Matrix-form Recursive Evaluation of the Aggregate Claims Distribution Revisited

By Kok Keng Siaw, Xueyuan Wu, David Pitt and Yan Wang

Abstract

This paper aims to evaluate the aggregate claims distribution under the collective risk model when the number of claims follows a so-called generalised $(a, b, 1)$ family distribution. The definition of the generalised $(a, b, 1)$ family of distributions is given first, then a simple matrix-form recursion for the compound generalised $(a, b, 1)$ distributions is derived to calculate the aggregate claims distribution with discrete non-negative individual claims. Continuous individual claims are discussed as well and an integral equation of the aggregate claims distribution is developed. Moreover, a recursive formula for calculating the moments of aggregate claims is also obtained in this paper. With the recursive calculation framework being established, members that belong to the generalised $(a, b, 1)$ family are discussed. As an illustration of potential applications of the proposed generalised $(a, b, 1)$ distribution family on modelling insurance claim numbers, two numerical examples are given. The first example illustrates the calculation of the aggregate claims distribution using a matrix-form Poisson for claim frequency with logarithmic claim sizes. The second example is based on real data and illustrates maximum likelihood estimation for a set of distributions in the generalised $(a, b, 1)$ family.

Keywords

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1. Introduction

In the risk theory literature, how to evaluate the distribution of aggregate claims arising from a portfolio of risks in a certain time period is one of the long-lasting interesting problems. The two models mostly used in addressing the problem are the collective risk model and the individual risk model. In this paper, we will only look at the former model, under which the aggregate claims amount, denoted by S, is defined as

$$S = \sum_{i=1}^{N} X_i.$$

In the above expression, N is a random variable (r.v.) denoting the number of claims incurred over a fixed time period and it is valued on the non-negative integers. It has probability function (p.f.) $p_n = \Pr(N = n)$, $n \geq 0$, and probability generating function (p.g.f.) $\hat{p}(z) = \Sigma_{n=0}^{\infty} z^n p_n$, $z \in \mathbb{C}$. $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) non-negative r.v.'s, either discrete or continuous, denoting the individual claim sizes. In addition, we assume that N is independent of $\{X\}_{i=1}^{\infty}$. In the following, we first consider the case of discrete individual claims X_i, with p.f. f_x, $x = 0, 1, 2, \ldots$

To evaluate the probability function of the aggregate claims S, one can use the convolution method. Let $g_x = \Pr(S = x)$. A well-known result is, for $x \geq 0$,

$$g_x = \sum_{n=0}^{\infty} p_n f_x^{n*}, \tag{1.1}$$

where f^{n*} is the n-fold convolution of f. Although the above formula provides us with a way of calculating the p.f. of S, it is not computationally efficient in practice as it often involves high-order convolutions of f.

To overcome this problem, Panjer (1981) developed a recursive formula to compute the aggregate claims distribution when $\{p_n\}$, the p.f. of N, belongs to the $(a, b, 0)$ family of distributions and $X_i > 0$ (i.e. $f_0 = 0$) for all $i = 1, 2, \ldots$. In particular, he states that if there exist constants a and b such that the p.f. p_n can be written as, for $n \geq 1$,

$$p_n = p_{n-1}\left(a + \frac{b}{n}\right), \tag{1.2}$$

then $\{p_n\}$ is said to belong to the $(a, b, 0)$ family and

$$g_x = \sum_{j=1}^{x}\left(a + \frac{bj}{x}\right)f_j g_{x-j}, \quad x > 0. \tag{1.3}$$

The starting value for the recursion is given by $g_0 = p_0$. As shown by Sundt & Jewell (1981), the only counting distributions belonging to the $(a, b, 0)$ family are the Poisson, negative binomial (with geometric distribution as a special case) and binomial distributions. The $(a, b, 0)$ family also contains the distribution concentrated at zero. Moreover, for a general f_0 Sundt & Jewell (1981) extended (1.3) to

$$g_x = \frac{1}{1 - af_0}\sum_{j=1}^{x}\left(a + \frac{bj}{x}\right)f_j g_{x-j}, \quad x > 0, \tag{1.4}$$

with the starting value $g_0 = \hat{p}(f_0)$. Sundt & Jewell (1981) further generalised the Panjer recursion (1.3) to the family of $(a, b, 1)$ distributions, sometimes known as the Sundt-Jewell class of distributions in the literature. In particular, the recursive structure (1.2), initiates at p_1 rather than p_0. One of their main results says that if N follows an $(a, b, 1)$ family distribution, then

$$g_x = \frac{1}{1 - af_0}\left[(p_1 - (a + b)p_0)f_x + \sum_{j=1}^{x}\left(a + \frac{bj}{x}\right)f_j g_{x-j}\right], \quad x > 0. \tag{1.5}$$

The starting value for the recursion is still $g_0 = \hat{p}(f_0)$. Willmot (1988) showed that the only non-degenerate members of the $(a, b, 1)$ family are the Poisson, negative binomial, binomial, logarithmic series, the extended truncated negative binomial (ETNB) distributions and the zero-modified versions of these distributions. This was also mentioned in Klugman *et al.* (1998, pp 229).

Since then, a large amount of research has been undertaken to allow for cases where N has other types of counting distributions. For example, Schröter (1990) derived a recursive formula for the distribution of S when p_n belongs to the Schröter family of distributions. A detailed review on the development of recursive evaluation of aggregate claims distribution can be found in Sundt (2002). Sundt (2003) studied how to calculate the higher moments of S recursively. A more recent resource is the book written by Sundt & Vernic in 2009, which conducted a thorough review on modelling the aggregate claims distribution as compound distributions and convolutions.

Wu & Li (2010) proposed the generalised $(a, b, 0)$ family of distributions to model the claim number N. Its definition is given as follows.

Definition 1. *Let $\{p_n\}_{n=0}^{\infty}$ be the p.f. of the r.v. N. If p_n can be written as*

$$p_n = \vec{\gamma} \mathbf{P}_n \vec{1}^{\top}, \quad n \geq 0,$$

where $\vec{1} = (1, 1, \ldots, 1)_{1 \times m}, \vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ is a row vector with $\gamma_i \geq 0$ and $\sum_{i=1}^{m} \gamma_i = 1$, and $\mathbf{P}_n$, $n = 0, 1, \ldots$, is a sequence of $m \times m$ matrices, satisfying the following recursion

$$\mathbf{P}_n = \mathbf{P}_{n-1} \left(\mathbf{A} + \frac{\mathbf{B}}{n} \right), \quad n \geq 1, \tag{1.6}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two $m \times m$ matrices, then $\{p_n\}_{n=0}^{\infty}$ is said to belong to the generalised $(a, b, 0)$ family.

The p.g.f. of a generalised $(a, b, 0)$ family distribution becomes

$$\hat{p}(z) = \sum_{n=0}^{\infty} z^n p_n = \sum_{n=0}^{\infty} z^n \left(\vec{\gamma} \mathbf{P}_n \vec{1}^{\top} \right) = \vec{\gamma} \left(\sum_{n=0}^{\infty} z^n \mathbf{P}_n \right) \vec{1}^{\top} = \vec{\gamma} \hat{\mathbf{P}}(z) \vec{1}^{\top}, \tag{1.7}$$

where $\hat{\mathbf{P}}(z) = \sum_{n=0}^{\infty} z^n \mathbf{P}_n$. We note that when $m = 1$, the recursion (1.6) reduces to recursion (1.2). Wu & Li (2010) provided a matrix-form of Panjer's recursive equation for the compound generalised $(a, b, 0)$ distributions. At the same time, the authors also considered matrix-form recursion for the compound discrete phase-type (DPH) distributions as DPH distributions are commonly used in modelling count data as well. Detailed discussions about the DPH distributions can be found in Neuts (1981) and Latouche & Ramaswami (1999) and the references therein. An interesting observation is that DPH distributions follow a similar recursive structure as (1.6) but starting from one step later. Motivated by this observation, it is natural to introduce the generalised $(a, b, 1)$ family of distributions with matrix parameters.

Definition 2. *Let $\{p_n\}_{n=0}^{\infty}$ be the p.f. of the r.v. N. If p_n can be written as*

$$p_n = \vec{\gamma} \mathbf{Q}_n \vec{1}^{\top}, \quad n \geq 0, \tag{1.8}$$

where $\vec{\gamma}$ and $\vec{1}$ are the same as defined in Definition 1 and $\mathbf{Q}_n$, $n = 1, 2, \ldots$, is a sequence of $m \times m$ matrices, satisfying the following recursion:

$$\mathbf{Q}_n = \mathbf{Q}_{n-1} \left(\mathbf{A} + \frac{\mathbf{B}}{n} \right), \quad n \geq 2, \tag{1.9}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two $m \times m$ matrices, then $\{p_n\}_{n=0}^{\infty}$ is said to belong to the generalised $(a, b, 1)$ family.

Analogous to (1.7), the p.g.f. of $\{p_n\}_{n=0}^{\infty}$, denoted by $\hat{p}(z)$, can be expressed as

$$\hat{p}(z) = \sum_{n=0}^{\infty} z^n p_n = \vec{\gamma}\, \hat{\mathbf{Q}}(z)\vec{\mathbf{1}}^{\top},$$

where $\hat{\mathbf{Q}}(z) = \sum_{n=0}^{\infty} z^n \mathbf{Q}_n$. Again, we see that if $m = 1$, the recursion (1.9) reduces to the $(a, b, 1)$ family case as mentioned before. Furthermore, it can be shown that all DPH distributions are members of this family, including many others. Members other than DPH distributions that belong to the generalised $(a, b, 1)$ family will be discussed in Section 4 of this paper.

Motivated by the idea and based on the results obtained in Wu & Li (2010), we will develop a recursive method to evaluate the distribution and moments of the aggregate claims S. Firstly, a matrix-form recursion is developed in Section 2 to evaluate the aggregate claims distribution when N belongs to the generalised $(a, b, 1)$ family of distributions and individual claims take non-negative integer values. An integral equation is also derived for the aggregate claims distribution with continuous claim amounts. Section 3 demonstrates how to compute the moments of S recursively based on the results from Section 2. Section 4 is devoted to discussion of some special members that belong to the generalised $(a, b, 1)$ family of distributions. Two numerical examples, one involving real data, are presented in Section 5 to illustrate the use of the matrix-form recursion derived in Section 2 and the estimation of parameters associated with a member of the generalised $(a, b, 1)$ family. And lastly, we provide a short summary of some technical issues regarding the generalised $(a, b, 1)$ family of distributions and some possible future investigations.

2. Compound Generalised $(a, b, 1)$ Distributions

In this section, we will derive a recursive formula to calculate the distribution of aggregate claims S defined at the beginning of this paper, given that the number of claims N follows a generalised $(a, b, 1)$ distribution. We first assume that the individual claim amounts are discrete random variables valued on the non-negative integers. Secondly, continuous claim amounts will be considered using similar procedures adopted for the discrete case. We will begin with a lemma that is useful in our subsequent derivations.

Lemma 1. *For a generalised* $(a, b, 1)$ *distribution,* $\{p_n\}_{n=0}^{\infty}$, *as defined in* Definition 2, $\hat{\mathbf{Q}}(z)$ *satisfies the following differential equation*

$$\hat{\mathbf{Q}}'(z) = \mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B}) + z\, \hat{\mathbf{Q}}'(z)\mathbf{A} + \hat{\mathbf{Q}}(z)(\mathbf{A} + \mathbf{B}). \tag{2.1}$$

Proof. We have

$$\begin{aligned}
\hat{\mathbf{Q}}'(z) &= \sum_{n=1}^{\infty} n z^{n-1} \mathbf{Q}_n \\
&= \mathbf{Q}_1 + \sum_{n=2}^{\infty} n z^{n-1} \mathbf{Q}_{n-1}\left(\mathbf{A} + \frac{\mathbf{B}}{n}\right) \\
&= \mathbf{Q}_1 + \sum_{n=1}^{\infty} (n+1) z^n \mathbf{Q}_n \mathbf{A} + \sum_{n=1}^{\infty} z^n \mathbf{Q}_n \mathbf{B} \\
&= \mathbf{Q}_1 + \left[z\, \hat{\mathbf{Q}}'(z) + \hat{\mathbf{Q}}(z) - \mathbf{Q}_0\right]\mathbf{A} + \left[\hat{\mathbf{Q}}(z) - \mathbf{Q}_0\right]\mathbf{B} \\
&= \mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B}) + z\, \hat{\mathbf{Q}}'(z)\mathbf{A} + \hat{\mathbf{Q}}(z)(\mathbf{A} + \mathbf{B}).
\end{aligned}$$

This completes the proof. □

166

2.1. Discrete Claim Amounts Distribution

Using the notation defined in Section 1, let the individual claims be i.i.d. r.v.'s with p.f. f. Let $\hat{f}(z) = \sum_{x=0}^{\infty} z^x f_x$ and $\hat{g}(z) = \sum_{x=0}^{\infty} z^x g_x$ be the p.g.f. of f and g respectively, where g is the p.f. of S. Under the generalised $(a, b, 1)$ framework, substituting equation (1.8) into equation (1.1) yields

$$g_x = \sum_{n=0}^{\infty} f_x^{n*}\left(\vec{\gamma}\,\mathbf{Q}_n \vec{\mathbf{1}}^{\top}\right) = \vec{\gamma}\left(\sum_{n=0}^{\infty} f_x^{n*}\mathbf{Q}_n\right)\vec{\mathbf{1}}^{\top} = \vec{\gamma}\mathbf{G}(x)\vec{\mathbf{1}}^{\top}, \qquad (2.2)$$

where $\mathbf{G}(x) = \sum_{n=0}^{\infty} f_x^{n*}\mathbf{Q}_n$. Therefore, $\hat{g}(z)$ can be expressed as

$$\hat{g}(z) = \sum_{x=0}^{\infty} z^x g_x = \sum_{x=0}^{\infty} z^x\left(\vec{\gamma}\mathbf{G}(x)\vec{\mathbf{1}}^{\top}\right) = \vec{\gamma}\left(\sum_{x=0}^{\infty} z^x\mathbf{G}(x)\right)\vec{\mathbf{1}}^{\top}.$$

On the other hand, we can rewrite $\hat{g}(z)$ as follows:

$$\hat{g}(z) = \hat{p}\left(\hat{f}(z)\right) = \sum_{n=0}^{\infty} p_n\left[\hat{f}(z)\right]^n = \vec{\gamma}\left[\sum_{n=0}^{\infty}\mathbf{Q}_n\left(\hat{f}(z)\right)^n\right]\vec{\mathbf{1}}^{\top}.$$

Let

$$\zeta(z) = \sum_{n=0}^{\infty}\mathbf{Q}_n\left(\hat{f}(z)\right)^n = \hat{\mathbf{Q}}\left(\hat{f}(z)\right). \qquad (2.3)$$

So alternatively, we have $\zeta(z) = \sum_{x=0}^{\infty} z^x\mathbf{G}(x)$.

The expression (2.2) shows us that in order to calculate the p.f. of g_x, we need to find a method to calculate $\mathbf{G}(x)$. If $\mathbf{G}(x)$ can be obtained for all x, then the calculation for g_x is trivial. In what follows, we will derive a recursive equation for $\mathbf{G}(x)$.

Theorem 1. *If the distribution of the number of claims, N, belongs to the generalised $(a, b, 1)$ family of distributions and the individual claim amounts are non-negative integer valued i.i.d. r.v.'s, then the matrix $\mathbf{G}(x)$ defined just after (2.2) satisfies the following recursive formula*

$$\mathbf{G}(x) = \left[(\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B}))f_x + \sum_{j=1}^{x} f_j\mathbf{G}(x-j)\left(\mathbf{A} + \frac{j}{x}\mathbf{B}\right)\right](\mathbf{I} - f_0\mathbf{A})^{-1} \qquad (2.4)$$

for $x \geq 1$, and the starting value is $\mathbf{G}(0) = \hat{\mathbf{Q}}(f_0)$.

Proof. From (2.3) we know $\zeta(z) = \hat{\mathbf{Q}}(\hat{f}(z))$. Differentiating with respect to z on both sides of the equation gives

$$\zeta'(z) = \hat{\mathbf{Q}}'\left(\hat{f}(z)\right)\hat{f}'(z). \qquad (2.5)$$

By insertion of (2.1), we obtain

$$\zeta'(z) = \left[\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B}) + \hat{f}(z)\,\hat{\mathbf{Q}}'\left(\hat{f}(z)\right)\mathbf{A} + \hat{\mathbf{Q}}\left(\hat{f}(z)\right)(\mathbf{A} + \mathbf{B})\right]\hat{f}'(z)$$

$$= \left[\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B})\right]\hat{f}'(z) + \hat{f}(z)\left[\hat{\mathbf{Q}}'\left(\hat{f}(z)\right)\hat{f}'(z)\right]\mathbf{A} + \hat{\mathbf{Q}}\left(\hat{f}(z)\right)(\mathbf{A} + \mathbf{B})\hat{f}'(z),$$

167

which gives

$$\zeta'(z) = [\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B})]\hat{f}'(z) + \hat{f}(z)\zeta'(z)\mathbf{A} + \zeta(z)(\mathbf{A} + \mathbf{B})\hat{f}'(z). \tag{2.6}$$

Expanding both sides of (2.6) in power series and comparing the coefficients of z^{x-1} on both sides yields, for $x > 0$,

$$
\begin{aligned}
x\mathbf{G}(x) &= [\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B})]xf_x + \sum_{j=0}^{x-1}(x-j)f_j\mathbf{G}(x-j)\mathbf{A} \\
&\quad + \sum_{j=1}^{x}jf_j\mathbf{G}(x-j)(\mathbf{A} + \mathbf{B}) \\
&= [\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B})]xf_x + \sum_{j=0}^{x}xf_j\mathbf{G}(x-j)\mathbf{A} + \sum_{j=1}^{x}jf_j\mathbf{G}(x-j)\mathbf{B} \\
&= [\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B})]xf_x + xf_0\mathbf{G}(x)\mathbf{A} + \sum_{j=1}^{x}f_j\mathbf{G}(x-j)(x\mathbf{A} + j\mathbf{B}).
\end{aligned}
$$

Solving for $\mathbf{G}(x)$ gives the recursion (2.4). The starting value $\mathbf{G}(0)$ of the recursion can be determined as follows:

$$\mathbf{G}(0) = \sum_{n=0}^{\infty} f_0^{n*}\mathbf{Q}_n = \sum_{n=0}^{\infty}(f_0)^n\mathbf{Q}_n = \hat{\mathbf{Q}}(f_0).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As commented by Wu & Li (2010), it is worthwhile to further obtain a vector version of recursive equation for g_x. We define $\vec{\mathbf{Q}}_0 = \vec{\gamma}\mathbf{Q}_0$, $\vec{\mathbf{Q}}_1 = \vec{\gamma}\mathbf{Q}_1$ and $\vec{\mathbf{G}}(x) = \vec{\gamma}\mathbf{G}(x)$ to be $1 \times m$ row vectors. The matrix-form recursive formula (2.4) has a version in terms of vectors

$$\vec{\mathbf{G}}(x) = \left[\left(\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A} + \mathbf{B})\right)f_x + \sum_{j=1}^{x}f_j\vec{\mathbf{G}}(x-j)\left(\mathbf{A} + \frac{j}{x}\mathbf{B}\right)\right](\mathbf{I} - f_0\mathbf{A})^{-1}, \tag{2.7}$$

for $x > 0$. The starting vector is $\vec{\mathbf{G}}(0) = \vec{\gamma}\hat{\mathbf{Q}}(f_0)$.

Remarks:

1. Equation (2.7) can save computational time when calculating the distribution of S.

2. If we assume that the individual claim amounts, X_i, can only take positive integers, i.e. $f_0 = 0$, then equation (2.7) can be further simplified as

$$\vec{\mathbf{G}}(x) = \left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A} + \mathbf{B})\right]f_x + \sum_{j=1}^{x}f_j\vec{\mathbf{G}}(x-j)\left(\mathbf{A} + \frac{j}{x}\mathbf{B}\right),$$

with the initial vector given by $\vec{\mathbf{G}}(0) = \vec{\gamma}\mathbf{Q}_0$.

3. When $m = 1$, we have $\mathbf{A} = a$, $\mathbf{B} = b$, $\vec{\gamma} = 1$ and $\vec{\mathbf{G}}(x) = g_x$. Then equation (2.7) reduces to recursive formula (1.5).

2.2. Continuous Claim Amounts Distribution

In this subsection, we consider continuous individual claim amounts. We still assume that X_i, $i = 1, 2, \ldots$ are i.i.d. r.v.'s with probability density function $f(x)$, where $x \in (0, \infty)$. Let $g(x)$ denote

the probability density function of S, and $\vec{G}(x)$ be defined as before with $f(x)$ replacing f_x. Then we have $g(x) = \vec{G}(x)\vec{1}^\top$. Further, we denote the moment generating functions of X_i and S by $M_X(z)$ and $M_S(z)$ respectively, and we obtain

$$M_S(z) = \hat{p}(M_X(z)) = \vec{\gamma}\,\overset{\wedge}{\mathbf{Q}}(\mathbf{M_X(z)})\vec{1}^\top = \vec{\eta}(z)\vec{1}^\top,$$

where $\vec{\eta}(z) = \vec{\gamma}\,\overset{\wedge}{\mathbf{Q}}(\mathbf{M_X(z)})$.

As a counterpart of the discrete claims case, we derive the following integral equation of the aggregate claims distribution.

Theorem 2. *If the distribution of the number of claims, N, belongs to the generalised $(a, b, 1)$ family and the individual claim amounts are i.i.d. continuous non-negative random variables, then for S, $\vec{G}(x)$ satisfies the following integral equation, for $x > 0$,*

$$\vec{G}(x) = \left[\vec{Q}_1 - \vec{Q}_0(\mathbf{A} + \mathbf{B})\right]f(x) + \int_0^x \vec{G}(x-y)\left[\mathbf{A} + \frac{y}{x}\mathbf{B}\right]f(y)dy. \tag{2.8}$$

Proof. Differentiating $\vec{\eta}(z)$ with respect to z gives

$$\vec{\eta}'(z) = \vec{\gamma}\,\overset{\wedge}{\mathbf{Q}}{}'(\mathbf{M_X(z)})\mathbf{M}'_X(z). \tag{2.9}$$

Applying Lemma 1 to the right hand side of (2.9) yields

$$\vec{\eta}'(z) = \left[\vec{Q}_1 - \vec{Q}_0(\mathbf{A} + \mathbf{B})\right]M'_X(z) + M_X(z)\vec{\eta}'(z)\mathbf{A} + M'_X(z)\vec{\eta}(z)(\mathbf{A} + \mathbf{B}). \tag{2.10}$$

Since $\vec{\eta}(z) = \int_0^\infty e^{zx}\vec{G}(x)dx$ and $M_X(z) = \int_0^\infty e^{zx}f(x)dx$, the above equation can be rewritten as

$$\begin{aligned}
\int_0^\infty xe^{zx}\vec{G}(x)dx &= \left[\vec{Q}_1 - \vec{Q}_0(\mathbf{A} + \mathbf{B})\right]\int_0^\infty xe^{zx}f(x)dx + \int_0^\infty e^{zy}f(y)dy\int_0^\infty xe^{zx}\vec{G}(x)\,dx\,\mathbf{A} \\
&\quad + \left(\int_0^\infty ye^{zy}f(y)dy\int_0^\infty e^{zx}\vec{G}(x)dx\right)(\mathbf{A} + \mathbf{B}) \\
&= \left[\vec{Q}_1 - \vec{Q}_0(\mathbf{A} + \mathbf{B})\right]\int_0^\infty xe^{zx}f(x)dx + \int_0^\infty\int_y^\infty (x-y)f(y)e^{zx}\vec{G}(x-y)\,dx\,dy\,\mathbf{A} \\
&\quad + \int_0^\infty\int_y^\infty yf(y)e^{zx}\vec{G}(x-y)dx\,dy(\mathbf{A} + \mathbf{B}). \tag{2.11}
\end{aligned}$$

Interchanging the order of integration on the right hand side of (2.11) gives

$$\begin{aligned}
\int_0^\infty xe^{zx}\vec{G}(x)dx &= \left[\vec{Q}_1 - \vec{Q}_0(\mathbf{A} + \mathbf{B})\right]\int_0^\infty xe^{zx}f(x)dx \\
&\quad + \int_0^\infty e^{zx}\int_0^x (x-y)f(y)\vec{G}(x-y)dy\,dx\,\mathbf{A} \\
&\quad + \int_0^\infty e^{zx}\int_0^x yf(y)\vec{G}(x-y)dx\,dy(\mathbf{A} + \mathbf{B}). \tag{2.12}
\end{aligned}$$

By comparing the coefficients of e^{zx} on both sides of equation (2.12), we obtain equation (2.8). □

Remark: Note that the result (2.8) is more of interest in theory. In practice, discretising the continuous claim amounts would be an easier option for calculations.

3. The Moments of the Aggregate Claims

Having discussed the recursive calculation for the p.f. of the aggregate claims S with a generalised $(a, b, 1)$ distributed number of claims, in this section we will consider how to evaluate the moments of S. As shown in the following, the moments of S can also be calculated recursively.

Firstly we define the row vector $\vec{\mathbf{H}}(r) = \sum_{x=0}^{\infty} x^r \vec{\mathbf{G}}(x)$, such that, for $r = 0, 1, 2, \ldots,$

$$\mathbb{E}(S^r) = \sum_{x=0}^{\infty} x^r g_x = \left(\sum_{x=0}^{\infty} x^r \vec{\mathbf{G}}(x) \right) \vec{\mathbf{1}}^{\top} = \vec{\mathbf{H}}(r) \vec{\mathbf{1}}^{\top}, \tag{3.1}$$

where $\vec{\mathbf{H}}(0) = \sum_{x=0}^{\infty} \vec{\mathbf{G}}(x)$. Formula (3.1) implies that if $\vec{\mathbf{H}}(r)$ can be calculated, so can $\mathbb{E}(S^r)$. In what follows a method is developed to calculate $\vec{\mathbf{H}}(r)$ recursively when the individual claim amounts are discrete.

Theorem 3. *The moment vectors $\vec{\mathbf{H}}(r)$, $r \geq 1$, defined above satisfy the following recursive equation*:

$$\vec{\mathbf{H}}(r) = \left[\left(\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A} + \mathbf{B}) \right) \mathbb{E}(X^r) \right. \\ \left. + \sum_{k=0}^{r-1} \mathbb{E}\left(X^{r-k} \right) \vec{\mathbf{H}}(k) \left(\binom{r}{k} \mathbf{A} + \binom{r-1}{k} \mathbf{B} \right) \right] [\mathbf{I} - \mathbf{A}]^{-1}. \tag{3.2}$$

Proof. Based on the recursive formula (2.7) and the definition of $\vec{\mathbf{H}}(r)$, we have

$$\vec{\mathbf{H}}(r) = \sum_{x=1}^{\infty} x^r \left[\left(\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A} + \mathbf{B}) \right) f_x + \sum_{j=1}^{x} f_j \vec{\mathbf{G}}(x-j) \left(\mathbf{A} + \frac{j}{x} \mathbf{B} \right) \right] [\mathbf{I} - f_0 \mathbf{A}]^{-1}.$$

Using a similar approach as in Wu & Li (2010) gives

$$\vec{\mathbf{H}}(r) [\mathbf{I} - f_0 \mathbf{A}]$$
$$= \left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A} + \mathbf{B}) \right] \mathbb{E}(X^r) + \sum_{x=1}^{\infty} x^r \sum_{j=1}^{x} f_j \vec{\mathbf{G}}(x-j) \mathbf{A} + \sum_{x=1}^{\infty} x^{r-1} \sum_{j=1}^{x} j f_j \vec{\mathbf{G}}(x-j) \mathbf{B}$$
$$= \left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A} + \mathbf{B}) \right] \mathbb{E}(X^r) + \sum_{j=1}^{\infty} f_j \sum_{x=0}^{\infty} (x+j)^r \vec{\mathbf{G}}(x) \mathbf{A} + \sum_{j=1}^{\infty} j f_j \sum_{x=0}^{\infty} (x+j)^{r-1} \vec{\mathbf{G}}(x) \mathbf{B}.$$

Applying the binomial expansion, we obtain

$$\vec{\mathbf{H}}(r) [\mathbf{I} - f_0 \mathbf{A}] = \left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A} + \mathbf{B}) \right] \mathbb{E}(X^r) + \sum_{j=1}^{\infty} f_j \sum_{x=0}^{\infty} \left(\sum_{k=0}^{r} \binom{r}{k} x^k j^{r-k} \right) \vec{\mathbf{G}}(x) \mathbf{A}$$
$$+ \sum_{j=1}^{\infty} j f_j \sum_{x=0}^{\infty} \left(\sum_{k=0}^{r-1} \binom{r-1}{k} x^k j^{r-1-k} \right) \vec{\mathbf{G}}(x) \mathbf{B}$$
$$= \left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A} + \mathbf{B}) \right] \mathbb{E}(X^r) + \sum_{j=1}^{\infty} f_j \sum_{k=0}^{r} \binom{r}{k} j^{r-k} \left(\sum_{x=0}^{\infty} x^k \vec{\mathbf{G}}(x) \right) \mathbf{A}$$
$$+ \sum_{j=1}^{\infty} j f_j \sum_{k=0}^{r-1} \binom{r-1}{k} j^{r-1-k} \left(\sum_{x=0}^{\infty} x^k \vec{\mathbf{G}}(x) \right) \mathbf{B},$$

170

which gives

$$\vec{\mathbf{H}}(r)\left[\mathbf{I} - f_0\,\mathbf{A}\right] = \left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0\,(\mathbf{A} + \mathbf{B})\right]\mathbb{E}(X^r) + \sum_{k=0}^{r-1}\binom{r}{k}\mathbb{E}\left(X^{r-k}\right)\vec{\mathbf{H}}(k)\,\mathbf{A}$$

$$+ (1 - f_0)\,\vec{\mathbf{H}}(r)\,\mathbf{A} + \sum_{k=0}^{r-1}\binom{r-1}{k}\mathbb{E}\left(X^{r-k}\right)\vec{\mathbf{H}}(k)\,\mathbf{B}.$$

Solving for $\vec{\mathbf{H}}(r)$ gives the recursion (3.2). This completes the proof. $\qquad\square$

4. Members of the Generalised $(a, b, 1)$ Family

We will now turn our attention to discussing the members of the generalised $(a, b, 1)$ family other than the DPH distributions. They are the matrix versions of the zero-modified Poisson distribution, the extended truncated negative binomial distributions with non-integer shape parameter and the logarithmic series distributions.

To begin with, notice that based on Definition 2, the matrix $\mathbf{Q}_0$ can be chosen independently from the subsequent matrices $\mathbf{Q}_n$ for $n \geq 1$. This particular feature allows one to tackle the issue of unusually high or low probability observed at $N = 0$, which usually arises in insurance count data as suggested by Klugman *et al.* (1998, pp. 225). This led us to consider the problem of how to determine the starting value in recursive formula (1.9), i.e., the matrix $\mathbf{Q}_1$. If there is an approach to determine $\mathbf{Q}_1$ (the obtained $\mathbf{Q}_1$ may not be unique), given $\vec{\gamma}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{Q}_0$, then it is equivalent to say that sequence $\{\mathbf{Q}_n\}_{n=0}^{\infty}$ is fully determined. To address this problem, we propose the following method.

Firstly, to make $\{p_n\}_{n=0}^{\infty}$ a proper distribution, we require

$$\sum_{n=0}^{\infty} p_n = \vec{\gamma}\sum_{n=0}^{\infty}\mathbf{Q}_n\vec{\mathbf{1}}^{\top} = 1,$$

or more specifically,

$$\vec{\gamma}\sum_{n=1}^{\infty}\mathbf{Q}_n\vec{\mathbf{1}}^{\top} = 1 - p_0. \tag{4.1}$$

Applying the recursive property of $\mathbf{Q}$ to the left-hand side of equation (4.1) gives

$$\vec{\gamma}\,\mathbf{Q}_1\left[\mathbf{I} + \sum_{n=1}^{\infty}\prod_{j=1}^{n}\left(\mathbf{A} + \frac{\mathbf{B}}{j+1}\right)\right]\vec{\mathbf{1}}^{\top} = 1 - p_0.$$

In general, there are an infinite number of solutions for $\mathbf{Q}_1$ in the above equation.

A special solution of $\mathbf{Q}_1$ is

$$\mathbf{Q}_1 = (1 - p_0)\left[\mathbf{I} + \sum_{n=1}^{\infty}\prod_{j=1}^{n}\left(\mathbf{A} + \frac{\mathbf{B}}{j+1}\right)\right]^{-1} \tag{4.2}$$

given that the matrix $\mathbf{D} = \mathbf{I} + \sum_{n=1}^{\infty}\prod_{j=1}^{n}\left(\mathbf{A} + \frac{\mathbf{B}}{j+1}\right)$ is invertible. Closed-form expressions can be obtained under some special circumstances.

Case 1: Let $\mathbf{A} = \boldsymbol{\Theta}$ and $\mathbf{B} = -\boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ is an $m \times m$ non-singular matrix. Then we have

$$\mathbf{D} = \mathbf{I} + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left(\boldsymbol{\Theta} - \frac{\boldsymbol{\Theta}}{j+1} \right) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\boldsymbol{\Theta}^n}{n+1} = \boldsymbol{\Theta}^{-1} \sum_{n=1}^{\infty} \frac{\boldsymbol{\Theta}^n}{n}.$$

We assume that $\mathbf{I} - \boldsymbol{\Theta}$ is invertible and each Jordan block belonging to a negative eigenvalue, which is a matrix having the following form

$$\begin{pmatrix} -c & 1 & 0 & \cdots & 0 & 0 \\ 0 & -c & 1 & \ddots & 0 & 0 \\ 0 & 0 & -c & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 & -c \end{pmatrix}$$

where $-c$ is the negative eigenvalue, occurs an even number of times. According to Culver (1966) we know that the above expression can be further simplified to $-\boldsymbol{\Theta}^{-1} \ln(\mathbf{I} - \boldsymbol{\Theta})$ and here $\ln(\mathbf{I} - \boldsymbol{\Theta})$ is a real matrix. If in addition, the matrix $\ln(\mathbf{I} - \boldsymbol{\Theta})$ is invertible, then we have found an explicit solution for $\mathbf{Q}_1$:

$$\mathbf{Q}_1 = -(1 - p_0) [\ln(\mathbf{I} - \boldsymbol{\Theta})]^{-1} \boldsymbol{\Theta},$$

and it gives

$$\mathbf{Q}_n = \left(-\frac{1}{n} \right) (1 - p_0) [\ln(\mathbf{I} - \boldsymbol{\Theta})]^{-1} \boldsymbol{\Theta}^n, \quad n \geq 1.$$

Therefore, on the other hand, if a given distribution $\{p_n\}_{n=0}^{\infty}$ can be written in the form

$$p_0 = \vec{\gamma} \mathbf{Q}_0 \vec{1}^{\top},$$
$$p_n = -\frac{1}{n} (1 - p_0) \vec{\gamma} [\ln(\mathbf{I} - \boldsymbol{\Theta})]^{-1} \boldsymbol{\Theta}^n \vec{1}^{\top}, \quad n \geq 1, \tag{4.3}$$

then $\{p_n\}_{n=0}^{\infty}$ belongs to the generalised $(a, b, 1)$ family with parameters $\mathbf{A} = \boldsymbol{\Theta}$ and $\mathbf{B} = -\boldsymbol{\Theta}$. The p.g.f. in this case is given by

$$\hat{p}(z) = \vec{\gamma} \hat{\mathbf{Q}}(z) \vec{1}^{\top} = \vec{\gamma} \left[\mathbf{Q}_0 + \sum_{n=1}^{\infty} z^n \mathbf{Q}_n \right] \vec{1}^{\top}$$

$$= \vec{\gamma} \left[\mathbf{Q}_0 + (1 - p_0) [\ln(\mathbf{I} - \boldsymbol{\Theta})]^{-1} \left(-\sum_{n=1}^{\infty} \frac{(z\boldsymbol{\Theta})^n}{n} \right) \right] \vec{1}^{\top}$$

$$= p_0 + (1 - p_0) \vec{\gamma} [\ln(\mathbf{I} - \boldsymbol{\Theta})]^{-1} \ln(\mathbf{I} - z\boldsymbol{\Theta}) \vec{1}^{\top}.$$

For $j > 0$, the jth factorial moment of N can be obtained by differentiating $\hat{p}(z)$ for j times and letting $z = 1$, shown as follows:

$$\mathbb{E}[N(N-1)\dots(N-j+1)] = -(j-1)!(1-p_0) \vec{\gamma} [\ln(\mathbf{I} - \boldsymbol{\Theta})]^{-1} (\mathbf{I} - \boldsymbol{\Theta})^{-j} \boldsymbol{\Theta}^j \vec{1}^{\top}.$$

Case 2: Let $\mathbf{A} = 0$ and $\mathbf{B} = \mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is an $m \times m$ invertible matrix. Then the expression for the matrix $\mathbf{D}$ reduces to

$$\mathbf{I} + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{\mathbf{\Lambda}}{j+1} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{\Lambda}^n}{(n+1)!}$$

$$= \mathbf{\Lambda}^{-1} \left(\sum_{n=0}^{\infty} \frac{\mathbf{\Lambda}^n}{n!} - \mathbf{I} \right) = \mathbf{\Lambda}^{-1} e^{\mathbf{\Lambda}} (\mathbf{I} - e^{-\mathbf{\Lambda}}).$$

If, in addition, $\mathbf{I} - e^{-\mathbf{\Lambda}}$ is invertible, then we have an explicit solution for $\mathbf{Q}_1$:

$$\mathbf{Q}_1 = (1 - p_0)(\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} e^{-\mathbf{\Lambda}} \mathbf{\Lambda},$$

and hence

$$\mathbf{Q}_n = (1 - p_0)(\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} e^{-\mathbf{\Lambda}} \frac{\mathbf{\Lambda}^n}{n!}, \quad n \geq 1.$$

Thus, a distribution $\{p_n\}_{n=0}^{\infty}$ with the following structure

$$p_0 = \vec{\gamma} \mathbf{Q}_0 \vec{\mathbf{1}}^{\top},$$
$$p_n = (1 - p_0)\vec{\gamma}(\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} e^{-\mathbf{\Lambda}} \frac{\mathbf{\Lambda}^n}{n!} \vec{\mathbf{1}}^{\top}, \quad n \geq 1, \tag{4.4}$$

belongs to the generalised $(a, b, 1)$ family with parameters $\mathbf{A} = 0$ and $\mathbf{B} = \mathbf{\Lambda}$. Its p.g.f. equals

$$\hat{p}(z) = p_0 + (1 - p_0)\vec{\gamma} \, (\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} e^{-\mathbf{\Lambda}} (e^{z\mathbf{\Lambda}} - \mathbf{I}) \vec{\mathbf{1}}^{\top}. \tag{4.5}$$

The jth factorial moment of N is

$$\mathbb{E}[N(N-1)\dots(N-j+1)] = (1 - p_0)\vec{\gamma} \, (\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} \mathbf{\Lambda}^j \vec{\mathbf{1}}^{\top}, \quad j > 0.$$

Case 3: Let $k > -1$, $\mathbf{A} = \mathbf{R}$ and $\mathbf{B} = (k - 1)\mathbf{R}$, where $\mathbf{R}$ is an $m \times m$ invertible matrix such that $\mathbf{I} - \mathbf{R}$ is invertible as well. The expression of the matrix $\mathbf{D}$ can be simplified as

$$\mathbf{I} + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left(\mathbf{R} + \frac{(k-1)\mathbf{R}}{j+1} \right)$$

$$= \mathbf{I} + \sum_{n=1}^{\infty} \mathbf{R}^n \prod_{j=1}^{n} \frac{j+k}{j+1} = \sum_{n=0}^{\infty} \frac{(n+k)!}{(n+1)!k!} \mathbf{R}^n$$

$$= \frac{1}{k} \sum_{n=1}^{\infty} \binom{n+k-1}{n} \mathbf{R}^{n-1} = \frac{1}{k} \mathbf{R}^{-1} \left[\sum_{n=0}^{\infty} \binom{n+k-1}{n} \mathbf{R}^n - \mathbf{I} \right].$$

To simplify the above expression, we use the generalised binomial theorem for the function $(\mathbf{I} - \mathbf{R})^{-k}$, which gives

$$(\mathbf{I} - \mathbf{R})^{-k} = \sum_{n=0}^{\infty} \binom{k+n-1}{n} \mathbf{R}^n.$$

As a result, the expression of $\mathbf{D}$ simplifies to

$$\frac{1}{k} \mathbf{R}^{-1} \left[(\mathbf{I} - \mathbf{R})^{-k} - \mathbf{I} \right] = \frac{1}{k} \mathbf{R}^{-1} (\mathbf{I} - \mathbf{R})^{-k} \left[\mathbf{I} - (\mathbf{I} - \mathbf{R})^k \right].$$

Assume that $\mathbf{I} - (\mathbf{I} - \mathbf{R})^k$ is invertible, then an explicit expression for $\mathbf{Q}_1$ is

$$\mathbf{Q}_1 = k(1 - p_0) \left[\mathbf{I} - (\mathbf{I} - \mathbf{R})^k \right]^{-1} \mathbf{R}(\mathbf{I} - \mathbf{R})^k.$$

Moreover, the distribution $\{p_n\}$ can be expressed as

$$p_0 = \vec{\gamma} \mathbf{Q}_0 \vec{1}^\top,$$

$$p_n = (1-p_0)\vec{\gamma} \left[\mathbf{I} - (\mathbf{I}-\mathbf{R})^k \right]^{-1} \binom{k+n-1}{n} \mathbf{R}^n (\mathbf{I}-\mathbf{R})^k \vec{1}^\top, \quad n \geq 1. \tag{4.6}$$

The p.g.f. of N is given by

$$\hat{p}(z) = p_0 + (1-p_0)\vec{\gamma} \left[\mathbf{I} - (\mathbf{I}-\mathbf{R})^k \right]^{-1} \left[(\mathbf{I}-z\mathbf{R})^{-k} - \mathbf{I} \right] (\mathbf{I}-\mathbf{R})^k \vec{1}^\top.$$

The jth factorial moment of N can be found in the usual way. However, due to the complexity of the results, only the expectation is shown below:

$$\mathbb{E}(N) = k(1-p_0)\vec{\gamma} \left[\mathbf{I} - (\mathbf{I}-\mathbf{R})^k \right]^{-1} (\mathbf{I}-\mathbf{R})^{-1} \mathbf{R} \vec{1}^\top.$$

Remarks:

1. If $m = 1$, then:

   (a) In Case 1, $\mathbf{\Theta} = \theta$ and $\vec{\gamma} = 1$. Equation (4.3) will become

   $$p_n = -\frac{1}{n} \frac{\theta^n}{\ln(1-\theta)}, \quad n \geq 1,$$

   which is the usual logarithmic distribution.

   (b) In Case 2, $\mathbf{\Lambda} = \lambda$ and $\vec{\gamma} = 1$. Equation (4.4) will become

   $$p_n = \frac{1-p_0}{1-e^{-\lambda}} \frac{e^{-\lambda}\lambda^n}{n!}, \quad n \geq 1,$$

   which is the zero-modified Poisson distribution.

   (c) In Case 3, $\mathbf{R} = r$ and $\vec{\gamma} = 1$. If $k > 0$, Equation (4.6) will become

   $$p_n = \frac{1-p_0}{1-(1-r)^k} \binom{k+n-1}{n} r^n (1-r)^k \quad n \geq 1,$$

   which is the zero-modified negative binomial distribution. However, if $-1 < k < 0$ and $p_0 = 0$, Equation (4.6) will become

   $$p_n = \binom{k+n-1}{n} \frac{r^n}{(1-r)^{-k}-1} = \binom{-k}{n} \frac{(-r)^n}{(1-r)^{-k}-1}$$

   which is the extended truncated negative binomial distribution.

2. We do not extend the parameter of the binomial distribution and its variations into matrix-form as we did for others. This is because the binomial distribution is actually a special case in the DPH family, as according to Latouche & Ramaswami (1999, pp. 54), every finite-support non-negative integer distribution can be written as a DPH distribution.

3. As shown by Wu & Li (2010), mixtures and linear combinations (less restrictions apply to the linear coefficients, i.e., some of them can be negative numbers as long as the total of coefficients equals 1) of $(a, b, 0)$ family distributions can be obtained by imposing certain restrictions on the matrix parameters of the generalised $(a, b, 0)$ distributions. For the generalised $(a, b, 1)$ family, the same restrictions on the matrix parameters will generate mixtures and linear combinations of

174

(*a*, *b*, 1) family distributions. To avoid repetition in the current paper, we refer interested readers to Wu & Li (2010)for those special cases of the matrix parameters.

4. For the special cases mentioned in point 3, the matrix-form recursion derived in Section 2.1 will not make the computation of the aggregate claims distribution more efficient. It would be easier to simply compute the aggregate distribution using the ordinary (*a*, *b*, 1) recursion for each of the components, and then combine the results later.

To end this section, we suggest an alternative approach to determine $\mathbf{Q}_1$ under some special conditions. It can be observed that both the generalised (*a*, *b*, 0) family and the generalised (*a*, *b*, 1) family of distributions share the same recursive structure for $n > 1$. Therefore, it is a plausible assumption that there exists an $m \times m$ constant matrix $\mathbf{C}$ such that

$$\mathbf{Q}_n = \mathbf{C}\mathbf{P}_n, \quad n \geq 1, \tag{4.7}$$

where $\mathbf{P}$ and $\mathbf{Q}$ form part of a generalised (*a*, *b*, 0) and a generalised (*a*, *b*, 1) distribution, with the same initial vector, respectively. A special solution for $\mathbf{C}$, assuming that $\sum_{n=0}^{\infty} \mathbf{P}_n = \mathbf{I}$ and $\mathbf{I} - \mathbf{P}_0$ is invertible, is

$$\mathbf{C} = (1 - p_0)(\mathbf{I} - \mathbf{P}_0)^{-1},$$

or equivalently,

$$\mathbf{Q}_n = (1 - p_0)(\mathbf{I} - \mathbf{P}_0)^{-1}\mathbf{P}_n, \quad n \geq 1. \tag{4.8}$$

This method is useful when one intends to extend a generalised (*a*, *b*, 0) distribution into the generalised (*a*, *b*, 1) one. As a result, Cases 2–3 above can all be obtained alternatively using this method.

5. Numerical Examples

In this section, we will present two examples. The first example shows how one can apply the recursive formulae (2.7) and (3.2) to calculate the distribution and moments of *S* using hypothetical claim number and claim size distributions. The second example compares the fit of various members of the matrix-form Poisson distribution which is a member of the generalised (*a*, *b*, 1) family discussed in Case 2 from Section 4 of the paper. This example uses real motor insurance claim frequency data.

Example 1. This example corresponds to Case 2 that is discussed in Section 4. Let $\vec{\gamma} = (0.1, 0.15, 0.25, 0.45, 0.05)$ and *N* follow a generalised (*a*, *b*, 1) distribution with parameters $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \mathbf{\Lambda}$ where $\mathbf{\Lambda}$ is an invertible matrix given as follows

$$\mathbf{\Lambda} = \begin{pmatrix} 0.4 & 0.1 & 0.2 & 0.0 & 0.2 \\ 0.1 & 0.35 & 0.0 & 0.25 & 0.2 \\ 0.2 & 0.0 & 0.3 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.1 & 0.55 & 0.05 \\ 0.2 & 0.1 & 0.1 & 0.15 & 0.3 \end{pmatrix}.$$

Also, we assume

$$\mathbf{Q}_0 = \begin{pmatrix} 0.3 & 0.0 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.0 & 0.3 & 0.0 \\ 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\ 0.0 & 0.6 & 0.0 & 0.3 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 & 0.4 \end{pmatrix}.$$

Table 1. Vector $\vec{\mathbf{G}}$ values and the p.f. of S in Example 1.

x	$\vec{\mathbf{G}}(x)$	g(x)
0	(0.055000, 0.340000, 0.100000, 0.180000, 0.070000)	0.745000
1	(0.003588, −0.057945, 0.036393, 0.069502, −0.006710)	0.044826
2	(0.004158, −0.028585, 0.020126, 0.037195, −0.003525)	0.029369
3	(0.003645, −0.018369, 0.013829, 0.025121, −0.002239)	0.021987
4	(0.003149, −0.013188, 0.010421, 0.018713, −0.001563)	0.017532
5	(0.002739, −0.010068, 0.008267, 0.014717, −0.001153)	0.014501
10	(0.001528, −0.003920, 0.003649, 0.006332, −0.000369)	0.007220
20	(0.000627, −0.001166, 0.001247, 0.002112, −0.000068)	0.002752
30	(0.000296, −0.000460, 0.000538, 0.000900, −0.000013)	0.001261
40	(0.000149, −0.000204, 0.000256, 0.000424, 0.000000)	0.000624
50	(0.000078, −0.000097, 0.000128, 0.000211, 0.000002)	0.000322
100	(0.000004, −0.000004, 0.000006, 0.000009, 0.000000)	0.000016

Table 2. Vector $\vec{\mathbf{H}}$ values and $\mathbb{E}(S^r)$ in Example 1.

r	$\vec{\mathbf{H}}(r)$	$\mathbb{E}(S^r)$
1	(0.572807, −1.222330, 1.228880, 2.106730, −0.080291)	2.60578
2	(16.26330, −24.17530, 29.92750, 48.24820, −0.355619)	68.9081
3	(762.4480, −924.9410, 1238.570, 2035.780, 28.66220)	7983.54
4	(50319.90, −53039.40, 77236.10, 125741.0, 3804.560)	204062

The individual claims follow a Logarithmic distribution with parameter $\theta = 0.95$, i.e., having a p.f. with the form

$$f_x = \left(-\frac{1}{x}\right)\frac{0.95^x}{\ln(0.05)}, \quad x > 0.$$

Since f_x is only defined for $x > 0$, the initial vector $\vec{\mathbf{G}}(0)$ can be determined using the reduced formula

$$\vec{\mathbf{G}}(0) = \vec{\gamma}\mathbf{Q}_0.$$

Table 1 includes values of the vector $\vec{\mathbf{G}}(x)$ and $g(x)$ for some x values from 0 to 100. The total tail probability for $S > 100$ is 0.000265. The first four moments of S are presented in Table 2. Vector results provided in these two tables are purely for illustration purposes.

Example 2. In this example, the data set that we employed is from a major Spanish insurance company. In particular, we are going to analyse the claim frequency experience for a portfolio of motor vehicle insurance policies for privately used cars. These data were also used in Boucher *et al.* (2009), Brouhns *et al.* (2003) and Pinquet *et al.* (2001). Bermúdez & Karlis (2011) and Bermúdez (2009) used these data to test new multivariate models for insurance claim counts based on the Poisson model with zero-inflation.

The data include the claims experience for 80,994 policyholders that stay in the company for seven complete yearly periods (1991 to 1997). For each policyholder we have information including age,

gender and vehicle power at the beginning of each period as well as the total number of claims incurred in each period.

As discussed in Section 1, the collective risk model is used to evaluate the distribution of aggregate claims arising from a portfolio of risks in a certain time period. We begin by forming individual portfolios each of 200 policyholders. To ensure homogeneity in each individual portfolio, the policyholders are grouped in such a way that they share identical features based on their age, gender, vehicle power, etc. We end up having 294 portfolios. We aggregate the total number of claims incurred in each portfolio across seven years. An important assumption that we made is that the characteristics of each policyholder do not change over time and hence the policyholders in a particular portfolio will continue to be in the portfolio for all years.

Firstly, we need to determine a counting distribution for the number of claims. Popular choices of distribution are the Poisson distribution and the mixed Poisson distribution, which are special cases of the matrix-form Poisson distribution we introduced in Section 4 Case 2. The matrix-form Poisson corresponds to the most general case of the distribution derived in Section 4 Case 2. We demonstrate the flexibility and versatility of the generalised $(a, b, 1)$ family in modelling count events. Using maximum likelihood estimation, we obtained the parameter estimates given in Table 3. In order to compare the models we have also included Akaike's Information Criterion (AIC) and the

Table 3. Parameters estimation for the Poisson, mixed-Poisson and the matrix-form Poisson distribution.

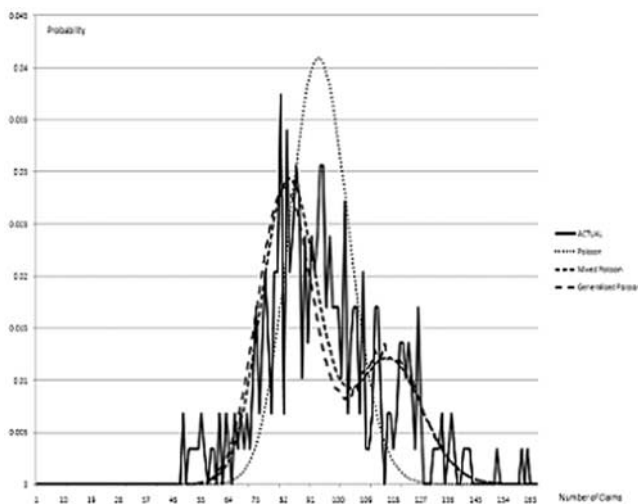| Distribution	γ_1	λ_{11}	λ_{12}	λ_{21}	λ_{22}	AIC	BIC
Poisson	1	94.88	0	0	0	3097.68	3101.36
Mixed Poisson	0.67	84.25	0	0	116.79	2679.92	2690.97
Matrix-form Poisson	0.83	83.72	5.52	5.35	115.69	2685.18	2703.60

Figure 1. Comparison between Poisson, Mixed Poisson and Matrix-form Poisson distribution. The black solid line represents the empirical distribution.

Bayesian Information Criterion (BIC) for each of the three models. Note that a low value of each of these information criteria indicates a better model in terms of adherence to data and overall model simplicity. Based on the AIC and BIC values given in the table, we see that the mixed Poisson is the optimal model here.

It is also of interest to compare graphically the fit of the Poisson, mixed Poisson and matrix-form Poisson. The figure 1 gives the fitted probabilities for each model along with the empirical probabilities based on the data. It is clear that the generalised (*a*, *b*, 1) family members, being the mixed and matrix-form Poisson models, outperform the Poisson distribution. The matrix-form Poisson provides a less smooth fit than the mixed Poisson. There are sections of the distribution where the matrix-form Poisson provides a superior fit to the mixed Poisson in this example. Given a fitted distribution from the generalised (*a*, *b*, 1) family, the next stage is to estimate parameters for a distribution for claim sizes. These two distributions can then be used together to calculate the aggregate claims distribution using the same technique as presented in Example 1.

6. Conclusions

In this paper, we proposed a generalised (*a*, *b*, 1) family of distributions that is a broad class of counting distributions, which employ matrices as parameters and satisfy a matrix version of recursive structure of the (*a*, *b*, 1) family. On the one hand, one can benefit from the great flexibility embedded in setting up matrix parameters. On the other hand, it makes it very difficult to propose appropriate conditions upon how to select proper matrices to build up claim number distributions under the structure. Also, the issue of identifying all members within the family remains unsolved. As a result, only some very special members of the generalised (*a*, *b*, 1) family are examined in this paper. It leaves a number of open problems for interested readers to explore.

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