Canad. Math. Bull. Vol. **26** (3), 1983

AN ALGEBRAIC CHARACTERIZATION OF REMAINDERS OF COMPACTIFICATIONS

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ABSTRACT. Let X be a locally compact, completely regular Hausdorff space. In this paper it is shown that all compact metric spaces are remainders of X if and only if the quotient ring $C^*(X)/C_\infty(X)$ contains a subring having no primitive idempotents.

1. **Introduction.** Let X be a locally compact, completely regular Hausdorff space and let αX be any Hausdorff compactification of X. Then $\alpha X - X$ is a remainder of X. In the theory of compactifications one of the major problems has been that of characterizing when all members of a certain class of spaces can serve as remainders for each member of another class of spaces (cf. [1], [2], [3], [5], [7], [9], etc.). In this paper we characterize when all compact metric spaces are remainders of X in terms of the ring $C^*(X)$ of bounded continuous real-valued functions on X.

Specifically, if $C_\infty(X)$ is the set of functions in $C^*(X)$ which "vanish at infinity," we show that all compact metric spaces are remainders of X if and only if the quotient $C^*(X)/C_\infty(X)$ contains a subring with no primitive idempotents (see [6], p. 74). Other characterizations of when all compact metric spaces are remainders of X may be found in [5].

2. **The characterization theorem.** Notation and terminology concerning the ring $C^*(X)$ and the Stone–Čech compactification βX of X will follow that of [4]. N denotes the positive integers and all rings under discussion are commutative rings with identity.

THEOREM. *Let X be a completely regular, locally compact Hausdorff space. Then all compact metric spaces are remainders of X if and only if $C^*(X)/C_\infty(X)$ contains a subring with no primitive idempotents.*

Proof. For each $f \in C^*(X)$, let f^β be the continuous extension of f to βX and let f^* be the restriction of f^β to $\beta X - X$. The mapping φ of $C^*(X)$ into $C^*(\beta X - X)$ defined by $\varphi(f) = f^*$ is a homomorphism. Since $\beta X - X$ is C^*-embedded in βX, φ is a surjection. Now $f \in \text{kernel } \varphi$ if and only if $f^*(p) = 0$, for

Received by the editors April 23, 1982 and, in revised form, September 28, 1982.

A.M.S. Classification Numbers: 54D30

https://doi.org/10.4153/CMB-1983-058-2 Published online by Cambridge University Press

all $p \in \beta X - X$. Thus, kernel $\varphi = C_\infty(X)$ since $C_\infty(X)$ is the intersection of all free maximal ideals in $C^*(X)$. (See 7.F.1 of [4]). Hence $C^*(X)/C_\infty(X)$ is isomorphic to $C^*(\beta X - X)$.

Now assume that $C^*(X)/C_\infty(X)$ contains a subring with no primitive idempotents. Then $C^*(\beta X - X)$ contains a subring R having no primitive idempotents. Let 1_R be the identity in R. It follows that $1_R = e_1^1 + e_2^1$, where e_1^1 and e_2^1 are non-trivial orthogonal idempotents in R.

Set $R_1^1 = Re_1^1$ and $R_2^1 = Re_2^1$. Then R_1^1 and R_2^1 are ideals of R and $R = R_1^1 \oplus R_2^1$. Now e_1^1 and e_2^1 are the respective identities of R_1^1 and R_2^1, so that the preceding argument can be applied to obtain decompositions $R_1^1 = R_1^2 \oplus R_2^2$ and $R_2^1 = R_3^2 \oplus R_4^2$. Thus, $R = R_1^2 \oplus \cdots \oplus R_4^2$. Inductively it follows that for each $n \in N$, $R = R_1^n \oplus \cdots \oplus R_{2^n}^n$, where each R_i^n is, in turn, a direct sum of non-trivial ideals R_{2i-1}^{n+1} and R_{2i}^{n+1}.

Since e_1^1 and e_2^1 are characteristic functions of disjoint clopen sets V_1^1 and V_2^1, respectively, in $\beta X - X$, it follows that $\beta X - X$ can be partitioned into disjoint non-empty clopen subsets U_1^1 and U_2^1, where $V_1^1 \subseteq U_1^1$ and $V_2^1 \subseteq U_2^1$.

Inductively, if e_i^n is the identity of R_i^n, then e_i^n is the characteristic function of some clopen subset V_i^n of $\beta X - X$, with $V_i^n \subseteq U_i^n$. Since $e_i^n = e_{2i-1}^{n+1} + e_{2i}^{n+1}$, it follows that U_i^n can be partitioned into non-empty clopen subsets U_{2i-1}^{n+1} and U_{2i}^{n+1}, where $V_{2i-1}^{n+1} \subseteq U_{2i-1}^{n+1}$ and $V_{2i}^{n+1} \subseteq U_{2i}^{n+1}$.

Now for each $n \in N$, set $K_n = \bigcup \{U_i^n \mid 1 \le i \le 2^n,\ i \text{ odd}\}$ and $K_n' = \bigcup \{U_i^n \mid 1 \le i \le 2^n,\ i \text{ even}\}$. Then each K_n and K_n' is open and compact. For each $n \in N$, let h_n be the characteristic function for K_n. Each h_n is continuous and we define a mapping h of $\beta X - X$ into the countable product $\times_{n \in N} \{0, 1\}$ of the discrete two-point space $\{0, 1\}$ by $h(x) = (h_n(x))$. Evidently, h is continuous.

Take $a = (a_n) \in \times_{n \in N} \{0, 1\}$. For each $n \in N$, define A_n by $A_n = K_n$ if $a_n = 1$ and $A_n = K_n'$ if $a_n = 0$. It follows that $\bigcap \{A_k \mid k = 1, \ldots, n\} = U_j^n$, for some j. Thus, for each n, $\bigcap \{A_k \mid k = 1, \ldots, n\} \ne \emptyset$. Since each A_n is compact, select $x \in \bigcap \{A_n \mid n \in N\}$. Now $h(x) = a$ so that h is a surjection. But $\times_{n \in N} \{0, 1\}$ is a homeomorph of the Cantor set $\mathscr{C}$ and each compact metric space is a continuous image of $\mathscr{C}$. Thus, it follows from Magill's Theorem [8] that all compact metric spaces are remainders of X.

Conversely, suppose that all compact metric spaces are remainders of X. Then $\mathscr{C}$ is a remainder and there is a continuous mapping t of $\beta X - X$ onto $\mathscr{C}$. Let R be the ring $C^*(\mathscr{C})$. If e is a non-zero idempotent in R, then e is the characteristic function of a clopen subset A of $\mathscr{C}$. Let B be a subset of $\mathscr{C}$ such that $A \cap B$ and $A - B$ are non-empty and open in $\mathscr{C}$. If g is the characteristic function of $A \cap B$ and if h is the characteristic function of $A - B$, then e is the sum of the orthogonal idempotents g and h. Hence e is not primitive.

Now t induces an isomorphism t' of R into $C^*(\beta X - X)$. (See 10.3A of [4].) It follows that $C^*(X)/C_\infty(X)$ contains a subring with no primitive idempotents.

This completes the proof.

3. **Examples.** (A) Consider any space X which is the (topological) free union of a locally compact space Y and an infinite discrete space D. Let S be the subring of $C^*(X)$ consisting of all functions f which map Y onto 0 and which are integer-valued on D. Then $R=\{f+C_\infty(X) \mid f\in S\}$ is a subring of $C^*(X)/C_\infty(X)$ which contains no primitive idempotents. Hence all compact metric spaces are remainders of X.

(B) Let X be the subset of the plane consisting of all points (x, y), where $-1\le x,\ y\le 1$, and with the set $S=\{(1/(n+1),\ 0) \mid n\in N\}\cup\{(0,0)\}$ deleted. Set $R=C^*(X)/C_\infty(X)$. To apply the theorem it suffices to consider subrings T of R where T contains the identity 1 in R. Suppose T contains no primitive idempotents. Now 1 is an idempotent, so that $1\equiv f+g(\text{mod } C_\infty(X))$, where f and g are non-trivial idempotents in T. Then $f^2\equiv f$ and $g^2\equiv g$, but $f, g\notin C_\infty(X)$.

Take $0<\varepsilon<\frac{1}{4}$. Then there is a compact subset K of X such that on $X-K$ we have $|1-(f+g)|<\varepsilon$, $|f^2-f|<\varepsilon/2$ and $|g^2-g|<\varepsilon/2$. From this it follows that either $|1-f(x)|<\varepsilon$ or $|f(x)|<\varepsilon$, for each $x\in X-K$, and a similar result holds for g. Moreover, the sets $A=\{x\in X-K \mid |f(x)|<\varepsilon\}$ and $B=\{x\in X-K \mid |g(x)|<\varepsilon\}$ are disjoint, non-empty, and partition $X-K$.

Since S is bounded away from K, there exists an open disc D_1 centered at $(0,0)$ which does not meet K and whose boundary (in the plane) contains no point of S. Let $P=(1/(m+1),\ 0)$ satisfy $1/(m+1)<\text{radius } D_1$ and $1/m>\text{radius } D_1$. Let $D_2,\ldots,D_m$ be pair-wise disjoint, open discs in $X-K$ centered at the points $(1/(n+1),0)$ of S with $n=1,\ldots,m-1$, and where for $n\ge 2$ each D_n is disjoint from D_1.

Now D_1 is connected so that both A and B cannot meet D_1. Suppose B does not meet D_1. Then $|f(x)|<\varepsilon$, for all $x\in D_1$. But $f\notin C_\infty(X)$ hence B covers at least one D_n, $n\ge 2$, so that $|1-f(x)|<\varepsilon$ on all such D_n. Since f is not primitive, $f\equiv h+k$, where h and k are non-trivial idempotents in T. Now f is close to zero at all points of D_1 so that h and k can be close to 1 only near those points $(1/(n+1),0)$, $n<m$, near which f is close to 1. Hence there are discs centered at a subset of the points $(1/(n+1),0)$, $n<m$, on each of which h and k are either close to zero or close to 1. Moreover, whenever h is close to 1 on such discs, then k is close to zero and vice-versa. Since $h\notin C_\infty(X)$, there is at least one such disc on which h is close to 1 and a similar result holds for k. Since the set of points $(1/(n+1),0)$, $n<m$, is finite, this process cannot continue indefinitely, which is a contradiction.

Thus, every subring of R contains a primitive idempotent so that not all metric spaces are remainders of X. (In particular, $\mathscr{C}$ is not.) However, we note that X has a countably infinite remainder so that $\beta X-X$ contains infinitely many components (see [7]) and since X is not pseudocompact, all Peano spaces are remainders of X (see [9]).

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