Probability in the Engineering and Informational Sciences, **34**, 2020, 583–604.
doi:10.1017/S0269964819000159

RELATIONSHIPS BETWEEN IMPORTANCE MEASURES AND REDUNDANCY IN SYSTEMS WITH DEPENDENT COMPONENTS

Jorge Navarro and Pedro Fernández-Martínez
Universidad de Murcia, Murcia, Spain
E-mail: jorgenav@um.es

Juan Fernández-Sánchez
Universidad de Almería, Almería, Spain

Antonio Arriaza
Universidad de Cádiz, Cádiz, Spain

The paper shows the connections between some importance indices for the components in an engineering coherent system and the performance of the system obtained when a redundancy mechanism is applied to a specific component. A copula approach is used to model the dependency among the components. This approach includes the popular case of independent components. Under some assumptions, it is proved that if component i is more important than component j, then the system obtained by applying a redundancy procedure to the ith component is better, under different stochastic criteria, than that obtained with the jth component. These results can be applied to several redundancy mechanisms. A new importance index is defined to study active redundancies. Some illustrative examples are provided.

Keywords: coherent systems, copula, distorted distributions, importance measures, redundancy

1. INTRODUCTION

Coherent systems are basic structures in reliability engineering. They can be used to represent both simple systems, with few components, and really complex systems. From a mathematical point-of-view, a two-states *system* with n components is a Boolean function $\phi : \{0,1\}^n \to \{0,1\}$. Here $\phi(x_1,\ldots,x_n)$ denotes the state of the system and x_i denotes the state of the ith component for $i = 1,\ldots,n$ ($x_i = 1$ means that the ith component is working and $x_i = 0$ that it is not). Note that the state of the system is completely determined by the states of the components. A system ϕ is *coherent* if it is increasing, that is, when the state of a component is improved, the state of the system cannot be worse, and every component is relevant for the system, that is, ϕ is strictly increasing in each variable in at least a point.

 0269-9648/19 $25.00

Several importance measures have been defined in the literature to study the influence of the components in the performance of the system. Many of them assume that the component lifetimes are independent. The most relevant importance measures and their applications can be seen in Barlow and Proschan [2] and Belzunce, Martínez-Riquelme, and Ruiz [13]. A combinatorial approach to compute component importance indices in coherent systems with independent components was given in Gertsbakh and Shpungin [9].

Note that, in many practical situations, the component lifetimes are dependent since they share the same environment. So, in these cases, the independence assumption is unrealistic. However, few importance measures have been proposed in the case of dependent components. For example, the extension of Barlow–Proschan importance index was studied in Iyer [11] and Marichal and Mathonet [14] and the Birnbaum measure in Miziula and Navarro [16] and Zhang and Wilson [22]. In the two last references, a copula approach was used to represent the dependence among the component lifetimes. A different dependence model based on sequential order statistics was proposed in Burkschat and Navarro [4]. Some information measures were also applied to systems with dependent components in Cali, Longobardi, and Navarro [5] and the references therein.

Another relevant problem in the theory of coherent systems is to determine the performance of the system when redundancy mechanisms are applied to some components. There exist several usual options such as active redundancy, standby redundancy, perfect, and imperfect repairs, etc. The literature on this topic is really wide (see, e.g., the recent papers [1,3] and the references therein).

In the present paper, we study the connections between the importance measure defined in Miziula and Navarro [16] and the systems obtained by applying redundancy options to the different components in the system. Additional importance measures are proposed for active redundancies as well. The same procedure can be used to define new indices for other redundancy options. Under some assumptions, we show that if component i is more important than component j, then the system obtained by applying a redundancy procedure to the ith component is better, under different stochastic criteria, than that obtained with the jth component. These results can be applied to several redundancy mechanisms.

The rest of the paper is organized as follows. The main results for the index defined in Miziula and Navarro [16] are placed in Section 2 where we also propose and study a new importance measure for active redundancies. Some illustrative examples and counterexamples can be found in Section 3. In Section 4 we include some conclusions and tasks for future research.

Throughout the paper, we say that a function $G : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is increasing (resp. decreasing) if $G(\mathbf{x}) \leq G(\mathbf{y})$ ($\geq$) for all $\mathbf{x} \leq \mathbf{y}$, where $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in A$ and $\mathbf{x} \leq \mathbf{y}$ means $x_i \leq y_i$ for all $i = 1, \ldots, n$. The partial derivative of G with respect to its ith variable will be represented as $\partial_i G$ (assuming tacitly that it exists). Furthermore, $\mathbf{e}_k$ is the kth vector of the canonical basis of $\mathbb{R}^n$, that is, $\mathbf{e}_k = (\delta_{1,k}, \ldots, \delta_{n,k})$ being $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

2. MAIN RESULTS

Let T be the lifetime of a coherent system based on n possibly dependent components with lifetimes $X_1, \ldots, X_n$. For the definition and basic properties of coherent systems, we refer the reader to the classic book [2]. The reliability functions of T and $X_1, \ldots, X_n$ will be represented as $\bar{F}_T(t) = \Pr(T > t)$ and $\bar{F}_i(t) = \Pr(X_i > t)$ for $i = 1, \ldots, n$. Whenever we assume that the component lifetimes are identically distributed (ID), the common reliability function will be represented as $\bar{F}$.

It is well known that the reliability function of a coherent system can be obtained from the reliability functions of its components as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)), \quad \textbf{(2.1)}$$

where $\bar{Q} : [0,1]^n \to [0,1]$ is an increasing continuous function, called *distortion function*, such that $\bar{Q}(0,\ldots,0) = 0$ and $\bar{Q}(1,\ldots,1) = 1$. The explicit expression for $\bar{Q}$ can be seen in, for example, formula (2.3) of Miziula and Navarro [15]. The distortion function $\bar{Q}$ depends on both the structure of the system and the dependence structure among the components, that is, the copula function associated to the component lifetimes. If the components are independent (i.e., the copula is the product copula), then $\bar{Q}$ is a multinomial (see, e.g., Barlow and Proschan [2], p. 21).

Moreover, since $\bar{Q}$ is a linear combination of Lipschitz functions (copulas), then the functions $\bar{q}_{\mathbf{x},i} : [0,1] \to [0,1]$, defined by

$$\bar{q}_{\mathbf{x},i}(u) := \bar{Q}(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n)$$

for $u \in [0,1]$, $i = 1, \ldots, n$ and $\mathbf{x} = (x_1, \ldots, x_n) \in (0,1)^n$ are Lipschitz functions with a common constant. Therefore, the subset D_i of $(0,1)^n$ where $\partial_i \bar{Q}$ exists has measure one. Hence, the subset $D = \cap_{i=1}^n D_i$ of $(0,1)^n$ where all the partial derivatives $\partial_1 \bar{Q}, \ldots, \partial_n \bar{Q}$ exist, has also measure one.

Recently, a new importance index based on (2.1) was defined in Miziula and Navarro [16] for systems with dependent components. This index extends the well-known Birnbaum index defined for systems with independent component (see, e.g., Barlow and Proschan [2]). It is defined as follows. The main properties of this index can be seen in Miziula and Navarro [16].

DEFINITION 2.1: *The* **importance index** I_j *for the* j*th component in a coherent system with distortion function* $\bar{Q}$ *is*

$$I_j(\mathbf{u}) = \partial_j \bar{Q}(\mathbf{u})$$

for all $\mathbf{u} = (u_1, \ldots, u_n) \in D_j$. *We say that component* i *is* **more important** *than component* j *(shortly written as* $i \geq_{mi} j$*) if* $I_i(\mathbf{u}) \geq I_j(\mathbf{u})$ *for all* $\mathbf{u} \in D_i \cap D_j$. *We say that component* i *is* **strictly more important** *than component* j *(shortly written as* $i >_{mi} j$*) if* $i \geq_{mi} j$ *and* $I_i(\mathbf{u}) > I_j(\mathbf{u})$ *for at least a point* $\mathbf{u} \in D_i \cap D_j$.

The reliability of a system can be improved by adding some redundancy mechanisms to some components. Typically, what we do is to improve the reliability function $\bar{F}_j$ of the jth component, obtaining a new reliability function $\bar{G}_j \geq \bar{F}_j$. In many cases, the new reliability function can be written as $\bar{G}_j(t) = \bar{q}(\bar{F}_j(t))$ for all t, where $\bar{q} : [0,1] \to [0,1]$ is an increasing continuous function such that $\bar{q}(0) = 0$, $\bar{q}(1) = 1$ and $\bar{q}(u) \geq u$ for all $u \in [0,1]$. We shall assume throughout the paper that this redundancy mechanism is the same for all the components, that is, $\bar{q}$ does not depend on j. Usually, we will have an explicit expression for the *redundancy function* $\bar{q}$. Some relevant and useful examples will be provided later.

Now we are ready to state the main result of the paper which proves that the importance index defined above determines the best way to apply any redundancy (as defined above) to a system with ID components. First, we need two technical lemmas. From now on we use the notation

$$[\mathbf{0}, \mathbf{1}] := [0,1]^n = [0,1] \times \cdots \times [0,1].$$

The set $(\mathbf{0}, \mathbf{1})$ is defined in a similar way. The first lemma contains the well-known Rademacher's theorem (see, e.g., Theorem 3.1.6 of Federer [8]).

Lemma 2.2 (Rademacher's theorem): *If A is an open subset of $\mathbb{R}^n$ and $f: A \to \mathbb{R}$ is Lipschitz continuous, then f is differentiable almost everywhere in A.*

Lemma 2.3: *Let $G: [\mathbf{0},\mathbf{1}] \to \mathbb{R}$ be a Lipschitz continuous function. Then, the following conditions are equivalent:*

(i) $\partial_i G(\mathbf{x}) \geq \partial_j G(\mathbf{x})$ *for all* $\mathbf{x} \in D_i \cap D_j$.

(ii) $G(\mathbf{x}+c\ \mathbf{e}_i) \geq G(\mathbf{x}+c\ \mathbf{e}_j)$ *for all* $\mathbf{x} \in [\mathbf{0},\mathbf{1}]$ *and all* $c>0$ *such that* $\mathbf{x}+c\ \mathbf{e}_i, \mathbf{x}+c\ \mathbf{e}_j \in [\mathbf{0},\mathbf{1}]$.

Proof: Without loss of generality, we may suppose (to simplify the notation) that $i=1$ and $j=2$.

(i) $\Rightarrow$ (ii) Assume that $\partial_1 G(\mathbf{z}) \geq \partial_2 G(\mathbf{z})$ for all $\mathbf{z} \in D_1 \cap D_2$.

Fix the values $x_3,\ldots,x_n \in [0,1]$ and define the function $H: [0,1]\times[0,1] \to \mathbb{R}$ as $H(x_1,x_2) := G(x_1,\ldots,x_n)$. Clearly, H is also a Lipschitz continuous function. So, from the preceding lemma, H is differentiable almost everywhere in $(0,1)\times(0,1)$. Let A be the set where H is differentiable. Then the Lebesgue measure of A is $\lambda(A)=1$. Note that for the points $(x_1,x_2)\in A$, the partial derivative with respect to the vector $v=(1,-1)$ exists and is given by $\partial_1 G(\mathbf{x}) - \partial_2 G(\mathbf{x})$ where $\mathbf{x}=(x_1,\ldots,x_n)$.

Consider now for $x \in [0,\sqrt{2}]$ the segment S_x perpendicular to the main diagonal of $[0,1]\times[0,1]$ (i.e., parallel to vector v) determined by the border points of this square and that passes through the point $(x/\sqrt{2}, x/\sqrt{2})$. The length of this segment is $f(x) = 2\min(x, \sqrt{2}-x)$. Note that

$$\int_0^{\sqrt{2}} f(x)dx = 1.$$

Define now the function $\ell: [0,\sqrt{2}] \to \mathbb{R}$ where $\ell(x)$ is the (unidimensional) measure of the points of A included in S_x. Hence

$$\int_0^{\sqrt{2}} \ell(x)dx = \lambda(A) = 1.$$

Moreover, since $0 \leq \ell(x) \leq f(x)$ for all $x \in [0,\sqrt{2}]$, then $\ell = f$ almost everywhere (a.e.). Let $B \subseteq [0,\sqrt{2}]$ be the set where these two functions coincide.

Fix now $x_1, x_2 \in [0,1]$ and $c>0$ such that $0 \leq x_1 < x_1+c \leq 1$ and $0 \leq x_2 < x_2 + c \leq 1$. Let S_x be the segment that contains the points (x_1, x_2+c) and (x_1+c, x_2) and let us assume that $x \in B$ (i.e., that H is differentiable a.e. in S_x). Then consider the function

$$g(t) := G(x_1+t, x_2+c-t, x_3,\ldots,x_n)$$

for $t\in[0,c]$. Clearly, g is Lipschitz continuous and so absolutely continuous in $[0,c]$, it is differentiable almost everywhere in $(0,c)$ and (when it exists) its derivative satisfies

$$g'(t) = \partial_1 G(x_1+t, x_2+c-t, x_3,\ldots,x_n) - \partial_2 G(x_1+t, x_2+c-t, x_3,\ldots,x_n) \geq 0.$$

Therefore, g is increasing in $[0,c]$. In particular,

$$g(0) = G(x_1, x_2+c, x_3,\ldots,x_n) \leq g(c) = G(x_1+c, x_2, x_3,\ldots,x_n)$$

and (ii) holds for all those points. This property can be extended to the rest of the points by using that G is continuous in $[\mathbf{0},\mathbf{1}]$ and that B is dense in $[0,\sqrt{2}]$.

https://doi.org/10.1017/S0269964819000159 Published online by Cambridge University Press

(ii) $\Rightarrow$ (i) Assume that for some $\mathbf{x} = (x_1, \ldots, x_n) \in D_1 \cap D_2$

$$G(x_1, x_2 + c, x_3, \ldots, x_n) \leq G(x_1 + c, x_2, \ldots, x_n)$$

holds for all $c > 0$ such that $0 < x_1 < x_1 + c \leq 1$ and $0 < x_2 < x_2 + c \leq 1$. Then

$$\frac{G(x_1, x_2 + c, x_3, \ldots, x_n) - G(x_1, \ldots, x_n)}{c} \leq \frac{G(x_1 + c, x_2, \ldots, x_n) - G(x_1, \ldots, x_n)}{c}$$

for all $c > 0$. Hence, since the partial derivatives exist for $\mathbf{x} \in D_1 \cap D_2$, taking limits as $c \to 0^+$ in the preceding expression, we get (i). ■

The preceding lemma can be extended to a general n-dimensional rectangle $[\mathbf{a}, \mathbf{b}]$. Also note that if G is differentiable in $(\mathbf{0}, \mathbf{1})$, then we just need the continuity in $[\mathbf{0}, \mathbf{1}]$ (i.e., we do not need the Lipschitz condition). Moreover, in this case, the proof of "(i) $\Rightarrow$ (ii)" can be shortened (we only need the second part of the proof since $A = (0, 1) \times (0, 1)$ and $B = (0, \sqrt{2})$).

As a consequence of the preceding lemmas, we obtain the following theorem.

THEOREM 2.4: *In a coherent system with ID components, if component i is more important than component j, then the system obtained by applying a redundancy to component i is more reliable than that obtained by applying the same redundancy to component j for any redundancy function $\bar{q}$.*

PROOF: Again, to simplify the notation, we suppose that $i = 1$ and $j = 2$. The proof for general i and j is analogous. Thus, let us assume that component 1 is more important than component 2, that is, $\partial_1 \bar{Q}(\mathbf{u}) \geq \partial_2 \bar{Q}(\mathbf{u})$ for all $\mathbf{u} \in D_1 \cap D_2$, where $\bar{Q}$ is the distortion function of the system.

Let $\bar{q}$ be the common redundancy function applied to both components. Then, from (2.1), the resulting reliability functions are

$$R_1(t) = \bar{Q}(\bar{q}(\bar{F}(t)), \bar{F}(t), \ldots, \bar{F}(t))$$

and

$$R_2(t) = \bar{Q}(\bar{F}(t), \bar{q}(\bar{F}(t)), \bar{F}(t), \ldots, \bar{F}(t)),$$

respectively, where $\bar{F}$ represents the common reliability function of the components. As $\bar{q}(u) \geq u$ for all $u \in [0, 1]$, we have

$$c_t = \bar{q}(\bar{F}(t)) - \bar{F}(t) \geq 0$$

for all t. Therefore, from Lemma 2.3, we have $R_1(t) \geq R_2(t)$ for all t. ■

The following proposition proves that the condition $i \geq_{mi} j$, assumed in the preceding theorem, can be considered as a strong condition. First we need some basic definitions extracted from Barlow and Proschan [2, p. 9]. A set $P \subseteq \{1, \ldots, n\}$ is a *path set* of a coherent system if the system works when all the components in P work. A path set P is a *minimal path set* if it does not contain other path sets. It is known that the structure of a system is completely determined by its minimal path sets (see Barlow and Proschan [2], p. 12).

PROPOSITION 2.5: *Given a coherent system with n independent components and distortion function $\bar{Q}$, there does not exist any component more important than another in the sense of Definition 2.1.*

https://doi.org/10.1017/S0269964819000159 Published online by Cambridge University Press

Proof: We compare the components 1 and 2 of the system. The same argument applies to any other pair of components.

First we recall that if the components are independent, then $\bar{Q}$ is a polynomial. Hence, all its partial derivatives exist and are continuous at any point $\mathbf{u} \in \mathbb{R}^n$. We want to prove that the importance indices $I_1(\mathbf{u})$ and $I_2(\mathbf{u})$ cannot be ordered for all $\mathbf{u} \in (\mathbf{0}, \mathbf{1})$. Let $P_1, \ldots, P_r$ be the minimal path sets of the system. We consider the following cases.

Case I: The sets $\{1\}$ *and* $\{2\}$ *are minimal path sets.* As $\bar{Q}$ is a polynomial, if $u_3 = \cdots = u_n = 0$, then $\bar{Q}(\mathbf{u}) = u_1 + u_2 - u_1u_2$, $I_1(\mathbf{u}) = 1 - u_2$ and $I_2(\mathbf{u}) = 1 - u_1$. Then, as I_1 and I_2 are continuous functions in $[\mathbf{0}, \mathbf{1}]$, we can find points $\mathbf{v}, \mathbf{w} \in (\mathbf{0}, \mathbf{1})$ such that $I_1(\mathbf{v}) > I_2(\mathbf{v})$ and $I_1(\mathbf{w}) < I_2(\mathbf{w})$.

Case II: There exist two minimal path sets P_1 *and* P_2 *such that* $1 \in P_1$, $2 \notin P_1$, $2 \in P_2$, $1 \notin P_2$. Let $P_1^* = P_1 - \{1\}$. Without loss of generality, we can assume that $P_1^* = \{3, \ldots, k\}$ for $k \leq n$. Then we choose the vectors $\mathbf{u} = (u_1, \ldots, u_n)$ such that $u_1, u_2 \in (0, 1)$, $u_3 = \cdots = u_k = 1$ and $u_{k+1} = \cdots = u_n = 0$ (when $k < n$).

Then we have two subcases:

- II.a) If $\{2\} \cup P$ is a path set for a $P \subseteq P_1^*$, then $\bar{Q}(\mathbf{u}) = u_1 + u_2 - u_1u_2$. So $I_1(\mathbf{u}) = 1 - u_2$ and $I_2(\mathbf{u}) = 1 - u_1$ and, as in Case I, we can prove that $I_1(\mathbf{u}) > I_2(\mathbf{u})$ for a $\mathbf{u} \in (\mathbf{0}, \mathbf{1})$.
- II.b) If $\{2\} \cup P$ is not a path set for all $P \subseteq P_1^*$, then $\bar{Q}(\mathbf{u}) = u_1$. So $I_1(\mathbf{u}) = 1 > I_2(\mathbf{u}) = 0$ and hence we can prove that $I_1(\mathbf{u}) > I_2(\mathbf{u})$ for a $\mathbf{u} \in (\mathbf{0}, \mathbf{1})$.

We have proved that there exist points of $(\mathbf{0}, \mathbf{1})$ such that $I_1(\mathbf{u}) > I_2(\mathbf{u})$. Analogously, we can prove that there exist points of $(\mathbf{0}, \mathbf{1})$ such that $I_1(\mathbf{u}) < I_2(\mathbf{u})$.

Case III: *Components 1 and 2 are in the same minimal path set.* Assume, without loss of generality, that $1, 2 \in P_1$. Then, we choose the vectors $\mathbf{u} = (u_1, \ldots, u_n)$ such that $u_1, u_2 \in (0, 1)$ and for $i \geq 3$, $u_i = 1$ if $i \in P_1$ or $u_i = 0$ if $i \notin P_1$. Note that all the minimal path sets $P_2, \ldots, P_r$ contain components not included in P_1. Hence $\bar{Q}(\mathbf{u}) = u_1u_2$, $I_1(\mathbf{u}) = u_2$, and $I_2(\mathbf{u}) = u_1$. Therefore, I_1 and I_2 are not ordered in $(\mathbf{0}, \mathbf{1})$. ■

Since condition $i \geq_{mi} j$ can be considered hard to be satisfied, we now propose a weaker condition. We will show later that this new condition is also related with the importance index given in Definition 2.1 and that it can be used to obtain a result similar to that included in Theorem 2.4, that is, to determine where to apply a redundancy mechanism in a given system.

Definition 2.6: *We say that component* i *is* **weakly more important** *than component* j *(shortly written as* $i \geq_{wmi} j$*) in a coherent system with* n *components and distortion function* $\bar{Q}$ *if*

$$\bar{Q}(\mathbf{u} + c\ \mathbf{e}_i) \geq \bar{Q}(\mathbf{u} + c\ \mathbf{e}_j) \qquad \textbf{(2.2)}$$

for all $\mathbf{u} = (u, \ldots, u) \in [0, 1)^n$ *and* $c \in (0, 1 - u]$.

It follows from Lemma 2.3 that if component i is more important than component j, then it is also weakly more important. Example 3.7 proves that the reverse property is not true (i.e., condition (2.2) is strictly weaker than $i \geq_{mi} j$).

Next we use this new condition to determine where a general redundancy mechanism should be applied in a coherent system with ID components.

Theorem 2.7: *In a coherent system with ID components, if component* i *is weakly more important than component* j*, then the system obtained by applying a redundancy to component* i *is more reliable than that obtained by applying the same redundancy to component* j *for any redundancy function* $\bar{q}$.

The proof is similar to that of Theorem 2.4. An application of this theorem is illustrated in Example 3.7.

The following result provides a simple sufficient condition to check whether $i \geq_{wmi} j$ holds.

THEOREM 2.8: *In a coherent system with distortion function $\bar{Q}$, if*

$$\partial_i \bar{Q}(\mathbf{u} + (p-u)\mathbf{e}_i) \geq \partial_j \bar{Q}(\mathbf{u} + (p-u)\mathbf{e}_j) \tag{2.3}$$

for all $\mathbf{u} = (u, \ldots, u) \in [0,1]^n$ and all $p \in [u,1]$ such that these partial derivatives exist, then $i \geq_{wmi} j$.

PROOF: If we integrate in (2.3) with respect to the variable p on the interval $[u,t]$ for $u \leq t \leq 1$, then

$$\int_u^t \partial_i \bar{Q}(\mathbf{u} + (p-u)\mathbf{e}_i)dp \geq \int_u^t \partial_j \bar{Q}(\mathbf{u} + (p-u)\mathbf{e}_j)dp,$$

or equivalently,

$$\bar{Q}(\mathbf{u} + (t-u)\mathbf{e}_i) \geq \bar{Q}(\mathbf{u} + (t-u)\mathbf{e}_j).$$

We define now $c = t - u \geq 0$, and then

$$\bar{Q}(\mathbf{u} + c\ \mathbf{e}_i) \geq \bar{Q}(\mathbf{u} + c\ \mathbf{e}_j)$$

for all $0 \leq c \leq 1-u$. Therefore $i \geq_{wmi} j$. ■

Condition (2.3) is sufficient to prove that component i is weakly more important than component j. Example 3.8 proves that the reverse is not true, that is, (2.3) is not a necessary condition for the property states in (2.2).

In the case of independent components, the condition given in (2.3) can be simplified as follows. Note that it is also related with the importance index provided in Definition 2.1.

COROLLARY 2.9: *In a coherent system with independent components and distortion function $\bar{Q}$, if*

$$\partial_i \bar{Q}(u, \ldots, u) \geq \partial_j \bar{Q}(u, \ldots, u) \tag{2.4}$$

for all $u \in [0,1]$, then $i \geq_{wmi} j$.

PROOF: If the components are independent, then the distortion $\bar{Q}$ associated to the system is a multinomial function and the exponent of each variable is 0 or 1 in each term. Therefore, the partial derivative with respect to the ith variable does not depend on that variable, that is,

$$\partial_i \bar{Q}(u, \ldots, u) = \partial_i \bar{Q}(u, \ldots, u, p, u, \ldots, u)$$

for all $u, p \in [0,1]$, where, in the above expression, p is placed at the ith position. Hence, (2.3) holds and the preceding theorem concludes the proof. ■

Note that the functions $\partial_i \bar{Q}(u, \ldots, u)$ and $\partial_j \bar{Q}(u, \ldots, u)$ in (2.4) are polynomials in u. So it is easy to check if (2.4) holds in $[0,1]$. Moreover, condition (2.4) can be used jointly with Theorem 2.7 to determine where to apply a general redundancy mechanism in a system with IID components. Example 3.7 below shows how to use this procedure.

The ordering properties obtained above can be expressed by using the usual stochastic (ST) order which compares the reliability functions. To obtain properties for other orders, we need to assume that the system structure and the dependence among its components are known, that is, we need the distortion function $\bar{Q}$ associated to the system. Furthermore, we will consider a fixed redundancy function $\bar{q}$ applicable to any component of the system. Some particular relevant cases are studied below. Next we shall study properties for the hazard rate (HR) order, the reversed hazard rate (RHR) order, the mean residual life (MRL) order, and the likelihood ratio (LR) order. Their definitions and basic properties can be seen in Shaked and Shanthikumar [21].

If a system has possibly dependent ID components, then the reliability function of the system obtained by applying the redundancy $\bar{q}$ to its ith component is

$$R_i(t) = \bar{Q}(\bar{F}(t), \ldots, \bar{F}(t), \bar{q}(\bar{F}(t)), \bar{F}(t), \ldots, \bar{F}(t)) = \bar{q}_i(\bar{F}(t)), \quad \textbf{(2.5)}$$

where $\bar{q}_i(u) = \bar{Q}(u, \ldots, u, \bar{q}(u), u, \ldots, u)$ and the $\bar{q}$ function is placed at the ith position. The function $\bar{q}_i$ is a distortion function, that is, it is a continuous increasing function in $[0,1]$ that satisfies $\bar{q}_i(0) = 0$ and $\bar{q}_i(1) = 1$. The relationships for the respective distribution functions are $1 - R_i(t) = 1 - \bar{q}_i(\bar{F}(t)) = q_i(F(t))$, where $q_i(u) = 1 - \bar{q}_i(1-u)$ for $i = 1, \ldots, n$. In this case, we can state the following result which allows us to obtain stronger comparisons between the improved systems. The proof is immediate from the representation given in (2.5) and the ordering results for distorted distributions obtained in Navarro and Gomis [17] and Navarro et al. [18].

Proposition 2.10: *Let $\bar{q}_i$ and $\bar{q}_j$ be the distortion functions obtained in (2.5) for a coherent system with ID components by applying a common redundancy $\bar{q}$ to components i and j, respectively. Let T_i and T_j be the respective system lifetimes. Then:*

(i) $T_i \le_{ST} T_j$ *for all* F *if and only if* $\overline{q}_i \le \overline{q}_j$ *in* $(0,1)$.
(ii) $T_i \le_{HR} T_j$ *for all* F *if and only if* $\overline{q}_j/\overline{q}_i$ *is decreasing in* $(0,1)$.
(iii) $T_i \le_{RHR} T_j$ *for all* F *if and only if* q_j/q_i *is increasing in* $(0,1)$.
(iv) $T_i \le_{LR} T_j$ *for all* F *if and only if* $\overline{q}'_j/\overline{q}'_i$ *is decreasing in* $(0,1)$.
(v) *If there exists* $u_0 \in (0,1]$ *such that* $\overline{q}_j/\overline{q}_i$ *is decreasing in* $(0,u_0)$ *and increasing in* $(u_0,1)$, *then* $T_i \le_{MRL} T_j$ *for all* F *such that* $E(T_i) \le E(T_j)$.

In the general case (i.e., when we do not assume that the component lifetimes are ID), the reliability function of the system obtained by applying redundancy $\bar{q}$ to the ith component is

$$R_i(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_{i-1}(t), \bar{q}(\bar{F}_i(t)), \bar{F}_{i+1}(t), \ldots, \bar{F}_n(t)) = \bar{Q}_i(\bar{F}_1(t), \ldots, \bar{F}_n(t)), \quad \textbf{(2.6)}$$

where $\bar{Q}_i(u_1, \ldots, u_n) = \bar{Q}(u_1, \ldots, u_{i-1}, \bar{q}(u_i), u_{i+1}, \ldots, u_n)$. The function $\bar{Q}_i$ is a distortion function, that is, it is a continuous increasing function in $[0,1]^n$ that satisfies $\bar{Q}_i(0, \ldots, 0) = 0$ and $\bar{Q}_i(1, \ldots, 1) = 1$. The relationships for the respective distribution functions are

$$1 - R_i(t) = 1 - \bar{Q}_i(1 - \bar{F}_1(t), \ldots, 1 - \bar{F}_n(t)) = Q_i(F_1(t), \ldots, F_n(t))$$

for $i = 1, \ldots, n$, where $Q_i(u_1, \ldots, u_n) = 1 - \bar{Q}_i(1 - u_1, \ldots, 1 - u_n)$. In this general case, we have the following result that can be used to improve the results proved above for the ST order. The proof is immediate from the representation obtained in (2.6) and the ordering results for distorted distributions obtained in Navarro et al. [19].

PROPOSITION 2.11: *Let $\bar{Q}_i$ and $\bar{Q}_j$ be the distortion functions obtained in (2.6) for a coherent system with possibly dependent components by applying a common redundancy $\bar{q}$ to components i and j, respectively. Let T_i and T_j be the respective system lifetimes. Then:*

(i) $T_i \le_{ST} T_j$ *for all* $F_1, \ldots, F_n$ *if and only if* $\overline{Q}_i \le \overline{Q}_j$ *in* $(\mathbf{0}, \mathbf{1})$.

(ii) $T_i \le_{HR} T_j$ *for all* $F_1, \ldots, F_n$ *if and only if* $\overline{Q}_j/\overline{Q}_i$ *is decreasing in* $(\mathbf{0}, \mathbf{1})$.

(iii) $T_i \le_{RHR} T_j$ *for all* $F_1, \ldots, F_n$ *if and only if* Q_j/Q_i *is increasing in* $(\mathbf{0}, \mathbf{1})$.

The two preceding propositions can be applied to the following typical and useful cases, as well as other redundancy mechanisms.

Case I: Active redundancies (hot standby).

Here it is assumed that a new component is added in parallel to one of the system components. If the lifetime of the new unit added to the ith component is Y_i and it has the same distribution as X_i, then the lifetime of the resulting component is $X_i^* = \max(X_i, Y_i)$ with reliability

$$\begin{aligned}\bar{F}_i^*(t) &= \Pr(X_i^* > t) = 2\Pr(X_i > t) - \Pr(X_i > t, Y_i > t)\\ &= 2\bar{F}_i(t) - K(\bar{F}_i(t), \bar{F}_i(t)) = \bar{q}(\bar{F}_i(t)),\end{aligned}$$

where K is the survival copula of (X_i, Y_i) and

$$\bar{q}(u) := 2u - K(u,u) \ge u$$

for $u \in [0,1]$. The basic properties of copulas can be seen in Durante and Sempi [6] and Nelsen [20]. Here we can choose any copula K. In particular, if X_i and Y_i are independent, which is a usual assumption in practice, then the survival copula K is the product copula and the associated distortion is given by

$$\bar{q}_{2:2}(u) = 2u - u^2. \tag{2.7}$$

The application of the preceding propositions to this case is illustrated in Example 3.9.

Case II: Inactive redundancies (cold standby) or perfect repair.

Here we assume that, when the ith component fails at time X_i, then it is replaced by another component with lifetime Y_i. If we assume that (X_i, Y_i) has an absolutely continuous joint distribution, then the lifetime of the resulting component is $X_i^* = X_i + Y_i$ with reliability

$$\bar{F}_i^*(t) = \Pr(X_i + Y_i > t) = \Pr(X_i > t) + \int_0^t \Pr(Y_i > t - x | X_i = x) f_i(x)dx \tag{2.8}$$

where f_i represents the probability density function (pdf) of X_i. If the joint reliability of (X_i, Y_i) is represented as

$$\bar{H}(x,y) = \Pr(X_i > x, Y_i > y) = K(\bar{F}_i(x), \bar{G}_i(y)),$$

where K is the survival copula and $\bar{G}_i$ is the reliability function of Y_i, then its joint pdf is

$$h(x,y) = f_i(x) g_i(y) \partial_2 \partial_1 K(\bar{F}_i(x), \bar{G}_i(y)),$$

where g_i is the pdf of Y_i. Hence the conditional pdf of $(Y_i|X_i = x)$ is

$$h_{2|1}(y|x) = \frac{h(x,y)}{f_i(x)} = g_i(y)\partial_2\partial_1 K(\bar{F}_i(x), \bar{G}_i(y))$$

and

$$\Pr(Y_i > t - x | X_i = x) = \int_{t-x}^{\infty} g_i(y)\partial_2\partial_1 K(\bar{F}_i(x), \bar{G}_i(y))dy = \partial_1 K(\bar{F}_i(x), \bar{G}_i(t-x)).$$

Therefore, from (2.8),

$$\bar{F}_i^*(t) = \Pr(X_i + Y_i > t) = \bar{F}_i(t) + \int_0^t f_i(x)\partial_1 K(\bar{F}_i(x), \bar{G}_i(t-x))dx. \tag{2.9}$$

In particular, if K is the product copula, then we get the convolution formula for the reliability

$$\bar{F}_i^*(t) = \bar{F}_i(t) + \int_0^t \bar{G}_i(t-x) f_i(x)dx. \tag{2.10}$$

If $\bar{F}_i$ is strictly decreasing and $\bar{F}_i^{-1}$ is its inverse function, we obtain from (2.9) that

$$\bar{F}_i^*(t) = \bar{F}_i(t) + \int_0^t f_i(x)\partial_1 K(\bar{F}_i(x), \bar{G}_i(t - \bar{F}_i^{-1}(\bar{F}_i(x))))dx,$$

and making the change of variables $v = \bar{F}_i(x)$

$$\bar{F}_i^*(t) = \bar{F}_i(t) + \int_{\bar{F}_i(t)}^1 \partial_1 K(v, \bar{G}_i(t - \bar{F}_i^{-1}(v)))dv.$$

If $F_i = G_i$, that is, the failed component is replaced by a new one with the same distribution (or it is perfectly repaired), then

$$\bar{F}_i^*(t) = \bar{F}_i(t) + \int_{\bar{F}_i(t)}^1 \partial_1 K(v, \bar{F}_i(t - \bar{F}_i^{-1}(v)))dv = \bar{q}_{cold}^{(i)}(\bar{F}_i(t)),$$

where

$$\bar{q}_{cold}^{(i)}(u) = u + \int_u^1 \partial_1 K(v, \bar{F}_i(\bar{F}_i^{-1}(u) - \bar{F}_i^{-1}(v)))dv$$

is a distortion function such that $\bar{q}_{cold}^{(i)}(u) \geq u$ for all $u \in [0,1]$ (since $X_i \leq X_i + Y_i$). Here we need to assume that $\bar{q}_i$ does not depend on i. This expression can be simplified if $F_1 = \cdots = F_n = F$, that is, when the components are ID. In this case, we obtain

$$\bar{q}_{\text{cold}}(u) = u + \int_u^1 \partial_1 K(v, \bar{F}(\bar{F}^{-1}(u) - \bar{F}^{-1}(v)))dv \geq u \tag{2.11}$$

for all $u \in [0,1]$.

Case III: Minimal repair.

As in the preceding case, we assume that the ith component is replaced (or repaired) when it fails at an age $x > 0$ by another component with lifetime Y_i and that (X_i, Y_i) has an absolutely continuous joint distribution. However, in this case, we assume that the new unit has age x, that is, when $X_i = x$, Y_i is replaced by $Y_x = (Y_i - x | Y_i > x)$. Then the lifetime of the resulting component is $X_i^* = X_i + Y_{X_i}$, where the reliability of Y_{X_i} is

$$\bar{G}_{x,i}(t) = \Pr(Y_i > t | X_i = x) = \Pr(Y_i > x + t | Y_i > x) = \frac{\bar{G}_i(x+t)}{\bar{G}_i(x)}$$

for $t \geq 0$. Hence, from (2.8), the reliability of X_i^* is

$$\bar{F}_i^*(t) = \Pr(X_i + Y_{X_i} > t) = \bar{F}_i(t) + \int_0^t \Pr(Y_i > t - x | X_i = x) f_i(x) dx$$
$$= \bar{F}_i(t) + \int_0^t \frac{\bar{G}_i(t)}{\bar{G}_i(x)} f_i(x) dx. \quad \textbf{(2.12)}$$

In particular, if $F_i = G_i$, then

$$\bar{F}_i^*(t) = \bar{F}_i(t) + \int_0^t \frac{\bar{F}_i(t)}{\bar{F}_i(x)} f_i(x) dx = \bar{F}_i(t) - \bar{F}_i(t) \ln \bar{F}_i(t) = \bar{q}_{(1)}(\bar{F}_i(t)),$$

where $\bar{q}_{(1)}(u) = u - u\ln(u) \geq u$ is a redundancy function. If the same component is repaired k times, then the resulting component has reliability $\bar{F}_i^{(k)}(t) = \bar{q}_{(k)}(\bar{F}_i(t))$, where

$$\bar{q}_{(k)}(u) = u \sum_{j=0}^{k} \frac{1}{j!} (-\ln(u))^j \quad \textbf{(2.13)}$$

for $k = 1, 2, \ldots$. This procedure is called *the relevation transform*; see, for example, Krakowski [12]. Ordering properties for this case were widely studied in Arriaza, Navarro, and Suarez-Llorens [1]. So we do not include examples here.

Note that $\bar{q}_{2:2}(u) \leq \bar{q}_{cold}(u)$ and $\bar{q}_{2:2}(u) \leq \bar{q}_{(1)}(u) \leq \bar{q}_{(2)}(u) \leq \cdots$ for all $u \in [0, 1]$, where $\bar{q}_{cold}$ and $\bar{q}_{(k)}$ are given in (2.11) and (2.13), respectively. The first inequality is obtained from $\max(X, Y) \leq X + Y$ and the second by a straightforward calculation. Hence, the systems obtained by applying active redundancy to the ith component are always worse than those obtained by cold standby with perfect or minimal repair. The following result compares the last two redundancies, that is, perfect and minimal repairs. To this end, we need the well-known concepts of *new better than used* (NBU) and *new worse than used* (NWU) defined as follows. We say that a reliability $\bar{G}$ is NBU (NWU) if $\bar{G}(t) \geq \bar{G}_x(t)$ $(\leq)$ for all $x, t \geq 0$, where $\bar{G}_x(t) = \bar{G}(t + x)/\bar{G}(x)$.

PROPOSITION 2.12: *If G_i is NBU (NWU), then to apply a perfect repair to the ith component is better (worse) than to apply a minimal repair to the same component.*

Since $\bar{Q}$ is increasing, the proof is immediate from (2.6), (2.10), and (2.12).

Case IV: Component redundancy versus system redundancy.

Some authors also consider comparisons between redundancy at a component level and redundancy at the system level (see, e.g., Hazra and Nanda [10]). In the first option, as above, the redundancy is assigned to a component. When the redundancy function $\bar{q}$ is applied to the ith component, the resulting reliability is given by (2.6). In the second option, the redundancy function $\bar{q}$ is applied to the system obtaining the following reliability

$$R^*(t) = \bar{q}(\bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t))) = \bar{Q}^*(\bar{F}_1(t), \ldots, \bar{F}_n(t)),$$

where $\bar{Q}^*(u_1, \ldots, u_n) = \bar{q}(\bar{Q}(u_1, \ldots, u_n))$. Note that these two cases can be compared proceeding as in Proposition 2.11.

Finally, we propose an alternative importance index closely related with active redundancy. Similar importance measures can be defined by using other redundancy mechanisms (as minimal repair).

Definition 2.13: *The* **importance index under active redundancy** *for the* j*th component in a system with a distortion function* $\bar{Q}$ *is defined by*

$$I_{2:2}^{(j)}(\mathbf{u}) = \bar{Q}(u_1, \ldots, u_{j-1}, \bar{q}_{2:2}(u_j), u_{j+1}, \ldots, u_n)$$

for all $\mathbf{u} = (u_1, \ldots, u_n) \in [0,1]^n$*, where* $\bar{q}_{2:2}$ *is the distortion function associated to a parallel system with two IID components given in* (2.7)*. We say that the* i*th component is* **more important under active redundancy** *than the* j*th component (shortly written as* $i \geq_{miar} j$*) if*

$$I_{2:2}^{(i)}(\mathbf{u}) \geq I_{2:2}^{(j)}(\mathbf{u}) \tag{2.14}$$

for all $\mathbf{u} = (u_1, \ldots, u_n) \in [0,1]^n$*. Analogously, we say that the* i*th component is* **weakly more important under active redundancy** *than the* j*th component (shortly written as* $i \geq_{wmiar} j$*) if*

$$I_{2:2}^{(i)}(u, \ldots, u) \geq I_{2:2}^{(j)}(u, \ldots, u) \tag{2.15}$$

for all $u \in [0,1]$.

Note that

$$i \geq_{mi} j \quad \Rightarrow \quad i \geq_{wmi} j \quad \Rightarrow \quad i \geq_{wmiar} j \quad \Leftarrow \quad i \geq_{miar} j.$$

It is easy to see that $1 \geq_{wmi} 2$ does not imply $1 \geq_{mi} 2$ and that $1 \geq_{wmiar} 2$ does not imply $1 \geq_{miar} 2$ (e.g., consider a series system with two independent components). Examples 3.4 and 3.10 below prove that $i \geq_{miar} j$ does not imply $i \geq_{mi} j$ and that $i \geq_{wmiar} j$ does not imply $i \geq_{wmi} j$, respectively.

The following proposition will be used to show that the importance index based on active redundancy can also be applied to study other redundancy mechanisms.

Proposition 2.14: *In a coherent system with a distortion function* $\bar{Q}$*, if* $i \geq_{wmiar} j$ *and*

$$\partial_i \bar{Q}(\mathbf{u} + (p-u)\mathbf{e}_i) \geq \partial_j \bar{Q}(\mathbf{u} + (p-u)\mathbf{e}_j) \tag{2.16}$$

for all $\mathbf{u} = (u, \ldots, u) \in [0,1]^n$ *and* $p \in [\bar{q}_{2:2}(u), 1]$*, then*

$$\bar{Q}(\mathbf{u} + (c-u)\mathbf{e}_i) \geq \bar{Q}(\mathbf{u} + (c-u)\mathbf{e}_j)$$

for all $c \in [\bar{q}_{2:2}(u), 1]$.

Proof: If we integrate in (2.16) with respect to the variable p on the interval $[\bar{q}_{2:2}(u), c]$ with $c \in [\bar{q}_{2:2}(u), 1]$, then we obtain

$$\int_{\bar{q}_{2:2}(u)}^{c} \partial_i \bar{Q}(\mathbf{u} + (p-u)\mathbf{e}_i)dp \geq \int_{\bar{q}_{2:2}(u)}^{c} \partial_j \bar{Q}(\mathbf{u} + (p-u)\mathbf{e}_j)dp,$$

or equivalently,

$$\bar{Q}(\mathbf{u} + (c-u)\mathbf{e}_i) - \bar{Q}(\mathbf{u} + (\bar{q}_{2:2}(u)-u)\mathbf{e}_i) \geq \bar{Q}(\mathbf{u} + (c-u)\mathbf{e}_j) - \bar{Q}(\mathbf{u} + (\bar{q}_{2:2}(u)-u)\mathbf{e}_j).$$

That is to say,

$$\bar{Q}(\mathbf{u} + (c-u)\mathbf{e}_i) - I_{2:2}^{(i)}(\mathbf{u}) \geq \bar{Q}(\mathbf{u} + (c-u)\mathbf{e}_j) - I_{2:2}^{(j)}(\mathbf{u}).$$

Hence

$$\bar{Q}(\mathbf{u} + (c-u)\mathbf{e}_i) \geq \bar{Q}(\mathbf{u} + (c-u)\mathbf{e}_j) + I_{2:2}^{(i)}(\mathbf{u}) - I_{2:2}^{(j)}(\mathbf{u}) \geq \bar{Q}(\mathbf{u} + (c-u)\mathbf{e}_j),$$

as we wanted to prove. ■

As a consequence of the preceding proposition, we obtain the following result for redundancy options whose distortion functions, $\bar{q}$, are greater than the distortion function of an active redundancy $\bar{q}_{2:2}$.

COROLLARY 2.15: *Under the same conditions of the previous proposition, if $\bar{q}$ is a distortion function such that $\bar{q}_{2:2}(u) \le \bar{q}(u)$ for all $u \in [0,1]$, then*

$$\bar{Q}(\mathbf{u} - (\bar{q}(u) - u)\mathbf{e}_i) \ge \bar{Q}(\mathbf{u} - (\bar{q}(u) - u)\mathbf{e}_j)$$

for all $\mathbf{u} = (u, \ldots, u) \in [0,1]^n$.

PROOF: The proof follows from Proposition 2.14 by using $c = \bar{q}(u)$. ■

Note that, in the case of systems with ID components, the redundancy mechanisms associated to cold standby and minimal repair can be expressed as the distortion functions $\bar{q}_{cold}$ and $\bar{q}_{(1)}$, respectively, which satisfy $\bar{q}_{2:2} \le \bar{q}_{cold}$ and $\bar{q}_{2:2} \le \bar{q}_{(1)}$. This means that, in the conditions of Proposition 2.14, if $i \ge_{wmiar} j$, then the lifetimes of the systems obtained by applying the same redundancy (cold or minimal repair) to the components i and j are ordered similarly.

3. SOME ILLUSTRATIVE EXAMPLES

In the first example, we study parallel systems with two components. We show that if a component is more important than the other, in the sense of Definition 2.1, then they actually are equally important. Moreover, the components should be dependent with the Fréchet–Hoeffding lower bound copula given by

$$W(u,v) = \max(u + v - 1, 0) \tag{3.1}$$

for $u, v \in [0,1]$.

PROPOSITION 3.1: *If in a parallel system with two components having a copula C, $1 \ge_{mi} 2$, then $1 =_{mi} 2$ and $C = W$.*

PROOF: The lifetime of the parallel system is $T = \max(X_1, X_2)$. Then, its distribution function is

$$F_T(t) = \Pr(\max(X_1, X_2) \le t) = \Pr(X_1 \le t, X_2 \le t) = C(F_1(t), F_2(t))$$

for all t, where F_1 and F_2 are the distribution functions of the component lifetimes. Hence, its reliability is

$$\bar{F}_T(t) = 1 - F_T(t) = 1 - C(F_1(t), F_2(t)) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t)),$$

where $\bar{F}_1$ and $\bar{F}_2$ are the reliability functions of the component lifetimes and

$$\bar{Q}(u,v) = 1 - C(1-u, 1-v)$$

for $u, v \in [0,1]$. Hence, if $I_1 \ge I_2$ in $D_1 \cap D_2$, then

$$\partial_1 C(u,v) \ge \partial_2 C(u,v)$$

for almost all $u, v \in (0,1)$. Therefore,

$$\begin{aligned} 1/2 &= \int_0^1 v dv \\ &= \int_0^1 \int_0^1 \partial_1 C(u,v) du dv \\ &\geq \int_0^1 \int_0^1 \partial_2 C(u,v) du dv \\ &= \int_0^1 \int_0^1 \partial_2 C(u,v) dv du \\ &= 1/2 \end{aligned}$$

since $C(u,1) = C(1,u) = u$ and $C(u,0) = C(0,u) = 0$ for all $u \in (0,1)$. Therefore $\partial_1 C(u,v) = \partial_2 C(u,v)$ and $I_1(u,v) = I_2(u,v)$ (a.e.) in $(0,1)^2$.

Moreover, for $u_0 \in (0,1)$, we can write

$$\begin{aligned} \frac{u_0^2}{2} &= \int_0^{u_0} u du \\ &= \int_0^{u_0} \int_0^1 \partial_2 C(u,v) dv du \\ &= \int_0^{u_0} \int_0^1 \partial_1 C(u,v) dv du \\ &= \int_0^1 \int_0^{u_0} \partial_1 C(u,v) du dv \\ &= \int_0^1 C(u_0, v) dv \end{aligned}$$

where the third equality follows from $I_1 = I_2$. A straightforward calculation shows that

$$\frac{u_0^2}{2} = \int_0^1 W(u_0, v) dv.$$

So

$$\int_0^1 [C(u_0, v) - W(u_0, v)] dv = 0$$

for all $u_0 \in (0,1)$, where $C(u_0, v) - W(u_0, v) \geq 0$ since W is the lower bound. Therefore, as C and W are continuous, we have $C = W$. ■

A similar result can be obtained for series systems. It can be stated as follows. The proof is similar.

PROPOSITION 3.2: *If in a series system with two components having a copula* C, $1 \geq_{mi} 2$, *then* $1 =_{mi} 2$ *and* $C = W$.

The following example shows that the preceding property cannot be extended to other parallel or series systems.

Example 3.3: Let us consider the parallel system with lifetime $T = \max(X_1, X_2, X_3)$ where the random vector of the component lifetimes (X_1, X_2, X_3) has the copula

$$C(u_1, u_2, u_3) = u_3 W(u_1, u_2)$$

and W is given in (3.1). Then

$$\bar{Q}(u_1, u_2, u_3) = 1 - (1 - u_3) W(1 - u_1, 1 - u_2)$$

and, by the symmetry, $I_1 = I_2$. Note that, in this case, C cannot be equal to the Fréchet–Hoeffding lower bound since that bound is not a copula.

The next example shows that a component can be more important than another one when they are dependent (with a given copula). It also shows that $i \le_{mi} j$ does not imply $i \le_{miar} j$.

Example 3.4: As in the preceding example let us consider a parallel system with three dependent components. In this case, we choose the copula with a uniform distribution over the segment which connects the point $(1, 0, 1)$ with the point $(0, 1, 0)$, that is, the function defined in $[0, 1]^3$ by

$$C(u_1, u_2, u_3) = \begin{cases} u_1 + u_2 - 1, & \text{for } 0 \le u_1 \le u_3, 1 - u_3 \le u_2 \le 1, u_1 + u_2 \ge 1, \\ u_2 + u_3 - 1, & \text{for } u_3 \le u_1 \le 1, 1 - u_3 \le u_2 \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\bar{Q}(u_1, u_2, u_3) = 1 - C(1 - u_1, 1 - u_2, 1 - u_3)$$

and

$$\partial_i \bar{Q}(u_1, u_2, u_3) = \partial_i C(1 - u_1, 1 - u_2, 1 - u_3)$$

for $i = 1, 2, 3$. Therefore, for $(u_1, u_2, u_3) \in D_1 \cap D_2$, we have

$$\partial_1 \bar{Q}(u_1, u_2, u_3) = \begin{cases} 1, & \text{for } 0 < u_1 < u_3, 1 - u_3 < u_2 < 1, u_1 + u_2 > 1, \\ 0, & \text{for } u_3 < u_1 < 1, 1 - u_3 < u_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\partial_2 \bar{Q}(u_1, u_2, u_3) = \begin{cases} 1, & \text{for } 0 < u_1 < u_3, 1 - u_3 < u_2 < 1, u_1 + u_2 > 1, \\ 1, & \text{for } u_3 < u_1 < 1, 1 - u_3 < u_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $1 <_{mi} 2$, that is, the second component is strictly more important than the first.

On the other hand, let us see that $1 \nleq_{miar} 2$. First, we provide the explicit expression of the distortion function associated to the system

$$\bar{Q}(u_1,u_2,u_3) = \begin{cases} u_1+u_2, & \text{for } u_3 \le u_1,\ u_2+u_3 \le 1,\ u_1+u_2 \le 1,\\ u_2+u_3, & \text{for } u_1 \le u_3,\ u_2+u_3 \le 1,\\ 1, & \text{otherwise.}\end{cases}$$

If we take, for example, $\mathbf{u} = (0.03, 0.01, 0.035) \in [0,1]^3$, then we have

$$I^{(1)}_{2:2}(u_1,u_2,u_3) = \bar{Q}(\bar{q}_{2:2}(u_1),u_2,u_3) = \bar{q}_{2:2}(u_1)+u_2 = 0.0691$$

and

$$I^{(2)}_{2:2}(u_1,u_2,u_3) = \bar{Q}(u_1,\bar{q}_{2:2}(u_2),u_3) = \bar{q}_{2:2}(u_2)+u_3 = 0.0549.$$

Therefore $1 \nleq_{miar} 2$. However, it is not difficult to prove that $1 \le_{wmiar} 2$ holds. Note that $\le_{mi}$ implies $\le_{wmi}$ and, from Theorem 2.7, $\bar{Q}(\bar{q}(u),u,u) \le \bar{Q}(u,\bar{q}(u),u)$ holds for all $u \in [0,1]$ and all distortion functions $\bar{q}$.

The next example shows that, in mixed systems, a component can be more important than another one for a wide family of copulas. This family includes the product copula which represents the case of independent components.

Example 3.5: The *mixed systems* are mixtures of coherent systems. They might represent situations in which a system has different structures (requirements) at different times. Clearly, their reliability functions admit a representation similar to (2.1) for a distortion function which is a linear combination of the distortion functions of the different coherent systems in the mixture. Hence, all the results obtained in this paper can also be applied to mixed systems. Let us see an example. Consider the mixed system with lifetime T and two possibly dependent components with lifetimes X_1 and X_2, where T is equal to X_1 with probability 1/2 and to $\min(X_1,X_2)$ with probability 1/2. This mixed system might represent a service that half of the time works if the first unit works, while during the other period both units are needed. Several real situations can be represented in this way. If K is the survival copula of (X_1,X_2), then

$$\Pr(X_1>t, X_2>t) = K(\bar{F}_1(t),\bar{F}_2(t)),$$

where $\bar{F}_1$ and $\bar{F}_2$ are the component reliability functions. Hence, the system reliability is

$$\bar{F}_T(t) = \frac{1}{2}\bar{F}_1(t) + \frac{1}{2}K(\bar{F}_1(t),\bar{F}_2(t)) = \bar{Q}(\bar{F}_1(t),\bar{F}_2(t)),$$

where

$$\bar{Q}(u_1,u_2) = \frac{1}{2}u_1 + \frac{1}{2}K(u_1,u_2)$$

for $u_1,u_2 \in [0,1]$. If the partial derivatives of K exist in $(0,1)^2$, then the importance of the first component is

$$I_1(u_1,u_2) = \partial_1\bar{Q}(u_1,u_2) = \frac{1}{2} + \frac{1}{2}\partial_1 K(u_1,u_2)$$

and that of the second one is

$$I_2(u_1,u_2) = \partial_2\bar{Q}(u_1,u_2) = \frac{1}{2}\partial_2 K(u_1,u_2).$$

Since K is increasing, the partial derivatives are non-negative. Moreover, from Theorem 2.2.7 in p. 13 of Nelsen [20], we have $\partial_2 K(u_1,u_2) \le 1$. Therefore, $I_1(u_1,u_2) \ge 1/2 \ge I_2(u_1,u_2)$ for all $u_1, u_2 \in [0,1]$. Hence, from Theorem 2.4, if the components are ID$\sim F$, then the system obtained by applying any redundancy $\bar{q}$ to the first component is ST − better (more reliable) than that obtained doing the same with the second component for all $K, \bar{q}, F$.

Remark 3.6: Note that the results obtained in this paper can be applied (as in the preceding example) to generalized distorted distributions, that is, to distribution functions F that can be written as

$$F(t) = Q(F_1(t), \ldots, F_n(t))$$

for a distortion function Q (see Navarro et al. [19]). Hence, the associated reliability function can be written as in (2.1) where

$$\bar{Q}(u_1, \ldots, u_n) = 1 - Q(1 - u_1, \ldots, 1 - u_n)$$

for all $u_1, \ldots, u_n \in [0,1]$. All the importance measures considered before for systems can also be applied to distorted distributions. This general case includes the particular case of a finite mixture with weights $p_1, \ldots, p_n \in (0,1)$ such that $p_1 + \cdots + p_n = 1$. In this case

$$Q(u_1, \ldots, u_n) = \bar{Q}(u_1, \ldots, u_n) = p_1 u_1 + \cdots + p_n u_n$$

and therefore the importance of the ith component is

$$I_i(u_1, \ldots, u_n) = \partial_i \bar{Q}(u_1, \ldots, u_n) = p_i$$

for all $u_1, \ldots, u_n \in [0,1]$. The importance indices can be ordered, just by the values of the respective weights in the mixture, as expected. Note that this case can also be seen as a mixed system.

The following example illustrates how to use Theorem 2.7 and Corollary 2.9 to determine where the redundancy should be applied.

Example 3.7: Let us consider the system with lifetime $T = \min(X_1, \max(X_2, X_3))$. If the components are independent, then the system reliability is

$$\bar{F}_T(t) = \bar{F}_1(t)\bar{F}_2(t) + \bar{F}_1(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t)),$$

where $\bar{Q}(u_1, u_2, u_3) = u_1 u_2 + u_1 u_3 - u_1 u_2 u_3$. Therefore, the respective importance indices of components 1 and 2 are

$$I_1(u_1, u_2, u_3) = \partial_1 \bar{Q}(u_1, u_2, u_3) = u_2 + u_3 - u_2 u_3$$

and

$$I_2(u_1, u_2, u_3) = \partial_2 \bar{Q}(u_1, u_2, u_3) = u_1 - u_1 u_3.$$

Note that, from Proposition 2.5, both indices are not ordered. However, using Corollary 2.9, it is not difficult to see that $1 \geq_{wmi} 2$ since

$$\partial_1 \bar{Q}(u, u, u) = 2u - u^2 \geq u - u^2 = \partial_2 \bar{Q}(u, u, u)$$

for all $u \in [0,1]$. Observe that, by the structure of the system, the first component is also weakly more important than the third component.

Therefore, from Theorem 2.7, if the components are IID$\sim F$, the system obtained by applying a redundancy $\bar{q}$ to the first component is more reliable than that obtained by applying the same redundancy to the second (or the third) component for all F and all $\bar{q}$.

Next, we include a counterexample which shows that conditions (2.2) and (2.3) are not equivalent.

Example 3.8: Let us consider a parallel system with two dependent components having the following copula

$$C(u,v) = \begin{cases} u, & \text{for } 0 \le u \le 1/3, 2/3 \le v \le 1, v-u \ge 2/3, \\ v-2/3, & \text{for } 0 \le u \le 1/3, 2/3 \le v \le 1, v-u < 2/3, \\ v, & \text{for } 1/3 < u \le 1, 0 \le v \le 2/3, u-v \ge 1/3, \\ u-1/3, & \text{for } 1/3 < u \le 1, 0 \le v \le 2/3, u-v < 1/3, \\ 0, & \text{for } 0 \le u \le 1/3, 0 \le v \le 2/3, \\ u+v-1, & \text{for } 1/3 < u \le 1, 2/3 \le v \le 1. \end{cases}$$

Hence, the distortion function is

$$\bar{Q}(u,v) = 1 - C(1-u, 1-v) = \begin{cases} u, & \text{for } 2/3 \le u \le 1, 0 \le v \le 1/3, u-v \ge 2/3, \\ v+2/3, & \text{for } 2/3 \le u \le 1, 0 \le v \le 1/3, u-v < 2/3, \\ v, & \text{for } 0 \le u < 2/3, 2/3 \le v \le 1, v-u \ge 1/3, \\ u+1/3, & \text{for } 0 \le u < 2/3, 2/3 \le v \le 1, v-u < 1/3, \\ 1, & \text{for } 2/3 \le u \le 1, 1/3 \le v \le 1, \\ u+v, & \text{for } 0 \le u < 2/3, 0 \le v \le 1/3. \end{cases}$$

A straightforward calculation shows that $1 \ge_{wmi} 2$, that is, (2.2) holds. However, (2.3) does not hold for $i=1, j=2, u=0.1$, and $p=0.7$ since

$$\partial_1 \bar{Q}(0.7, 0.1) = 0 < 1 = \partial_2 \bar{Q}(0.1, 0.7).$$

The following example shows how to apply Propositions 2.10 and 2.11. It also proves that $i \ge_{miar} j$ and $i \ge_{wmiar} j$ are not equivalent.

Example 3.9: Let us consider the coherent system $\phi(x_1,x_2,x_3) = \max(x_1, \min(x_2,x_3))$. Its lifetime is $T = \max(X_1, \min(X_2,X_3))$. Let us assume first that the components are independent. Then the system reliability function is

$$\bar{F}_T(t) = \bar{F}_1(t) + \bar{F}_2(t)\bar{F}_3(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t)),$$

where $\bar{F}_1, \bar{F}_2, \bar{F}_3$ are the component reliability functions and $\bar{Q}(u,v,w) = u + vw - uvw$ is its distortion function.

Also let us assume that we can apply an active redundancy to the first or the second components with redundancy function $\bar{q}_{2:2}(u) = 2u - u^2$. If the components are IID with a common reliability $\bar{F}$, then the resulting reliability functions are that given in (2.5) with

$$\bar{q}_1(u) = \bar{Q}(\bar{q}_{2:2}(u), u, u) = 2u - 2u^3 + u^4$$

and

$$\bar{q}_2(u) = \bar{Q}(u, \bar{q}_{2:2}(u), u) = u + 2u^2 - 3u^3 + u^4,$$

respectively. A straightforward calculation shows that $\bar{q}_1 \ge \bar{q}_2$ in $[0,1]$ and so, from Proposition 2.10, (i), $T_1 \ge_{ST} T_2$ holds for all $\bar{F}$, that is, $1 \ge_{wmiar} 2$ holds, see (2.15). Even more, as

$$\partial_1 \bar{Q}(u,u,u) = 1 - u^2 \ge u - u^2 = \partial_2 \bar{Q}(u,u,u)$$

for all $u \in [0,1]$, (2.4) holds and we have that $1 \ge_{wmi} 2$. Therefore $T_1 \ge_{ST} T_2$ for all $\bar{F}$ and all $\bar{q}$. However, if we want to obtain the hazard rate (HR) ordering, we need to assume

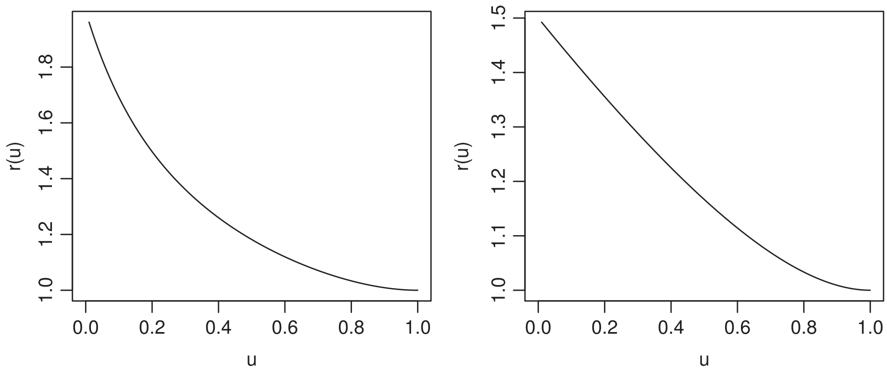

FIGURE 1. Plots of the ratios $r = \bar{q}_1/\bar{q}_2$ for the system studied in Example 3.9 when the components are IID (left) or DID with a Clayton copula (right).

again that we use active redundancy (i.e., $\bar{q}_{2:2}$) and to study the monotonicity of the ratio $r = \bar{q}_1/\bar{q}_2$ on $(0, 1)$. It is easy to prove analytically that it is decreasing in $(0, 1)$ (see the plot in Figure 1, left). Hence, from Proposition 2.10, (ii), we have $T_1 \geq_{HR} T_2$ for all $\bar{F}$. Analogously, to study the likelihood ratio (LR) order, we need to study the ratio of the derivatives $\bar{q}_1'/\bar{q}_2'$. Since it is also decreasing we have from Proposition 2.10, (iv), $T_1 \geq_{LR} T_2$ for all $\bar{F}$.

If we assume now that the components are independent, but not identically distributed (ID), then the resulting reliability functions are those given in (2.6) with

$$\bar{Q}_1(u, v, w) = \bar{Q}(\bar{q}_{2:2}(u), v, w)$$

and

$$\bar{Q}_2(u, v, w) = \bar{Q}(u, \bar{q}_{2:2}(v), w).$$

Thus

$$\bar{Q}_1(0.1, 0.5, 0.5) = 0.3925 < 0.4375 = \bar{Q}_2(0.1, 0.5, 0.5)$$

and

$$\bar{Q}_1(0.5, 0.5, 0.5) = 0.8125 > 0.6875 = \bar{Q}_2(0.5, 0.5, 0.5).$$

Hence, from Proposition 2.11, T_1 and T_2 are not ST-ordered for all $\bar{F}_1, \bar{F}_2, \bar{F}_3$. That is to say, $1 \geq_{miar} 2$ does not hold, see (2.14).

Let us assume now that the components are ID, that component 1 is independent from components 2 and 3 and that these last two components are dependent with the following Clayton survival copula

$$K(v, w) = \frac{vw}{v + w - vw}$$

(see, e.g., Nelsen [20], p. 118). Then the distortion function of the system is

$$\bar{Q}(u, v, w) = u + K(v, w) - uK(v, w) = u + \frac{(1-u)vw}{v + w - vw}.$$

Hence, the resulting reliability functions are those given in (2.5) with

$$\bar{q}_1(u) = \bar{Q}(\bar{q}_{2:2}(u), u, u) = 2u - u^2 + \frac{u - 2u^2 + u^3}{2 - u}$$

https://doi.org/10.1017/S0269964819000159 Published online by Cambridge University Press

and

$$\bar{q}_2(u) = \bar{Q}(u, \bar{q}_{2:2}(u), u) = u + \frac{u(1-u)(2-u)}{3-3u+u^2}.$$

A straightforward calculation shows that $\bar{q}_1 \geq \bar{q}_2$ in $[0,1]$, that is, $1 \geq_{wmiar} 2$ holds for this copula. Therefore, from Proposition 2.10, (i), $T_1 \geq_{ST} T_2$ holds for all $\bar{F}$. In order to get the hazard rate (HR) ordering, we need to study the ratio $r = \bar{q}_1/\bar{q}_2$ in $[0,1]$. This ratio is plotted in Figure 1 (right). Since it is decreasing we have from Proposition 2.10, (ii), that $T_1 \geq_{HR} T_2$ for all $\bar{F}$.

We know from Theorem 2.7 that $i \geq_{wmi} j$ implies $i \geq_{wmiar} j$. The last example shows that the reverse property is not true.

Example 3.10: Let us consider a parallel system with lifetime $T = \max(X_1, X_2)$ and with two dependent components having a copula C. Hence, the system reliability is

$$\bar{F}_T(t) = \Pr(T > t) = 1 - \Pr(X_1 \leq t, X_2 \leq t) = 1 - C(F_1(t), F_2(t)) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t)),$$

where $\bar{Q}(u,v) = 1 - C(1-u, 1-v)$.

Therefore, $1 \geq_{wmi} 2$ holds, if and only if, $\bar{Q}(u+c, u) \geq \bar{Q}(u, u+c)$ for all $u, c \in (0,1)$, that is, if and only if,

$$C(x,y) \geq C(y,x) \quad \text{for all } 0 \leq y \leq x \leq 1. \tag{3.2}$$

Analogously, $1 \geq_{wmiar} 2$ holds, if and only if, $\bar{Q}(2u-u^2, u) \geq \bar{Q}(u, 2u-u^2)$ for all $u \in (0,1)$, that is, if and only if,

$$C(x,x^2) \geq C(x^2,x) \quad \text{for all } 0 \leq x \leq 1. \tag{3.3}$$

Clearly, (3.2) implies (3.3), that is, $i \geq_{wmi} j$ implies $i \geq_{wmiar} j$. However, there exist copulas such that (3.3) holds but (3.2) does not hold.

To construct one of these copulas, we are going to use the rectangular patchwork method developed in Durante, Saminger-Platz, and Sarkoci [7]. We start with the product copula $\Pi(u,v) = uv$, that is, with a uniform distribution over $[0,1]^2$. Then we modify this copula just in the squares A and B in Figure 2. In these squares, Π is replaced by the Fréchet–Hoeffding upper bound copula given by

$$M(u,v) = \min(u,v)$$

for $u, v \in [0,1]$, that is, the mass in A and B is concentrated in the diagonals of these squares. Doing so we obtain the copula D. If $(u,v) \in [0,1] - (A \cup B)$, then $D(u,v) = \Pi(u,v) = uv$.

However, for $(0.1, 0.9) \in A$, we have,

$$D(0.9, 0.1) = 0.9 \cdot 0.1 = 0.09 < D(0.1, 0.9) = 0.1 \cdot 0.8 + 0.5 \cdot \frac{1}{25} = 0.10.$$

Hence (3.2) does not hold, that is, component 1 is not weakly more important than component 2 for this copula. The same happens for all the points in the interior part of A. Analogously, for the points (u,v) in the interior part of B, we have $D(u,v) > D(v,u)$. Moreover, as the curve $y = x^2$ used in (3.3) does not cut the region A, then (3.3) holds, that is, $1 \geq_{wmiar} 2$. Even more, as the curve $y = x^2$ cuts the region B, $1 \leq_{wmiar} 2$ does not hold. To summarize, for the copula D, we have proved that $1 \geq_{wmiar} 2$ holds (with $1 \neq_{wmiar} 2$) but that $1 \geq_{wmi} 2$ does not hold.

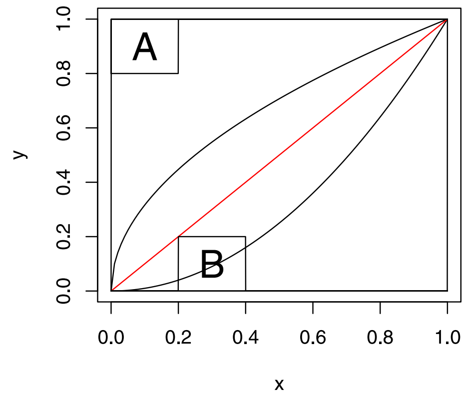

FIGURE 2. Regions for the rectangular patchwork copula considered in Example 3.10.

4. CONCLUSIONS

We have provided several results connecting importance measures in coherent systems with the performance of the systems obtained by adding redundancy in some components. These connections can be applied to decide where the redundant components should be placed. Our results hold for systems with both dependent and independent components and for different redundancy mechanisms, including the most usual ones. To this purpose, we have used known importance measures and we have also defined new ones for specific redundancy options.

The main disadvantage of our results is that, in many of them, we need to assume that the system has identically distributed components. However, it is a usual assumption in engineering systems. In fact, a simple way to get a lower bound for the reliability of a system is to consider as a common reliability function $\bar{F} = \min(\bar{F}_1, \ldots, \bar{F}_n)$, where $\bar{F}_i$ represents the reliability function of the ith component for $i = 1, \ldots, n$.

An important task for future research is to extend the present results to systems with non-identically distributed components. Another interesting problem is to apply the preceding results to specific dependence structures (copulas) and/or systems. Also numerical techniques should be developed for complex systems.

Acknowledgments

The first author (JN) was supported in part by Ministerio de Economía, Industria y Competitividad of Spain (AEI/FEDER, UE) under grant MTM2016-79943-P and the other authors were supported in part by Ministerio de Economía y Competitividad of Spain (AEI/FEDER, UE) under respective grants MTM2017-84058-P (PFM), MTM2014-60594-P (JFS), and MTM2017-89577-P (AA).

References

1. Arriaza, A., Navarro, J., & Suárez-Llorens, A. (2018). Stochastic comparisons of replacement policies in coherent systems under minimal repair. *Naval Research Logistics* 65: 550–565.
2. Barlow, R.E. & Proschan, F. (1975). *Statistical theory of reliability and life testing.* International Series in Decision Processes. New York: Holt, Rinehart and Winston, Inc.
3. Belzunce, F., Martínez-Riquelme, C., & Ruiz, J.M. (2018). Allocation of a relevation in redundancy problems. To appear in *Applied Stochastic Models in Business and Industry.* Published online first April 6, 2018. DOI: 10.1002/asmb.2328.

4. Burkschat, M. & Navarro, J. (2018). Stochastic comparisons of systems based on sequential order statistics via properties of distorted distributions. *Probability in the Engineering and Informational Sciences* 32: 246–274.
5. Calì, C., Longobardi, M., & Navarro, J. (2018). Properties for generalized cumulative past measures of information. To appear in *Probability in the Engineering and Informational Sciences*. Published online first October 29, 2018. DOI: 10.1017/S0269964818000360.
6. Durante, F. & Sempi, C. (2016). *Principles of Copula Theory*. London: CRC/Chapman & Hall.
7. Durante, F., Saminger-Platz, S., & Sarkoci, P. (2009). Rectangular patchwork for bivariate copulas and tail dependence. *Communications in Statistics Theory and Methods* 38: 2515–2527.
8. Federer, H. (1969). *Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften 153*. Berlin-Heidelberg-New York: Springer-Verlag.
9. Gertsbakh, I.B. & Shpungin, Y. (2012). Combinatorial approach to computing component importance indexes in coherent systems. *Probability in the Engineering and Informational Sciences* 26: 117–128.
10. Hazra, N.K. & Nanda, A.K. (2014). Component redundancy versus system redundancy in different stochastic orderings. *IEEE Transactions on Reliability* 63: 567–582.
11. Iyer, S. (1992). The Barlow–Proschan importance and its generalizations with dependent components. *Stochastic Processes and their Applications* 42: 353–359.
12. Krakowski, M. (1973). The relevation transform and a generalization of the Gamma distribution function. *Revue française d'automatique, informatique, recherche opérationnelle,* tome 7 (no. V-2): 107–120.
13. Kuo, W. & Zhu, X. (2012). *Importance measures in reliability, risk, and optimization.* Principles and applications. West Sussex, UK: John Wiley & Sons Ltd.
14. Marichal, J.-L. & Mathonet, P. (2013). On the extensions of Barlow–Proschan importance index and system signature to dependent lifetimes. *Journal of Multivariate Analysis* 115: 48–56.
15. Miziuła, P. & Navarro, J. (2017). Sharp bounds for the reliability of systems and mixtures with ordered components. *Naval Research Logistics* 64: 108–116.
16. Miziuła, P. & Navarro, J. (2019). A new importance measure for reliability systems with dependent components. *To appear in IEEE Transactions on Reliability*, Published online first February 2019. DOI: 10.1109/TR.2019.2895400.
17. Navarro, J. & Gómis, M.C. (2016). Comparisons in the mean residual life order of coherent systems with identically distributed components. *Applied Stochastic Models in Business and Industry* 32: 33–47.
18. Navarro, J., del Águila, Y., Sordo, M.A., & Suárez-Llorens, A. (2013). Stochastic ordering properties for systems with dependent identically distributed components. *Applied Stochastic Models in Business and Industry* 29: 264–278.
19. Navarro, J., del Águila, Y., Sordo, M.A., & Suárez-Llorens, A. (2016). Preservation of stochastic orders under the formation of generalized distorted distributions. *Applications to coherent systems. Methodology and Computing in Applied Probability* 18: 529–545.
20. Nelsen, R.B. (2006). *An introduction to copulas*, 2nd ed., New York: Springer.
21. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders.* Springer series in statistics. New York: Springer.
22. Zhang, X. & Wilson, A. (2017). System reliability and component importance under dependence. *Technometrics* 59: 215–224.