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TRIVIAL SOURCE CHARACTER TABLES OF $\mathrm{SL}_2(q)$, PART II

NIAMH FARRELL[1] AND CAROLINE LASSUEUR[2]

[1]*Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Hannover, Germany* (farrell@math.uni-hannover.de)
[2]*FB Mathematik, RPTU Kaiserslautern-Landau, Kaiserslautern, Germany* (lassueur@mathematik.uni-kl.de)

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Abstract We compute the trivial source character tables (also called species tables of the trivial source ring) of the infinite family of finite groups $\mathrm{SL}_2(q)$ for q even over a large enough field of odd characteristics. This article is a continuation of our article *Trivial Source Character Tables of* $\mathrm{SL}_2(q)$, where we considered, in particular, the case in which q is odd in non-defining characteristic.

Keywords: special linear group; trivial source modules; p-permutation modules; species tables; character theory; block theory; Brauer correspondence; Green correspondence

2020 Mathematics subject classification: Primary 19A22; 20C05; 20C15; 20C20; 20C33

1. Introduction

Let G be a finite group, let ℓ be a prime number dividing $|G|$ and let k be an algebraically closed field of characteristic ℓ. Permutation kG-modules and their direct summands, the trivial source modules, are omnipresent in the modular representation theory of finite groups. They are, for example, elementary building blocks for the construction and for the understanding of different categorical equivalences between block algebras, such as source-algebra equivalences, Morita equivalences with endo-permutation source, splendid Rickard equivalences or ℓ-permutation equivalences. A deep understanding of the structure of these modules is therefore essential.

The *trivial source character table of G at the prime ℓ*, denoted by $\mathrm{Triv}_\ell(G)$, is by definition the species table of the trivial source ring of kG in the sense of [3]. The present article is a sequel to [4], in which Böhmler and the authors calculate the trivial source character tables for the special linear group $\mathrm{SL}_2(q)$ over the finite field $\mathbb{F}_q$ when q and ℓ are odd and $\ell \nmid q$ and when q is odd, $\ell = 2$ and $\mathrm{SL}_2(q)$ has quaternion Sylow 2-subgroups. We refer the reader to the latter article for a complete introduction to trivial source character tables. We emphasize here that these table encapsulate, in a very compact way, a lot of information about the ordinary and Brauer characters of the trivial source kG-modules, as well as of those of their Brauer quotients.

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In this article, we calculate the trivial source character tables of $\mathrm{SL}_2(2^f) = \mathrm{PSL}_2(2^f)$ in non-defining characteristic for any integer $f \geq 2$. Note that if $f = 1$, then $\mathrm{SL}_2(2) \cong S_3$ and the trivial source character tables are easily calculated using elementary arguments (see e.g. [1]). Our main results, the generic trivial source character tables of $\mathrm{SL}_2(2^f)$, appear in Tables 8 and 9 for $\ell \mid (2^f - 1)$ and in Tables 14, 15 and 16 for $\ell \mid (2^f + 1)$.

The character table of $\mathrm{SL}_2(2^f)$ and the block distributions are given in [6]. However, it is more convenient for our purposes to interpret these data in terms of Harish-Chandra and Deligne-Lusztig induction, as [5] does for $\mathrm{SL}_2(q)$ with q odd. This done, one of the main issues we solve in this article is the explicit calculation of the Brauer correspondence in the normalizer N of a Sylow ℓ-subgroup of $\mathrm{SL}_2(2^f)$ and the explicit calculation of the Green correspondents in N of the trivial source $k\,\mathrm{SL}_2(2^f)$-modules.

The paper is organized as follows. In § 2, we recall the notation and definitions for trivial source character tables and for blocks with cyclic defect groups, which were established in [4]. Section 3 contains notation and preliminary results on the structure of $\mathrm{SL}_2(2^f)$. The trivial source character tables are calculated in § 4 for $\ell \mid (2^f - 1)$ and in § 5 for $\ell \mid (2^f + 1)$.

2. Notation and definitions

2.1. General notation

Throughout, unless otherwise stated, we adopt the notation and conventions given below. We let ℓ denote a prime number and G denote a finite group of order divisible by ℓ. We let $(K, \mathcal{O}, k)$ be an ℓ-modular system, where $\mathcal{O}$ denotes a complete discrete valuation ring of characteristic zero with unique maximal ideal $\mathfrak{p} := J(\mathcal{O})$, algebraically closed residue field $k = \mathcal{O}/\mathfrak{p}$ of characteristic ℓ and field of fractions $K = \mathrm{Frac}(\mathcal{O})$, which we assume to be large enough for G and its subgroups.

Given a positive integer n, we denote by C_n the cyclic group of order n. By an ℓ-block of G, we mean a block algebra of kG. We denote by $\mathrm{Irr}(G)$ (respectively, $\mathrm{Irr}(\mathbf{B})$) the set of irreducible K-characters of G (respectively, of the block of $\mathcal{O}G$ corresponding to the ℓ-block $\mathbf{B}$). We write $\mathbf{B}_0(G)$ for the principal ℓ-block of G. For a subgroup $H \leq G$, we let $[H]$ denote a set of representatives of the conjugacy classes of H, $[H/\equiv]$ a set of representatives of the conjugacy classes of H up to inverse, and $[H]_{\ell'}$ a set of representatives of the conjugacy classes of H of order prime to ℓ. If H is abelian, then we write $H^\wedge := \mathrm{Irr}(H)$ and $H = H_\ell \times H_{\ell'}$ is the decomposition of H into the product of its ℓ-part and of its ℓ'-part.

For $R \in \{\mathcal{O}, k\}$, RG-modules are assumed to be finitely generated left RG-lattices, that is, free as R-modules, and we let R denote the trivial RG-lattice. If M is a kG-module and $Q \leq G$, then the Brauer quotient (or Brauer construction) of M at Q is the k-vector space $M[Q] := M^Q / \sum_{R<Q} \mathrm{tr}_R^Q(M^R)$, where M^Q denotes the fixed points of M under Q and tr_R^Q for $R < Q$ denotes the relative trace map. This vector space has a natural structure of a $kN_G(Q)$-module, but also of a $kN_G(Q)/Q$-module, and is equal to zero if Q is not an ℓ-group. The abbreviation PIM means "projective indecomposable module". We refer the reader to [11, 14] for further standard notation and background results in the modular representation theory of finite groups.

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2.2. Trivial source character tables

Given $R \in \{\mathcal{O}, k\}$, an RG-lattice M is called a *trivial source* RG-lattice if it is isomorphic to an indecomposable direct summand of an induced lattice $\mathrm{Ind}_Q^G R$ and if Q is of minimal order subject to this property, then Q is a vertex of M. Any trivial source kG-module M lifts in a unique way to a trivial source $\mathcal{O}G$-lattice $\widehat{M}$ (see, e.g. [2, Corollary 3.11.4]), and we denote by $\chi_{\widehat{M}}$ the K-character afforded by $\widehat{M}$. Up to isomorphism, there are only finitely many trivial source kG-modules (see, e.g. [4, Proposition 2.2 (d)]), and we will study them vertex by vertex. We denote by $\mathrm{TS}(G;Q)$ the set of isomorphism classes of indecomposable trivial source kG-modules with vertex Q. We let $a(kG, \mathrm{Triv})$ be the *trivial source ring* of kG, which is defined to be the subring of the Grothendieck ring of kG generated by the set of all isomorphism classes of indecomposable trivial source kG-modules.

By definition, the *trivial source character table of the group G at the prime ℓ*, denoted $\mathrm{Triv}_\ell(G)$, is the species table (or representation table) of the trivial source ring of kG in the sense of Benson and Parker; see [3]. However, as in [4], we follow [12, Section 4.10] and consider $\mathrm{Triv}_\ell(G)$ as the block square matrix defined by the following notational convention.

Convention 2.1. *First, fix a set of representatives $Q_1, \ldots, Q_r$ ($r \in \mathbb{N}$) for the conjugacy classes of ℓ-subgroups of G, where $Q_1 := \{1\}$ and $Q_r \in \mathrm{Syl}_\ell(G)$. For each $1 \leq v \leq r$, set $N_v := N_G(Q_v)$, $\overline{N}_v := N_G(Q_v)/Q_v$. Then, for each pair (Q_v, s) with $1 \leq v \leq r$ and $s \in [\overline{N}_v]_{\ell'}$ there is a ring homomorphism*

$$\begin{array}{cccc} \tau^G_{Q_v,s}: & a(kG, Triv) & \longrightarrow & K \\ & [M] & \mapsto & \varphi_{M[Q_v]}(s) \end{array}$$

mapping the class of a trivial source kG-module M to the value at s of the Brauer character $\varphi_{M[Q_v]}$ of the Brauer quotient $M[Q_v]$. For each $1 \leq i, v \leq r$, define a matrix

$$T_{i,v} := \left[\tau^G_{Q_v,s}([M])\right]_{M \in \mathrm{TS}(G;Q_i), s \in [\overline{N}_v]_{\ell'}}.$$

The trivial source character table of G at the prime ℓ is then the block matrix

$$Triv_\ell(G) := [T_{i,v}]_{1 \leq i,v \leq r}.$$

Moreover, the rows of $Triv_\ell(G)$ are labelled with the ordinary characters $\chi_{\widehat{M}}$ instead of the isomorphism classes of trivial source modules M themselves.

We note that the group G acts by conjugation on the pairs (Q_v, s), and the values of $\tau^G_{Q_v,s}$ do not depend on the choice of (Q_v, s) in its G-orbit.

We refer the reader to Section 2 of our previous paper [4] for details and further properties of trivial source modules and trivial source character tables. Moreover, a more detailed and elementary introduction to this class of modules is available in the survey article [9, Sections 3–4].

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2.3. Blocks with cyclic defect groups

In the cases considered in this article, we need to describe trivial source modules lying in blocks with cyclic defect groups. Therefore, we recall the following essential notions about cyclic blocks. Further details can be found in the first part of our paper [4] and also in [7].

Given an ℓ-block $\mathbf{B}$ of kG with a non-trivial cyclic defect group $D \cong C_{\ell^n}$ $(n \geq 1)$, we let $D_1 < D$ denote the subgroup of D of order ℓ, and we let e denote the inertial index of $\mathbf{B}$. We write

$$\mathrm{Irr}(\mathbf{B}) = \mathrm{Irr}'(\mathbf{B}) \sqcup \{\chi_\lambda \mid \lambda \in \Lambda\},$$

where $\mathrm{Irr}'(\mathbf{B})$ is the set of the non-exceptional K-characters (there are e of them) of $\mathbf{B}$ and $|\Lambda| = \frac{|D|-1}{e}$. If $|\Lambda| > 1$, then $\{\chi_\lambda \mid \lambda \in \Lambda\}$ is the set of exceptional K-characters of $\mathbf{B}$, which all restrict in the same way to the ℓ-regular conjugacy classes of G. Further, we set $\chi_\Lambda := \sum_{\lambda\in\Lambda} \chi_\lambda$. The Brauer tree of $\mathbf{B}$ is then the graph $\sigma(\mathbf{B})$ with vertices labelled by $\mathrm{Irr}^\circ(\mathbf{B}) := \mathrm{Irr}'(\mathbf{B}) \sqcup \{\chi_\Lambda\}$ and edges labelled by the simple $\mathbf{B}$-modules. If $|\Lambda| > 1$, the vertex corresponding to χ_Λ is called the *exceptional vertex* and is indicated with a filled black circle in our drawings of $\sigma(\mathbf{B})$.

Vertices and sources of indecomposable modules are encoded in a source algebra of a block, and hence so are the trivial source $\mathbf{B}$-modules. We recall that by the work of Linckelmann [10], a source algebra of $\mathbf{B}$ is determined up to isomorphism of interior D-algebras by three parameters:

(1) $\sigma(\mathbf{B})$, understood with its planar embedding;
(2) a *type function* associating a sign to each vertex in an alternating way as follows: if x is a generator of D_1, then a vertex $\chi \in \mathrm{Irr}^\circ(\mathbf{B})$ of $\sigma(\mathbf{B})$ is said to be *positive* if $\chi(x) > 0$, whereas it is said to be *negative* if $\chi(x) < 0$;
(3) an indecomposable capped endo-permutation kD-module $W(\mathbf{B})$; more precisely letting $\mathbf{b}$ be the Brauer correspondent of $\mathbf{B}$ in $N_G(D_1)$, then $W(\mathbf{B})$ is defined to be a source of the simple $\mathbf{b}$-modules.

It turns out that $\mathbf{B}$ contains precisely e trivial source kG-modules for each possible vertex $Q \leq D$. These trivial source modules are explicitly classified by [7, Theorem 5.3] as a function of the three parameters above. We refer to [4, Remark 2.2] for a summary of this classification, relevant to the cyclic blocks of $\mathrm{SL}_2(q)$.

3. Structure and characters of $\mathbf{SL_2}(q)$ when q is even

From now on, and until the end of this article, we assume that $\ell \neq 2$. Moreover, we let $G := \mathrm{SL}_2(q) = \mathrm{PSL}_2(q)$ be the special linear group of degree 2 over the finite field $\mathbb{F}_q$, where $q = 2^f$ for some integer $f \geq 2$. Given a positive integer r, let μ_r be the group of the rth roots of unity in an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$. For a subset of $S \subseteq \mathbb{F}^\times$, let $[S/\equiv]$ denote a set of representatives for the elements of S up to inverse.

In this section, we collect all necessary information about G and its subgroups needed in order to calculate the trivial source character tables of G in cross-characteristic. Our

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aim is to use notation analogous to that used in [5]. However, [5] cannot be cited directly as it is assumed throughout the book that q is odd. We note that the notation and some arguments need small adjustments when q is even. We also refer to the unpublished master thesis of Schulte [13] where further details, but not all, can be found.

3.1. Tori, centralizers and normalizers of ℓ-elements

We let $T := \{\mathrm{diag}(a, a^{-1}) \mid a \in \mathbb{F}_q^\times\}$ be the maximally split torus of G consisting of the diagonal matrices. Then, there exists an isomorphism

$$\mathbf{d} : \mu_{q-1} = \mathbb{F}_q^\times \longrightarrow T, a \mapsto \mathrm{diag}(a, a^{-1}).$$

Fixing an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^2}$ induces a group isomorphism $\mathbf{d}' : \mathrm{GL}_{\mathbb{F}_q}(\mathbb{F}_{q^2}) \longrightarrow \mathrm{GL}_2(\mathbb{F}_q)$. The image $T' := \mathbf{d}'(\mu_{q+1})$ of μ_{q+1} under this isomorphism is a non-split torus of G. We will identify T with μ_{q-1} and T' with μ_{q+1} via $\mathbf{d}$ and $\mathbf{d}'$ without further mention. As q is even, both T and T' are cyclic groups of odd order. We let S_ℓ and S'_ℓ denote Sylow ℓ-subgroups of T and T', respectively, and thus we have a decomposition into direct products $T = S_\ell \times T_{\ell'}$ and $T' = S'_\ell \times T'_{\ell'}$.

Finally, we consider the Frobenius automorphism $F : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}, x \mapsto x^q$, which we see as an element of $\mathrm{GL}_{\mathbb{F}_q}(\mathbb{F}_{q^2})$. We fix

$$\sigma := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \quad \text{and} \quad \sigma' := \mathbf{d}'(F),$$

which are clearly both of order 2.

The following two lemmas are well known and can be proved using elementary arguments similar to those used in [5, Sections 1.3 and 1.4].

Lemma 3.1.

(a) *If* $g = \mathbf{d}(a)$ *with* $a \in \mu_{q-1} \setminus \{1\}$, *then* $C_G(g) = T$. *In particular,* $C_G(T) = T$.
(b) *If* $g = \mathbf{d}'(\xi)$ *with* $\xi \in \mu_{q+1} \setminus \{1\}$, *then* $C_G(g) = T'$. *In particular,* $C_G(T') = T'$.

Lemma 3.2.

(a) *If* $\ell \mid (q-1)$, *then* $S_\ell \in \mathrm{Syl}_\ell(G)$ *and* $N_G(Q) = N_G(T) = \langle T, \sigma\rangle =: N$ *for any* $1 \lneq Q \leq S_\ell$.
(b) *If* $\ell \mid (q+1)$, *then* $S'_\ell \in \mathrm{Syl}_\ell(G)$ *and* $N_G(Q) = N_G(T') = \langle T', \sigma'\rangle =: N'$ *for any* $1 \lneq Q \leq S'_\ell$.

3.2. Characters and conjugacy classes of G

The conjugacy classes, the ordinary characters and the character table of G were known to Schur and are given in [6, Sections I and II]. We use notation analogous to that used in [5] for the case where q is odd, however, as this is more convenient for our purposes. In order to do this, we fix the following.

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Table 1. Character table of $\mathrm{SL}_2(q)$ *when* q *is even.*

	$\boldsymbol{I_2}$	$\mathbf{d}(a)$ $(a \in \Gamma)$	$\mathbf{d}'(\xi)$ $(\xi \in \Gamma')$	$\boldsymbol{u}$
No. of classes	1	$\frac{q-2}{2}$	$\frac{q}{2}$	1
Order of g	1	$o(a)$	$o(\xi)$	2
Class size	1	$q(q+1)$	$q(q-1)$	$(q-1)(q+1)$
1_G	1	1	1	1
St	q	1	-1	0
$R(\alpha)$ $(\alpha \in [T^\wedge/\equiv], \alpha \neq 1)$	$q+1$	$\alpha(a)+\alpha(a^{-1})$	0	1
$R'(\theta)$ $(\theta \in [T'^\wedge/\equiv], \theta \neq 1)$	$q-1$	0	$-\theta(\xi)-\theta(\xi^{-1})$	-1

- Let 1_G and St denote the trivial character and the Steinberg character of G, respectively.
- Let $R : \mathbb{Z}\,\mathrm{Irr}(T) \longrightarrow \mathbb{Z}\,\mathrm{Irr}(G)$ and $R' : \mathbb{Z}\,\mathrm{Irr}(T') \longrightarrow \mathbb{Z}\,\mathrm{Irr}(G)$ denote Harish-Chandra induction and Deligne-Lusztig induction, respectively.
- Set $\Gamma := [(\mu_{q-1} \setminus \{1\})/\equiv]$, $\Gamma' := [(\mu_{q+1} \setminus \{1\})/\equiv]$.
- Fix the following set of representatives for the conjugacy classes of G

$$\{I_2\} \;\dot\cup\; \{u\} \;\dot\cup\; \{\mathbf{d}(a) \mid a \in \Gamma\} \;\dot\cup\; \{\mathbf{d}'(\xi) \mid \xi \in \Gamma'\},$$

where $u := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is an element of order 2.

There are $q-2$ non-trivial characters $\alpha \in \mathrm{Irr}(T)$, all satisfying $R(\alpha) = R(\alpha^{-1}) \in \mathrm{Irr}(G)$, giving us $\frac{q-2}{2}$ irreducible characters in $\mathrm{Irr}(G)$. Similarly, the q non-trivial characters $\theta \in \mathrm{Irr}(T')$ satisfy $R'(\theta) = R'(\theta^{-1}) \in \mathrm{Irr}(G)$, giving us $\frac{q}{2}$ irreducible characters of G. Hence,

$$\mathrm{Irr}(G) = \{1_G, \mathrm{St}\} \cup \{R(\alpha) \mid \alpha \in [T^\wedge/\equiv], \alpha \neq 1\} \cup \{R'(\theta) \mid \theta \in [T'^\wedge/\equiv], \theta \neq 1\},$$

and the character table of G is as given in Table 1.

3.3. Characters and conjugacy classes of N and N'

We adopt here notation for the character theory of N and N' analogous to the notation used in [5, Sections 6.2.1 and 6.2.2] for the case in which q is odd. First, we fix the following sets of representatives for the conjugacy classes of N and N', respectively.

$$\{I_2\} \cup \{\sigma\} \cup \{\mathbf{d}(a) \mid a \in \Gamma\} \qquad \{I_2\} \cup \{\sigma'\} \cup \{\mathbf{d}'(\xi) \mid \xi \in \Gamma'\}.$$

The difference with the odd case is that when q is even, T has no non-trivial N-invariant characters and T' has no non-trivial N'-invariant characters. For $\alpha \in \mathrm{Irr}(T) \setminus \{1\}$ (respectively, $\theta \in \mathrm{Irr}(T') \setminus \{1\}$), we let χ_α be the unique element of $\mathrm{Irr}(N)$ such that $\chi_\alpha = \mathrm{Ind}_T^N(\alpha) = \mathrm{Ind}_T^N(\alpha^{-1})$ (respectively, we let χ'_θ be the unique element of $\mathrm{Irr}(N')$

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Table 2. Character table of N.

	I_2	$\mathbf{d}(a)$ $(a \in \Gamma)$	σ
No. of classes	1	$\frac{q-2}{2}$	1
Order of g	1	$o(a)$	2
Class size	1	2	$q-1$
$\mathbf{C_N(g)}$	N	T	$\langle\sigma\rangle$
1_N	1	1	1
ε	1	1	-1
χ_α	2	$\alpha(a)+\alpha(a^{-1})$	0

Table 3. Character table of N′.

	I_2	$\mathbf{d}'(\xi)$ $(\xi \in \Gamma')$	σ'
No. of classes	1	$\frac{q}{2}$	1
Order of g	1	$o(\xi)$	2
Class size	1	2	$q+1$
$\mathbf{C_{N'}(g)}$	N'	T'	$\langle\sigma'\rangle$
$1_{N'}$	1	1	1
ε'	1	1	-1
χ'_θ	2	$\theta(\xi)+\theta(\xi^{-1})$	0

such that $\chi'_\theta = \mathrm{Ind}_{T'}^{N'}(\theta) = \mathrm{Ind}_{T'}^{N'}(\theta^{-1}))$. We let ε (respectively, ε') denote the linear character of N (respectively, of N') of order 2. With this notation, the character tables of N and N' are as follows.

4. Trivial source character table of G when $\ell \mid (q-1)$

Notation 4.1. In order to describe $\mathrm{Triv}_\ell(G)$ according to Convention 2.1, we adopt the following notation. We fix $Q_{n+1} := S_\ell \cong C_{\ell^n}$ and for each $1 \le i \le n$, we let Q_i denote the unique cyclic subgroup of Q_{n+1} of order ℓ^{i-1}. The chain of subgroups

$$\{1\} = Q_1 \le \cdots \le Q_{n+1} \in \mathrm{Syl}_\ell(G)$$

is then our fixed set of representatives for the conjugacy classes of ℓ-subgroups of G.

We fix $\Gamma_{\ell'} = [((\mu_{q-1})_{\ell'} \setminus \{1\})/\equiv]$ and $\Gamma'_{\ell'} = [((\mu_{q+1})_{\ell'} \setminus \{1\})/\equiv]$. Note that here $\Gamma'_{\ell'} = \Gamma'$ as $\ell \nmid q+1$. We fix the following set of representatives for the ℓ'-conjugacy classes of G:

$$[G]_{\ell'} := \{I_2\} \cup \{u\} \cup \{\mathbf{d}(a) \mid a \in \Gamma_{\ell'}\} \cup \{\mathbf{d}'(\xi) \mid \xi \in \Gamma'_{\ell'}\}.$$

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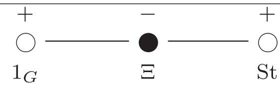
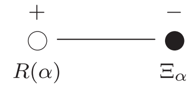
Table 4. The ℓ-blocks of $\mathrm{SL}_2(q)$ when $\ell \mid (q-1)$ and q are even.

Block $\boldsymbol{B}$	*Number of blocks (type)*	*Defect groups*	*Brauer tree with type function or* $\mathrm{Irr}(\mathbf{B})$
$\mathbf{B}_0(G)$	*1 (Principal)*	C_{ℓ^n}	 $\Xi := \sum_{\eta \in [S_\ell^\wedge/\equiv], \eta \neq 1} R(\eta)$
A_α ($\alpha \in [T_{\ell'}^\wedge/\equiv]$, $\alpha \neq 1$)	$\frac{(q-1)_{\ell'}-1}{2}$ *(Nilpotent)*	C_{ℓ^n}	 $\Xi_\alpha := \sum_{\eta \in S_\ell^\wedge \setminus \{1\}} R(\alpha\eta)$
A'_θ ($\theta \in [T'^\wedge_{\ell'}/\equiv]$, $\theta \neq 1$)	$\frac{(q+1)_{\ell'}-1}{2} = \frac{q}{2}$ *(Defect zero)*	$\{1\}$	$\mathrm{Irr}(A'_\theta) = \{R'(\theta)\}$

For any $2 \leq v \leq n+1$, $1 \leq i \leq n+1$, the columns of $T_{i,v}$ are labelled by a set of representatives for the ℓ'-conjugacy classes of $\overline{N}_v = N_G(Q_v)/Q_v = N/Q_v$ as $N_G(Q_v) = N_G(T) = N$ for each $2 \leq v \leq n+1$ by Lemma 3.2(a). However, since Q_v is an ℓ-group, we will simply label the columns of $T_{i,v}$ by the following fixed set of representatives for the ℓ'-conjugacy classes of N

$$[N]_{\ell'} := \{I_2\} \cup \{\sigma\} \cup \{\mathbf{d}(a) \mid a \in \Gamma_{\ell'}\}.$$

Moreover, in order to describe the exceptional characters occurring as constituents of the trivial source characters, for each $0 \leq i \leq n$, we fix

$$\pi_{q,i} := \frac{(q-1)_\ell \cdot \ell^{-i} - 1}{2},$$

we let $\pi_q := \pi_{q,0}$ and note that $\pi_{q,n} = 0$. These numbers arise naturally from the classification of the trivial source modules in cyclic blocks in [7].

4.1. The $\boldsymbol{\ell}$-blocks and trivial source characters of $\boldsymbol{G}$

Lemma 4.2. *When $\ell \mid (q-1)$, the ℓ-blocks of G, their defect groups and their Brauer trees with type function are as given in Table 4.*

Proof. All of the information in the table comes directly from [6, Section I] and the character table of G (Table 1), except for the type functions on the Brauer trees, which we compute according to Equation (2) in § 2.3. The trivial character is clearly positive so the type function for the principal block is immediate. For each block A_α, the ℓ'-character α takes the value 1 on ℓ-elements and therefore $R(\alpha)$ is positive. □

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Table 5. Trivial source characters of $\mathrm{SL}_2(q)$ *when* $\ell \mid (q-1)$.

Vertices of M	*Character* $\chi_{\widehat{M}}$	*Block containing M*
$\{1\}$	$1_G + \Xi, \mathrm{St} + \Xi$	$\mathbf{B}_0(G)$
	$R(\alpha) + \Xi_\alpha$	$A_\alpha\ (\alpha \in [T_{\ell'}^{\wedge}/\equiv], \alpha \neq 1)$
	$R'(\theta)$	$A'_\theta\ (\theta \in [T_{\ell'}'^{\wedge}/\equiv], \theta \neq 1)$
$C_{\ell^i}\ (1 \leq i < n)$	$1_G + \Xi_i,\ \mathrm{St} + \Xi_i$	$\mathbf{B}_0(G)$
	$R(\alpha) + \Xi_{\alpha,i}$	$A_\alpha\ (\alpha \in [T_{\ell'}^{\wedge}/\equiv], \alpha \neq 1)$
C_{ℓ^n}	$1_G,\ \mathrm{St}$	$\mathbf{B}_0(G)$
	$R(\alpha)$	$A_\alpha\ (\alpha \in [T_{\ell'}^{\wedge}/\equiv], \alpha \neq 1)$

Lemma 4.3. *When* $\ell \mid (q-1)$*, the ordinary characters* $\chi_{\widehat{M}}$ *of the trivial source kG-modules M are as given in Table 5, where for each* $1 \leq i \leq n$,

$$\Xi_i = \sum_{j=1}^{\pi_{q,i}} R(\eta_j)$$

is a sum of $\pi_{q,i}$ *pairwise distinct exceptional characters in* $\mathbf{B}_0(G)$*, and for any non-trivial character* $\alpha \in \mathrm{Irr}(T_{\ell'})$,

$$\Xi_{\alpha,i} = \sum_{j=1}^{2\pi_{q,i}} R(\alpha\eta_j)$$

is a sum of $2\pi_{q,i}$ *pairwise distinct exceptional characters in* A_α.

Proof. First, the ordinary characters of the PIMs lying in blocks of defect zero are immediate from Table 4, and the characters of the PIMs lying in blocks with a non-trivial cyclic defect group can also be read off from Table 4, e.g. using [4, Remark 2.6(a)].

The trivial source kG-modules with non-trivial vertices $C_{\ell^i} (1 \leq i \leq n)$ all belong to ℓ-blocks $\mathbf{B}$ with a non-trivial cyclic defect group. By § 2.3, each such block contains precisely e trivial source kG-modules with vertex C_{ℓ^i}, where e is the inertial index of the block. Moreover, in order to determine these modules up to isomorphism, we need parameters (1), (2) and (3) of § 2.3, namely the Brauer trees with their type function, which are given in Table 4, and the module $W(\mathbf{B})$, which is always trivial in our case by [8, Proposition 6.5(a)]. Thus, the characters $\chi_{\widehat{M}}$ listed in Table 5 are obtained by applying the classification of the trivial source modules given in [7, Theorem 5.3(b)(2) and Theorem A.1(d)], exactly as in [4, Lemma 3.3]. □

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Table 6. The ℓ-blocks of N *when $\ell \mid (q-1)$ and* q *are even.*

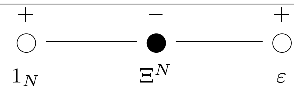
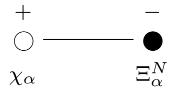
Block $\mathbf{b}$	Number of blocks (type)	Defect groups	Brauer tree $\sigma(\mathbf{b})$ with type function
$\mathbf{B}_0(N)$	1 (Principal)	C_{ℓ^n}	 $\Xi^N := \sum_{\eta \in [S_\ell^\wedge/\equiv], \eta \neq 1} \chi_\eta$
$\mathbf{b}_\alpha(\alpha \in [T_{\ell'}^\wedge/\equiv], \alpha \neq 1)$	$\frac{(q-1)_{\ell'}-1}{2}$ (Nilpotent)	C_{ℓ^n}	 $\Xi_\alpha^N := \sum_{\eta \in S_\ell^\wedge \setminus \{1\}} \chi_{\alpha\eta}$

Table 7. The trivial source characters of the kN-*Green correspondents when $\ell \mid (q-1)$.*

Vertices of M	Character $\chi_{\widehat{M}}$	Character $\chi_{\widehat{f(M)}}$ of the Green correspondent
$C_{\ell^i} (1 \leq i < n)$	$1_G + \Xi_i$	$1_N + \Xi_i^N$
	$\mathrm{St} + \Xi_i$	$\varepsilon + \Xi_i^N$
	$R(\alpha) + \Xi_{\alpha,i} \quad (\alpha \in [T_{\ell'}^\wedge/\equiv], \alpha \neq 1)$	$\chi_\alpha + \Xi_{\alpha,i}^N$
C_{ℓ^n}	1_G	1_N
	St	ε
	$R(\alpha)\ (\alpha \in [T_{\ell'}^\wedge/\equiv],\ \alpha \neq 1)$	χ_α

4.2. The ℓ-blocks and trivial source characters of N

Lemma 4.4. *When $\ell \mid (q-1)$, the ℓ-blocks of N, their defect groups and their Brauer trees with type function are as given in Table 6.*

Proof. We determine the partition of $\mathrm{Irr}(N)$ into ℓ-blocks of N by examining the central characters of N modulo ℓ, and we find that

- $\mathrm{Irr}(\mathbf{B}_0(N))$ contains 1_N, ε and $\{\chi_\eta \mid \eta \in \mathrm{Irr}(S_\ell) \setminus \{1\}\}$; and
- for each non-trivial $\alpha \in \mathrm{Irr}(T_{\ell'})$, there exists a block $\mathbf{b}_\alpha$ containing $\chi_{\alpha\eta}$ for all $\eta \in \mathrm{Irr}(S_\ell)$.

Since all the blocks have maximal normal defect groups, their Brauer trees are star-shaped with a central exceptional vertex (see, e.g. [2, Proposition 6.5.4]). The Brauer trees are therefore fully determined because the ℓ-rational characters 1_N, ε and $\chi_\alpha(\alpha \in \mathrm{Irr}(T_{\ell'}))$

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must be non-exceptional. The type functions, as defined in Equation (2) of § 2.3, are immediate in this case, as the cyclic subgroups of order ℓ of the defect groups are normal in N by Lemma 3.2(a). □

Lemma 4.5. *When* $\ell \mid (q-1)$*, the ordinary characters* $\chi_{\widehat{f(M)}}$ *of the kN-Green correspondents f(M) of the trivial source kG-modules M with a non-trivial vertex are as given in Table 7, where, for each* $1 \le i \le n$*,*

$$\Xi_i^N := \sum_{j=1}^{\pi_{q,i}} \chi_{\eta_j}$$

is a sum of $\pi_{q,i}$ *pairwise distinct exceptional characters in* $\mathrm{Irr}(\mathbf{B}_0(N))$*, and for any non-trivial* $\alpha \in \mathrm{Irr}(T_{\ell'})$*,*

$$\Xi_{\alpha,i}^N := \sum_{j=1}^{2\pi_{q,i}} \chi_{\alpha\eta_j}$$

is a sum of $2\pi_{q,i}$ *pairwise distinct exceptional characters in* $\mathrm{Irr}(\mathbf{b}_\alpha)$*.*

Proof. We first determine the Brauer correspondents in N of the blocks of G. We claim that for a fixed $\alpha \in T_{\ell'}^{\wedge}$, $\alpha \neq 1$, the block $\mathbf{b}_\alpha$ is the Brauer correspondent in N of the A_α. Let i_α be the central primitive idempotent of $\mathcal{O}G$ such that $\mathcal{O}Gi_\alpha$ is the block of $\mathcal{O}G$ corresponding to the ℓ-block A_α. Thus,

$$i_\alpha := \sum_{\eta \in S_\ell^\wedge} \sum_{g \in G} \frac{1}{|G|} R(\alpha\eta)(1) R(\alpha\eta)(g) g^{-1} = \sum_{g \in G} \frac{q+1}{|G|} \sum_{\eta \in S_\ell^\wedge} R(\alpha\eta)(g) g^{-1}$$

When we apply the Brauer homomorphism to $\bar{i}_\alpha$, the image of i_α in kG, the only terms in the sum that survive are those for $g \in C_G(S_\ell) = T$. The coefficient in i_α of a non-trivial element $\mathbf{d}(a^{-1}) \in T$ for some $a \in \mu_{q-1} \setminus \{1\}$ is

$$\frac{q+1}{|G|} \sum_{\eta \in S_\ell^\wedge} R(\alpha\eta)(\mathbf{d}(a)) = \frac{q+1}{|G|} \left(\alpha(a) \sum_{\eta \in S_\ell^\wedge} \eta(a) + \alpha(a^{-1}) \sum_{\eta \in S_\ell^\wedge} \eta(a^{-1}) \right).$$

If a has non-trivial ℓ-part, then the second orthogonality relations show that $\sum_{\eta \in S_\ell^\wedge} \eta(a) = 0$. Thus, the only elements in T with non-zero coefficients in i_α are the ℓ'-elements, and they have coefficients

$$\begin{cases} \frac{(q+1)^2 |S_\ell|}{|G|} = \frac{q+1}{q|T_{\ell'}|} & \text{if } a = 1, \text{ and} \\ \frac{(q+1)|S_\ell|}{|G|} \left(\alpha(a) + \alpha(a^{-1})\right) = \frac{1}{q|T_{\ell'}|} \left(\alpha(a) + \alpha(a^{-1})\right) & \text{if } 1 \neq a \in (\mu_{q-1})_{\ell'}. \end{cases}$$

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Now let i_α^N denote the central primitive idempotent of $\mathcal{O}N$ such that $\mathcal{O}Ni_\alpha^N$ is the block of $\mathcal{O}N$ corresponding to the ℓ-block $\mathbf{b}_\alpha$. Then

$$i_\alpha^N = \sum_{\eta\in S_\ell^\wedge}\sum_{n\in N}\frac{1}{|N|}\chi_{\alpha\eta}(1)\chi_{\alpha\eta}(n)n^{-1} = \sum_{n\in N}\frac{2}{|N|}\sum_{\eta\in S_\ell^\wedge}\chi_{\alpha\eta}(n)n^{-1}.$$

Since $\chi_{\alpha\eta}$ is linear and $\chi_{\alpha\eta}(\sigma)=0$, in fact, this sum only has non-zero terms for $n\in T$. By the same arguments as above, the only elements $\mathbf{d}(a^{-1})\in T$ with non-zero coefficients in i_α^N are the ℓ'-elements and they have coefficients

$$\begin{cases} \frac{4|S_\ell|}{|N|}=\frac{2}{|T_{\ell'}|} & \text{if } a=1, \text{ and}\\ \frac{2|S_\ell|}{|N|}\left(\alpha(a)+\alpha(a^{-1})\right)=\frac{1}{|T_{\ell'}|}\left(\alpha(a)+\alpha(a^{-1})\right) & \text{if } 1\neq a\in(\mu_{q-1})_{\ell'}.\end{cases}$$

Let $\bar{i}_\alpha^N$ denote the image of i_α^N in kN. Then since $\ell\mid(q-1)$, we have $q\equiv 1 \mod \mathfrak{p}$. Therefore, $\bar{i}_\alpha^N \equiv \mathrm{Br}_{S_\ell}(\bar{i}_\alpha) \mod \mathfrak{p}$, where Br_{S_ℓ} denotes the Brauer homomorphism. In particular, $\mathbf{b}_\alpha$ is the Brauer correspondent in N of A_α.

Next, we recall that the Brauer correspondence and the Green correspondence commute, so if a trivial source kG-module lies in the block $\mathbf{B}$, then its Green correspondent lies in the Brauer correspondent $\mathbf{b}$ of $\mathbf{B}$. Moreover, by definition, $W(\mathbf{b})=W(\mathbf{B})$, which as we already noticed is trivial in all cases. Therefore, the characters of the trivial source $\mathbf{b}$-modules can be determined using [7, Theorem 5.3] in all cases, as we did for G. This yields the list of characters in the third column of Table 7, up to reordering. Therefore, it only remains to check that the characters printed on the same lines of the second and third columns are the characters of Green correspondent modules. For $\mathbf{B}_0(G)$, it is enough to notice that if a trivial source module of kG has the trivial character as a constituent of its ordinary character, then so does its kN-Green correspondent. Since both A_α and $\mathbf{b}_\alpha$ have to contain a unique trivial source module with a given vertex, there is only one possibility for the blocks of type A_α, as required. □

4.3. The trivial source character table of G

Theorem 4.6. *Let $G=\mathrm{SL}_2(q)$ with $q=2^f$ for an integer $f\geq 2$ and suppose that $\ell\mid(q-1)$. Then, with notation as in Notation 4.1, the trivial source character table $\mathrm{Triv}_\ell(G)=[T_{i,v}]_{1\leq i,v\leq n+1}$ is given as follows:*

(a) $T_{i,v}=\mathbf{0}$ *if* $v>i$;
(b) *the matrices* $T_{i,1}$ *are as given in Table 8 for each* $1\leq i\leq n+1$;
(c) *the matrices* $T_{i,i}$ *are as given in Table 9 for each* $2\leq i\leq n+1$; *and*
(d) $T_{i,v}=T_{i,i}$ *for all* $2\leq v<i\leq n+1$.

Proof. By Convention 2.1, the labels for the rows of $\mathrm{Triv}_\ell(G)$ are the ordinary characters of the trivial source kG-modules determined in Lemma 4.3.

(a) It follows from [4, Remark 2.5(c)] and Notation 4.1 that $T_{i,v}=\mathbf{0}$ whenever $v>i$.

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Table 8. $T_{i,1}$ *for* $1 \leq i \leq n+1$.

		$\boldsymbol{I_2}$	$\mathbf{d}(a)$ $(a \in \Gamma_{\ell'})$	$\mathbf{d}'(\xi)$ $(\xi \in \Gamma'_{\ell'})$	$\boldsymbol{u}$
$T_{1,1}$	$1_G + \Xi$	$1+(q+1)\pi_q$	$1+2\pi_q$	1	$1+\pi_q$
	$\mathrm{St} + \Xi$	$q+(q+1)\pi_q$	$1+2\pi_q$	-1	π_q
	$R(\alpha) + \Xi_\alpha$ $\left(\alpha \in [T_{\ell'}^{\wedge}/\equiv], \alpha \neq 1\right)$	$(q+1)(1+2\pi_q)$	$(\alpha(a)+\alpha(a^{-1}))(1+2\pi_q)$	0	$1+2\pi_q$
	$R'(\theta)$ $\left(\theta \in [T'^{\wedge}_{\ell'}/\equiv], \theta \neq 1\right)$	$q-1$	0	$-\theta(\xi)-\theta(\xi^{-1})$	-1
$T_{i,1}(1 \leq i \leq n)$	$1_G + \Xi_{i-1}$	$1+(q+1)\pi_{q,i-1}$	$1+2\pi_{q,i-1}$	1	$1+\pi_{q,i-1}$
	$\mathrm{St} + \Xi_{i-1}$	$q+(q+1)\pi_{q,i-1}$	$1+2\pi_{q,i-1}$	-1	$\pi_{q,i-1}$
	$R(\alpha) + \Xi_{\alpha,i-1}$ $(\alpha \in [T_{\ell'^{\wedge}}/\equiv], \alpha \neq 1)$	$(q+1)(1+2\pi_{q,i-1})$	$(\alpha(a)+\alpha(a^{-1}))(1+2\pi_{q,i-1})$	0	$1+2\pi_{q,i-1}$
$T_{n+1,1}$	1_G	1	1	1	1
	St	q	1	-1	0
	$R(\alpha)$ $(\alpha \in [T_{\ell'^{\wedge}}/\equiv], \alpha \neq 1)$	$q+1$	$\alpha(a)+\alpha(a^{-1})$	0	1

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Table 9. $T_{i,i}$ for $2 \le i \le n+1$.

	$\boldsymbol{I_2}$	$\mathbf{d}(a)$ $(a \in \Gamma_{\ell'})$	$\boldsymbol{\sigma}$
$1_G + \Xi_{i-1}$	$1 + 2\pi_{q,i-1}$	$1 + 2\pi_{q,i-1}$	1
$\mathrm{St} + \Xi_{i-1}$	$1 + 2\pi_{q,i-1}$	$1 + 2\pi_{q,i-1}$	-1
$R(\alpha)$ + $\Xi_{\alpha,i-1}$ $\left(\alpha \in [T_{\ell'}^{\wedge}/\equiv], \alpha \neq 1\right)$	$2(1 + 2\pi_{q,i-1})$	$(\alpha(a) + \alpha(a^{-1}))(1 + 2\pi_{q,i-1})$	0

Table 10. The ℓ-blocks of $\mathrm{SL}_2(q)$ when $\ell \mid (q+1)$.

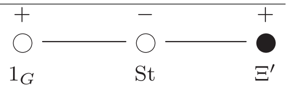
Block $\mathbf{B}$	Number of blocks (type)	Defect groups	Brauer tree with type function or $\mathrm{Irr}(\mathbf{B})$
$\mathbf{B}_0(G)$	1 (Principal)	C_{ℓ^n}	 $\Xi' := \sum_{\eta \in [S_{\ell}'^{\wedge}/\equiv], \eta \neq 1} R'(\eta)$
$A'_\theta(\theta \in [T_{\ell'}'^{\wedge}/\equiv], \theta \neq 1)$	$\frac{(q+1)_{\ell'}-1}{2}$ (Nilpotent)	C_{ℓ^n}	 $\Xi'_\theta := \sum_{\eta \in S_{\ell}'^{\wedge} \setminus \{1\}} R'(\theta\eta)$
$A_\alpha(\alpha \in [T_{\ell'}^{\wedge}/\equiv], \alpha \neq 1)$	$\frac{(q-1)_{\ell'}-1}{2} = \frac{q-2}{2}$ (Defect zero)	$\{1\}$	$\mathrm{Irr}(A_\alpha) = \{R(\alpha)\}$

Table 11. Trivial source characters of $\mathrm{SL}_2(q)$ when $\ell \mid (q+1)$.

Vertices of M	Character $\chi_{\widehat{M}}$	Block containing M
$\{1\}$	$1_G + \mathrm{St}$, $\mathrm{St} + \Xi'$	$\mathbf{B}_0(G)$
	$R'(\theta) + \Xi'_\theta$	A'_θ $(\theta \in [T_{\ell'}'^{\wedge}/\equiv], \theta \neq 1)$
	$R(\alpha)$	A_α $(\alpha \in [T_{\ell'}^{\wedge}/\equiv], \alpha \neq 1)$
C_{ℓ^i} $(1 \le i < n)$	$1_G + \mathrm{St} + \Xi'_i$, $\mathrm{St} + \Xi'_i$	$\mathbf{B}_0(G)$
	$\Xi'_{\theta,i}$	A'_θ $(\theta \in [T_{\ell'}'^{\wedge}/\equiv], \theta \neq 1)$
C_{ℓ^n}	1_G, Ξ'	$\mathbf{B}_0(G)$
	Ξ'_θ	A'_θ $(\theta \in [T_{\ell'}'^{\wedge}/\equiv], \theta \neq 1)$

(b) By [4, Remark 2.5(d)], the values in $T_{i,1}$ for $1 \le i \le n+1$ (Table 8) are calculated by evaluating the character of each trivial source module given in Table 5 at the relevant representatives of the ℓ'-conjugacy classes of G using the character table of G (Table 1).

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(c) By Convention 2.1, the values in $T_{i,i}$ for $2 \leq i \leq n+1$ (Table 9) are given by the values of the species $\tau^G_{Q_i,s}$, with s running through $[\overline{N}_i]_{\ell'}$ (identified here with $[N]_{\ell'}$), evaluated at the trivial source modules $[M] \in \mathrm{TS}(G;Q_i)$. By definition of the species and [4, Proposition 2.2(d)], these are calculated by evaluating the ordinary character of the kN-Green correspondent given in Table 7 of the trivial source kG-module labelling the relevant row, at the representatives of the ℓ'-conjugacy classes of N using the character table of N given in Table 2.

(d) For each $2 \leq v \leq i \leq n+1$, by Convention 2.1, the matrix $T_{i,v}$ consists of the values of the species $\tau^G_{Q_v,s}$, with s running through $[\overline{N}_v]_{\ell'}$ (identified here with $[N]_{\ell'}$), evaluated at the trivial source modules $[M] \in \mathrm{TS}(G;Q_i)$. However, by definition of the species, $\tau^G_{Q_v,s}([M]) = \chi_{\widehat{M[Q_v]}}(s)$ and [4, Lemma 2.8] together with Lemma 3.2(a) show that $M[Q_v]$ is the kN-Green correspondent of M. Hence, $T_{i,v} = T_{i,i}$ for all $2 \leq v < i \leq n+1$. □

5. Trivial source character table of G when $\ell \mid (q+1)$

We now adopt notation analogous to Notation 4.1 in order to describe $\mathrm{Triv}_\ell(G)$ according to Convention 2.1. Here, we fix $Q_{n+1} := S'_\ell \cong C_{\ell^n}$. Then, as before, for each $1 \leq i \leq n$, we let Q_i denote the unique cyclic subgroup of Q_{n+1} of order ℓ^{i-1}, and

$$\{1\} = Q_1 \leq \cdots \leq Q_{n+1} \in \mathrm{Syl}_\ell(G)$$

is our fixed set of representatives for the conjugacy classes of ℓ-subgroups of G. We keep the same set of representatives for the ℓ'-conjugacy classes of G:

$$[G]_{\ell'} := \{I_2\} \cup \{u\} \cup \{\mathbf{d}(a) \mid a \in \Gamma_{\ell'}\} \cup \{\mathbf{d}'(\xi) \mid \xi \in \Gamma'_{\ell'}\},$$

where $\Gamma_{\ell'}$ and $\Gamma'_{\ell'}$ are as defined in Notation 4.1. Note that in this case $\Gamma_{\ell'} = \Gamma$ as $\ell \nmid q-1$. We fix the following set of representatives for the ℓ'-conjugacy classes of N':

$$[N']_{\ell'} := \{I_2\} \cup \{\sigma'\} \cup \{\mathbf{d}'(\xi) \mid \xi \in \Gamma'_{\ell'}\}.$$

By the same arguments as in Notation 4.1, for any $2 \leq v \leq n+1$ and any $1 \leq i \leq n+1$, we can label the columns of $T_{i,v}$ by this fixed set of representatives for the ℓ'-conjugacy classes of N'. Finally, for each $0 \leq i \leq n$, we fix

$$\pi'_{q,i} := \frac{(q+1)_\ell \cdot (1-\ell^{-i})}{2}, \quad \pi''_{q,i} := \frac{(q+1)_\ell - 1}{2} - \pi'_{q,i} = \frac{(q+1)_\ell \cdot \ell^{-i} - 1}{2},$$

and let $\pi'_q := \pi'_{q,n}$. Again, these numbers arise naturally in the classification of the trivial source modules in blocks with cyclic defect goups in [7].

5.1. The ℓ-blocks and trivial source characters of G

Lemma 5.2. *When $\ell \mid (q+1)$, the ℓ-blocks of G, their defect groups and their Brauer trees with type function are as given in Table 10.*

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Proof. As in Lemma 4.2, all of the information in the table comes directly from [6, Section II] and the character table of G (Table 1), except for the type functions on the Brauer trees, which we compute according to Equation (2) in § 2.3. The trivial character is once again positive so the type function for the principal block is immediate, and for each block A'_θ, the character $R'(\theta)$ takes a negative value on all non-trivial ℓ-elements and is therefore negative. □

Lemma 5.3. *When $\ell \mid (q+1)$, the ordinary characters $\chi_{\widehat{M}}$ of the trivial source kG-modules M are as given in Table 11, where for each $1 \leq i < n$,*

$$\Xi'_i := \sum_{j=1}^{\pi'_{q,i}} R'(\eta_j)$$

is a sum of $\pi'_{q,i}$ pairwise distinct exceptional characters in $\mathrm{Irr}(\mathbf{B}_0(G))$ and for any non-trivial $\theta \in \mathrm{Irr}(T'_{\ell'})$,

$$\Xi'_{\theta,i} := \sum_{j=1}^{2\pi'_{q,i}} R'(\theta\eta_j)$$

is a sum of $2\pi'_{q,i}$ pairwise distinct exceptional characters in $\mathrm{Irr}(A'_\theta)$.

Proof. The arguments are analogous to those given in the proof of Lemma 4.3. In this case, the parameters (1), (2) and (3) of § 2.3 necessary to apply the classification of the trivial source modules given in [7, Theorem 5.3] are the Brauer trees with their type function given in Table 10 and the module $W(\mathbf{B})$, which is also always trivial in this case by [8, Proposition 6.5(a)]. The characters $\chi_{\widehat{M}}$ are then obtained exactly as in the proof of [4, Lemma 4.3]. □

5.2. The ℓ-blocks and trivial source characters of N'

Lemma 5.4. *When $\ell \mid (q+1)$, the blocks of N', their defect groups and their Brauer trees with type function are as given in Table 12.*

Proof. The distribution of the characters of N' into blocks can be determined by examining the values of the central characters of N' modulo ℓ:

- the principal block $\mathbf{B}_0(N')$ contains $1_{N'}$, ε' and χ'_η for each $\eta \in S'^{\wedge}_\ell \setminus \{1\}$; and
- for each non-trivial $\theta \in \mathrm{Irr}(T_{\ell'})$, there is a block $\mathbf{b}'_\theta$ containing $\chi'_{\theta\eta}$ for all $\eta \in \mathrm{Irr}(S'_\ell)$.

The Brauer trees and their type functions are determined using arguments analogous to those in Lemma 4.4, where in this case we note that $1_{N'}$, ε' and χ'_θ ($\theta \in \mathrm{Irr}(T'_{\ell'})$) are ℓ-rational characters and therefore cannot be exceptional. □

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Table 12. The ℓ*-blocks of* N' *when* $\ell \mid (q+1)$ *and* q *is even.*

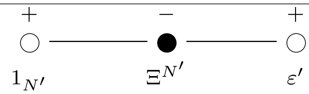
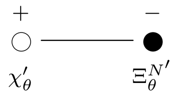
Block $\mathbf{b}$	Number of blocks (type)	Defect groups	Brauer tree $\sigma(\mathbf{b})$ with type function
$\mathbf{B}_0(N')$	1 (Principal)	C_{ℓ^n}	 $\Xi^{N'} := \sum_{\eta \in [S_\ell'^{\wedge}/\equiv], \eta \neq 1} \chi'_\eta$
$\mathbf{b}'_\theta(\theta \in [T_{\ell'}'^{\wedge}/\equiv], \theta \neq 1)$	$\frac{(q+1)_{\ell'}-1}{2}$ (Nilpotent)	C_{ℓ^n}	 $\Xi_\theta^{N'} := \sum_{\eta \in S_\ell'^{\wedge} \setminus \{1\}} \chi'_{\theta\eta}$

Table 13. The trivial source characters of the kN'*-Green correspondents when* $\ell \mid (q+1)$.

Vertices of M	Character $\chi_{\widehat{M}}$	Character $\chi_{\widehat{f(M)}}$ of the Green correspondent
C_{ℓ^i} $(1 \leq i < n)$	$1_G + \mathrm{St} + \Xi'_i$	$1_{N'} + \Xi_i^{N'}$
	$\mathrm{St} + \Xi'_i$	$\varepsilon' + \Xi_i^{N'}$
	$\Xi'_{\theta,i}$ $(\theta \in [T_{\ell'}'^{\wedge}/\equiv], \theta \neq 1)$	$\chi'_\theta + \Xi_{\theta,i}^{N'}$
C_{ℓ^n}	1_G	$1_{N'}$
	Ξ'	ε'
	Ξ'_θ $(\theta \in [T_{\ell'}'^{\wedge}/\equiv],\ \theta \neq 1)$	χ'_θ

Lemma 5.5. *When* $\ell \mid (q+1)$, *the ordinary characters* $\chi_{\widehat{f(M)}}$ *of the* kN'*-Green correspondents f(M) of the trivial source kG-modules M with a non-trivial vertex are as given in Table 13, where for each* $1 \leq i < n$,

$$\Xi_i^{N'} := \sum_{j=1}^{\pi''_{q,i}} \chi'_\eta$$

is a sum of $\pi''_{q,i}$ *pairwise distinct exceptional characters in* $\mathrm{Irr}(\mathbf{B}_0(N'))$, *and for any non-trivial* $\theta \in \mathrm{Irr}(T'_{\ell'})$,

$$\Xi_{\theta,i}^{N'} := \sum_{j=1}^{2\pi''_{q,i}} \chi'_{\theta\eta}$$

is a sum of $2\pi''_{q,i}$ *pairwise distinct exceptional characters of* $\mathrm{Irr}(\mathbf{b}'_\theta)$.

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Proof. This proof is analogous to the proof of Lemma 4.5. We first determine the Brauer correspondents in N' of the nilpotent blocks of G. Fix a non-trivial $\theta \in \mathrm{Irr}(T'_{\ell'})$. Let i'_θ denote the central primitive idempotent of $\mathcal{O}G$ such that $\mathcal{O}Gi'_\theta$ is the block of $\mathcal{O}G$ corresponding to the ℓ-block A_θ, and let $i_\theta^{N'}$ denote the central primitive idempotent of $\mathcal{O}N'$ such that $\mathcal{O}N'i_\theta^{N'}$ is the block of $\mathcal{O}N'$ corresponding to the ℓ-block $\mathbf{b}'_\theta$. As in Lemma 4.5, we need only compare the coefficients of elements $\mathbf{d}'(\xi^{-1}) \in T'$ in i'_θ and $i_\theta^{N'}$. For any $\xi \in \mu_{q+1}$ with non-trivial ℓ-part, the coefficient of $\mathbf{d}'(\xi^{-1}) \in T'$ is 0 in both i'_θ and $i_\theta^{N'}$. If ξ is an ℓ'-element, then the coefficient of $\mathbf{d}'(\xi^{-1})$ in i'_θ is

$$\begin{cases} \frac{(q-1)^2|S'_\ell|}{|G|} = \frac{q-1}{q|T'_{\ell'}|} & \text{for } \xi = 1, \\ \frac{(q-1)|S'_\ell|}{|G|}\left(-\theta(\xi) - \theta(\xi^{-1})\right) = \frac{1}{q|T'_{\ell'}|}\left(-\theta(\xi) - \theta(\xi^{-1})\right) & \text{for } \xi \neq 1, \xi \in (\mu_{q+1})_{\ell'}, \end{cases}$$

and the coefficient in $i_\theta^{N'}$ is

$$\begin{cases} \frac{4|S'_\ell|}{|N'|} = \frac{2}{|T'_{\ell'}|} & \text{for } \xi = 1, \\ \frac{2|S'_\ell|}{|N'|}\left(\theta(\xi) + \theta(\xi^{-1})\right) = \frac{1}{|T'_{\ell'}|}\left(\theta(\xi) + \theta(\xi^{-1})\right) & \text{for } \xi \neq 1, \xi \in (\mu_{q+1})_{\ell'}. \end{cases}$$

Since $\ell \mid (q+1)$, we have $\frac{q-1}{q} \equiv 2 \mod \mathfrak{p}$ and $\frac{1}{q} \equiv -1 \mod \mathfrak{p}$, and therefore $\mathrm{Br}_{S'_\ell}(\overline{i'_\theta}) \equiv \bar{i}_\theta^{N'} \mod \mathfrak{p}$ so $\mathbf{b}'_\theta$ is the Brauer correspondent of A'_θ in N'.

As mentioned in Lemma 4.5, the Green correspondent of a trivial source kG-module in a block $\mathbf{B}$ of G lies in the Brauer correspondent block $\mathbf{b}$ of N', and since $W(\mathbf{B}) = W(\mathbf{b})$ is trivial in all cases, the characters of the trivial source $\mathbf{b}$-modules can be determined using [7, Theorem 5.3]. Moreover, having determined the Brauer correspondent blocks, in each case, there is only one possible choice for the Green correspondent of a trivial source module of kG with a fixed vertex. The characters of these Green correspondent modules are as in Table 13. □

5.3. The trivial source character table of G

Theorem 5.6. *Let $G = \mathrm{SL}_2(q)$ with $q = 2^f$ for some integer $f \geq 2$, and suppose that $\ell \mid (q+1)$. Then, with notation as in Notation 5.1, the trivial source character table $\mathrm{Triv}_\ell(G) = [T_{i,v}]_{1 \leq i,v \leq n+1}$ is given as follows:*

(a) $T_{i,v} = \mathbf{0}$ if $v > i$;
(b) the matrices $T_{i,1}$ are as given in Table 14 for each $1 \leq i \leq n+1$;
(c) the matrices $T_{i,i}$ are as given in Table 15 for each $2 \leq i \leq n$;
(d) the matrix $T_{n+1,n+1}$ is as given in Table 16; and
(e) $T_{i,v} = T_{i,i}$ for all $2 \leq v \leq i \leq n+1$.

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Table 14. $T_{i,1}$ *for* $1 \le i \le n+1$.

		I_2	$\mathbf{d}(a)(a \in \Gamma_{\ell'})$	$\mathbf{d}'(\xi)(\xi \in \Gamma'_{\ell'})$	u
$T_{1,1}$	$1_G + \mathrm{St}$	$1+q$	2	0	1
	$\mathrm{St} + \Xi'$	$q+(q-1)\pi'_q$	1	$-1-2\pi'_q$	$-\pi'_q$
	$R'(\theta)$ + Ξ'_θ $\left(\theta \in [T'^{\wedge}_{\ell'}/\equiv], \theta^2 \neq 1\right)$	$(q-1)(1+2\pi'_q)$	0	$-(\theta(\xi)+\theta(\xi^{-1}))(1+2\pi'_q)$	$-(1+2\pi'_q)$
	$R(\alpha)$ $\left(\alpha \in [T^{\wedge}_{\ell'}/\equiv], \alpha^2 \neq 1\right)$	$q+1$	$\alpha(a)+\alpha(a^{-1})$	0	1
$T_{i,1}(2 \le i \le n)$	$1_G + \mathrm{St} + \Xi'_{i-1}$	$1+q+(q-1)\pi'_{q,i-1}$	2	$-2\pi'_{q,i-1}$	$1-\pi'_{q,i-1}$
	$\mathrm{St} + \Xi'_{i-1}$	$q+(q-1)\pi'_{q,i-1}$	1	$-1-2\pi'_{q,i-1}$	$-\pi'_{q,i-1}$
	$\Xi'_{\theta,i-1}$ $\left(\theta \in [T'^{\wedge}_{\ell'}/\equiv], \theta \neq 1\right)$	$2(q-1)\pi'_{q,i-1}$	0	$-2\left(\theta(\xi)+\theta(\xi^{-1})\right)\pi'_{q,i-1}$	$-2\pi'_{q,i-1}$
$T_{n+1,1}$	1_G	1	1	1	1
	Ξ'	$(q-1)\pi'_q$	0	$-2\pi'_q$	$-\pi'_q$
	Ξ'_θ $\left(\theta \in [T'^{\wedge}_{\ell'}/\equiv], \theta \neq 1\right)$	$2(q-1)\pi'_q$	0	$-2\pi'_q(\theta(\xi)+\theta(\xi^{-1}))$	$-2\pi'_q$

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Table 15. $T_{i,i}$ *for* $2 \leq i \leq n$.

	$\boldsymbol{I_2}$	$\mathbf{d}'(\xi) \quad (\xi \in \Gamma'_{\ell'})$	$\boldsymbol{\sigma'}$
$1_G + \mathrm{St} + \Xi'_{i-1}$	$1 + 2\pi''_{q,i-1}$	$1 + 2\pi''_{q,i-1}$	1
$\mathrm{St} + \Xi'_{i-1}$	$1 + 2\pi''_{q,i-1}$	$1 + 2\pi''_{q,i-1}$	-1
$\Xi'_{\theta,i-1}\left(\theta \in [T'^{\wedge}_{\ell'}/\equiv], \theta \neq 1\right)$	$2(1 + 2\pi''_{q,i-1})$	$\left(\theta(\xi) + \theta(\xi^{-1})\right)(1 + 2\pi''_{q,i-1})$	0

Table 16. $T_{n+1,n+1}$.

	$\boldsymbol{I_2}$	$\mathbf{d}'(\xi)\ (\xi \in \Gamma')$	$\boldsymbol{\sigma'}$
1_G	1	1	1
Ξ'	1	1	-1
$\Xi'_\theta\left(\theta \in [T'^{\wedge}_{\ell'}/\equiv], \theta \neq 1\right)$	2	$\theta(\xi) + \theta(\xi^{-1})$	0

Proof. The calculations are completely analogous to those in the proof of Theorem 4.6, except that we take the trivial source kG-modules from Table 11, their Green correspondents from Table 13 and use the character table of N' from Table 3. □

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References

(1) D. J. Benson, *Modular representation theory: new trends and methods*, Lecture Notes in Mathematics, Volume **1081** (Springer-Verlag, Berlin, 1984).

(2) D. J. Benson, *Representations and cohomology I*, Cambridge Studies in Advanced Mathematics, 2nd edn., Volume **30** (Cambridge University Press, Cambridge, 1998).

(3) D. J. Benson and R. A. Parker, The Green ring of a finite group, *J. Algebra* **87** (1984), 290–331.

(4) B. Böhmler, N. Farrell and C. Lassueur, Trivial source character tables of $\mathrm{SL}_2(\mathrm{q})$, *J. Algebra* **598** (2022), 308–350.

(5) C. Bonnafé, *Representations of* $\mathrm{SL}_2(\mathbb{F}_\mathrm{q})$, Algebra and Applications, Volume 13 (Springer-Verlag London, Ltd, London, 2011).

(6) R. Burkhardt, Die Zerlegungsmatrizen der Gruppen $\mathrm{PSL}(2, \mathrm{p}^\mathrm{f})$, *J. Algebra* **40** (1976), 75–96.

(7) G. Hiss and C. Lassueur, The classification of the trivial source modules in blocks with cyclic defect groups, *Algebr. Represent. Theory* **24** (2021), 673–698.

(8) G. Hiss and C. Lassueur, On the source algebra equivalence class of blocks with cyclic defect groups, I, *Beitr. Algebra Geom.* (2023), 10.1007/s13366-022-00681-9.

(9) C. Lassueur, A tour of p-permutation modules and related classes of modules, *Jahresber. Dtsch. Math.-Ver.* (2023), 10.1365/s13291-023-00266-y.

(10) M. Linckelmann, The isomorphism problem for cyclic blocks and their source algebras, *Invent. Math.* **125** (1996), 265–283.

(11) M. Linckelmann, *The block theory of finite group algebras*, London Mathematical Society Student Texts 92, Volume II (Cambridge University Press, Cambridge, 2018).

(12) K. Lux and H. Pahlings, Representations of groups, Cambridge Studies in Advanced Mathematics, Volume **124** (Cambridge University Press, Cambridge, 2010).

(13) E. Schulte, Simple endotrivial modules for finite simple groups. Diplomarbeit, TU Kaiserslautern, 2012.

(14) J. Thévenaz, *G-algebras and modular representation theory*, Oxford Mathematical Monographs (The Clarendon Press, Oxford University Press, New York, 1995).

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