}(|\nabla w|+|\nabla u|)^{p-2}|\nabla(w-u)^+|^2\,\mathrm{d}x\\ &\leqslant\int_{B_R}(|\nabla w|^{p-2}\nabla w-|\nabla u|^{p-2}\nabla u,\nabla(w-u)^+)\,\mathrm{d}x\leqslant 0\end{aligned}$$

then $(w-u)^+=0$ on B_R, i.e. $w\leqslant u$ on B_R.

Therefore, since $w(r)\to+\infty$ for $r\to 0$, we have that $\lim_{|x|\to 0}u(x)=+\infty$. □

3. Symmetry and monotonicity results

3.1. Weak comparison principle

Let us first prove the following result:

Theorem 3.1 Weak comparison principle. *Let $\lambda<0$ and $\tilde{\Omega}$ be a bounded domain such that $\tilde{\Omega}\Subset\Omega_\lambda$. Assume that $u\in C^1(\bar{\Omega}\backslash\{0\})$ is a solution to (1.1) such that $u\leqslant u_\lambda$ on $\partial\tilde{\Omega}$. Then there exists a positive constant $\delta=\delta(\lambda,p,q,g,dist(\tilde{\Omega},\partial\Omega))$ such that if $|\tilde{\Omega}|<\delta$, then it holds*

$$u\leqslant u_\lambda\ \text{in}\ \tilde{\Omega}.$$

Proof. We have (in the weak sense) that u and u_λ satisfy

$$-\mathrm{div}(A(|\nabla u|)\nabla u) = \vartheta \frac{u^q}{|x|^p} + g(x,u) \quad \text{in } \Omega \tag{3.1}$$

$$-\mathrm{div}(A(|\nabla u_\lambda|)\nabla u_\lambda) = \vartheta \frac{u_\lambda^q}{|x_\lambda|^p} + g(x_\lambda,u_\lambda) \quad \text{in } R_\lambda(\Omega). \tag{3.2}$$

If $(u-u_\lambda)^+ \equiv 0$ in $\tilde{\Omega}$, we have the claim. Then, we assume by contradiction that $(u-u_\lambda)^+ \not\equiv 0$ in $\tilde{\Omega}$ and we want to use $\varphi := (u-u_\lambda)^+ \chi_{\tilde{\Omega}}$ as test function.

In order to do this, we notice that, by lemma 2.11, $\lim_{|x|\to 0} u(x) = +\infty$.

Then, since $u \in C^1(\bar{\Omega}\backslash\{0\})$ (see remark 1.2) and $0 \notin \tilde{\Omega}$, we have that $u \in L^\infty(\tilde{\Omega})$; therefore $u_\lambda \in L^\infty(\tilde{\Omega})$ in $\mathrm{supp}(u-u_\lambda)^+$.

By lemma 2.11, we have that

$$\lim_{|x|\to 0} u(x) = +\infty \Leftrightarrow \lim_{|y|\to 0_\lambda} u_\lambda(y) = +\infty,$$

hence

$$0_\lambda = R_\lambda(0) \notin \mathrm{supp}(u-u_\lambda)^+. \tag{3.3}$$

From the assumption $u \leqslant u_\lambda$ on $\partial\tilde{\Omega}$, it follows $\varphi := (u-u_\lambda)^+\chi_{\tilde{\Omega}} \in W_0^{1,p}(\Omega)$, then, we can choose φ as test function in weak formulation of (3.1) and (3.2) obtaining

$$\int_{\tilde{\Omega}} A(|\nabla u|)(\nabla u, \nabla(u-u_\lambda)^+)\,\mathrm{d}x = \int_{\tilde{\Omega}} \vartheta \frac{u^q}{|x|^p}(u-u_\lambda)^+ \mathrm{d}x + \int_{\tilde{\Omega}} g(x,u)(u-u_\lambda)^+ \mathrm{d}x \tag{3.4}$$

$$\int_{\tilde{\Omega}} A(|\nabla u_\lambda|)(\nabla u_\lambda, \nabla(u-u_\lambda)^+)\,\mathrm{d}x = \int_{\tilde{\Omega}} \vartheta \frac{u_\lambda^q}{|x_\lambda|^p}(u-u_\lambda)^+ \mathrm{d}x + \int_{\tilde{\Omega}} g(x_\lambda,u_\lambda)(u-u_\lambda)^+ \mathrm{d}x. \tag{3.5}$$

Then, if we set $\Omega^+ := \mathrm{supp}(u-u_\lambda)^+ \cap \tilde{\Omega}$ and we subtract (3.4) and (3.5), we get

$$\begin{aligned}\int_{\Omega^+} &(A(|\nabla u|)\nabla u - A(|\nabla u_\lambda|)\nabla u_\lambda, \nabla(u-u_\lambda))\,\mathrm{d}x \\ &= \int_{\Omega^+} \vartheta \left(\frac{u^q}{|x|^p} - \frac{u_\lambda^q}{|x_\lambda|^p}\right)(u-u_\lambda)\,\mathrm{d}x + \int_{\Omega^+} [g(x,u) - g(x_\lambda,u_\lambda)](u-u_\lambda)\,\mathrm{d}x.\end{aligned} \tag{3.6}$$

Noticing $|x| \geqslant |x_\lambda|$, (3.6) becomes

$$\begin{aligned}\int_{\Omega^+} &(A(|\nabla u|)\nabla u - A(|\nabla u_\lambda|)\nabla u_\lambda, \nabla(u-u_\lambda))\,\mathrm{d}x \\ &\leqslant \vartheta \int_{\Omega^+} \frac{1}{|x|^p}\left(\frac{u^q - u_\lambda^q}{u-u_\lambda}\right)(u-u_\lambda)^2\,\mathrm{d}x \\ &+ \int_{\Omega^+} \frac{g(x,u)-g(x,u_\lambda)}{u-u_\lambda}(u-u_\lambda)^2\,\mathrm{d}x + \int_{\Omega^+} [g(x,u_\lambda) - g(x_\lambda,u_\lambda)](u-u_\lambda)\,\mathrm{d}x.\end{aligned}$$

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Using monotonicity of $g(\cdot,u)$, the locally Lipschitz continuity of $g(x,\cdot)$ and taking in account that for $\lambda<0$ one has that $|x|\geqslant C$ in Ω_λ for some positive constant C, one has

$$\begin{aligned}&\int_{\Omega^+}(A(|\nabla u|)\nabla u-A(|\nabla u_\lambda|)\nabla u_\lambda,\nabla(u-u_\lambda))\,\mathrm{d}x\\&\quad\leqslant c\vartheta\int_{\Omega^+}\left(\frac{u^q-u_\lambda^q}{u-u_\lambda}\right)(u-u_\lambda)^2\,\mathrm{d}x+K_L\int_{\Omega^+}(u-u_\lambda)^2\,\mathrm{d}x.\end{aligned}\tag{3.7}$$

We recall that y^q is locally Lipschitz continuous in $(0,+\infty)$ and the solution u is strictly positive in $\tilde\Omega$ and also in Ω^+. Then, (3.7) becomes

$$\begin{aligned}&\int_{\Omega^+}(A(|\nabla u|)\nabla u-A(|\nabla u_\lambda|)\nabla u_\lambda,\nabla(u-u_\lambda))\,\mathrm{d}x\\&\leqslant\vartheta C_L\int_{\Omega^+}(u-u_\lambda)^2\,\mathrm{d}x+K_L\int_{\Omega^+}(u-u_\lambda)^2\,\mathrm{d}x=:c\int_{\Omega^+}(u-u_\lambda)^2\,\mathrm{d}x.\end{aligned}\tag{3.8}$$

Let us now consider separately the following cases.

- $1<p<2$.
- $p\geqslant 2$, $m_A\geqslant 0$.
- $p\geqslant 2$, $m_A<0$ and $A(t)\geqslant\tau(C)>0$ with τ positive constant that depends on $C\subset[0,+\infty)$ compact.
- $p\geqslant 2$, $m_A<0$, $\exists\tilde C\subset[0,+\infty)$ compact such that $\inf_{\tilde C}A(t)=0$.

Case: $\mathbf{1<p<2}$.

Using (2.2) and classic Poincaré inequality, (3.8) becomes

$$\begin{aligned}&c_2\bar c\int_{\Omega^+}(|\nabla u|+|\nabla u_\lambda|)^{p-2}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x\leqslant c\int_{\Omega^+}(u-u_\lambda)^2\,\mathrm{d}x\\&\leqslant cC_p^2(|\Omega^+|)\int_{\Omega^+}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x\\&=cC_p^2(|\Omega^+|)\int_{\Omega^+}|\nabla(u-u_\lambda)|^2(|\nabla u|+|\nabla u_\lambda|)^{p-2}(|\nabla u|+|\nabla u_\lambda|)^{2-p}\,\mathrm{d}x.\end{aligned}\tag{3.9}$$

Now, from (3.3), we infer that $|\nabla u_\lambda|\in L^\infty(\Omega^+)$.

Then, since $|\nabla u|\in L^\infty(\Omega^+)$, using also the fact that $2-p>0$, $(|\nabla u|+|\nabla u_\lambda|)^{2-p}$ is bounded in Ω^+. Thus, equation (3.9) becomes

$$\begin{aligned}&\int_{\Omega^+}(|\nabla u|+|\nabla u_\lambda|)^{p-2}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x\\&\leqslant cC_p^2(|\Omega^+|)\int_{\Omega^+}(|\nabla u|+|\nabla u_\lambda|)^{p-2}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x,\end{aligned}$$

which gives a contradiction if $cC_p^2(|\Omega^+|)<1$.

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This occurs if we consider $\delta := \delta(\lambda, q, p, g, \mathrm{dist}(\tilde{\Omega}, \partial\Omega))$ sufficiently small such that $|\tilde{\Omega}| \leqslant \delta$ (that satisfies our assumption) and $C_p(|\Omega^+|) = C_p(|\tilde{\Omega} \cap \mathrm{supp}(u-u_\lambda)^+|) < \sqrt{\frac{1}{c}}$ jointly.

Then, taking into account the boundary condition,

$$u - u_\lambda = 0 \quad \text{in } \Omega^+ := \tilde{\Omega} \cap \mathrm{supp}(u-u_\lambda)^+.$$

This shows that actually $(u-u_\lambda)^+ = 0$ on $\tilde{\Omega}$, that is

$$u \leqslant u_\lambda \text{ in } \tilde{\Omega}.$$

Case: $\mathbf{p \geqslant 2}$ and $\mathbf{m_A \geqslant 0}$.

Since $m_A \geqslant 0$, by (2.6) we get that $A(t)$ is a non-decreasing function; then we can use [**5**, lemma 4.3], namely

$$(A(|\xi|)\xi - A(|\eta|)\eta, \xi - \eta) \geqslant \frac{1}{3}(A(|\xi|) + A(|\eta|)|\xi-\eta|^2, \quad \forall \xi, \eta \in \mathbb{R}^N. \tag{3.10}$$

Hence, (3.8) becomes

$$\begin{aligned}\frac{1}{3}\int_{\Omega^+} A(|\nabla u|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x &\leqslant \frac{1}{3}\int_{\Omega^+} A(|\nabla u| + |\nabla u_\lambda|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x \\ &\leqslant \int_{\Omega^+} (A(|\nabla u|)\nabla u - A(|\nabla u_\lambda|)\nabla u_\lambda, \nabla(u-u_\lambda))\,\mathrm{d}x \leqslant c\int_{\Omega^+} (u-u_\lambda)^2\,\mathrm{d}x.\end{aligned} \tag{3.11}$$

Moreover, exploiting theorem 2.4, we can use in the right-hand side of (3.11) the weighted Poincaré inequality (see corollary 2.8) with $\rho = A(|\nabla u|)$. Therefore, we get

$$\begin{aligned}\frac{1}{3}\int_{\Omega^+} A(|\nabla u|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x &\leqslant c\int_{\Omega^+} (u-u_\lambda)^2\,\mathrm{d}x \\ &\leqslant cC_{pp}(|\Omega^+|)\int_{\Omega^+} A(|\nabla u|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x\end{aligned} \tag{3.12}$$

which gives a contradiction if $cC_{pp}^2(|\Omega^+|) < \frac{1}{3}$.

Arguing as in the case $1 < p < 2$, we obtain

$$u \leqslant u_\lambda \text{ in } \tilde{\Omega}.$$

Case: $\mathbf{p \geqslant 2}$, $\mathbf{m_A < 0}$ and $\mathbf{A(t) \geqslant \tau(C) > 0}$ with τ positive constant that depends on $\mathbf{C \subset [0, +\infty)}$ compact.

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Using (2.1) in the left-hand side of (3.8), one has

$$\int_{\Omega^+} (A(|\nabla u|)\nabla u - A(|\nabla u_\lambda|)\nabla u_\lambda, \nabla(u-u_\lambda))\ \mathrm{d}x$$
$$\geqslant \bar{c}\int_{\Omega^+} |\nabla(u-u_\lambda)|^2 \left(\int_0^1 A\left(|\nabla u_\lambda + s\left(\nabla(u-u_\lambda)\right)|\right)\mathrm{d}s\right)\mathrm{d}x \tag{3.13}$$

As we remarked in the first part of the proof, $|\nabla u|$ and $|\nabla u_\lambda|$ are in $L^\infty(\Omega^+)$; hence there exists $\tilde{K}\in\mathbb{R}^+$ such that

$$|\nabla u_\lambda + s\nabla(u-u_\lambda)| \leqslant (1-s)|\nabla u_\lambda| + s|\nabla u| \leqslant \max\{|\nabla u|, |\nabla u_\lambda|\}$$
$$\leqslant \max\{\|\nabla u\|_\infty, \|\nabla u_\lambda\|_\infty\} \leqslant \tilde{K}. \tag{3.14}$$

Therefore, if we consider the compact set $[0,\tilde{K}]$, equation (3.13) becomes

$$\int_{\Omega^+} (A(|\nabla u|)\nabla u - A(|\nabla u_\lambda|)\nabla u_\lambda, \nabla(u-u_\lambda))\ \mathrm{d}x$$
$$\geqslant \bar{c}\int_{\Omega^+} |\nabla(u-u_\lambda)|^2 \left(\int_0^1 A\left(|\nabla u_\lambda + s\left(\nabla(u-u_\lambda)\right)|\right)\mathrm{d}s\right)\mathrm{d}x$$
$$\geqslant \bar{c}\tau(\tilde{K})\int_{\Omega^+} |\nabla(u-u_\lambda)|^2\,\mathrm{d}x$$

Hence, substituting in (3.8) and using classic Poincaré inequality, we have

$$\bar{c}\tau(\tilde{K})\int_{\Omega^+} |\nabla(u-u_\lambda)|^2\,\mathrm{d}x \leqslant cC_p(|\Omega^+|)\int_{\Omega^+} |\nabla(u-u_\lambda)|^2\,\mathrm{d}x$$

which gives a contradiction if $cC_p(|\Omega^+|) < \bar{c}\tau(\tilde{K})$.

Arguing as in the case $1<p<2$, we obtain

$$u \leqslant u_\lambda \text{ in } \tilde{\Omega}.$$

Case: $\mathbf{p \geqslant 2}$, $\mathbf{m_A < 0}$ and there exists $\mathbf{\tilde{C} \subset [0,+\infty)}$ compact such that $\mathbf{\inf_{\tilde{C}} A(t) = 0}$.

By (1.4), $A(t)\to+\infty$ for $t\to+\infty$, then if there exists $\tilde{C}\subset[0,+\infty)$ compact set such that $\inf_{\tilde{C}} A(t) = 0$, we have that necessarily $\liminf_{t\to 0^+} A(t) = 0$.

Therefore, by (1.5), there exists $\delta>0$ such that $A(t)$ is a non-decreasing function on $I_0 := (0,\delta)$.

Let us set

$$\Omega_1^+ := \{x\in\Omega^+ : |\nabla u(x)| < \delta\}$$
$$\Omega_2^+ := \{x\in\Omega^+ : \delta \leqslant |\nabla u(x)| < K\}$$
$$\Omega_3^+ := \{x\in\Omega^+ : |\nabla u(x)| \geqslant K\}$$

where K is chosen as in (1.3).

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Using the weighted Poincaré inequality in the right-hand side of (3.8) with $\rho = A(|\nabla u|)$, we get

$$
\begin{aligned}
& c\int_{\Omega^+}(u-u_\lambda)^2\,\mathrm{d}x \leqslant cC_{pp}(|\Omega^+|)\int_{\Omega^+}A(|\nabla u|)|\nabla(u-u_\lambda)^+|^2\,\mathrm{d}x \\
& \leqslant cC_{pp}(|\Omega^+|)\left[\int_{\Omega_1^+}A(|\nabla u|)|\nabla(u-u_\lambda)^+|^2\,\mathrm{d}x \right.\\
& \quad \left. + \sup_{\delta\leqslant|\nabla u|<K}A(|\nabla u|)\int_{\Omega_2^+}|\nabla(u-u_\lambda)^+|^2\,\mathrm{d}x + \int_{\Omega_3^+}A(|\nabla u|)|\nabla(u-u_\lambda)^+|^2\,\mathrm{d}x\right]
\end{aligned} \tag{3.15}
$$

For the left-hand side of (3.8) we use (3.10) on the set Ω_1^+ since $A(t)$ is non-decreasing, while we use (2.2) on Ω_2^+ and Ω_3^+. Then

$$
\begin{aligned}
& \int_{\Omega^+}(A(|\nabla u|)\nabla u - A(|\nabla u_\lambda|)\nabla u_\lambda, \nabla(u-u_\lambda))\,\mathrm{d}x \\
& \geqslant \frac{1}{3}\int_{\Omega_1^+}A(|\nabla u|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x + c_2\bar{c}\int_{\Omega_2^+}(|\nabla u|+|\nabla u_\lambda|)^{p-2}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x \\
& \quad + c_2\bar{c}\int_{\Omega_3^+}(|\nabla u|+|\nabla u_\lambda|)^{p-2}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x \\
& \geqslant \frac{1}{3}\int_{\Omega_1^+}A(|\nabla u|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x + c_2\bar{c}\int_{\Omega_2^+}|\nabla u|^{p-2}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x \\
& \quad + c_2\bar{c}\int_{\Omega_3^+}|\nabla u|^{p-2}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x \\
& \geqslant \frac{1}{3}\int_{\Omega_1^+}A(|\nabla u|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x + c_2\bar{c}\delta^{p-2}\int_{\Omega_2^+}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x \\
& \quad + \frac{c_2\bar{c}}{c_1}\int_{\Omega_3^+}A(|\nabla u|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x
\end{aligned} \tag{3.16}
$$

Using (3.15) and (3.16) in (3.8), we get

$$
\begin{aligned}
& \left(\frac{1}{3} - cC_{pp}(|\Omega^+|)\right)\int_{\Omega_1^+}A(|\nabla u|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x \\
& \quad + \left(c_2\bar{c}\delta^{p-2} - cC_{pp}(|\Omega^+|)\left(\sup_{\delta<|\nabla u|\leqslant K}A(|\nabla u|)\right)\right)\int_{\Omega_2^+}|\nabla(u-u_\lambda)|^2\,\mathrm{d}x \\
& \quad + \left(\frac{c_2\bar{c}}{c_1} - cC_{pp}(|\Omega^+|)\right)\int_{\Omega_3^+}A(|\nabla u|)|\nabla(u-u_\lambda)|^2\,\mathrm{d}x \leqslant 0
\end{aligned} \tag{3.17}
$$

and arguing as for $1 < p < 2$ we get that $u \leqslant u_\lambda$ in $\tilde{\Omega}$. □

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3.2. Symmetry result

Now we can prove our symmetry and monotonicity result.

Proof of theorem 1.3. The proof follows via the moving planes technique [**2**, **14**, **15**, **18**, **21**]. First we define

$$\Lambda_0 := \{a < \lambda < 0 : u \leqslant u_t \text{ in } \Omega_t \text{ for all } t \in (a, \lambda]\}.$$

We start showing that

$$\Lambda_0 \neq \emptyset.$$

To prove this, we observe that, since $u(a) = 0$ and Ω is smooth and strictly convex, by Höpf Lemma [**27**, theorem 5.5.1]

$$\exists \delta > 0 \text{ such that } \frac{\partial u}{\partial x_1}(x) > 0 \text{ for all } x \in I_\delta(a).$$

Moreover we note that, by the Höpf Lemma applied to a solution u of problem (1.1), we know that

$$Z_u \subset \Omega.$$

Hence, we can consider $a < \lambda < a + \varepsilon$ with ε a small positive constant such that

$$(\Omega_\lambda \cup R_\lambda(\Omega_\lambda)) \subset I_\delta(a).$$

So, by monotonicity, we have that

$$u \leqslant u_\lambda \text{ in } \Omega_\lambda.$$

Now we define

$$\bar\lambda := \sup \Lambda_0.$$

We want to show that $u \leqslant u_\lambda$ in Ω_λ for every $\lambda \in (a, 0]$, namely that:

$$\bar\lambda = 0.$$

Assume by contradiction that $\bar\lambda < 0$. We will prove that $u \leqslant u_{\bar\lambda+\tau}$ in $\Omega_{\bar\lambda+\tau}$ for any $0 < \tau < \bar\tau$ with $\bar\tau$ small enough.

By continuity we have that $u \leqslant u_{\bar\lambda}$ in $\Omega_{\bar\lambda} \backslash 0_{\bar\lambda}$.

Let us consider $Z^R_{u,\bar\lambda} := \{x \in \Omega_{\bar\lambda} : \nabla u(x) = 0 \vee \nabla u_{\bar\lambda}(x) = 0\}$ and open set $A_{\bar\lambda} \subset \Omega_{\bar\lambda}$ such that

$$Z^R_{u,\bar\lambda} \subset A_{\bar\lambda} \Subset \Omega.$$

We note that by Höpf Lemma, we can assume that $A_{\bar\lambda} \Subset \Omega$ and since $|Z^R_{u,\bar\lambda}| = 0$, we can take $A_{\bar\lambda}$ of arbitrarily small measure. Since we are working in $\Omega_{\bar\lambda}$, with $\bar\lambda < 0$, the weight $\frac{1}{|x|^p}$ is not singular there. Moreover, in a neighbourhood of 0_λ, by lemma 2.11, we have that $u < u_\lambda$. Since elsewhere $\frac{1}{|x_\lambda|^p}$ is not singular and u, u_λ

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are bounded, we can use Strong Comparison Principle (see [**27**, theorem 2.5.2]) to get that, if $\mathcal{C}$ is a connected component of $\Omega_{\bar\lambda}\backslash Z^R_{u,\bar\lambda}$, then

$$u < u_{\bar\lambda} \text{ or } u \equiv u_{\bar\lambda} \text{ in } \mathcal{C}.$$

Actually, we prove that the latter case is not possible. In fact, supposing that $u \equiv u_{\bar\lambda}$ in $\mathcal{C}$, if we reflect $\mathcal{C}$ through the hyperplane $T_{\bar\lambda}$, we obtain that

$$u \equiv u_{\bar\lambda} \text{ in } \mathcal{C} \cup R_{\bar\lambda}(\mathcal{C}) \text{ connected component of } \Omega\backslash Z_u. \tag{3.18}$$

We note that $\bar{\mathcal{C}} \cap \partial\Omega \neq 0$ follows by Dirichlet datum. Then, (3.18) would imply the existence of a local symmetry phenomenon where $(\partial\mathcal{C}\backslash T_{\bar\lambda}) \cup R_{\bar\lambda}(\partial\mathcal{C}\backslash T_{\bar\lambda}) \subset Z_u$; therefore $\Omega\backslash Z_u$ would be not connected, contradicting lemma 2.9.

Then

$$u < u_{\bar\lambda} \text{ in } \mathcal{C}. \tag{3.19}$$

Now, let us consider a compact set $K \subset \Omega_{\bar\lambda}\backslash A_{\bar\lambda}$. We get that:

- By lemma 2.11, in a neighbourhood of 0_λ, $u < u_\lambda$ for every $\lambda \in [\bar\lambda, \bar\lambda + \tau]$ for any $0 < \tau < \bar\tau$ with $\bar\tau > 0$ small.

  By uniform continuity of u we have that $u < u_{\bar\lambda+\tau}$ in K; it is equivalent to say that

  $$w := (u - u_{\bar\lambda+\tau})^+ = 0 \text{ in } K \tag{3.20}$$

  then

  $$\operatorname{supp}(w) \subset \Omega_{\bar\lambda+\tau}\backslash K. \tag{3.21}$$

- Taking into account the zero Dirichlet boundary datum and using Höpf Lemma, it is easy to show that, for some $\bar\delta > 0$, there exists a tubular neighbourhood such that

  $$u < u_{\bar\lambda+\tau} \text{ in } I_{\bar\delta}(\partial\Omega) \cap \Omega_{\bar\lambda+\tau} \tag{3.22}$$

  for any $0 < \tau < \bar\tau$.

  Instead, for the region near $\partial\Omega \cap T_{\bar\lambda+\tau}$, we use the monotonicity properties of solutions obtained by Höpf Lemma. In fact, since Ω is smooth and strictly convex, if we consider $p \in \partial\Omega \cap T_{\bar\lambda+\tau}$, we get that

  $$\exists \bar\delta > 0 \text{ such that } \frac{\partial u}{\partial x_1}(x) > 0 \text{ for all } x \in I_{\bar\delta}(p).$$

  So we have obtained that

  $$u < u_{\bar\lambda+\tau} \text{ in } I_{\bar\delta}(\partial\Omega) \cap T_{\bar\lambda+\tau}. \tag{3.23}$$

  If we denote with $\mathcal{N}_{\bar\lambda+\tau}$ a neighbourhood of $\partial\Omega_{\bar\lambda+\tau} \cap \partial\Omega$, from (3.22) and (3.23), we obtain that

  $$u < u_{\bar\lambda+\tau} \text{ in } \mathcal{N}_{\bar\lambda+\tau}. \tag{3.24}$$

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Summarizing up, in particular we get that

$$u < u_{\bar{\lambda}+\tau} \text{ in } K \cup \mathcal{N}_{\bar{\lambda}+\tau}. \tag{3.25}$$

Moreover, equation (3.21), using (3.24), becomes

$$\operatorname{supp}(w) \Subset \Omega_{\bar{\lambda}+\tau} \backslash K.$$

Now , we choose $|K|$ big enough such that $|\Omega_{\bar{\lambda}+\tau} \backslash (K \cup \mathcal{N}_{\bar{\lambda}+\tau})|$ is sufficiently small so that theorem 3.1 works.

Since K is a closed set, using (3.20) and (3.24), it follows

$$u \leqslant u_{\bar{\lambda}+\tau} \text{ on } \partial\left(\Omega_{\bar{\lambda}+\tau} \backslash \left(K \cup \mathcal{N}_{\bar{\lambda}+\tau}\right)\right). \tag{3.26}$$

Therefore, by theorem 3.1 it follows that

$$u \leqslant u_{\bar{\lambda}+\tau} \text{ in } \Omega_{\bar{\lambda}+\tau} \backslash \left(K \cup \mathcal{N}_{\bar{\lambda}+\tau}\right)$$

for any $0 < \tau < \bar{\tau}$.

Exploiting also (3.25), we have that

$$u \leqslant u_{\bar{\lambda}+\tau} \quad \text{in } \Omega_{\bar{\lambda}+\tau}$$

for any $0 < \tau < \bar{\tau}$, but this gives a contradiction with the definition of $\bar{\lambda}$. So we have that $\bar{\lambda} = 0$ and so

$$u \leqslant u_0 \text{ in } \Omega_0.$$

If we perform the moving plane technique in the opposite direction, we obtain that

$$u \geqslant u_0 \text{ in } \Omega_0.$$

Then u is symmetric with respect to the hyperplane $\{x_1 = 0\}$.

Moreover, the fact that the solution is non-decreasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$ is implicit in the moving plane procedure. Finally, if Ω is a ball centred at origin, repeating this argument along any direction, it follows that u is radially symmetric. The fact the $\frac{\partial u}{\partial r} < 0$ for $r \neq 0$, follows by Höpf boundary lemma which works in this case since the level sets are balls and therefore fulfill the interior sphere condition.

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Appendix A.

In this appendix we study the existence and the asymptotic behaviour of solutions to the problem

$$\begin{cases} -\mathrm{div}\,(A(|\nabla w|)\nabla w) = \frac{C}{|x|^p} & \text{in } B_R, \\ w > 0 & \text{in } B_R, \\ w = 0 & \text{on } \partial B_R. \end{cases} \tag{A.1}$$

Taking into account [5], the problem (A.1) is the Euler equation of the minimization problem for the functional $J : W_0^{1,p}(B_R) \to \mathbb{R}$ defined as

$$J(w) = \int_{B_R} G(|\nabla w|)\,\mathrm{d}x - \int_{B_R} \frac{C}{|x|^p}\, w\,\mathrm{d}x$$

where $G : [0, +\infty) \to [0, +\infty)$ is defined as

$$G(t) := \int_0^t sA(s)\,\mathrm{d}s. \tag{A.2}$$

By (1.4), $sA(s) \geqslant c_2 s^{p-1}$ for $s > 0$. Integrating

$$\int_0^{|\nabla w|} sA(s)\,\mathrm{d}s \geqslant \int_0^{|\nabla w|} c_2 s^{p-1}\,\mathrm{d}s \Rightarrow G(|\nabla w|) \geqslant c_2 \frac{|\nabla w|^p}{p}. \tag{A.3}$$

Our goal is to apply Weierstrass Theorem to prove that the functional J has a global minimum point.

We set $h(x) := \frac{C}{|x|^p} \in L^q(B_R)$ with $1 \leqslant q < \frac{N}{p}$.

- First, we prove that J is coercive.

  Let us consider a minimizing sequence $w_m \in W_0^{1,p}(B_R)$ for J. Using (A.3) we get

$$\begin{aligned} J(w_m) &= \int_{B_R} G(|\nabla w_m|)\,\mathrm{d}x - \int_{B_R} h(x)\; w_m\,\mathrm{d}x \\ &\geqslant \frac{c_2}{p} \int_{B_R} |\nabla w_m|^p\,\mathrm{d}x - \int_{B_R} h(x)|w_m|\,\mathrm{d}x \\ &\geqslant \frac{c_2}{p} \|w_m\|^p_{W_0^{1,p}(B_R)} - \|h(x)\|_{p^{*\prime}} \|w_m\|_{p^*} \\ &\geqslant \frac{c_2}{p} \|w_m\|^p_{W_0^{1,p}(B_R)} - c\,\|h(x)\|_{p^{*\prime}} \|w_m\|_{W_0^{1,p}(B_R)}. \end{aligned}$$

  Since $p^{*'} < \frac{N}{p}$ and $p > 1$, if $\|w_m\|_{W_0^{1,p}(B_R)} \to +\infty$ we have $J(w_m) \to +\infty$.

- To prove the weak lower semi-continuity of J, we write the functional as

$$J(w_m) = \int_{B_R} G(|\nabla w_m|)\,\mathrm{d}x - \int_{B_R} h(x)\; w_m\,\mathrm{d}x = J_1(w_m) - J_2(w_m).$$

  Let us consider $w_m \rightharpoonup w$ in $W_0^{1,p}(B_R)$. We take $s < p^*$ and s', conjugate exponent of s, such that $s' < \frac{N}{p}$. By compact embedding, up to a subsequence,

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$w_m \to w$ in $L^s(B_R)$, hence

$$|J_2(w_m) - J_2(w)| \leqslant \int_{B_R} |h(x)w_m - h(x)w| \, \mathrm{d}x \leqslant \|h(x)\|_{s'} \, \|w_m - w\|_s \to 0$$

for $m \to +\infty$. Therefore, we have proved that

$$\lim_{m\to+\infty} \int_{B_R} h(x)w_m \, \mathrm{d}x = \int_{B_R} h(x)w \, \mathrm{d}x. \tag{A.4}$$

While for the functional $J_1(w_m)$, we first prove the strict convexity.

For $w_m \neq z_m$,

$$\begin{aligned}
&(J_1'(w_m) - J_1'(z_m))(w_m - z_m) \\
&= \int_{B_R} (A(|\nabla w_m|)\nabla w_m - A(|\nabla z_m|)\nabla z_m, \nabla(w_m - z_m)) \, \mathrm{d}x \\
&\geqslant c_2\bar{c} \int_{B_R} (|\nabla w_m| + |\nabla z_m|)^{p-2} |\nabla(w_m - z_m)|^2 \, \mathrm{d}x > 0
\end{aligned}$$

then J_1 is strictly convex.

Now, let us consider $w_m \to w$ in $W_0^{1,p}(B_R)$. Using Lagrange Theorem, (A.2), (2.5), Hölder inequality and (1.3), for $\xi \in \{\min\{|\nabla w_m|, |\nabla w|\}, \max\{|\nabla w_m|, |\nabla w|\}\}$, we have

$$\begin{aligned}
&|J_1(w_m) - J_1(w)| \leqslant \int_{B_R} |G(|\nabla w_m|) - G(|\nabla w|)| \, \mathrm{d}x \\
&= \int_{B_R} |G'(|\xi|)| \, ||\nabla w_m| - |\nabla w|| \, \mathrm{d}x = \int_{B_R} |\xi| A(|\xi|) \, ||\nabla w_m| - |\nabla w|| \, \mathrm{d}x \\
&\leqslant C_K \int_{B_R \cap \{|\xi| < K\}} ||\nabla w_m| - |\nabla w|| \, \mathrm{d}x \\
&\quad + c_1 \int_{B_R \cap \{|\xi| \geqslant K\}} \max\{|\nabla w_m|^{p-1}, |\nabla w|^{p-1}\} \, ||\nabla w_m| - |\nabla w|| \, \mathrm{d}x \\
&\leqslant C_K \|w_m - w\|_{W_0^{1,1}(B_R)} \\
&\quad + c_1 \left(\int_{B_R} \max\{|\nabla w_m|^p, |\nabla w|^p\} \, \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{B_R} |\nabla(w_m - w)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\leqslant C_K \|w_m - w\|_{W_0^{1,1}(B_R)} + c_1 c \, \|w_m - w\|_{W_0^{1,p}(B_R)} \to 0, \text{ for } m \to +\infty.
\end{aligned}$$

Then $J_1(w_m)$ is strong continuous. Since $J_1(w_m)$ is strictly convex, $J_1(w_m)$ is weakly lower semi-continuous.

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Hence, using (A.4), we get

$$
\begin{aligned}
J(w) = J_1(w) - J_2(w) &\leqslant \liminf_{m\to+\infty} J_1(w_m) - \lim_{m\to+\infty} J_2(w_m)\\
&\leqslant \liminf_{m\to+\infty} (J_1(w_m) - J_2(w_m)) = \liminf_{m\to+\infty} J(w_m).
\end{aligned}
$$

Thus J has a global maximum point w that is a weak solution to the problem (A.1). In particular, since J_1 is strictly convex and J_2 is linear, the functional J is strictly convex. Then, the solution w is unique.

Moreover we point out that the solution w is radial.

In fact, recalling the definition of x_λ and u_λ, since $|x| = |x_{\lambda=0}|$, if we consider w solution to (A.1) and w_0, we get

$$
\int_{B_R} (A(|\nabla w|)\nabla w - A(|\nabla w_0|)\nabla w_0, \nabla(w - w_0))\,\mathrm{d}x = 0.
$$

Then, by (2.2) we obtain

$$
c_2\bar{c}\int_{B_R} (|\nabla w| + |\nabla w_0|)^{p-2}|\nabla(w - w_0)|^2\,\mathrm{d}x \leqslant 0
$$

and $w = w_0$. Repeating this argument along any direction, it follows that w is radially symmetric. The radiality can be obtained also by the uniqueness of solution.

Let us now study the asymptotic behaviour of w near the origin.

First, we observe that w is non-increasing with respect to r.

In fact, for $\lambda < 0$, if we consider w and w_λ that respectively satisfy the following

$$
\int_{B_{R_\lambda}} A(|\nabla w|)(\nabla w, \nabla(w - w_\lambda)^+)\,\mathrm{d}x = \int_{B_{R_\lambda}} \frac{C}{|x|^p}(w - w_\lambda)^+\mathrm{d}x
$$

$$
\int_{B_{R_\lambda}} A(|\nabla w_\lambda|)(\nabla w_\lambda, \nabla(w - w_\lambda)^+)\,\mathrm{d}x = \int_{B_{R_\lambda}} \frac{C}{|x_\lambda|^p}(w - w_\lambda)^+\mathrm{d}x
$$

we get, using the fact that $|x| > |x_\lambda|$ and (2.2), that

$$
\begin{aligned}
c_2\bar{c}\int_{B_{R_\lambda}} &(|\nabla w| + |\nabla w_\lambda|)^{p-2}|\nabla(w - w_\lambda)^+|^2\,\mathrm{d}x\\
&\leqslant C\int_{B_{R_\lambda}} \left(\frac{1}{|x|^p} - \frac{1}{|x_\lambda|^p}\right)(w - w_\lambda)^+\mathrm{d}x \leqslant 0.
\end{aligned}
$$

Then, $(w - w_\lambda)^+ = 0$, i.e. $w \leqslant w_\lambda\ \forall \lambda < 0$.

If we choose $\varphi = \varphi(|x|)$ as test function in weak formulation of (A.1), we get

$$
\int_{B_R} A(|\nabla w|)(\nabla w, \nabla\varphi(|x|))\,\mathrm{d}x = \int_{B_R} \frac{C}{|x|^p}\varphi(|x|)\,\mathrm{d}x.
$$

Passing in radial coordinates, for $r = |x|$, we get

$$
-(A(|w'|)w'r^{N-1})' = Cr^{N-1-p}, \quad r \in (0, R]. \tag{A.5}
$$

Since w is radial, we have proved that w is non-increasing with respect to r. Then w' is negative and $|w'(r)| = -w'(r)$.

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We rewrite (A.5) as

$$\frac{-(A(|w'|)w'r^{N-1})'}{(r^{N-p})'}(N-p) = C, \quad r \in (0,R).$$

Then

$$\lim_{r\to 0^+} \frac{-\left(A(|w'|)w'r^{N-1}\right)'}{(r^{N-p})'} = c \neq 0.$$

From (A.5), since $Cr^{N-1-p} \geqslant 0$, we have that $(-A(|w'|)w'r^{N-1})$ is a non-decreasing non negative function, hence

$$\lim_{r\to 0^+} -A(|w'|)w'r^{N-1} = \alpha \geqslant 0. \tag{A.6}$$

If $\alpha = 0$, we can apply de l'Hospital Theorem and we obtain

$$\lim_{r\to 0^+} \frac{-A(|w'|)w'r^{N-1}}{r^{N-p}} = c \tag{A.7}$$

that we can rewrite as

$$\lim_{r\to 0^+} A(|w'|)|w'|r^{p-1} = c. \tag{A.8}$$

Since $A \in C^1(\mathbb{R}^+)$, $c \neq 0$ and (2.4) holds, we have to consider only the cases $0 < |w'| < \tau$ with τ enough small and $|w'| \geqslant K$.

Using (A.8), we get

$$-A(|w'|)w'r^{p-1} = c + o(1), \text{ for } r \to 0^+$$

then, there exists $\tilde{c} > 0$ such that

$$-A(|w'|)w' \geqslant \frac{\tilde{c}}{r^{p-1}}, \text{ for } r \to 0^+. \tag{A.9}$$

If $|w'| \geqslant K$, using (1.3) in (A.9) one has

$$c_1|w'|^{p-1} \geqslant A(|w'|)|w'| = -A(|w'|)w' \geqslant \frac{\tilde{c}}{r^{p-1}}.$$

Let us consider $\varepsilon > 0$. Integrating on $[\varepsilon, R]$ the following

$$|w'| \geqslant \frac{\tilde{c}}{r},$$

we get

$$\int_\varepsilon^R -w' \,\mathrm{d}r \geqslant \int_\varepsilon^R \frac{\tilde{c}}{r} \,\mathrm{d}r \;\Rightarrow\; w(\varepsilon) \geqslant \tilde{c}\log\left(\frac{R}{\varepsilon}\right).$$

then $w(\varepsilon) \to +\infty$ if $\varepsilon \to 0^+$.

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If $0 < |w'| < \tau$, using (2.3) in (A.9) we get

$$A(1)|w'|^{m_A+1} \geqslant A(|w'|)|w'| = -A(|w'|)w' \geqslant \frac{\tilde{c}}{r^{p-1}}.$$

Let us consider $\varepsilon > 0$. Integrating on $[\varepsilon, R]$, one has that

$$|w'| \geqslant \frac{c}{r^{\frac{p-1}{m_A+1}}}$$

and, using (2.7), we obtain that $\frac{p-1}{m_A+1} \geqslant 1$ and then $w(\varepsilon) \to +\infty$ if $\varepsilon \to 0^+$.

If $\alpha > 0$, arguing in (A.6) as in (A.8), we consider only the cases $0 < |w'| < \tau$ with τ enough small and $|w'| \geqslant K$.

From (A.6), there exists $\tilde{\alpha} > 0$ such that

$$-A(|w'|)w' \geqslant \frac{\tilde{\alpha}}{r^{N-1}}. \tag{A.10}$$

If $|w'| \geqslant K$, using (1.3) in (A.10) one has

$$c_1|w'|^{p-1} \geqslant A(|w'|)|w'| = -A(|w'|)w' \geqslant \frac{\tilde{\alpha}}{r^{N-1}}.$$

Integrating on $[\varepsilon, R]$ the following

$$-w' = |w'| \geqslant \frac{c}{r^{\frac{N-1}{p-1}}},$$

since $\frac{N-1}{p-1} > 1$ we obtain $w(\varepsilon) \to +\infty$ if $\varepsilon \to 0^+$.

If $0 < |w'| < \tau$, using (2.3) in (A.10) we get

$$A(1)|w'|^{m_A+1} \geqslant A(|w'|)|w'| = -A(|w'|)w' \geqslant \frac{\tilde{\alpha}}{r^{N-1}}.$$

Considering $\varepsilon > 0$. Integrating on $[\varepsilon, R]$, one has

$$|w'| \geqslant \frac{c}{r^{\frac{N-1}{m_A+1}}},$$

and, using (2.7), we obtain that $\frac{N-1}{m_A+1} \geqslant 1$ and then $w(\varepsilon) \to +\infty$ if $\varepsilon \to 0^+$.

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