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Theories of analytic monads

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In this paper we characterise the categories of Lawvere theories and equational theories that correspond to the categories of analytic and polynomial monads on Set, and hence also to the categories of the symmetric and rigid operads in Set. We show that the category of analytic monads is equivalent to the category of regular-linear theories. The category of polynomial monads is equivalent to the category of rigid theories, that is, regular-linear theories satisfying an additional global condition. This solves a problem posed by A. Carboni and P. T. Johnstone. The Lawvere theories corresponding to these monads are identified *via* some factorisation systems. We also show that the categories of analytic monads and finitary endofunctors on Set are monadic over the category of analytic functors. The corresponding monad for analytic monads distributes over the monad for finitary endofunctors and hence the category of (finitary) monads on Set is monadic over the category of analytic functors. This extends a result of M. Barr.

1. Introduction

The category of algebras of a (finitary) equational theory can be equivalently described as a category of models of a Lawvere theory or as a category of algebras of a finitary monad on the category Set. In some cases there are also two other descriptions available. Some categories of algebras can be also described as algebras for a symmetric operad, and some can be described as algebras for a rigid† operad (Hermida *et al.* 2000; Hermida *et al.* 2001; Hermida *et al.* 2002; Zawadowski 2011). It is well known that the categories of equational theories **ET**, Lawvere theories **LT** and monads (on Set) **Mnd** are equivalent.

† What we call a 'rigid operad' was earlier called an 'operad with non-standard amalgamation', that is, a one-object version of the multicategories considered by C. Hermida, M. Makkai and J. Power in Hermida *et al.* (2000; 2001; 2002). We decided to change the name because it is a very important notion deserving a simpler name. The choice of this name was motivated by the property of the equational theories that correspond to such operads. The category of rigid operads can be identified with the full subcategory of the symmetric operads such that the actions of symmetric groups on their operations are free.

In fact, the multicategories considered in Hermida *et al.* (2000; 2001; 2002) have yet another feature in that they have two kinds of objects (upper and lower). This additional complication was necessary for the Hermida–Makkai–Power construction of opetopic sets, but in Szawiel and Zawadowski (2013b) we gave a more conceptual and simpler construction of opetopic sets based on multicategories with objects of one kind only. A more detailed discussion of the comparison of rigid operads/multicategories with the multicategories considered in Hermida *et al.* (2000; 2001; 2002) is presented in Zawadowski (2011, Sections 6.5 and 6.5). Sections 5.5, 5.6 and 5.7 of the same paper presents yet another construction of opetopic sets based on the relative version of T-categories (Burroni 1971) and a construction of the set of opetopes given in Leinster (2004).

It is also known that the categories of symmetric and rigid operads are equivalent to the categories of analytic and polynomial monads, respectively (Zawadowski 2011). In the current paper we give a description of the subcategories of **ET** and **LT** that correspond to the categories of symmetric and rigid operads.

The equational theories corresponding to analytic monads are linear-regular theories. A linear-regular theory is an equational theory that can be axiomatised by equations having the same variables on both sides, each variable occurring exactly once. A linear-regular theory T is rigid if and only if whenever a linear-regular equation

$$t(x_1, \dots x_n) = t(x_{\sigma(1)}, \dots x_{\sigma(n)})$$

is provable in T, the permutation σ is the identity permutation. In the above equation, $t(x_1, \dots x_n)$ denotes any term with n different variables $x_1, \dots x_n$, each of which occurs exactly once, and $t(x_{\sigma(1)}, \dots x_{\sigma(n)})$ denotes the same term t but with variables permuted according to σ. For example, the theory of monoids is rigid but the theory of commutative monoids is not since it contains the equation

$$m(x_1, x_2) = m(x_2, x_1).$$

The category of polynomial monads **PolyMnd** is equivalent to the category of rigid theories **RiET**. The notion of a linear-regular theory was considered in universal algebra but the notion of a rigid theory, as well as that of a linear-regular interpretation, seems to be new. If all the axioms of an equational theory are linear-regular, then the theory is linear-regular. However, the problem of whether a finite set of linear-regular equations defines a rigid theory is undecidable (Bojańczyk *et al.* 2014).

We also give a characterisation of the categories of Lawvere theories that correspond to the categories of analytic and polynomial monads. The category $\mathbb{F}^{op}$, which is the opposite of the skeleton of the category of finite sets, is the initial Lawvere theory. Thus, it has a unique morphism into any other Lawvere theory $\pi : \mathbb{F}^{op} \to \mathbf{T}$. The class of morphisms in the image of π closed under isomorphisms is called the class of structural morphisms in $\mathbf{T}$. The analytic morphisms are defined as those that are right orthogonal to the structural ones. A Lawvere theory $\mathbf{T}$ is analytic if and only if the classes of structural and analytic morphisms form a factorisation system and the group of automorphisms of any object n in $\mathbf{T}$ is uniquely determined by the group of automorphisms of 1. A Lawvere theory is rigid if it is analytic and the symmetric group actions act freely on analytic operations. We show that the categories **AnLT** of analytic Lawvere theories and **RiLT** of rigid Lawvere theories correspond to the categories of analytic and polynomial monads, respectively.

Figure 1 illustrates the relations between the categories mentioned above. The vertical lines denote adjoint equivalences. Thus, up to equivalence, it only contains three categories, one on each level: at the top level, we have the category of finitary monads; in the middle, the category of analytic monads; and at the bottom, the category of (finitary) polynomial monads. Thus, all four columns of equational theories, Lawvere theories, monads and operads[†] are 'level-wise' equivalent. These columns are denoted by the letters e, l, m and o, respectively. The vertical functors going up are inclusions of subcategories. The

[†] The column for operads is a bit shorter in this paper, but it can be extended as we show in Szawiel and Zawadowski (2013a).

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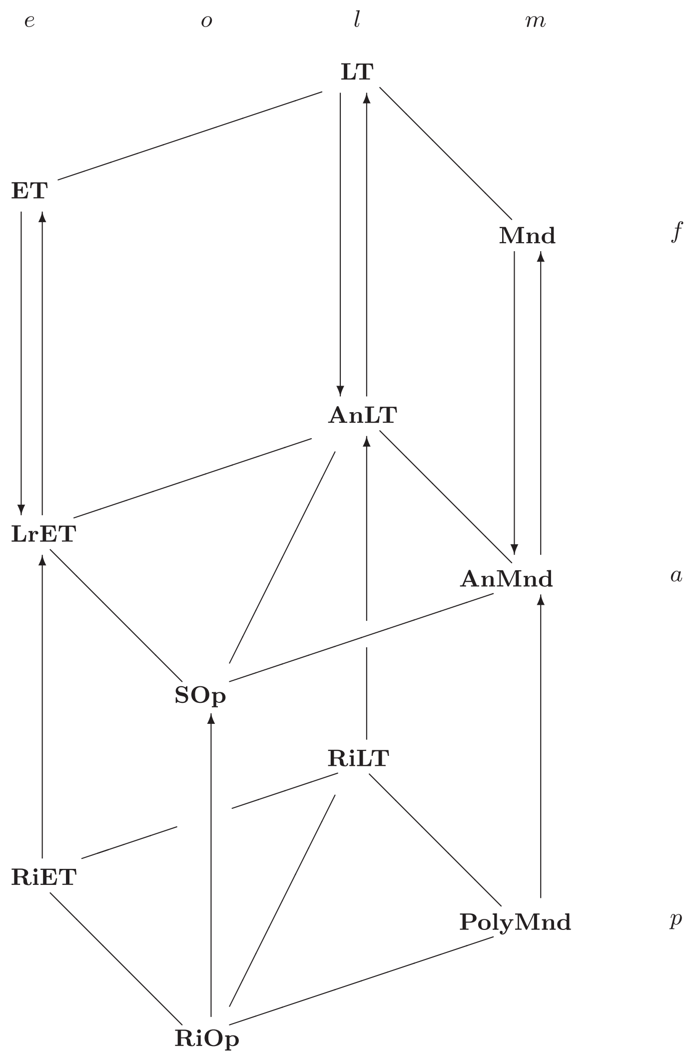

Fig. 1. Relations between categories

lower functors are full inclusions and the upper ones are inclusions that are full on isomorphisms. The vertical functors going down, which are the right adjoints to those going up, are monadic. All the squares in the diagram commute up to isomorphism.

The notation for the categories involved is shown in Figure 1. The notation for functors is not on the diagram, but it refers systematically to the levels and columns they 'connect'. The horizontal functors are denoted using letters from the two columns they connect (the codomain by the script letter and the domain by a subscript), and the level is denoted by a superscript. For example, the functor **AnMnd** → **AnLT** is denoted by $\mathcal{L}^a_m$. We will usually drop the superscripts and will often drop the subscripts when there is no risk of confusion, so we can write $\mathcal{E} = \mathcal{E}_o = \mathcal{E}^p_o$: **RiOp** → **RiET**. The vertical functors going up are denoted by the script letter $\mathcal{P}$ with a superscript indicating the column and a subscript indicating the level of the codomain. The vertical functors going down are denoted by the script

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letter $\mathcal{Q}$ with the same subscript and superscript as those going up. Thus, for example, we have functors $\mathcal{P} = \mathcal{P}^o = \mathcal{P}^o_a$: **RiOp** → **SOp** and $\mathcal{Q} = \mathcal{Q}_f = \mathcal{Q}^m_a$: **Mnd** → **AnMnd**. We will also refer to various diagonal morphisms, so we extend the notation for vertical functors by specifying the columns of both the domain and the codomain. For example, we write $\mathcal{P}^{ol}_f$: **SOp** → **LT** to denote one such functor; its right adjoint is denoted by $\mathcal{Q}^{lo}_a$: **LT** → **SOp**. In principle, this notation will not always specify the codomain uniquely, but in practice it is sufficient, in fact, much less is usually needed, and each time it is used it will be recalled explicitly.

1.1. *Organisation of the paper*

In Section 2, we recall categories of equational theories, Lawvere theories, monads on *Set* and operads. We also discuss some of their subcategories. In Section 3, we study relations between Lawvere theories and operads. We define a functor $\mathcal{L}_o$: **SOp** → **LT** from the category of symmetric operads to the category of Lawvere theories, identify its image and show that its right adjoint is monadic. We also identify the image of the category of the rigid operads **RiOp** in **LT**. In Section 4, we relate the result from Section 3 to monads. We note that finitary monads are monadic over analytic ones, but also explain that this is a consequence of the even more fundamental fact that there is a lax monoidal monad on the category of analytic functors. From this we get that finitary monads are monadic over analytic functors, which extends a result from Barr (1970). In Section 5, we define the embedding **SOp** in **ET** and characterise the images of both **SOp** and **RiOp**. This gives the characterisations described at the beginning of the introduction that solve a problem stated in Carboni and Johnstone (2004). In Section 6, we provide some examples and comments. Finally, we recall the correspondence between equational theories, Lawvere theories and monads in an appendix.

1.2. *Notation*

Throughout the paper, we will use the following notation. ω denotes the set of natural numbers. For $n \in \omega$, we have $n = \{0, \dots, n-1\}$, $[n] = \{0, \dots, n\}$, $(n] = \{1, \dots, n\}$. The set X^n is interpreted as $X^{(n]}$ and it has a (natural) right action of the symmetric group S_n by composition. The skeletal category equivalent to the category of finite sets will be denoted by $\mathbb{F}$. The objects of $\mathbb{F}$ are sets $(n]$, for $n \in \omega$. The subcategories of $\mathbb{F}$ with the same objects as $\mathbb{F}$ but having as morphisms bijections, surjections and injections will be denoted by $\mathbb{B}$, $\mathbb{S}$ and $\mathbb{I}$, respectively. When S_n acts on the set A on the right and on the set B on the left, the set $A \otimes_n B$ is the usual tensor product of S_n-sets.

2. Presentations of categories of algebras

In this section we collect together several categories whose objects describe (some) categories of algebras of finitary equational theories and whose morphisms induce functors between such categories of algebras.

2.1. *Equational theories*

By an *equational theory* we mean a pair of sets $T = (L, A)$, where

$$L = \bigcup_{n \in \omega} L_n$$

and L_n is the set of n-ary operations of T. The sets of operations of different arities are disjoint. The set $\mathcal{T}r(L, \vec{x}^n)$ of terms of L in context $\vec{x}^n = \langle x_1, \ldots, x_n \rangle$ is the usual set of terms over L built using variables from $\vec{x}^n$. We write $t : \vec{x}^n$ for the term t in context $\vec{x}^n$. Thus, all the variables occurring in t are among those in $\vec{x}^n$. The set A is a set of equations in context $t = s : \vec{x}^n$, that is, both $t : \vec{x}^n$ and $s : \vec{x}^n$ are terms in context.

A morphism of equational theories, an *interpretation*

$$I : (L, A) \to (L', A'),$$

is given by a set of functions

$$I_n : L_n \to \mathcal{T}r(L', \vec{x}^n),$$

for $n \in \omega$. The I_ns extend to functions

$$\bar{I}_n : \mathcal{T}r(L, \vec{x}^n) \to \mathcal{T}r(L', \vec{x}^n)$$

in an obvious way. We require that for any $t = s : \vec{x}^n$ in A, we have

$$A' \vdash \bar{I}(t) = \bar{I}(s) : \vec{x}^n$$

where $A' \vdash$ is the provability in the equational logic from axioms in the set A'. We identify two such interpretations if they are provably equal. In this way we have defined the category of equational theories **ET**.

A term in context $t : \vec{x}^n$ is *regular* if every variable in $\vec{x}^n$ occurs in t at least once. A term in context $t : \vec{x}^n$ is *linear* if every variable in $\vec{x}^n$ occurs in t at most once. A term in context $t : \vec{x}^n$ is *linear-regular* if it is both linear and regular. An equation $s = t : \vec{x}^n$ is *linear-regular* if and only if both $s : \vec{x}^n$ and $t : \vec{x}^n$ are linear-regular terms in contexts.

A *simple ϕ-substitution of a term in context* $t : \vec{x}^n$ along a function $\phi : (n] \to (k]$ is a term in context denoted $\phi \cdot t : \vec{x}^k$ such that every occurrence of the variable x_i is replaced by the occurrence of $x_{\phi(i)}$. An *α-conversion of a term in context* $t : \vec{x}^n$ is a simple ϕ-substitution of a term in context along a monomorphism $\phi : (n] \to (k]$.

An equational theory $T = (L, A)$ is a *linear-regular theory* if and only if every equation $s = t : \vec{x}^n$ that is a consequence of the theory T is a consequence of the set of linear-regular consequences of T. An interpretation is *linear-regular* if and only if it interprets function symbols as linear-regular terms.

A theory $T = (L, A)$ is a *rigid theory* if and only if it is linear-regular and for any linear-regular term in context $t : \vec{x}^n$, if

$$A \vdash t = \tau \cdot t : \vec{x}^n,$$

then τ is the identity permutation. $\tau \cdot t$ is the simple τ-substitution of a term in context $t : \vec{x}^n$ along a permutation $\tau \in S_n$.

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Remark 2.1. Every equation in a linear-regular theory T is a simple substitution of a linear-regular equation. By this we mean that if T proves an equation $s = t : \vec{x}^n$, then there are linear-regular terms $s' : \vec{x}^m$ and $t' : \vec{x}^m$ and a function $\phi : (m] \to (n]$ such that $T \vdash s' = t' : \vec{x}^n$ and $s = \phi \cdot s'$ and $t = \phi \cdot t'$ (as terms). If T is rigid, then such a ϕ is unique. This can be proved by showing that any proof from linear-regular axioms in equational logic can be replaced by a proof in which the substitutions leading to repetition of the variables are simple substitutions (variable to variable) and are all moved to the end of the proof.

We write **LrET** to denote the subcategory of **ET** consisting of linear-regular theories and linear-regular interpretations, and **RiET** to denote the full subcategory of **LrET** whose objects are rigid theories. We have inclusion functors

$$\mathbf{RiET} \longrightarrow \mathbf{LrET} \longrightarrow \mathbf{ET}$$

with the first inclusion being full and the second being full on isomorphisms (Zawadowski 2011).

2.2. *Lawvere theories*

A *Lawvere theory* (Lawvere 1963; Kock and Reyes 1977) is a category whose objects are natural numbers such that n is a product 1^n with *chosen* projections $\pi_i^n : n \to 1$, for $n \in \omega$ and $i \in (n]$. An interpretation (or a morphism) of Lawvere theories is a functor constant on objects that preserves the chosen projections. Lawvere theories and their morphisms form a category, which is denoted by **LT**.

The initial object in the category **LT** is the category $\mathbb{F}^{op}$ with the obvious inclusions as projections – see the introduction. The unique morphism from $\mathbb{F}^{op}$ into any Lawvere theory $\mathbf{T}$ will be denoted by $\pi = \pi_{\mathbf{T}} : \mathbb{F}^{op} \longrightarrow \mathbf{T}$. Thus, for $\phi : (n] \to (m]$ in $\mathbb{F}$, we have

$$\pi_\phi = \langle \pi_{\phi(i)} \rangle_{i \in (n]} : m \to n$$

in $\mathbf{T}$. The functor π_T is faithful unless $\mathbf{T}$ is the terminal theory or its unique proper subtheory. Both theories are not regular.

The class of *structural morphisms* in $\mathbf{T}$ is the closure under isomorphism of the image under π of all morphisms in $\mathbb{F}^{op}$. A morphism in $\mathbf{T}$ is *analytic* if and only if it is right orthogonal to all structural morphisms.

By a factorisation system in a category $\mathcal{C}$, we mean the factorisation system in the sense of Freyd and Kelly (1972), see also Carboni *et al.* (1997, Section 2.8), that is, it consists of two classes of morphisms in $\mathcal{C}$ closed under isomorphisms, say $\mathcal{E}$ and $\mathcal{M}$, such that the morphisms in $\mathcal{E}$ are left orthogonal to those in $\mathcal{M}$, and each morphism f in $\mathcal{C}$ factors as $f = m \circ e$ where $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

$Aut_{\mathbf{T}}(n)$ is the set of automorphisms of n in $\mathbf{T}$. In any Lawvere theory $\mathbf{T}$ we have a function

$$\rho_n^{\mathbf{T}} : S_n \times Aut_{\mathbf{T}}(1)^n \longrightarrow Aut_{\mathbf{T}}(n)$$

such that

$$(\sigma, a_1, \ldots, a_n) \mapsto a_1 \times \ldots \times a_n \circ \pi_\sigma,$$

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that is, ρ_n sends a permutation σ and n isomorphisms of 1 to an isomorphism of $n = 1^n$ in $\mathbf{T}$. We say that $\mathbf{T}$ has *simple automorphisms* if and only if $\rho_n^{\mathbf{T}}$, for $n \in \omega$, is a bijection. Clearly, if $\mathbf{T}$ has simple automorphisms, then 2 is not initial in $\mathbf{T}$.

A Lawvere theory $\mathbf{T}$ is *analytic* if and only if structural morphisms and analytic morphisms form a factorisation system in $\mathbf{T}$ and $\mathbf{T}$ has simple automorphisms. A Lawvere theory $\mathbf{T}$ is *rigid* if and only if it is analytic and the symmetric groups S_n, for $n \in \omega$, acting on $\mathbf{T}(n, 1)$ by permuting factors act freely on analytic morphisms.

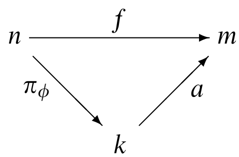
An *analytic interpretation* of Lawvere theories is an interpretation of Lawvere theories that preserves analytic morphisms. Thus, we have a non-full subcategory of analytic Lawvere theories and analytic interpretations **AnLT**. The latter has as a full subcategory the category **RiLT** of rigid Lawvere theories. We have inclusion functors

$$\mathbf{RiLT} \longrightarrow \mathbf{AnLT} \longrightarrow \mathbf{LT}$$

with the first one being a full inclusion.

We have the following easy lemma.

Lemma 2.2. In any analytic Lawvere theory $\mathbf{T}$, any morphism $f : n \to m$ has a factorisation



with a being an analytic morphism in $\mathbf{T}$ and $\phi : (k] \to (n]$ being a function. Such a factorisation is unique up to a permutation, that is, if $f = a' \circ \pi_{\phi'}$ is another such factorisation, there is $\sigma \in S_k$ such that

$$\phi \circ \sigma = \phi'$$
$$a = a' \circ \pi_\sigma.$$

Proof. When $\mathbf{T}$ has simple automorphisms, any structural morphism $s : n \to m$ in $\mathbf{T}$ can be presented as $(a_1 \times \ldots, a_m) \circ \pi_\phi$ for some function $\phi : (m] \to (n]$ and $a_i \in Aut(1)$ for $i \in (m]$. Thus, if $f = a \circ s$ is a structural-analytic factorisation of f, with s as above, then

$$f = (a \circ (a_1 \times \ldots, a_m)) \circ \pi_\phi$$

is one also. □

2.3. *Monads*

We shall consider three categories of finitary monads on Set. The category of all finitary monads with the usual morphisms of monads will be denoted by **Mnd**. A morphism of monads

$$\tau : (M, \eta, \mu) \to (M', \eta', \mu')$$

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is a natural transformation $\tau : M \to M'$ such that

$$\tau \circ \eta^M = \eta^{M'}$$
$$\tau \circ \mu^M = \mu^{M'} \circ \tau_{M'} \circ M(\tau).$$

Recall that a finitary monad (M, η, μ) on Set is *analytic* if and only if M weakly preserves wide pullbacks and both η and μ are weakly cartesian natural transformations. A *morphism of analytic monads* on Set

$$\tau : (M, \eta, \mu) \to (M', \eta', \mu')$$

is a weakly cartesian natural transformation τ that is a morphism of monads (Joyal 1986; Zawadowski 2011). Recall that a finitary monad (M, η, μ) is a *polynomial monad* on Set if and only if M preserves wide pullbacks and both η and μ are cartesian natural transformations. However, both types of functors and monads have a much more explicit description (Joyal 1986; Zawadowski 2011).

The categories of analytic and polynomial monads with the suitable morphisms will be denoted by **AnMnd** and **PolyMnd**, respectively. We have two inclusion functors

$$\mathbf{PolyMnd} \longrightarrow \mathbf{AnMnd} \longrightarrow \mathbf{Mnd}$$

where the first is full and the second is full on isomorphisms (Zawadowski 2011).

The equivalence of the three categories **ET**, **LT** and **Mnd** is briefly recalled in the appendix.

2.4. *Operads*

The symmetric operads provide yet another way of presenting models of equational theories. This kind of presentation is usually very convenient, but the models defined by such operads are more specific.

Recall that a *symmetric operad* $\mathcal{O}$ (in Set) consists of:

— a family of sets $\mathcal{O}_n$, for $n \in \omega$;
— a unit element $\iota \in \mathcal{O}_1$;
— for any $k, n, n_1, \ldots, n_k \in \omega$ with $n = \sum_{i=1}^{k} n_i$, a composition operation

$$* : \mathcal{O}_{n_1} \times \ldots \times \mathcal{O}_{n_k} \times \mathcal{O}_k \longrightarrow \mathcal{O}_n;$$

— a left action of the symmetric groups

$$\cdot : S_n \times \mathcal{O}_n \longrightarrow \mathcal{O}_n$$

for $n \in \omega$;

such that the composition is associative with unit ι and compatible with the group actions. A *morphism of symmetric operads* $f : \mathcal{O} \to \mathcal{O}'$ is a function that respects arities of operations, unit, compositions and group actions. For more on symmetric operads and their history, see, for example, Leinster (2004)[†].

[†] Note that permutations act on the right in Leinster (2004), while, because of our convention for compositions, we have left actions.

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The symmetric operad of symmetries $\mathcal{S}ym$ is defined as follows:

— the set of n-ary operations of $\mathcal{S}ym$ is the symmetric group S_n on which S_n act on the left by multiplication;

— the composition

$$\star : S_{n_1} \times \ldots \times S_{n_k} \times S_k \longrightarrow S_n$$

for

$$(\sigma_1, \ldots, \sigma_k ; \tau) \in S_{n_1} \times \ldots \times S_{n_k} \times S_k$$

is the permutation

$$\langle \sigma_1, \ldots, \sigma_k \rangle \star \tau : n = \sum_{i=1}^{k} n_{\tau(i)} \longrightarrow n = \sum_{i=1}^{k} n_i$$

given by

$$\langle i, r \rangle \mapsto \langle \tau(i), \sigma_{\tau(i)}(r) \rangle$$

where we consider the obvious lexicographic order on both $\sum_{i=1}^{k} n_{\tau(i)}$ and $\sum_{i=1}^{k} n_i$.

Note that although composition consists of functions between groups, these functions are not homomorphisms of groups in general.

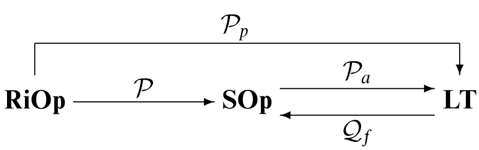
The category of rigid operads **RiOp** can be identified with the full subcategory of symmetric operads **SOp** whose objects are those operads that have all the actions of symmetric groups free. Thus, we have a full embedding

$$\mathcal{P} : \mathbf{RiOp} \longrightarrow \mathbf{SOp}.$$

For further details, see Hermida *et al.* (2000; 2001; 2002) and Zawadowski (2011).

3. Lawvere theories versus operads

In this section we study the relations between Lawvere theories and operads, both symmetric and rigid. We shall describe the adjunction $\mathcal{P}_a \dashv \mathcal{Q}_f$ and the properties of the embeddings $\mathcal{P}_a$ and $\mathcal{P}_p$:



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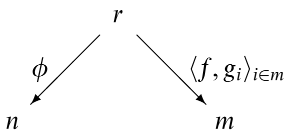
3.1. *The functor* $\mathcal{P}_a : \mathbf{SOp} \to \mathbf{LT}$

Let $\mathcal{O}$ be a symmetric operad and let ι, $\cdot$ and $*$ denote the unit, symmetric groups actions and compositions in $\mathcal{O}$, respectively. We define a Lawvere theory $\mathcal{P}_a(\mathcal{O})$ as follows. A morphism from n to m in $\mathcal{P}_a(\mathcal{O})$ is an equivalence class of spans



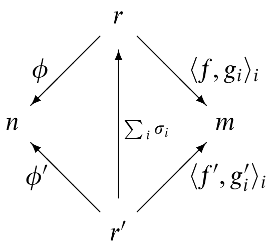
such that:

— $\phi : (r] \to (n]$ is a function;
— $f : (r] \to (m]$ is a monotone function;
— $r_i = |f^{-1}(i)|$;
— $g_i \in \mathcal{O}_{r_i}$

for $i \in (m]$ and

$$r = \sum_{i=1}^{m} r_i.$$

Two spans $\langle \phi, f, g_i \rangle_{i \in m}$ and $\langle \phi', f', g'_j \rangle_{j \in m'}$ are equivalent if and only if $f = f'$ and there are permutations $\sigma_i \in S_{r_i}$ for $i \in (m]$



such that

$$g_i = \sigma_i \cdot g'_i$$
$$\phi \circ \sum_i \sigma_i = \phi'$$

where we write

$$\sum_i \sigma_i : r \to r$$

to mean the permutation formed by placing permutations σ_i 'one after another'. Thus, it respects the fibre of f, that is,

$$f \circ \sum_i \sigma_i = f.$$

We shall deal with the spans when we perform constructions on morphisms in $\mathcal{P}_a(\mathcal{O})$, but when we consider equalities between spans, we shall invoke the above equivalence relation.

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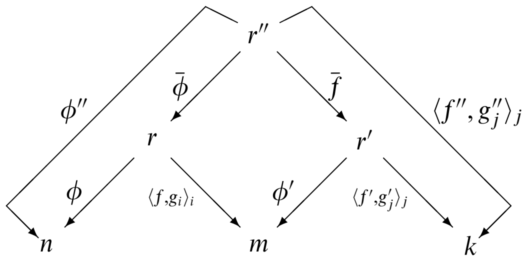
The composition

$$\langle \phi'', f'', g''_j \rangle_{i\in(k]} : n \to k$$

of two morphisms

$$\langle \phi, f, g_i \rangle_{i\in(m]} : n \to m$$
$$\langle \phi', f', g'_j \rangle_{j\in(k]} : m \to k$$

is defined as follows. In the diagram

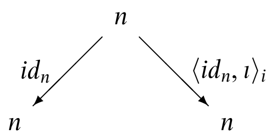


(1)

the square is a pullback of f along ϕ' and the function $\bar{f}$ is chosen so that it is monotone. We put

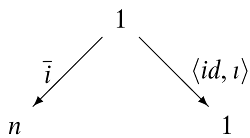
$$f'' = f' \circ \bar{f}$$
$$\phi'' = \phi \circ \bar{\phi}$$
$$g''_j = g'_j * \langle g_{\phi(l)} \rangle_{l\in f^{-1}(j)}.$$

The identity on n is the span



As S_1 contains the identity permutation only, any span equivalent to an identity span is actually equal to it.

The projection $\pi^n_i : n \to 1$ on the ith coordinate is the span



where $i \in (n]$ and $\bar{i}(1) = i$.

For a morphism of symmetric operads $h : \mathcal{O} \to \mathcal{O}'$, we define a functor

$$\mathcal{P}_a(h) : \mathcal{P}_a(\mathcal{O}) \longrightarrow \mathcal{P}_a(\mathcal{O}')$$

so for a morphism $\langle \phi, f, g_i \rangle_{i\in(m]} : n \to m$ in $\mathcal{P}_a(\mathcal{O})$, we define a morphism

$$\mathcal{P}_a(h)(\langle \phi, f, g_i \rangle_{i\in(m]}) = \langle \phi, f, h(g_i) \rangle_{i\in(m]} : n \to m$$

in $\mathcal{P}_a(\mathcal{O}')$.

This concludes the definition of the functor $\mathcal{P}_a$.

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Remark 3.1. Note that since we require that the function f in any span $\langle \phi, f, g_i \rangle_{i \in m}$ representing a morphism in $\mathcal{P}_a$ is monotone, the arities of the operations $\{g_i\}_{i \in m}$ determine f. Thus, we could have either dropped f or not required f to be monotone, but we chose this definition as it makes the arguments simpler.

3.2. *The functor* $\mathcal{Q}_f : \mathbf{LT} \longrightarrow \mathbf{SOp}$

Let $\mathbf{T}$ be a Lawvere theory. The operad $\mathcal{Q}_f(\mathbf{T})$ consists of operations of $\mathbf{T}$, that is, morphisms with codomain 1. It can be described in detail as follows. The set of n-ary operations $\mathcal{Q}_f(\mathbf{T})_n$ is the set of n-ary operations $\mathbf{T}(n, 1)$ of $\mathbf{T}$ for $n \in \omega$. The action

$$\cdot : S_n \times \mathcal{Q}_f(\mathbf{T})_n \longrightarrow \mathcal{Q}_f(\mathbf{T})_n,$$

for $f \in \mathbf{T}(n, 1)$ and $\sigma \in S_n$, is given by

$$\sigma \cdot f = f \circ \pi_\sigma.$$

The identity of $\mathcal{Q}_f(\mathbf{T})$ is $\iota = id_1 \in \mathbf{T}(1, 1)$. The composition

$$* : \mathcal{Q}_f(\mathbf{T})_{n_1} \times \ldots \times \mathcal{Q}_f(\mathbf{T})_{n_k} \times \mathcal{Q}_f(\mathbf{T})_k \longrightarrow \mathcal{Q}_f(\mathbf{T})_n,$$

for $f \in \mathcal{Q}_f(\mathbf{T})_k$ and $f_i \in \mathcal{Q}_f(\mathbf{T})_{n_i}$, where $i \in (k]$, $n = \sum_{i \in k} n_i$, is given by

$$\langle f_1, \ldots, f_k \rangle * f = f \circ (f_1 \times \ldots, \times f_k)$$

where $f_1 \times \ldots, \times f_k$ is defined using the chosen projections in $\mathbf{T}$ and $\circ$ is the composition in $\mathbf{T}$.

If $F : \mathbf{T} \to \mathbf{T}'$ is a morphism of Lawvere theories, then the map of symmetric operads

$$\mathcal{Q}_f(F) : \mathcal{Q}_f(\mathbf{T}) \to \mathcal{Q}_f(\mathbf{T}'),$$

for $f \in \mathcal{Q}_f(\mathbf{T})_n$, is defined by

$$\mathcal{Q}_f(F)(f) = F(f).$$

This concludes the definition of the functor $\mathcal{Q}_f$.

3.3. *The adjunction* $\mathcal{P}_a \dashv \mathcal{Q}_f$ *and the properties of the functor* $\mathcal{P}_a$

We have the following easy lemma.

Lemma 3.2. Let $\mathcal{O}$ be a symmetric operad and $n \in \omega$. An automorphism on n in $P_a(\mathcal{O})$ is represented by a span of the following form

$$\begin{array}{ccc} & n & \\ {\scriptstyle \phi} \swarrow & & \searrow {\scriptstyle \langle id_n, a_i \rangle_i} \\ n & & n \end{array}$$

where $\phi : (n] \to (n]$ is a bijection and $a_i \in \mathcal{O}_1$ is an invertible operation, that is, there is $b_i \in \mathcal{O}_1$ such that

$$a_i * b_i = \iota = b_i * a_i$$

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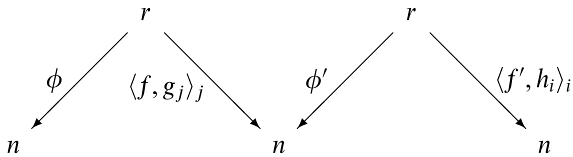
for $i \in (n]$. This is the unique span in its equivalence class.

Proof. Consider a pair of morphisms in $\mathcal{P}_a(\mathcal{O})$



that are inverse to each other. As the above composition is an identity, it follows that ϕ and f' are epi. Thus, because of the other composition, ϕ' and f are also surjections. As pulling back along a surjection reflects injections, all functions ϕ, f, ϕ' and f' must also be injective and hence bijective. Then it is easy to see that $g_{\phi'(j)}$ is an inverse of h_j for $j \in (n]$. □

Proposition 3.3. We have an adjunction $\mathcal{P}_a \dashv \mathcal{Q}_f$ and the functor $\mathcal{P}_a$ is faithful.

Proof. We first show that $\mathcal{P}_a \dashv \mathcal{Q}_f$. For a symmetric operad $\mathcal{O}$ the unit is

$$\eta_{\mathcal{O}} : \mathcal{O} \longrightarrow \mathcal{Q}_f(\mathcal{P}_a(\mathcal{O}))$$

$$\mathcal{O}_n \ni g \mapsto \langle id_n, !, g \rangle.$$

For Lawvere theory $\mathbf{T}$ the counit is

$$\varepsilon_{\mathbf{T}} : \mathcal{P}_a\mathcal{Q}_f(\mathbf{T}) \longrightarrow \mathbf{T}$$

$$\langle \phi, f, g_i \rangle_{i \in (m]} \mapsto (g_1 \times \ldots \times g_m) \circ \pi_\phi.$$

We verify the triangular equalities. For $g \in \mathcal{Q}_f(\mathbf{T})_n = \mathbf{T}(n, 1)$ we have

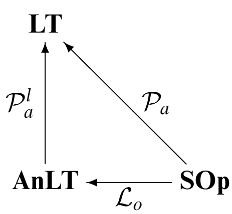
$$\begin{aligned} \mathcal{Q}_f(\varepsilon_{\mathbf{T}}) \circ \eta_{\mathcal{Q}_f(\mathbf{T})}(g) &= \mathcal{Q}_f(\varepsilon_{\mathbf{T}})(\langle id_n, !, g \rangle) \\ &= g \circ \pi_{id_n} \\ &= g. \end{aligned}$$

For $\langle \phi, f, g_i \rangle_{i \in (m]} \in \mathcal{P}_a(\mathcal{O})$, we have

$$\begin{aligned} \varepsilon_{\mathcal{P}_a(\mathcal{O})} \circ \mathcal{P}_a(\eta_{\mathcal{O}})(\langle \phi, f, g_i \rangle_{i \in (m]}) &= \varepsilon_{\mathcal{P}_a(\mathcal{O})}(\langle \phi, f, \langle id_{r_i}, !, g_i \rangle \rangle_{i \in (m]}) \\ &= (\langle id_{r_1}, !, g_1 \rangle \times \ldots \times \langle id_{r_m}, !, g_m \rangle) \circ \pi_\phi \\ &= \langle \phi, f, g_i \rangle_{i \in (m]}. \end{aligned}$$

Since the unit $\eta_{\mathcal{O}}$ is mono, $\mathcal{P}_a$ is faithful. □

Theorem 3.4. The functor $\mathcal{P}_a$ is faithful, full on isomorphisms and its essential image is the category of analytic Lawvere theories **AnLT**, that is, it factorises as an equivalence of categories $\mathcal{L}_o$ followed by $\mathcal{P}_a^l$:



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Proof. Recall that we have a unique morphism of Lawvere theories from the initial theory $\pi : \mathbb{F}^{op} \to \mathcal{P}_a(\mathcal{O})$. For a function $\phi : (m] \to (n]$, π_ϕ, the morphism π_ϕ is represented by the span of the form

$$n \xleftarrow{\ \phi\ } m \xrightarrow{\ \langle id_m, \iota\rangle_{i\in(m]}\ } m$$

The class of the structural morphisms in $\mathcal{P}_a(\mathcal{O})$ is the closure under isomorphism of the class of morphisms $\{\pi_\phi : \phi \in \mathbb{F}\}$. Using Lemma 3.2, it is easy to see that the structural morphisms in $\mathcal{P}_a(\mathcal{O})$ are (represented by) the spans of the form

$$n \xleftarrow{\ \phi\ } m \xrightarrow{\ \langle id_m, a_i\rangle_{i\in(m]}\ } m$$

where ϕ is any function and a_i is an invertible unary operation for $i \in (m]$.

The analytic morphisms in $\mathcal{P}_a(\mathcal{O})$ are (represented by) the spans of the form

$$n \xleftarrow{\ \phi\ } n \xrightarrow{\ \langle f, g_i\rangle_{i\in(m]}\ } m$$

where ϕ is a bijection.

Clearly, both classes contain isomorphisms and are closed under composition.

Any morphism $\langle \phi, f, g_i\rangle_{i\in(m]} : n \to m$ in $\mathcal{P}_a(\mathcal{O})$ has a structural-analytic factorisation as follows

$$n \xleftarrow{\ \phi\ } r \xrightarrow{\ \langle id_r, \iota\rangle_j\ } r \xleftarrow{\ id_r\ } r \xrightarrow{\ \langle f, g_i\rangle_i\ } n$$

Thus, in order to show that structural and analytic morphisms form a factorisation system, we need to show that structural morphisms are left orthogonal to the analytic morphisms. Let

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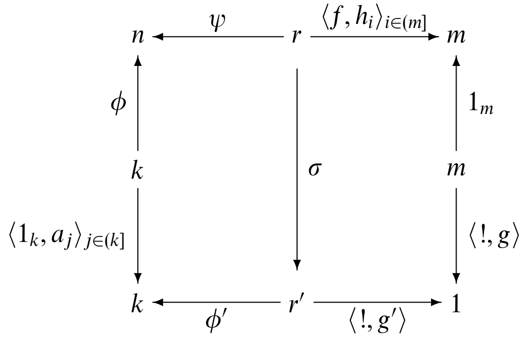


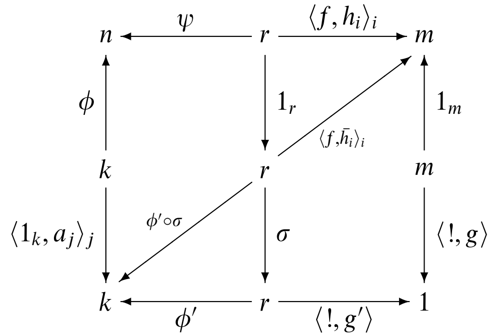
be a commutative square in $\mathcal{P}_a(\mathcal{O})$ where the left vertical morphism $\langle \phi, 1_r, a_i \rangle_{j\in(k]}$ is a structural map and the right vertical morphism $\langle 1_m, !, g\rangle$ is an analytic map. We have chosen the right bottom to be 1 to simplify the notation, but the general case is just a product of such instances. The commutation means that $r = r'$ and there is a permutation $\sigma \in S_r$ such that

$$\psi = \phi \circ \phi' \circ \sigma$$

and

$$\langle a_{\phi'(1)}, \dots, a_{\phi'(r)}\rangle * g' = \sigma \cdot (\langle h_1, \dots, h_m\rangle * g).$$

Putting a diagonal morphism $\langle \phi' \circ \sigma, f, \bar{h}_i\rangle_{i\in(m]}$ into the square, we get



where

$$\bar{h}_i = \langle a^{-1}_{\phi'\circ\sigma(l)}\rangle_{l\in f^{-1}(i)} * h_i$$

and it can be seen that the permutations 1_r and σ show that both triangles commute. It is not difficult to see that this diagonal filling is unique. Thus, the analytic morphisms are indeed right orthogonal to the structural ones and $\mathcal{P}_a(\mathcal{O})$ is an analytic Lawvere theory.

From the description of the functor $\mathcal{P}_a(h) : \mathcal{P}_a(\mathcal{O}) \to \mathcal{P}_a(\mathcal{O}')$ and the description of the structure of $\mathcal{P}_a(\mathcal{O})$, it is clear that $\mathcal{P}_a(h)$ sends the analytic (structural) morphisms to the analytic (structural) morphisms, so $\mathcal{P}_a(h)$ is an analytic interpretation of Lawvere theories.

Now let $\mathbf{T}$ be any Lawvere theory. As the class of analytic morphisms in $\mathbf{T}$ is right orthogonal to a class of morphisms, it is closed under finite products and isomorphisms. In particular, a composition of an analytic morphism $f : n \to 1$ in $\mathbf{T}$ with a permutation morphism π_σ with $\sigma \in S_n$ is again an analytic morphism. Thus, the analytic operations of

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any Lawvere theory $\mathbf{T}$ form a symmetric operad. The composition $\langle f_1, \ldots, f_n \rangle * f$ is defined to be $f \circ (f_1 \times \ldots \times f_n)$ and the action of $\sigma \in S_n$ on an analytic morphism $f : n \to 1$ is $\sigma \cdot f = f \circ \pi_\sigma$. The unit is the identity morphism on 1. Defined in this way, the 'symmetric operad part' of Lawvere theory $\mathbf{T}$ will be denoted by $\mathbf{T}^s$. We have an inclusion morphism of symmetric operads

$$\mathbf{T}^s \to \mathcal{Q}_f(\mathbf{T}).$$

By adjunction, we get a morphism

$$\psi_{\mathbf{T}} : \mathcal{P}_a(\mathbf{T}^s) \longrightarrow \mathbf{T}.$$

Clearly, $\psi_{\mathbf{T}}$ is bijective on objects. If $\mathbf{T}$ is analytic, then $\psi_{\mathbf{T}}$ is full (faithful) since the structural-analytic factorisation exists (is unique and $\pi : \mathbb{F}^{op} \to \mathbf{T}$ is faithful) – see Lemma 2.2.

If $I : \mathbf{T} \to \mathbf{T}'$ is an analytic interpretation between any Lawvere theories, then the diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathbf{T}^s) & \xrightarrow{\ \mathcal{P}(I^s)\ } & \mathcal{P}(\mathbf{T}'^s) \\
{\scriptstyle \psi_{\mathbf{T}}}\downarrow & & \downarrow{\scriptstyle \psi_{\mathbf{T}'}} \\
\mathbf{T} & \xrightarrow[\ I\ ]{} & \mathbf{T}
\end{array}$$

commutes, where I^s is the obvious restriction of I to $\mathbf{T}^s$. Thus, the essential image of $\mathcal{P}_a$ is indeed the category of analytic Lawvere theories and analytic interpretations. An isomorphic interpretation of Lawvere theories is always analytic. Therefore $\mathcal{P}_a$ is full on isomorphisms. $\square$

Proposition 3.5. The functor $\mathcal{Q}_f : \mathbf{LT} \to \mathbf{SOp}$ is monadic.

Proof. We shall verify that $\mathcal{Q}_f$ satisfies the assumptions of Beck's monadicity theorem. By Proposition 3.3, $\mathcal{Q}_f$ has a left adjoint. It is easy to see that $\mathcal{Q}_f$ reflects isomorphisms. We shall verify that $\mathbf{LT}$ has and $\mathcal{Q}_f$ preserves $\mathcal{Q}_f$-contractible coequalizers.

Let $I, I' : \mathbf{T}' \to \mathbf{T}$ be a pair of interpretations between Lawvere theories such that

$$\mathcal{Q}_f(\mathbf{T}') \underset{\mathcal{Q}_f(I')}{\overset{\mathcal{Q}_f(I)}{\rightrightarrows}} \mathcal{Q}_f(\mathbf{T}) \underset{s}{\overset{q}{\rightleftarrows}} \mathcal{O}, \qquad r : \mathcal{Q}_f(\mathbf{T}) \to \mathcal{Q}_f(\mathbf{T}')$$

is a split coequalizer in $\mathbf{SOp}$. We define a Lawvere theory $\mathbf{T}_{\mathcal{O}}$ so that a morphism from n to m in $\mathbf{T}_{\mathcal{O}}$ is an m-tuple $\langle g_1, \ldots, g_m \rangle$ with $g_i \in \mathcal{O}_n$, for $i = 1, \ldots, m$. Projections, compositions and identities in $\mathbf{T}_{\mathcal{O}}$ are defined from the corresponding projections, compositions and identities in $\mathbf{T}$. Thus, the projections $\bar{\pi}_i^n$ in $\mathbf{T}_{\mathcal{O}}$ are the images of the projections π_i^n in $\mathbf{T}$, that is, $\bar{\pi}_i^n = q(\pi_i^n)$. The identity on n in $\mathbf{T}_{\mathcal{O}}$ is $\langle \bar{\pi}_1^n, \ldots, \bar{\pi}_n^n \rangle$. The composition of $\langle g_i \rangle_{i \in (m]} : n \to m$ with $\langle h_j \rangle_{j \in (k]} : m \to k$ in $\mathbf{T}_{\mathcal{O}}$ is

$$\langle q(s(h_j) \circ \langle s(q_i) \rangle_{i \in (m]}) \rangle_{j \in (k]} : n \to k.$$

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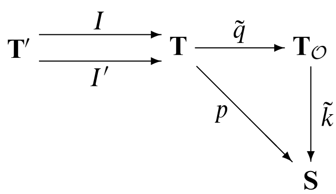
The functor $\tilde{q} : \mathbf{T} \to \mathbf{T}_{\mathcal{O}}$ is defined, for $f : n \to m$ in $\mathbf{T}$, by

$$\tilde{q}(f) = \langle q(\pi_1^m \circ f), \ldots, q(\pi_m^m \circ f) \rangle.$$

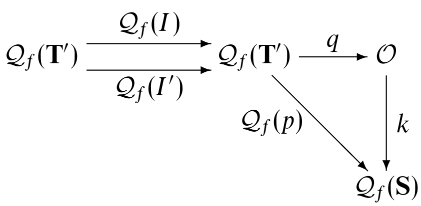
We first verify that the morphisms $\bar{\pi}$ do specify finite products in $\mathbf{T}_{\mathcal{O}}$. To do this it is enough to verify that $\bar{\pi}_i^m \circ \langle g_1, \ldots, g_m \rangle = g_i$. The uniqueness of the morphism into the product is obvious from the construction. By routine calculations,

$$\begin{aligned}
\bar{\pi}_i^m \circ \langle g_1, \ldots, g_m \rangle &= q(sq(\pi_i^m) \circ \langle s(g_1), \ldots, s(g_m) \rangle) \\
&= qsq(\pi_i^m) \circ \langle qs(g_1), \ldots, qs(g_m) \rangle \\
&= q(\pi_i^m) \circ \langle qs(g_1), \ldots, qs(g_m) \rangle \\
&= q(\pi_i^m \circ \langle s(g_1), \ldots, s(g_m) \rangle) \\
&= q(s(g_i)) \\
&= g_i.
\end{aligned}$$

It is obvious that $\tilde{q}$ is a morphism of Lawvere theories and that $\mathcal{Q}_f(\tilde{q}) = q$. We still need to verify that $\tilde{q}$ is a coequalizer in $\mathbf{LT}$. Let $p : \mathbf{T} \to \mathbf{S}$ be a morphism in $\mathbf{LT}$ coequalizing I and I':



The morphism $\mathcal{Q}_f(p)$ coequalizes $\mathcal{Q}_f(I)$ and $\mathcal{Q}_f(I')$ in $\mathbf{SOp}$. Thus, there is a unique morphism k in $\mathbf{SOp}$ making the triangle on the right of



commute. We define the functor $\tilde{k}$ so that

$$\tilde{k}(\langle f_1, \ldots, f_n \rangle) = \langle k(f_1), \ldots, k(f_n) \rangle$$

for any morphism $\langle f_1, \ldots, f_n \rangle$ in $\mathbf{T}_{\mathcal{O}}$. The verification that $\tilde{k}$ is the required unique functor is left as an exercise. □

3.4. *The functor* $\mathcal{P}_p^{ol} = \mathcal{P}_p : \mathbf{RiOp} \to \mathbf{LT}$

The functor $\mathcal{P}_p$ is defined as the composition of the functors $\mathcal{P}_a \circ \mathcal{P}$.

Theorem 3.6. The essential image of the functor $\mathcal{P}_p : \mathbf{RiOp} \longrightarrow \mathbf{LT}$ is the category of $\mathbf{RiLT}$ of rigid Lawvere theories and analytic morphisms between them.

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Proof. Since $\mathcal{P}$ is full and faithful, $\mathcal{P}_p$ is faithful and full on analytic morphisms. The image of $\mathcal{P}$ consists of those symmetric operads for which the symmetric group actions are free. Thus, the image of $\mathcal{P}_p$ consists of those analytic Lawvere theories in which the symmetric actions are free on analytic operations, that is, it consists of the rigid Lawvere theories. □

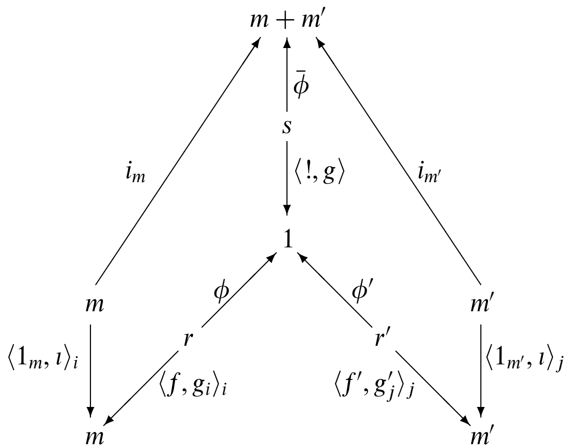
We conclude this section by pointing out yet another property of analytic Lawvere theories. Let $\mathbf{T}$ be a category with finite products. A morphism $p : n \to m$ in $\mathbf{T}$ is a projection if and only if there is a morphism $p' : n \to m'$ such that the diagram

$$m \xleftarrow{\ p\ } n \xrightarrow{\ p'\ } m'$$

is a product in $\mathbf{T}$. We call such a diagram a *decomposition* of n. A decomposition is *trivial* if and only if m or m' is the terminal object (that is, 0 if $\mathbf{T}$ is a Lawvere theory), otherwise it is non-trivial. An object is *indecomposable* if it does not have a non-trivial decomposition.

Proposition 3.7. 1 is indecomposable in any analytic Lawvere theory.

Proof. It is enough to show that for any symmetric operad $\mathcal{O}$, 1 is indecomposable in $\mathcal{P}_a(\mathcal{O})$. Consider the diagram



We assume that the morphisms $\langle \phi, f, g_i \rangle_i$, $\langle \phi', f', g'_j \rangle_j$ are projections making 1 into a product in $\mathcal{P}_a(\mathcal{O})$. We also have two canonical projections from $m + m'$ to m and m'. The morphism $\langle \bar{\phi}, !, g \rangle$ is the unique morphism into the product making both triangles commute.

From the commutations of the triangles, it easily follows that

$$g_i * \langle g, \ldots, g \rangle = \iota = g'_j * \langle g, \ldots, g \rangle$$

for $i \in (m]$ and $j \in (m']$. This means that

$$g_i = g'_j = g^{-1} \in \mathcal{O}_1$$

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for $i \in (m]$ and $j \in (m']$, and hence

$$\begin{aligned} r &= m \\ r' &= m' \\ f &= 1_m \\ f' &= 1_{m'}. \end{aligned}$$

Moreover,

$$\begin{aligned} s &= 1 \\ ! &= 1_1. \end{aligned}$$

Now, commutativity says that there are $\sigma \in S_m$ and $\sigma' \in S_{m'}$ such that

$$\begin{aligned} i_m \circ \sigma &= \bar{\phi} \circ \phi \\ i_{m'} \circ \sigma' &= \bar{\phi} \circ \phi', \end{aligned}$$

but this is only possible if $m + m' = 1$. □

It follows from the last proposition that the Lawvere theory of Jonsson–Tarski algebras is not analytic.

4. Finitary monads versus operads

We begin by explaining the diagram

$$\begin{array}{ccc} \mathbf{LT} & \xrightarrow{\mathcal{M}_l} & \mathbf{Mnd} \\ {\scriptstyle \mathcal{P}_a = \mathcal{P}_a^{ol}}\uparrow & & \uparrow{\scriptstyle \mathcal{P}_a^m} \\ \mathbf{SOp} & \xrightarrow{\mathcal{M}_o^a} & \mathbf{AnMnd} \\ {\scriptstyle \mathcal{P} = \mathcal{P}_p^o}\uparrow & & \uparrow{\scriptstyle \mathcal{P}_p^m} \\ \mathbf{RiOp} & \xrightarrow[\mathcal{M}_o^p]{} & \mathbf{PolyMnd} \end{array} \tag{2}$$

commuting up to isomorphism, where $\mathcal{P}_a^m$ and $\mathcal{P}_p^m$ are inclusions and $\mathcal{M}_l$ is the equivalence of categories defined in the appendix. The remaining two horizontal functors are also equivalences of categories (Zawadowski 2011), which we recall below.

For a set X, the set X^n is the set of functions $X^{(n]}$. The symmetric group S_n then acts naturally on X^n, on the right, by composition. For a symmetric operad $\mathcal{O}$, the monad $\mathcal{M}_o^a(\mathcal{O})$ on a set X is defined by

$$\mathcal{M}_o^a(\mathcal{O})(X) = \sum_{n \in \omega} X^n \otimes_n \mathcal{O}_n.$$

In $X^n \otimes_n \mathcal{O}_n$, we identify $\langle \vec{x} \circ \sigma, f \rangle$ with $\langle \vec{x}, \sigma \cdot f \rangle$ for $f \in \mathcal{O}_n$, $\vec{x} : (n] \to X$ and $\sigma \in S_n$.

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For a rigid operad $\mathcal{O}$, the monad $\mathcal{M}^p_o(\mathcal{O})$ on a set X is defined by

$$\mathcal{M}^p_o(\mathcal{O})(X) = \sum_{n\in\omega} X^n \times \mathcal{O}_n.$$

For a more detailed description, see, for example, Zawadowski (2011), which also shows the commutation of the lower square in the above diagram; the commutation of the upper square is the content of the following proposition.

Proposition 4.1. The square of categories and functors

$$\begin{array}{ccc} \mathbf{LT} & \xrightarrow{\mathcal{M}_l} & \mathbf{Mnd} \\ {\scriptstyle \mathcal{P}_a}\uparrow & & \uparrow{\scriptstyle \mathcal{P}^m_a} \\ \mathbf{SOp} & \xrightarrow[\mathcal{M}^a_o]{} & \mathbf{AnMnd} \end{array}$$

commutes up to isomorphism.

Proof. Let $\mathcal{O}$ be a symmetric operad. We need to define a natural isomorphism κ such that

$$\kappa^{\mathcal{O}} : \mathcal{M}^a_o(\mathcal{O}) \longrightarrow \mathcal{M}_l\mathcal{P}_a(\mathcal{O})$$

is an isomorphism of monads natural in $\mathcal{O}$. The component of $\kappa^{\mathcal{O}}$ at a set X

$$\kappa^{\mathcal{O}}_X : \sum_{n\in\omega} X^n \otimes \mathcal{O}_n \longrightarrow \int^{n\in\mathbb{F}} X^n \times \mathcal{P}(\mathcal{O})(n,1)$$

is given by

$$[\vec{x}, a] \mapsto [\vec{x}, (1_n, !, a)]$$

where $\vec{x} : (n] \to X$, $a \in \mathcal{O}_n$ and $(1_n, !, a)$ is a span

$$\begin{array}{ccc} & n & \\ {\scriptstyle 1_n}\swarrow & & \searrow{\scriptstyle \langle !, a\rangle} \\ n & & 1 \end{array}$$

The verification that κ defined in this way is indeed a natural isomorphism is left as an exercise. □

4.1. *The functor* $\mathcal{Q}^m_f$: **Mnd** → **AnMnd**

Since the horizontal functors in diagram (2) are equivalences of categories, it follows from Proposition 3.5 that the embedding functor i : **AnMnd** → **Mnd** has a right adjoint

$$\mathcal{Q}^m_f : \mathbf{Mnd} \to \mathbf{AnMnd}$$

that is monadic. In other words, any finitary monad on *Set* is an algebra for a monad on the category of analytic monads. We could define the functor $\mathcal{Q}^m_f$ and the related monad $\bar{\mathbb{V}}$ on **AnMnd** directly, but we shall derive it from a more fundamental situation.

Let $\beta : \mathbb{B} \to \mathbb{F}$ be the inclusion functor. It induces the following diagram of categories and functors, which we describe below,

$$\begin{array}{ccccccccc}
\mathbf{Mnd} & = & \mathbf{mon}(\mathbf{End}) & \xrightarrow{\;\widehat{U}\;} & \mathbf{End} & \xleftarrow{\;i_{\mathbb{F}}\;} & Set^{\mathbb{F}} & & \mathbb{F} \\
\mathcal{P}^m_a \uparrow \downarrow \mathcal{Q}^m_f & & \uparrow \downarrow \mathbf{mon}((-)^a) & & i^a \uparrow \downarrow (-)^a & & Lan_\beta \uparrow \downarrow \beta^* & & \uparrow \beta \\
\mathbf{AnMnd} & = & \mathbf{mon}(\mathbf{An}) & \xrightarrow{\;U\;} & \mathbf{An} & \xleftarrow{\;i_{\mathbb{B}}\;} & Set^{\mathbb{B}} & & \mathbb{B} \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & & & \\
(\bar{\mathbb{V}}, \bar{\eta}, \bar{\mu}) & = & \mathbf{mon}(\mathbb{V}, \eta, \mu) & & (\mathbb{V}, \eta, \mu) & & & &
\end{array} \qquad (3)$$

β^* is the functor of composing with β. It has a left adjoint Lan_β, which is the left Kan extension along β, and for $C \in Set^{\mathbb{B}}$, it is given by the coend formula

$$Lan_\beta(C)(X) = \int^{n \in \mathbb{F}} X^n \times C(n].$$

The equivalences

$$i_{\mathbb{F}} : Set^{\mathbb{F}} \longrightarrow \mathbf{End}$$
$$i_{\mathbb{B}} : Set^{\mathbb{B}} \longrightarrow \mathbf{An}$$

are defined by left Kan extensions, which are given by the formulae

$$i_{\mathbb{F}}(G)(X) = \int^{n \in \mathbb{F}} X^n \times G(n]$$
$$i_{\mathbb{B}}(C)(X) = \sum_{n \in \omega} X^n \otimes_n C(n]$$

where

$$G \in Set^{\mathbb{F}}$$
$$C \in Set^{\mathbb{B}}.$$

The functor $i^a : \mathbf{An} \to \mathbf{End}$ is just an inclusion, and its right adjoint $(-)^a$ is given for $F \in \mathbf{End}$ by the formula

$$F^a(X) = \sum_{n \in \omega} X^n \otimes_n F(n].$$

Note that both $\mathbf{An}$ and $\mathbf{End}$ are strict monoidal categories with tensor given by composition, and i^a is a strict monoidal functor. Thus, its right adjoint $(-)^a$ has a unique lax monoidal structure making the adjunction $i^a \dashv (-)^a$ a monoidal adjunction. This in turn gives us a monoidal monad $(\mathbb{V}, \eta, \mu)$ on $\mathbf{An}$.

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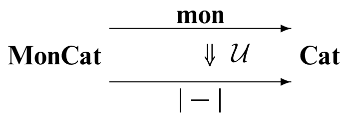
We have a 2-natural transformation $\mathcal{U}$

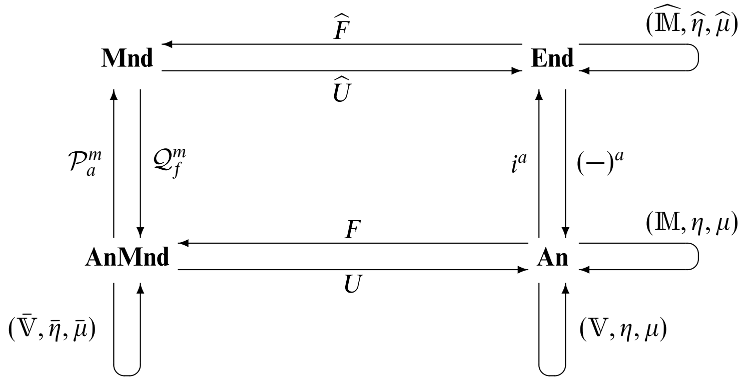


where:

— **MonCat** is the 2-category of monoidal categories, lax monoidal functors and monoidal transformations (Zawadowski 2012);
— **mon** is the 2-functor associating the categories of monoids to monoidal categories;
— $|-|$ is the forgetful functor forgetting the monoidal structure; and
— $\mathcal{U}$ is a 2-natural transformation whose component at a monoidal category M is the forgetful functor $\mathcal{U}_M : \mathbf{mon}(M) \to |M|$ (Sienkiewicz and Zawadowski 2013).

Note that the monoids in **End** and **An** are monads. Applying **mon** to the monoidal adjunction $i^a \dashv (-)^a$, we get an adjunction between categories of monoids, and hence the left most adjunction $\mathcal{Q}^m_f \dashv \mathcal{P}^m_a$. Similarly, applying **mon** to the monoidal monad $\mathbb{V}$, we get a monad on **mon**(**An**), and hence the monad $(\bar{\mathbb{V}}, \bar{\eta}, \bar{\mu})$ on the category of analytic monads. The unnamed arrow in the above diagram is **mon**(i^a).

There are free monads on finitary functors (Barr 1970) and free analytic monads on analytic functors (Zawadowski 2011). Therefore, the functors $\widehat{U}$ and U have left adjoints $\widehat{F}$ and F, respectively. The adjunctions $F \dashv U$ and $\widehat{F} \dashv \widehat{U}$ induce monads $\mathbb{M}$ and $\widehat{\mathbb{M}}$, respectively. $\widehat{\mathbb{M}}$ is the finitary version of what is called 'the monad for all monads' in Barr (1970). Adding this additional data to diagram (3) and simplifying it, we get the diagram



In this diagram, the square of the right adjoints commutes, so the square of the left adjoint commutes too. This shows, in particular, that the free monad on an analytic functor is analytic.

The monad $\bar{\mathbb{V}}$ is a lift of a monad $\mathbb{V}$ to the category of $\mathbb{M}$-algebras **AnMnd** and, by Beck (1969), we get the following theorem.

Theorem 4.2. The monad $\mathbb{M}$ for analytic monads distributes over the monad $\mathbb{V}$ for finitary functors, that is, we have a distributive law

$$\lambda : \mathbb{M}\mathbb{V} \longrightarrow \mathbb{V}\mathbb{M}.$$

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The category of algebras of the composed monad $\mathbb{V}\mathbb{M}$ on **An** is equivalent to the category **Mnd** of all finitary monads on *Set*.

Remark 4.3. We arrived at the above theorem with essentially no calculations. It has obvious positive aspects, but it does not give any idea about what the above distributive law is like. We shall present below some explicit formulae for how to calculate the values of some of the functors mentioned above, and we shall also describe the coherence morphism φ on the monoidal monad $\mathbb{V}$. This coherence morphism generates the distributive law λ, which is an analogue of the distributive law of combing trees (Szawiel and Zawadowski 2010; Baez and Dolan 1998).

We will begin by describing the adjunction $i^a \dashv (-)^a$. We shall drop the inclusion i^a when possible. Let $A \in An$ and $G \in \mathbf{End}$, and X be a set. The analytic functor A is given by its coefficients. Its value at X is

$$A(X) = \sum_{n\in\omega} X^n \otimes_n A_n$$

where A_n is a (left) S_n-set for $n \in \omega$. The value of G^a at X is

$$G^a(X) = \sum_{n\in\omega} X^n \otimes_n G(n].$$

Thus

$$\begin{aligned}\mathbb{V}(A)(X) &= A^a(X)\\ &= \sum_{n,m\in\omega} X^n \otimes_n (n]^m \otimes_m A_m.\end{aligned}$$

The unit of the adjunction $i^a \dashv (-)^a$ at X,

$$(\eta_A)_X : A(X) \longrightarrow A^a(X),$$

is given by

$$[\vec{x}, a] \mapsto [\vec{x}, 1_n, a]$$

where $\vec{x} : (n] \to X$ and $a \in A_n$.

The counit of the adjunction at X,

$$(\varepsilon_G)_X : \sum_{n\in\omega} X^n \otimes_n G(n] \longrightarrow G(X),$$

is given by

$$[\vec{x}, t] \mapsto G(\vec{x})(t)$$

where $\vec{x} : (n] \to X$ and $t \in G(n]$.

The multiplication in the monad $\mathbb{V}$,

$$(\mu_A)_X : \sum_{n,m,k\in\omega} X^n \otimes_n (n]^m \otimes_m (m]^k \otimes_k A_k \longrightarrow \sum_{n,k\in\omega} X^n \otimes_n (n]^k \otimes_k A_k,$$

is given by composition

$$[\vec{x}, g, f, a] \mapsto [\vec{x}, g \circ f, a]$$

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where

$$\vec{x} : (n] \to X$$
$$g : (m] \to (n]$$
$$f : (k] \to (m]$$

and

$$a \in A_k.$$

This concludes the definition of the monad $\mathbb{V}$.

We shall now describe the monoidal structure on $\mathbb{V}$. If B is another analytic functor, the nth coefficient of the composition $A \circ B$ is given by

$$(A \circ B)_n = \sum_{m,n_1,\ldots,n_m \in \omega, \sum_{i=1}^{m} n_i = n} (S_n \times B_{n_1} \times \ldots \times B_{n_m} \times A_m)_{/\sim_n}$$

where the equivalence relation $\sim_n$ is such that for $\sigma \in S_n$, $\sigma_i \in S_{n_i}$, $\tau \in S_m$, $b_i \in B_i$, for $i \in (m]$ and $a \in A_m$ we have

$$\langle \sigma, \sigma_1 \cdot b_1, \ldots, \sigma_m \cdot b_m, \tau \cdot a \rangle \sim_n \langle \sigma \circ (\langle \sigma_1, \ldots, \sigma_m \rangle \star \tau), b_{\tau(1)}, \ldots, b_{\tau(m)}, a \rangle$$

where $\star$ is the composition in the operad of symmetries $\mathcal{S}ym$.

The nth coefficient of $\mathbb{V}(A) \circ \mathbb{V}(A)$ is

$$\sum_{m,m',n_i,k_i \in \omega, \sum_{i=1}^{m} n_i = n} (S_n \times (n_1]^{k_1} \otimes_{k_1} A_{k_1} \times \ldots \times (n_m]^{k_m} \otimes_{k_m} A_{k_m} \times (m]^{m'} \otimes_{m'} A_{m'})_{/\sim_n}$$

and the nth coefficient of $\mathbb{V}(A \circ A)$ is given by

$$(\mathbb{V}(A \circ A))_n = \sum_{m,k,k_i \in \omega, \sum_{i=1}^{m} k_i = k} ((n]^k \times A_{k_1} \times \ldots \times A_{k_m} \times A_m)_{/\approx_k}$$

where the equivalence relation $\approx_k$ is defined in a similar way to $\sim_n$, except that $(n]^k$ replaces S_n. The coherence morphism φ for $\mathbb{V}$ at the nth coefficient of the functor A is

$$\varphi_n : (\mathbb{V}(A) \circ \mathbb{V}(A))_n \longrightarrow (\mathbb{V}(A \circ A))_n$$

given by

$$\langle \sigma, [\sigma_1, a_1], \ldots, [\sigma_m, a_m], \tau, a \rangle \mapsto \langle \sigma \circ (\langle \sigma_1, \ldots, \sigma_m \rangle \bar{\star} \tau), a_{\tau(1)} \ldots a_{\tau(m')}, a \rangle.$$

The operation $\bar{\star}$ is defined as multiplication in the operad of symmetries, except that now neither τ nor the σ_is are bijections but rather functions between finite sets. This is, in fact, the composition in what we call full operads, which was implicitly introduced in Johnstone and Wraith (1978), and explicitly discussed in Szawiel and Zawadowski (2013a) – see also Tronin (2002). Note that φ_n is well defined at the level of equivalence classes.

As the functor $(-)^a$: **End** $\to$ **An** is monadic, every finitary functor is a $\mathbb{V}$-algebra on an analytic functor. For G in **End**, the corresponding algebra map α_G at X,

$$\alpha_G(X) : \mathbb{V}((G)^a)(X) = \sum_{n,m \in \omega} X^n \otimes_n (n]^m \otimes_m G(m) \longrightarrow \sum_{n \in \omega} X^n \otimes_n G(n) = (G)^a(X),$$

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is given by

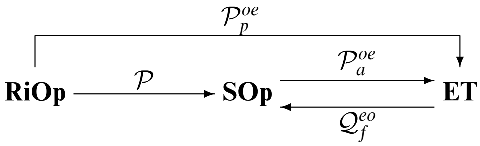
$$(\vec{x}, f, t) \mapsto (\vec{x}, G(f)(t))$$

where

$$\begin{aligned} \vec{x} &: (n] \to X \\ f &: (m] \to (n] \\ t &\in G(m). \end{aligned}$$

5. Equational theories versus operads

In this section we study the relations between equational theories and operads, both symmetric and rigid. We shall describe the diagram



5.1. *The functor* $\mathcal{P}_a^{oe} : \mathbf{SOp} \to \mathbf{ET}$

We begin by defining the functor $\mathcal{P}_a^{oe}$. Let $\mathcal{O}$ be a symmetric operad. We define an equational theory $\mathcal{P}_a^{oe}(\mathcal{O}) = (L, A)$. As the set of n-ary function symbols, we put $L_n = \mathcal{O}_n$ for $n \in \omega$. The set of axioms A contains the following equations in context:

(1) $\iota(x_1) = x_1 : \vec{x}^1$
where $\iota \in \mathcal{O}_1$ is the unit of the operad $\mathcal{O}$;

(2) $f(f_1(x_1, \ldots, x_{k_1}), \ldots, f_m(x_{k_{m-1}+1}, \ldots, x_{k_m})) = (\langle f_1, \ldots, f_m \rangle * f)(x_1, \ldots x_k) : \vec{x}^k$
where $f \in \mathcal{O}_m$, $f_i \in \mathcal{O}_{k_i}$ for $i \in 1, \ldots, m$, $k = \sum_{i=1}^m k_i$;

(3) $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (\sigma \cdot f)(x_1, \ldots, x_n) : \vec{x}^n$
for all $f \in \mathcal{O}_n$ and $\sigma \in S_n$.

Clearly, all equations are linear-regular, so the theory $\mathcal{P}_a^{oe}(\mathcal{O})$ is linear-regular.

Suppose $h : \mathcal{O} \to \mathcal{O}'$ is a morphism of symmetric operads. We define the interpretation

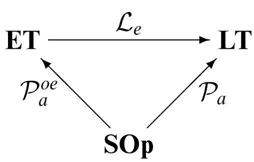
$$\mathcal{P}_a^{oe}(h) : \mathcal{P}_a^{oe}(\mathcal{O}) \longrightarrow \mathcal{P}_a^{oe}(\mathcal{O}').$$

For $f \in \mathcal{O}_n$, we put

$$\mathcal{P}_a^{oe}(h)(f) = (h(f)(x_1, \ldots, x_n) : \vec{x}^n),$$

for $n \in \omega$.

Proposition 5.1. The triangle



commutes up to a natural isomorphism.

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Proof. Let $\mathcal{O}$ be a symmetric operad. We define a natural transformation

$$\psi_{\mathcal{O}} : \mathcal{P}_a(\mathcal{O}) \longrightarrow \mathcal{L}_e \mathcal{P}_a^{oe}(\mathcal{O})$$

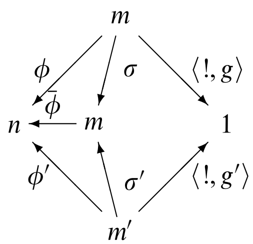
by

$$[\phi, !, f] : n \to 1 \mapsto [f(x_{\phi(1)}, \dots x_{\phi(m)}) : \vec{x}^n]$$

where $\phi : (m] \to (n]$, $f \in \mathcal{O}_m$. The extension of this definition to morphisms with arbitrary codomains is obvious.

$\psi_{\mathcal{O}}$ is clearly bijective on objects. Since every term in $\mathcal{P}_a^{oe}(\mathcal{O})$ is provably equal to a simple term (the $=$ operation applied to variables), $\psi_{\mathcal{O}}$ is full.

We shall show that $\psi_{\mathcal{O}}$ is faithful – this is where combinatorics meets equational logic. Suppose we have two morphisms $\langle \phi, !, g \rangle$ and $\langle \phi', !, g' \rangle$ in $\mathcal{P}_a(\mathcal{O})$



such that

$$\psi_{\mathcal{O}}(\phi, !, g) = \psi_{\mathcal{O}}(\phi', !, g').$$

This means that the theory $\mathcal{P}_a^{oe}(\mathcal{O})$ proves

$$g(x_{\phi(1)}, \dots, x_{\phi(m)}) = g'(x_{\phi'(1)}, \dots, x_{\phi'(m')}) : \vec{x}^n.$$

Since $\mathcal{P}_a^{oe}(\mathcal{O})$ is a linear-regular theory, every equation is a simple substitution of a linear-regular equation (see Remark 2.1), so $m = m'$ and there are permutations $\sigma, \sigma' \in S_m$ and a function $\bar{\phi} : (m] \to (n]$ such that $\mathcal{P}_a^{oe}(\mathcal{O})$ proves

$$g(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = g'(x_{\sigma'(1)}, \dots, x_{\sigma'(m)}) : \vec{x}^m$$

and

$$\phi = \bar{\phi} \circ \sigma$$
$$\phi' = \bar{\phi} \circ \sigma'.$$

Thus, $\mathcal{P}_a^{oe}(\mathcal{O})$ proves

$$g(x_1, \dots, x_m) = g'(x_{\sigma^{-1}\sigma'(1)}, \dots, x_{\sigma^{-1}\sigma'(m)}) : \vec{x}^m$$

and

$$g(x_1, \dots, x_m) = (\sigma^{-1}\sigma') \cdot g'(x_1, \dots, x_m) : \vec{x}^m.$$

The second equality only holds if

$$g = (\sigma^{-1}\sigma') \cdot g'$$

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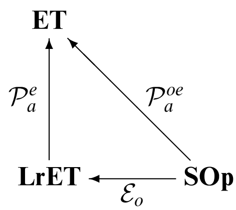
holds in $\mathcal{O}$. But this, together with $\phi' = \phi \circ \sigma^{-1} \circ \sigma'$, means that

$$(\phi, !, g) = (\phi', !, g')$$

in $\mathcal{P}_a(\mathcal{O})$, so $\psi_{\mathcal{O}}$ is faithful too. □

Next we identify the image of the functor $\mathcal{P}_a^{oe}$. The 'object part' of the following theorem was conjectured in Leinster (2004) and proved in Gould (2010).

Theorem 5.2. The functor $\mathcal{P}_a^{oe}$ is faithful, full on isomorphisms, and its essential image is the category of linear-regular theories **LrET**, that is, it factorises as an equivalence of categories $\mathcal{E}_o$ followed by $\mathcal{P}_a^{e}$:



Proof. Since $\mathcal{L}_e$ is an equivalence of categories, the fact that $\mathcal{P}_a^{oe}$ is faithful and full on isomorphisms follows from Proposition 5.1 and the same properties of the functor $\mathcal{P}_a$ stated in Proposition 3.3.

Let

$$I : \mathcal{P}_a^{oe}(\mathcal{O}) \longrightarrow \mathcal{P}_a^{oe}(\mathcal{O}')$$

be a linear-regular interpretation. We shall define $h_I : \mathcal{O} \longrightarrow \mathcal{O}'$ such that $\mathcal{P}_a^{oe}(h_I) = I$. For $f \in \mathcal{O}_n$, we have $I(f) : \vec{x}^n$ is a linear-regular term in $\mathcal{P}_a^{oe}(\mathcal{O}')$. In $\mathcal{P}_a^{oe}(\mathcal{O}')$ every (linear-regular) term is provably equal to a simple (linear-regular) term (just one function symbol), so we can assume that

$$I(f) = \bar{f}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \vec{x}^n$$

already holds, where $\bar{f} \in \mathcal{O}'$. We put

$$h_I(f) = \sigma \cdot \bar{f}.$$

The verification that $\mathcal{P}_a^{oe}(h_I) = I$ is left as an exercise.

Let $T = (L, A)$ be a linear-regular theory. We shall define a symmetric operad $\mathcal{O}$ such that T is isomorphic to $\mathcal{E}_o(\mathcal{O})$. The set of n-ary operations $\mathcal{O}_n$ is the set of linear-regular terms in context $\vec{x}^n$ modulo provable equations from the set of axioms A. The group S_n acts on $\mathcal{O}_n$ by permuting variables

$$\sigma \cdot [t(x_1, \ldots, x_n) : \vec{x}^n] = [t(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \vec{x}^n].$$

The unit in $\mathcal{O}_1$ is the term $[x_1 : \vec{x}^1]$. The composition in $\mathcal{O}$ is defined by 'disjoint substitution', that is, before substituting terms we need to perform α-conversion to make the result of the substitution a linear-regular term. For example, substituting terms in

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contexts

$$[t_1(x_1, x_2) : \vec{x}^2]$$
$$[t_2 : \vec{x}^0]$$
$$[t_3(x_1, x_2, x_3) : \vec{x}^3]$$

into the term

$$[t(x_1, x_2, x_3) : \vec{x}^3]$$

we get

$$[t(t_1(x_1, x_2), t_2, t_3(x_3, x_4, x_5)) : \vec{x}^5].$$

We hope that this explains the composition in $\mathcal{O}$ better than a formal definition. It should be clear that $\mathcal{O}$ is a symmetric operad.

There is an interpretation $I : T \to \mathcal{P}_a^{oe}(\mathcal{O})$ sending an operation $f \in L_n$ to the term in context

$$[[f(x_1, \ldots, x_n) : \vec{x}^n](\vec{x}^n) : \vec{x}^n].$$

Note that the term in context $[f(x_1, \ldots, x_n) : \vec{x}^n]$ is just a symbol of the theory $\mathcal{P}_a^{oe}(\mathcal{O})$. There is also an interpretation $I' : \mathcal{P}_a^{oe}(\mathcal{O}) \to T$ sending an operation $[f(x_1, \ldots, x_n) : \vec{x}^n] \in \mathcal{O}_n$ to the same 'thing', but considered this time as a term in context $[f(x_1, \ldots, x_n) : \vec{x}^n]$ of the theory T. These two interpretations are mutually inverse, so T is isomorphic to $\mathcal{P}_a^{oe}(\mathcal{O})(= \mathcal{E}_o(\mathcal{O}))$ in **ET**, as required. □

5.2. *The functor* $\mathcal{P}_p^{oe}$: **RiOp** → **ET**

The functor $\mathcal{P}_p^{oe}$ is defined as the composition of the functors $\mathcal{P}_a^{oe} \circ \mathcal{P}$.

Theorem 5.3. The functor $\mathcal{P}_p^{oe}$: **RiOp** → **ET** is faithful and full on isomorphisms, and its essential image is the category of rigid theories **RiET**.

Proof. Since $\mathcal{L}_e$: **ET** → **LT** is an equivalence of categories and $\mathcal{P}$: **RiOp** → **SOp** is full and faithful, the fact that $\mathcal{P}_p^{oe}$ is faithful and full on analytic morphisms (and hence also on isomorphisms) follows from Proposition 5.1 and the same properties of the functor $\mathcal{P}_p^{ol}$: **RiOp** → **LT** stated in Theorem 3.6.

It remains to show that for any symmetric operad $\mathcal{O}$, the equational theory $\mathcal{P}_p^{oe}(\mathcal{O})$ is rigid if and only if $\mathcal{O}$ is rigid.

Every linear-regular term $t(x_1, \ldots, x_n) : \vec{x}^n$ is provably equal in $\mathcal{P}_p^{oe}(\mathcal{O})$ to a simple term, that is, there is an operation $f \in \mathcal{O}_n$ such that

$$\mathcal{P}_p^{oe}(\mathcal{O}) \vdash f(x_1, \ldots, x_n) = t(x_1, \ldots, x_n) : \vec{x}^n.$$

Thus, $\mathcal{P}_p^{oe}(\mathcal{O})$ is a rigid theory if and only if for every n, we have $f \in \mathcal{O}_n$, and $\sigma \in S_n$

$$\text{if} \quad \mathcal{P}_p^{oe}(\mathcal{O}) \vdash f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \vec{x}^n, \quad \text{then} \quad \sigma = id. \tag{4}$$

The operad $\mathcal{O}$ is rigid if and only if for every n, we have $f \in \mathcal{O}_n$, and $\sigma \in S_n$

$$\text{if} \quad f = \sigma \cdot f, \quad \text{then} \quad \sigma = id. \tag{5}$$

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Now, if (5) holds, (4) is an axiom of $\mathcal{P}_p^{oe}(\mathcal{O})$. On the other hand, if (5) does not hold, the equality in (4) does not hold in the free model of $\mathcal{P}_p^{oe}(\mathcal{O})$ on n generators. Thus, $\mathcal{P}_p^{oe}(\mathcal{O})$ is rigid if and only if $\mathcal{O}$ is, as required. □

The following Corollary corrects a statement in Carboni and Johnstone (1995) and Carboni and Johnstone (2004) concerning the characterisation of equational theories corresponding to polynomial monads.

Corollary 5.4. The equivalence of categories

$$\mathbf{ET} \xrightarrow{\mathcal{M}_l \circ \mathcal{L}_e} \mathbf{Mnd}$$

restricts to the equivalence between the category of rigid equational theories and the category of finitary polynomial monads on Set

$$\mathbf{RiET} \xrightarrow{\mathcal{M}_e} \mathbf{PolyMnd}.$$

6. Examples and comments

In this section we provide some examples and make some comments:

(1) The equations expressing commutation of two operations are linear-regular, so all operations in a theory T commute if and only if they do in its analytic part T^a. However, this need not simplify the problem as the analytic part of an equational theory (or its monad on Set) is usually much bigger than the original equational theory. For example, if $\mathbf{T}$ is a finitary monad on Set, then the value of its analytic part on a one element set is the coproduct of the symmetrised free $\mathbf{T}$ algebras on finitely many generators:

$$\begin{aligned} \mathbf{T}^a(1) &= \sum_{n\in\omega} 1^n \otimes_n T(n) \\ &= \sum_{n\in\omega} T(n)_{/S_n}. \end{aligned}$$

Thus, it is not so surprising that theories arising in this way might only be of interest in special circumstances, and preferably when the theory we start with is 'very small'.

(2) The categories **SOp** and **LT** are complete and cocomplete. Since it is a left adjoint, the functor $\mathcal{P}_a : \mathbf{SOp} \to \mathbf{LT}$ preserves all colimits and also preserves all connected limits. However, it does not preserve the terminal object. The terminal object is the value of $\mathcal{Q}_f : \mathbf{LT} \to \mathbf{SOp}$ on the terminal Lawvere theory. We describe this below.

(3) Recall that $\mathbb{1}$ denotes the terminal equational theory. It has one constant, say e, and can be axiomatised by a single axiom: $x_1 = e : \vec{x}^1$. As a Lawvere theory, it is the category that has exactly one morphism between any two objects. Since $\mathcal{Q}_f^e : \mathbf{ET} \to \mathbf{LrET}$ is a right adjoint, $\mathcal{Q}_f^e(\mathbb{1})$, the linear-regular part $\mathbb{1}$ is the terminal linear-regular theory. The fact that it is the theory of commutative monoids is best seen at the level of Lawvere theories. Both theories, that is, $\mathcal{Q}_f^e(\mathbb{1})$ and the theory of commutative monoids, are linear-regular and, for any n, have exactly one analytic

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morphism

$$a : n \to 1.$$

In the case of the theory of monoids, it is given by

$$x_1, \ldots, x_n \mapsto x_1 \cdot \ldots \cdot x_n.$$

(4) The terminal Lawvere theory $\mathbb{1}$ has a proper subtheory in which $0 \ncong 1$. As an equational theory, it has no function symbols, and can be axiomatised by a single axiom:

$$x_1 = x_2 : \vec{x}^2.$$

The analytic part of this theory is the theory of commutative semigroups.

(5) The embedding of the strongly regular theories into all equational theories has a right adjoint $\mathcal{Q}$. The value of $\mathcal{Q}$ on the terminal equational theory $\mathbb{1}$ is the terminal strongly regular theory, that is, the theory of monoids. So the theory of monoids is rigid too. It is easy to show that any analytic morphism

$$a : n \to 1$$

in the Lawvere theory for monoids, $\mathbf{T}_{mon}$ is of the form

$$x_1, \ldots, x_n \mapsto x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)},$$

that is, it is a multiplication of all variables in the order given by some $\sigma \in S_n$. Thus, the symmetric operad $\mathbf{T}^s_{mon}$ (see the proof of Proposition 3.4 for the notation $(-)^s$) is the operad of symmetries, $\mathcal{S}ym$, and hence the theory of monoids $\mathbf{T}_{mon}$ is $\mathcal{L}_o(\mathcal{S}ym)$.

(6) The theory of monoids with anti-involution is the theory of monoids with an additional unary operation s and the additional two (linear-regular but not strongly regular) axioms

$$m(s(x_1), s(x_2)) = s(m(x_2, x_1))$$
$$s(s(x_1)) = x_1.$$

Thus, the Lawvere theory for monoids with anti-involution $\mathbf{T}_{mai}$ is analytic. Any analytic morphism

$$a : n \to 1$$

in $\mathbf{T}_{mai}$ is of the form

$$x_1, \ldots, x_n \mapsto s^{\varepsilon_1}(x_{\sigma(1)}) \cdot \ldots \cdot s^{\varepsilon_n}(x_{\sigma(n)})$$

where $\sigma \in S_n$ and $\varepsilon_i \in \{0, 1\}$, for $i = 1, \ldots, n$, and

$$s^0(x) = x$$
$$s^1(x) = s(x).$$

The actions of symmetric groups on such operations are free. This theory is rigid even though it is not strongly regular (Carboni and Johnstone 2004).

(7) The theory of sup-semilattices has two operations $\vee$ and $\perp$ of arity 2 and 0, respectively, and equations

$$\begin{aligned} x_1 \vee (x_2 \vee x_3) &= (x_1 \vee x_2) \vee x_3 \\ x_1 \vee e &= x_1 \\ &= e \vee x_1 \\ x_1 \vee x_1 &= x_1 \\ x_1 \vee x_2 &= x_2 \vee x_1. \end{aligned}$$

This theory is regular but not linear. The theory of groups is neither regular nor linear.

7. Appendix

In this appendix we recall the functors that exhibit equivalences of the categories **ET**, **LT** and **Mnd** defined in Section 2:

$$\mathbf{ET} \xrightarrow{\mathcal{L}_e} \mathbf{LT} \xrightarrow{\mathcal{M}_l} \mathbf{Mnd}.$$

7.1. *The functor* $\mathcal{L}_e = \mathcal{L} : \mathbf{ET} \longrightarrow \mathbf{LT}$

Let $T = (L, A)$ be an equational theory. A morphism $n \to m$ in $\mathcal{L}(T)$ is an m-tuple

$$\langle [t_1 : \vec{x}^n], \dots, [t_m : \vec{x}^n] \rangle : n \to m$$

where $[t_i : \vec{x}^n]$ is an equivalence class of terms in context $\vec{x}^n$ modulo provable equivalence from axioms in A. The identity on n is

$$\langle [x_1 : \vec{x}^n], \dots, [x_n : \vec{x}^n] \rangle : n \to n.$$

The composition is given by simultaneous substitution as follows:

$$n \xrightarrow{\langle [t_i : \vec{x}^n] \rangle_{i \in (m]}} m \xrightarrow{\langle [s_j : \vec{x}^m] \rangle_{j \in (k]}} k$$
$$n \xrightarrow{\langle [s_j(\langle x_i \backslash t_i \rangle_{i \in (m]}) : \vec{x}^m] \rangle_{j \in (k]}} k$$

The ith projection is

$$\pi_i^n = \langle [x_i : \vec{x}^n] \rangle.$$

Let $I : T \to T'$ be an interpretation. The functor $\mathcal{L}(I)$ is defined on a morphism

$$\langle [t_1 : \vec{x}^n], \dots, [t_m : \vec{x}^n] \rangle : n \to m$$

in $\mathcal{L}(T)$ by

$$\langle [\bar{I}(t_1) : \vec{x}^n], \dots, [\bar{I}(t_m) : \vec{x}^n] \rangle : n \longrightarrow m.$$

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7.2. *The functor* $\mathcal{M}_l = \mathcal{M} : \mathbf{LT} \longrightarrow \mathbf{Mnd}$

For a Lawvere theory $\mathbf{T}$, we define the monad $\mathcal{M}(\mathbf{T})$ using coends. We put

$$\mathcal{M}(\mathbf{T})(X) = \int^{n\in\mathbb{F}} X^n \times \mathbf{T}(n,1)$$

for $X \in Set$. The unit of $\mathcal{M}(\mathbf{T})$

$$\eta_X^{\mathbf{T}} : X \to \mathcal{M}(\mathbf{T})(X)$$

sends $x \in X$ to the class of the element $\langle id_1, \bar{x}\rangle$ where id_1 is the identity on 1 in $\mathbf{T}$ and $\bar{x} : (1] \to X$ is the function picking x, that is, $\bar{x}(1) = x$. The iterated functor $\mathcal{M}^2(\mathbf{T})$ is given, for X in Set, by

$$\mathcal{M}^2(\mathbf{T})(X) = \int^{m,n_1,\ldots,n_m\in\mathbb{F}} X^n \times \mathbf{T}(n_1,1) \times \ldots \times \mathbf{T}(n_m,1) \times \mathbf{T}(m,1)$$

where

$$n = \sum_{i=1}^{m} n_i.$$

The multiplication of the monad $\mathcal{M}(\mathbf{T})$

$$\mu_X^{\mathbf{T}} : \mathcal{M}^2(\mathbf{T})(X) \longrightarrow \mathcal{M}(\mathbf{T})(X)$$

is defined on components

$$X^n \times \mathbf{T}(n_1,1) \times \ldots \mathbf{T}(n_m,1) \times \mathbf{T}(m,1) \longrightarrow X^n \times \mathbf{T}(n,1)$$

by composition, that is, for

$$f : m \to 1, f_1 : n_1 \to 1, \ldots, f_m : n_m \to 1$$

in $\mathbf{T}$ and $\vec{x} : (n] \to X$, we have

$$\mu_X^{\mathbf{T}}(\vec{x}, f_1, \ldots, f_m, f) = \langle \vec{x}, f \circ (f_1 \times \ldots \times f_m)\rangle$$

where again

$$n = \sum_{i=1}^{m} n_i.$$

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