Intimate Connections: Symmetries and Conservation Laws in Quantum versus Classical Mechanics

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In this article, I use a number of remarks made by Eugene Wigner to defend the claim that the nature of the connection between symmetries and conservation laws is different in quantum and in classical mechanics. In particular, I provide a list of three differences that obtain between the Hilbert space formulation of quantum mechanics and the Lagrangian formulation of classical mechanics. I also show that these differences are due to the fact that conservation laws are not the only consequence that symmetries have in quantum mechanics and to the fact that, in classical mechanics, the connection between symmetries and conservation laws does not always obtain.

1. Introduction. Throughout his career, Eugene Wigner made a number of remarks suggesting that the role of symmetries and conservation laws is different in quantum and in classical mechanics. Although Wigner made the same point on several occasions, he never went on to elaborate on what these differences were and why they mattered. In this article, I take a close look at Wigner's remarks and try to elucidate their meaning. As I will argue, the basic claim behind Wigner's comments is that the *nature of the connection* between symmetries and conservation laws is different in quantum and in classical mechanics.

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Although I take Wigner's comments as my starting point, the article is not merely of historical interest. My purpose in elucidating Wigner's remarks, indeed, is to learn about the manner in which symmetries and conservation laws relate to each other. My goal in this article, in other words, is not only to determine what Wigner said but also to determine whether what he said is true and why.

My main conclusions will be as follows. First, I will argue that the nature of the connection between symmetries and conservation laws is different in quantum and in classical mechanics. I will provide, in particular, a list of three specific differences that obtain for the Hilbert space formulation of quantum mechanics and for the Lagrangian formulation of classical mechanics. Second, I provide an account of the reasons why these differences obtain. These will be related to the following two facts. In quantum mechanics, on the one hand, symmetries have consequences different from the presence of a conservation law. In classical mechanics, on the other hand, there are restrictions that limit the circumstances under which the connection between symmetries and conservation laws obtains.

The article is organized as follows. I start by introducing Wigner's comments about the different role that symmetries and conservation laws play in quantum and in classical mechanics in section 2. I single out, in particular, three differences whose importance was repeatedly emphasized by Wigner. After that, I introduce the elements of quantum and classical mechanics necessary for making sense of Wigner's remarks. Section 3 thus accounts for the manner in which symmetries and conservation laws relate to each other in quantum mechanics, while section 4 examines the case of classical mechanics. In section 5, I return to Wigner's comments, and I use them to shed light on the nature of the connection between symmetries and conservation laws. Finally, in section 6, I summarize the contents of this article and state its main conclusions.

2. Wigner's Remarks. Over the course of his career, Wigner repeatedly claimed that symmetries and conservation laws play different roles in quantum and in classical mechanics. These kinds of claims can be found in most of Wigner's writings on symmetry, including his essays in the collection *Symmetries and Reflections* and his talk and multiple interventions in a conference on "Symmetries in Physics" held at the Universitat Autònoma de Barcelona in 1983 (Wigner 1967; García Doncel et al. 1983). These materials were published over the course of more than 30 years, and, taken together, they virtually exhaust Wigner's writings on the subject. I have examined these works and identified a number of basic claims that, with some variations, are present in all of them.

The basic structure of Wigner's comments is as follows. Wigner often started by mentioning some of the similarities between the role that symmetry principles play in quantum and in classical mechanics. Thus, he sometimes said that invariance principles function as "touchstones" that the laws of both theories have to satisfy (Wigner 1964a, 998). On some other occasions, he pointed out that both in quantum and in classical mechanics, the conservation laws for energy, linear and angular momentum, and the motion of the center of mass of a system can be derived from invariance under translations, rotations, and Lorentz boosts (Wigner 1949, 523; 1964a, 998; 1964b, 959).

After having established that there are similarities between the two cases, Wigner always went on to make the point that the status of symmetry principles remains different in quantum and in classical mechanics. There are, in particular, three differences that Wigner kept mentioning in his writings on symmetry. I list them in order here, so as to facilitate my discussion of Wigner's remarks later in the article:

1. *Intimate connection*: Wigner often claimed that the connection between symmetries and conservation laws is "more intimate" (Wigner 1949, 523) or "most evident" (Wigner 1964b, 959–60) in quantum mechanics, the situation being "much more complex in classical theory" (Garcia Doncel et al. 1983, 163).
2. *Consequences of symmetry*: Wigner sometimes said that invariance principles permit "further-reaching conclusions" in quantum than in classical mechanics (Wigner 1964a, 998).
3. *Discrete symmetries*: some other times, Wigner mentioned the fact that discrete symmetries, which have "very little role in classical theory," play a much more significant role in quantum mechanics (Wigner 1949, 524).

Wigner did not always articulate all of these points in detail or mention all three differences in every single one of his writings. Still, the sequence above constitutes a pattern that can be discerned in most of his philosophical writings on symmetry. Although he repeated the same basic claims several times over the course of his career, he never went on to elaborate on the meaning behind these remarks. He never specified, for instance, how is it that the relation between symmetries and conservation laws is "more intimate" in quantum than in classical mechanics or what the "further-reaching conclusions" that symmetries have in this theory might be. My purpose in this article consists of elucidating Wigner's remarks by clarifying how exactly differences 1–3 are to be understood and by accounting for the precise

sense in which they are correct. In order to do this, I now turn to the relevant elements of quantum and classical mechanics.

3. Symmetries and Conservation Laws in Quantum Mechanics

3.1. Wigner's Definition of Symmetry. In this section, I account for the manner in which symmetries and conservation laws relate to each other in quantum mechanics. In order to do this, I will follow Wigner's own writings on the subject. My discussion will be based, in particular, on four journal articles that Wigner published in 1927, on an additional paper that he published the year after that, and on his 1931 classic *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Wigner 1927a, 1927b, 1927c, 1927d, 1928, 1959).

Wigner's first step in his work of the late 1920s and early 1930s was to provide a notion of symmetry that was appropriate for the context of quantum mechanics. Wigner drew from the familiar conception of symmetries as transformations of the dynamical variables of a theory that leave the explicit form of its equations of motion unchanged. This requirement, as is well known, also acts as a necessary and sufficient condition for a group of transformations to map the set of solutions of the equations of motion of a theory onto itself. Owing to this, the symmetries of the equations of motion of a theory are sometimes known as "dynamical" symmetries (Brown and Holland 2004).

In order to make use of the notion of a dynamical symmetry in his work on quantum mechanics, Wigner had to refine this concept in the following ways. First, he had to specify the symmetries of what equation he was going to study. Wigner invariably turned to the time-independent Schrödinger equation

$$\hat{H}\psi_E = E\psi_E, \tag{1}$$

which tells us that the eigenfunctions ψ_E of the Hamiltonian operator $\hat{H}$ are energy eigenfunctions.

Second, Wigner had to clarify what was the space that symmetry transformations were going to act on, in the case of quantum mechanics. Wigner's writings typically start with him considering transformations R of the position coordinates on which the wave function depends. In quantum mechanics, however, it is much more natural to think in terms of operators acting on the wave function itself. In order to be able to formulate his definition of symmetry in these terms, Wigner showed that for any transformation R^{-1} that acts on the arguments of the wave function we can always define an operator $\hat{P}_R$,

$$\hat{P}_R \psi(\vec{r}, t) = \psi(R^{-1}\vec{r}, t), \tag{2}$$

that acts on the wave function ψ.

We thus have that in the context of quantum mechanics, dynamical symmetries can be understood as operators that act on the Hilbert space of wave functions and map the set of solutions of the time-independent Schrödinger equation onto itself. If the notion of a dynamical symmetry is understood in this manner, it is easy to show that the necessary and sufficient condition for an operator $\hat{P}_R$ to implement a symmetry transformation is given by the condition

$$\left[\hat{H}, \hat{P}_R\right] = 0. \tag{3}$$

In quantum mechanics, that is to say, an operator $\hat{P}_R$ implements a symmetry transformation if and only if it commutes with the Hamiltonian.

3.2. Conservation Laws. Equipped with this definition of symmetry, Wigner could go on to explore the consequences that symmetries have in quantum mechanics. To see that conservation laws are among these consequences we need to introduce the time-dependent Schrödinger equation,

$$ih\frac{\partial}{\partial t}\psi(\vec{r}, t) = \hat{H}\psi(\vec{r}, t), \tag{4}$$

which describes the time evolution of an arbitrary wave function $\psi(\vec{r}, t)$.

Second, and more important, it follows from (4) that symmetries and conservation laws are related to each other in quantum mechanics. If a linear operator $\hat{P}_R$ satisfies the commutation condition (3), its eigenstates $\psi_{R_1}, ..., \psi_{R_n}$, with eigenvalues $a_{R_1}, ..., a_{R_n}$, will be stationary states. The following set of conservation laws will then obtain:

$$\frac{d}{dt}\left|c_{R_i}(t)\right|^2 = 0, \text{ for } i = 1, ..., n, \tag{5}$$

for the coefficients c_{R_i} that result from expressing an arbitrary wave function ψ with respect to the basis provided by the eigenstates of $\hat{P}_R$ as

$$\psi = \sum_{i=0}^{n} c_{R_i}(t)\psi_{R_i}. \tag{6}$$

3.3. Further Consequences of Symmetry. The above shows that there is a relation between symmetries and conservation laws in quantum mechanics. As it turns out, however, there is a second kind of consequence that

symmetries have in this theory, which obtains whenever (3) is satisfied and involves the transformation properties of the energy eigenfunctions.

To see how the presence of a symmetry affects the manner in which the energy eigenstates transform, assume that a group G of operators $\hat{P}_R$ satisfies (3). In this case, as we have seen, the operators of the group will implement dynamical symmetry transformations that map energy eigenstates onto energy eigenstates. For a nondegenerate eigenvalue E with eigenstate ψ_E and an operator $\hat{P}_R \in G$, therefore, we have that

$$\hat{P}_R \psi_E = \alpha_E \psi_E, \tag{7}$$

where α_E is a complex number that will in general be different for different energy eigenvalues.

The above is no longer true if the eigenvalue E is degenerate among l different eigenfunctions $\{\psi_{E_1}, ..., \psi_{E_l}\}$. In this case, however, the linearity of the Schrödinger equation gives us the following result. Since the time-independent Schrödinger equation is linear, the eigenfunctions ψ_{E_i} form a vector space. It must therefore be the case that

$$\hat{P}_R \psi_{E_i} = \sum_{j=1}^{l} D_R(G)_{ij} \psi_{E_j}, \tag{8}$$

where the coefficients $D_R(G)_{ij}$ will depend on the properties of the specific operator $\hat{P}_R$ under consideration and on the properties of the symmetry group G more generally. As Wigner shows, indeed, the coefficients $D_R(G)_{ij}$ form a representation of G.

We thus have that, whenever (3) obtains, the energy eigenfunctions transform as representations of the relevant symmetry group. This shows that, as I advanced before, conservation laws are not the only kind of consequence that symmetries have in quantum mechanics: there is a second kind of consequence that concerns the transformation properties of the energy eigenfunctions. This second kind of consequence, furthermore, was systematically deployed by Wigner and used for various practical purposes. In his work of the late 1920s and early 1930s, for example, Wigner used result (8) to substitute explicit calculations by qualitative considerations, thereby sidestepping computational difficulties in atomic physics. This fact will turn out to be important, as we will see, in interpreting Wigner's comments in section 5.

4. Symmetries and Conservation Laws in Classical Mechanics

4.1. Noether's Theorems. After having described the manner in which symmetries and conservation laws relate to each other in quantum mechanics, we can now turn to the case of classical mechanics. My discussion will

focus on the Lagrangian formalism, and it will follow the available litera-ture on Noether's theorems.[1] I will follow, in particular, Katherine Brading's and Harvey Brown's derivation of both of Noether's theorems and Harvey Brown's and Peter Holland's work on the first theorem (Brading and Brown 2003; Brown and Holland 2004). My treatment of the second theorem, addi-tionally, is based on Katherine Brading's work on this subject (Brading 2002, 2005).

Although Noether's theorems provide us with a very general account of the consequences that symmetries have in classical mechanics, seeing how this is the case is not immediately obvious. Noether's primary goal in prov-ing her two theorems, in fact, was to settle a dispute about the status of the conservation of energy in general relativity (Noether 1918; Brading 2005).

To see how Noether's work relates to the topic of symmetry, consider the manner in which her two theorems are proved. Noether took the action functional of a classical field theory

$$S[\psi_i, \partial_\mu \psi_i, x^\mu] = \int_R \mathcal{L}(\psi_i, \partial_\mu \psi_i, x^\mu)\, dx^4, \tag{9}$$

with Lagrangian density $\mathcal{L}$, and she required that its numerical value re-mains invariant under joint transformations of the x^μ and the ψ_i:

$$x^\mu \rightarrow x^\mu + \delta x^\mu$$
$$\psi_i \rightarrow \psi_i + \delta \psi_i, \tag{10}$$

where the $\delta \psi_i$ and the δx^μ can be parametrized in terms of r continuous pa-rameters ω^k as

$$\delta x^\mu = \eta_k^\mu \Delta \omega^k$$
$$\delta_0 \psi_i = \xi_{ik} \Delta \omega^k. \tag{11}$$

Note that the quantities $\Delta \omega^k$ above represent infinitesimal increments of the group parameters and that δ_0 is the so-called Lie drag, which accounts for the part of the variation in the ψ_i that is not due to the variation of the inde-pendent variables x^μ.

Noether first showed that the numerical value of the action remains in-variant under transformations of the form of (10), if and only if the follow-ing r identities are satisfied, one for each parameter ω^k in (11):

$$E^i \delta_0 \psi_i = j_k^\mu, \text{ for } k = 1, ..., r, \tag{12}$$

1. For an overview of the manner in which symmetries and conservation laws relate to each other in the Hamiltonian formalism, which includes a comparison with the La-grangian formalism, the reader is refereed to Butterfield (2006).

where the E^i above are the so-called Euler expressions, which vanish whenever the equations of motion of their associated fields are satisfied.[2] The quantities j_k^μ are known as "Noether currents," and they are given by the expression

$$j_k^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \xi_{ik} - L\eta_k^\mu.$$ (13)

Noether then derived two further sets of identities from (12), which correspond to her two theorems. The case of a global symmetry group with constant parameters ω^k gives us the first theorem, which consists of the following r identities:

$$E^i \xi_{ik} = \partial_\mu j_k^\mu, \text{ for } k = 1, ..., r.$$ (14)

For the case of a local symmetry in which the ω^k depend on the x^μ one obtains the second theorem, which consists of another set of r identities, and a further result that is sometimes known as the "boundary theorem" (Brading 2005). Although these two results are important in their own right, I focus on the first theorem in what follows, as it will turn out to be the most important for our purposes.

The relevance of the first theorem for the topic of symmetry derives from the fact that the requirement of numerical invariance of the action functions as a sufficient condition for a group of transformations to leave the equations of motion of the relevant theory invariant. A group of transformations that leaves the numerical value of the action functional of a theory invariant, that is to say, is a symmetry group of the theory in question, in the sense of a dynamical symmetry that was introduced before. Expression (14), therefore, provides us with an account of the consequences that symmetries have in classical mechanics. If a symmetry group leaves the numerical value of the action invariant, that is to say, then the identities in (14) obtain.

4.2. Conservation Laws.   To see how the above leads to the presence of a conservation law, assume that the Euler-Lagrange equations for all the fields involved in a symmetry transformation are satisfied. In this case, the left-hand side of (14) vanishes and the first theorem gives us r continuity equations for the Noether currents:

2. As it turns out, the requirement that the action remains numerically invariant under transformations of the form of (10) can be relaxed to require that the action remains invariant up to the integral of a divergence term. This was shown by Bessel-Hagen for the first time (Bessel-Hagen 1921; Brading and Brown 2003). I disregard this nuance in what follows, however, as it is not strictly relevant for my purposes.

$$\partial_\mu j_k^\mu = 0, \text{ for } k = 1, ..., r. \tag{15}$$

The case of a local symmetry introduces a few nuances, but one obtains a set of continuity equations of the form of (15) also in this case (Brading 2002).

Although the continuity equations in (15) are sometimes referred to as "local conservation laws," I reserve the term "conservation law" for expressions that explicitly assert the invariance in time of some quantity. In order to obtain these kinds conservation laws, one may simply integrate the continuity equations in (15) over a three-dimensional volume V to obtain

$$\int_V \left(\vec{\nabla} \cdot \vec{j_k} + \frac{d}{dt} j_k^0 \right) dx^3 = 0, \text{ for } k = 1, ..., r. \tag{16}$$

The first term in the left-hand side of (16) can be turned into a surface integral by making use of Gauss's theorem. If the $|\vec{j_k}|$ satisfy appropriate boundary conditions, then the surface term can be made arbitrarily small for a sufficiently large volume, and we obtain r conservation laws

$$\frac{dQ_k}{dt} = 0, \text{ for } k = 1, ..., r, \tag{17}$$

for the "Noether charges" given by

$$Q_k = \int_V j_k^0 \, dx^3. \tag{18}$$

4.3. Restrictions. The above shows that, just like in the case of quantum mechanics, there is a relation between symmetries and conservation laws in classical mechanics. This relation, however, is mediated by Noether's theorems, and it will only obtain as long as the two theorems are valid. As it turns out, there are limitations to the domain of applicability of Noether's theorems, and these impose restrictions on the circumstances under which symmetries and conservation laws are related to each other in classical mechanics. The derivation of Noether's theorems that I provided in section 4.1 will prove useful in making sense of these restrictions.

First, keep in mind that Noether's proof was cast in the framework of the Lagrangian formalism. Her starting point, as we saw above, was the action functional of a Lagrangian field theory. Because of this, Noether's theorems apply to Lagrangian theories only. And since not all classical theories can be written in Lagrangian form, Noether's use of the Lagrangian formalism constrains the domain of applicability of the two theorems. As an example, consider the case of theories with velocity-dependent forces, which are in

general not amenable to a Lagrangian treatment. In these kinds of theories, as Wigner (1954) pointed out, one cannot derive a conservation law from a symmetry.

Second, recall that Noether's derivation of the two theorems proceeded by requiring that the action functional remains invariant under infinitesimal transformations of both the dependent and the independent variables of a classical field theory. Noether's focus on infinitesimal transformations is not coincidental, however, as this is necessary for her proof to go through. Her derivation, indeed, proceeded by deploying the calculus of variations, which can only be used for transformations that can be written in infinitesimal form. Since only continuous symmetries can be expressed in this manner, discrete symmetries fall outside of the domain of applicability of Noether's theorems. Both theorems, that is to say, remain silent about the consequences that follow from invariance of the action under discrete symmetries such as reflections.

Finally, note that the relevance of Noether's work for the topic of symmetry derived from the fact that the requirement of invariance of the action functions as a sufficient condition for a group of transformations to count as dynamical symmetries. This condition, however, turns out to be sufficient but not necessary, and we thus have that there are symmetries that leave the equations of motion of a theory invariant but fail to leave the numerical value of its action functional unchanged. The first theorem does not apply to these kinds of symmetries, which have no conservation law attached to them. Brown and Holland (2004) have given examples of these kinds of symmetries, which include "rescaling" transformations that leave the equations of motion unchanged but introduce a multiplicative constant in the action.

The point is sometimes made by appealing to a difference between variational and dynamical symmetries (Olver 1986; Brown and Holland 2004). Variational symmetries are defined as transformations that leave the numerical value of the action invariant, while the definition of a dynamical symmetry remains as in section 3. While all variational symmetries are dynamical symmetries, the converse is not true. Using this terminology, therefore, we may say that Noether's theorems tell us about the consequences of some but not all dynamical symmetries. They only tell us, that is to say, about the consequences of those dynamical symmetries that are variational symmetries too.

We thus see that, although there is a relation between symmetries and conservation laws in classical mechanics, this relation does not obtain for all classical theories and all symmetry transformations. It only obtains, as we have just seen, for Lagrangian theories and for continuous, variational symmetries. This supports my previous claim that, in classical mechanics, a num-

ber of restrictions exist, which limit the circumstances under which symmetries and conservation laws relate to each other.

5. Wigner Explained. In the previous two sections, I described the manner in which symmetries and conservation laws relate to each other in quantum and in classical mechanics. We are now in a position to use these materials to shed light on Wigner's remarks. My goal in this section will consist of showing that Wigner's remarks can be construed as comments about the relation between symmetries and conservation laws and that, when so interpreted, Wigner's claims are correct. This will allow me to achieve my goal of getting clear on the specific ways in which the nature of the connection between symmetries and conservation laws is different in quantum and in classical mechanics and on the reasons why these differences obtain.

Let me start by going back to Wigner's claims 1–3, which I introduced in section 2, and showing the precise sense in which they are correct:

1*. *Intimate connection*: Wigner's first claim was that the connection between symmetries and conservation laws is "more intimate" in quantum mechanics than in the "much more complex case" of classical mechanics. Our discussion of Noether's theorems will help us understand the sense in which this is the case. As we saw in section 4, Noether's theorems introduce a number of restrictions on the kinds of theories and the kinds of transformations for which the connection between symmetries and conservation laws obtains. In particular, we saw that the connection only obtains for Lagrangian theories and for continuous, variational symmetries. What theories can be written in Lagrangian form and what dynamical symmetries are variational symmetries too, however, turn out to be difficult questions to give a general answer to, due to the subtleties behind the Lagrangian formalism. As we saw in section 3, however, none of these restrictions apply in quantum mechanics. In this sense, the connection between symmetries and conservation laws is "more intimate" and "less complex" in quantum than in classical mechanics.

2*. *Consequences of symmetry*: Wigner's second point was that symmetry principles allow for "further-reaching consequences" in quantum than in classical mechanics. Our discussion in section 3 makes clear the sense in which this is true. As we saw there, conservation laws are but one of the consequences that follow from the presence of a symmetry in quantum mechanics. In this theory, symmetries have a further kind of consequence that concerns the transformation properties of the energy eigenfunctions and was used by Wigner for various practical purposes. These are, if my analysis is correct, the

"further-reaching consequences" that he made reference to in his writings. As we saw in section 4, no such consequences obtain in classical mechanics, and it is in this sense that Wigner's second claim is also correct.

3*. *Discrete symmetries*: Finally, Wigner claimed that discrete symmetries play "very little role" in classical mechanics and are much more important in quantum mechanics. This is true in the following sense. We have already seen how, in classical mechanics, discrete symmetries do not have conservation laws attached to them. In quantum mechanics, by contrast, discrete symmetries are not only related to conservation laws, but they also give rise to the second kind of consequence that was discussed above. As Wigner emphasized multiple times, furthermore, discrete symmetries such as reflections play a particularly prominent role in atomic physics, via the associated notion of parity.

Items 1*–3* above account for the manner in which Wigner's remarks need to be interpreted in order to be correct. When construed this way, however, Wigner's remarks become claims about the different nature that the connection between symmetries and conservation laws takes in quantum and in classical mechanics. Item 1* becomes a difference about the kinds of symmetry transformations that are related to conservation laws in each case. Item 2* becomes a difference about the kinds of consequences that accompany, or fail to accompany, conservation laws. Finally, item 3* involves both kinds of differences. Because Wigner's comments can be interpreted as claims about the different nature that the connection between symmetries and conservation laws has in quantum and in classical mechanics, and because when so interpreted Wigner's comments are correct, we have that the nature of this connection is in fact different in the two theories.

As for the reasons why these differences obtain, they boil down to the two main facts that I mentioned in section 1. As we saw in section 4, there are restrictions in classical mechanics, which limit the kinds of circumstances under which the connection between symmetries and conservation laws obtains. These restrictions account for difference 1*. In quantum mechanics, additionally, symmetries have further consequences, which are different from the presence of a conservation law. Section 3 was devoted to introducing these kinds of further consequences, which are responsible for difference 2*. Finally, difference 3* is a result of both the restrictions that limit the circumstances under which the connection between symmetries and conservation laws obtains in classical mechanics and the further consequences that accompany symmetry principles in quantum mechanics. Taken together, therefore, these two considerations account for all three of the differences 1*–3* above.

6. Conclusion. My starting point in this article was a number of remarks that Eugene Wigner made at different times over the course of his career, which I labeled as 1–3. In order to make sense of Wigner's remarks, I introduced some elements of quantum and classical mechanics in sections 3 and 4. This allowed me to construe Wigner's remarks as three claims 1*–3* about the different ways in which symmetries and conservation laws relate to each other in quantum and in classical mechanics.

Construed as 1*–3*, Wigner's remarks tell us about the specific ways in which the nature of the connection between symmetries and conservation laws is different in quantum and in classical mechanics. The upshot is that, on the one hand, the connection between symmetries and conservation laws does not always obtain in classical mechanics. This is only the case, as we saw, for Lagrangian theories and for continuous, variational symmetries. On the other hand, symmetries have additional consequences in quantum mechanics, which are different from the presence of a conservation law. These consequences concern the transformation properties of the energy eigenfunctions, and they sometimes allow for various practical applications.

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