THE α-DIMENSIONAL MEASURE OF THE GRAPH AND SET OF ZEROS OF A BROWNIAN PATH

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In a recent joint paper (1) with Prof. Besicovitch we announced the conjecture that for almost all one-dimensional Brownian paths, the set of zeros has dimensional number $\frac{1}{2}$, and zero $\Lambda^{\frac{1}{2}}$-measure. It is the purpose of this paper to give a proof of this result. In doing so we consider the graph $C(\omega)$ of a Brownian path ω as a point set in the plane, and prove that, with probability 1, $C(\omega)$ has dimensional number $\frac{3}{2}$ and zero $\Lambda^{\frac{3}{2}}$-measure.

In obtaining the value of the dimensional number of the sets considered, the upper bound is easier to find than the lower bound. For both the sets considered we find the lower bound by using the concept of capacity. The first section will contain a definition of capacity, and a summary of the results in this subject which will be needed.

Ω will denote the space of one-dimensional paths ω. A probability measure μ is defined in Ω. For notation and definitions relevant to this space see (11).

1. *The α-capacity of a bounded closed set.* Let E be a bounded closed set in a Euclidean space $\mathfrak{X}$. Let m be a countably additive measure defined for all Borel sets in $\mathfrak{X}$, such that

$$m(\mathfrak{X}) = m(E) = 1.$$

For $\alpha > 0$, let

$$I_\alpha(m) = \int_{x\in\mathfrak{X}} dm(x) \int_{y\in\mathfrak{X}} \frac{dm(y)}{|x-y|^\alpha},$$

where $|x-y|$ denotes the distance from x to y. $I_\alpha(m)$ is known as the integral of energy. Let

$$W_\alpha(E) = \inf I_\alpha(m),$$

where the infimum is taken for all measures m satisfying the above conditions. The α-capacity of E, denoted by $C_\alpha(E)$, is defined by

$$\begin{aligned} C_\alpha(E) &= [W_\alpha(E)]^{-1/\alpha} \quad \text{if } W_\alpha(E) \text{ is finite,} \\ &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

Clearly $C_\alpha(E)$ decreases as α increases. The capacity dimension of E, denoted by C-dim (E) may be defined by

$$\text{C-dim}\,(E) = 0 \quad \text{if} \quad C_\alpha(E) = 0 \text{ for all } \alpha > 0;$$

$$\text{C-dim}\,(E) = s > 0 \quad \text{if} \quad \begin{cases} C_\alpha(E) = 0, & \alpha > s, \\ C_\alpha(E) > 0, & 0 < \alpha < s. \end{cases}$$

We shall use the following

THEOREM A. *For a bounded closed set E in a Euclidean space, the capacity dimension is equal to the dimensional number.*

This theorem follows immediately from Theorems 11 and 13 of (5).

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For the two sets considered we shall need two distinct methods of obtaining the capacity.

(i) If the set E is the continuous image of a closed interval $[a, b]$ of real numbers, it may be written
$$E = \{x(t), a \leqslant t \leqslant b\}.$$

THEOREM B. *If $x(t)$ is a continuous function of the real variable t taking values in a Euclidean space $\mathfrak{X}$,*
$$E = \{x(t), a \leqslant t \leqslant b\},$$
and
$$\int_a^b \int_a^b \frac{ds\,dt}{|x(t)-x(s)|^{\alpha}} < \infty;$$
then $C_\alpha(E)$ is positive.

This theorem follows immediately from the definition, since the image of Lebesgue measure in $[a, b]$ under the transformation $x(t)$ is a measure in $\mathfrak{X}$ with
$$m(E) = m(\mathfrak{X}) = b - a.$$

(ii) The idea of transfinite diameter (see, for example, (3)) may be generalized for any closed set E as follows. Let
$$D_n^{(\alpha)} = \inf \frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} |x_i - x_j|^{-\alpha}, \tag{1}$$
where the infimum extends over all n-systems of points $x_1, x_2, \ldots, x_n$ in E.

THEOREM C. *For a bounded closed set E with $D_n^{(\alpha)}$ defined by* (1), $D_\alpha(E) = \lim_{n\to\infty} D_n^{(\alpha)}$ *exists and satisfies $D_\alpha(E) = W_\alpha(E)$.*

This result is obtained on p. 47 of (4).

It follows from Theorem C, that if $D_\alpha(E)$ is finite, then $C_\alpha(E)$ is positive; and it is this result that we shall use in the sequel.

2. *The dimensional number of the set of roots.* Given $\omega \in \Omega$, the set of values of t for which $x(t, \omega) = 0$ will be denoted by $R(\omega)$. For $0 \leqslant t_1 \leqslant t_2 \leqslant \infty$, the part of this set for which $t_1 \leqslant t \leqslant t_2$ will be denoted by $R(t_1, t_2, \omega)$. It is known (see, for example, section 7 of (6)), that for almost all ω of Ω, $R(t_1, t_2, \omega)$ is a perfect linear set.

With probability 1, paths ω are completely determined by the values of $x(t, \omega)$ at an enumerable everywhere-dense set. For $\alpha > 0$, $\Lambda^\alpha R(t_1, t_2, \omega)$ is a measurable function of the end-points of the open intervals complementary to $R(\omega)$. Thus $\Lambda^\alpha R(t_1, t_2, \omega)$ is a measurable function of ω with respect to measure μ. As a start we prove a result of the 'zero-one' type for this function.

LEMMA 1. *If $\alpha > 0$ and $E = \{\omega: \Lambda^\alpha R(0, 1, \omega) > 0\}$, then $\mu(E) = 0$ or 1.*

Since a change of scale in which t is replaced by $\lambda^2 t$ and $x(t, \omega)$ by $\lambda x(t, \omega)$ does not have any effect on measure in Ω, it follows that, for any $\tau > 0$, $\mu(E) = \mu(E_\tau)$, where $E_\tau = \{\omega: \Lambda^\alpha R(0, \tau, \omega) > 0\}$; and, for $0 < \tau_1 < \tau_2$, if it is known that $R(\tau_1, \tau_2, \omega)$ is non-empty, the conditional probability of the event $\{\omega: \Lambda^\alpha R(\tau_1, \tau_2, \omega) > 0\}$ is again $\mu(E)$.

Suppose, if possible, that $\mu(E_\tau) = \delta$ with $0 < \delta < 1$. Let τ_1, τ_2 satisfy $0 < \tau_1 < \tau_2 < 1$. Let $E'_{\tau_1} = \{\omega: \Lambda^\alpha R(0, \tau_1, \omega) = 0\}$. Then $\mu(E'_{\tau_1}) = 1 - \delta > 0$. By the stationarity of the process, the conditional probability for $\{\omega: \Lambda^\alpha R(\tau_2, 1, \omega) > 0\}$ when it is known that there is at least one point in $R(\tau_2, 1, \omega)$ is δ. Now for $\omega \in E'_{\tau_1}$, $x(\tau_1, \omega)$ has some finite value, and the behaviour of $x(t, \omega)$ $(t \geqslant \tau_1)$ depends only on this value. Whatever the value stipulated for $x(\tau_1, \omega)$, the conditional probability that $R(\tau_2, 1, \omega)$ be non-empty is strictly positive. Hence, if

$$Q_2 = \{\omega\colon \Lambda^\alpha R(0, \tau_1, \omega) = 0;\ \Lambda^\alpha R(\tau_2, 1, \omega) > 0\},$$

Q_2 is a subset of E'_{τ_1} and $\mu(Q_2) > 0$.

Thus E_{τ_1}, Q_2 are disjoint sets such that

$$E_{\tau_1} + Q_2 \subset E, \quad \mu(E) = \mu(E_{\tau_1}) = \delta, \quad \mu(Q_2) > 0.$$

Since this is a contradiction, $\delta = 0$ or 1, and the lemma is proved.

THEOREM 1. *There exists an absolute real constant γ such that, for almost all paths $\omega \in \Omega$,*

$$\dim R(\omega) = \gamma.$$

Let γ be the exact lower bound of the values of α for which

$$\{\omega\colon \Lambda^\alpha R(0, 1, \omega) > 0\} = 0.$$

Then, by Lemma 1, if $\gamma > 0$ and $\beta < \gamma$,

$$\mu\{\omega\colon \Lambda^\beta R(0, 1, \omega) > 0\} = 1.$$

Let
$$\alpha_p = \gamma + 1/p, \quad \beta_p = \max[\tfrac{1}{2}\gamma, \gamma - 1/p] \quad (p = 1, 2, \ldots),$$
$$X_p = \{\omega\colon \Lambda^{\alpha_p} R(\omega) = 0\}, \quad Y_p = \{\omega\colon \Lambda^{\beta_p} R(\omega) > 0\}.$$

Then $\mu(X_p) = \mu(Y_p) = 1$ $(p = 1, 2, \ldots)$. Hence if $Z = \bigcap_{p=1}^{\infty} X_p Y_p$, then $\mu(Z) = 1$. Clearly, if $\omega \in Z$, the dimension of $R(\omega)$ is γ.

We now proceed to obtain the value of the absolute constant γ. It follows from Theorem 2 of (1) that $\gamma \leqslant \frac{1}{2}$. This may also be deduced from the last section of the present paper in which we prove that $\mu\{\omega\colon \Lambda^{\frac{1}{2}} R(0, 1, \omega) = 0\} = 1$. The remainder of the present section will be devoted to obtaining the opposite inequality $\gamma \geqslant \frac{1}{2}$. This result is obtained by showing that, for $0 < \alpha < \frac{1}{2}$, the α-capacity of $R(\omega)$ is positive with probability 1. For any path $\omega \in \Omega$ let

$$M(t, \omega) = \sup_{0 \leqslant \tau \leqslant t} x(\tau, \omega); \tag{2}$$

and let $S(\omega)$ be the set of values of t for which

$$M(t, \omega) = x(t, \omega).$$

Since with probability 1 $x(t, \omega)$ is continuous as a function of t, it follows that (i) $S(\omega)$ is a closed linear set, (ii) $y = M(t, \omega)$ is a monotonic increasing continuous function of t for almost all ω of Ω. Hence for almost all fixed $\omega \in \Omega$, there is an inverse function $T(y, \omega)$ defined by

$$T(y, \omega) = t \quad \text{if} \quad M(t, \omega) = y. \tag{3}$$

For almost all values of y (in the sense of Lebesgue measure), $T(y, \omega)$ is uniquely defined by ω. For fixed ω there is a countable set of values of y at which $T(y, \omega)$ has a 'jump'. However, if $\{y_i\}_{i=1,2,\ldots}$ is any preassigned countable set of values of y, there is probability 1 that at all of these values $T(y, \omega)$ is uniquely defined. Thus, for almost all ω, $\{T(y_i, \omega)\}_{i=1,2,\ldots}$ defines a countable set of points in $S(\omega)$. We need the following results due to P. Levy:

LEMMA A. *The sets $R(\omega)$ and $S(\omega)$ satisfy the same probability laws: that is, the complementary open intervals of $R(\omega)$ and $S(\omega)$ satisfy the same probability laws with regard to both position and magnitude.*

This lemma is Theorem 47·1 of (7). It will allow us to deduce results about $\Lambda^\alpha R(\omega)$ from $\Lambda^\alpha S(\omega)$.

LEMMA B. *For each value of t the functions* $|x(t,\omega)|$, $M(t,\omega)$ *and* $[M(t,\omega)-x(t,\omega)]$ *have the same probability distribution as* $y(t,\omega)$, *where, for* $k>0$,

$$\mu\{\omega\colon y(t,\omega)<k\} = 2(2\pi t)^{-\frac{1}{2}}\int_0^k \exp(-\xi^2/2t)\,d\xi.$$

This combines Theorems 42·1 and 42·4 of (7).

Since $M(t,\omega)$ is a monotonic function, we have by (3)

$$\mu\{\omega\colon T(y,\omega)<u\} = \mu\{\omega\colon M(u,\omega)>y\}$$
$$= 2(2\pi u)^{-\frac{1}{2}}\int_y^\infty \exp(-\xi^2/2u)\,d\xi,$$

by Lemma B. Differentiating with respect to u gives

$$\mu\{\omega\colon T(y,\omega)<u\} = y(2\pi)^{-\frac{1}{2}}\int_0^u \exp(-y^2/2\sigma)\,\sigma^{-\frac{3}{2}}\,d\sigma \tag{4}$$

for $y>0$, $u>0$. Suppose now that α is fixed with $0<\alpha<\frac{1}{2}$. Let

$$Q_{m,n} = \left\{\omega\colon \left[T\left(\frac{m+1}{n},\omega\right)-T\left(\frac{m}{n},\omega\right)\right]^{-\alpha} > \frac{n}{(\log n)^2}\right\} \quad (n=2,3,\ldots;\ m=0,1,2,\ldots,n-1). \tag{5}$$

By the stationarity of the Gaussian process,

$$\mu(Q_{m,n}) = \mu(Q_{0,n}) \quad (m=0,1,2,\ldots,n-1),$$
$$= \mu\{\omega\colon [T(1/n,\omega)]^{-\alpha} > n(\log n)^{-2}\}$$
$$= \mu\{\omega\colon T(1/n,\omega) < n^{-1/\alpha}(\log n)^{2/\alpha}\}$$
$$= (2\pi n^2)^{-\frac{1}{2}}\int_0^{n^{-1/\alpha}(\log n)^{2/\alpha}} \exp(-1/2\sigma n^2)\,\sigma^{-\frac{3}{2}}\,d\sigma,$$

by (4). Making the substitution $\sigma n^2 = w$, we have

$$\mu(Q_{m,n}) = (2\pi)^{-\frac{1}{2}}\int_0^{n^{2-1/\alpha}(\log n)^{2/\alpha}} \exp(-1/2w)\,w^{-\frac{3}{2}}\,dw. \tag{6}$$

The integrand in (6) is an increasing function of w in the range $(0,\frac{1}{3})$; and there exists an integer n_0 such that, for $n>n_0$,

$$n^{2-1/\alpha}(\log n)^{2/\alpha} < \tfrac{1}{3}.$$

Hence, for $n>n_0$, there is a positive constant k_1 with

$$\mu(Q_{m,n}) < k_1 n^{-1+1/2\alpha}(\log n)^{-3/\alpha}\exp\{-\tfrac{1}{2}n^{1/\alpha-2}(\log n)^{-2/\alpha}\}. \tag{7}$$

Let
$$E_p = \bigcup_{m=0}^{n-1} Q_{m,n}, \quad n=2^p \quad (p=1,2,\ldots). \tag{8}$$

By (7) we have
$$\mu(E_p) < k_2\,.\,2^{p/2\alpha}p^{-3/\alpha}\exp\{-k_3\,.\,2^{p(1/\alpha-2)}p^{-2/\alpha}\} \tag{9}$$

for $p>p_0$, where k_2, k_3 are positive constants. We can now prove

THEOREM 2. *For* $0<\alpha<\frac{1}{2}$, *and almost all* ω *of* Ω, *the set* $S(\omega)$ *has positive* α-*capacity.*

Consider a fixed ω of Ω. By (9) above, $\sum_{p=p_0}^{\infty}\mu(E_p)$ converges. Hence, by the Borel-

Cantelli lemma, there exists, with probability 1, an integer p_1 such that ω is in the complement of E_p for $p > p_1$. Thus

$$\left[T\left(\frac{m+1}{n}, \omega\right) - T\left(\frac{m}{n}, \omega\right)\right]^{-\alpha} < n(\log n)^{-2} \tag{10}$$

for $m = 0, 1, 2, \ldots, n-1$; $n = 2^p$, $p > p_1$. Let

$$\lambda_1 = \max_{\substack{0 \leqslant m \leqslant n-1 \\ n = 2^{p_1}}} \left[T\left(\frac{m+1}{n}, \omega\right) - T\left(\frac{m}{n}, \omega\right)\right]^{-\alpha}. \tag{11}$$

For $p > p_1$, write

$$t_{m,p} = T(m/n, \omega) \quad (n = 2^p;\ m = 0, 1, \ldots, n-1). \tag{12}$$

With probability 1, each of the numbers $t_{m,p}$ is uniquely defined and lies in $S(\omega)$. Now any interval of the form $(i.2^{-p}, j.2^{-p})$, where i, j, p are non-negative integers, contains at least one interval of the form $(k.2^{-q}, (k+1).2^{-q})$ with

$$q = p + 2 - [\log_2 (j-i)].\dagger$$

Further, for $s = 0, 1, 2, \ldots$, there are not more than 2^{s+p} combinations of integers i, j such that

$$[\log_2 (j-i)] = s \quad (0 \leqslant i < j \leqslant n-1;\ n = 2^p).$$

Thus, if

$$\phi_{s,p} = \frac{2}{n(n-1)} \sum_{\substack{0 \leqslant i < j \leqslant n-1 \\ [\log_2 (j-i)] = s}} |t_{i,p} - t_{j,p}|^{-\alpha}, \tag{13}$$

we have, for $q = p + 2 - s > p_1$, by (10),

$$\phi_{s,p} < \frac{2^{s+p}.2^q}{(q \log 2)^2} \frac{2}{n(n-1)}.$$

That is,

$$\phi_{s,p} < k_4 q^{-2}, \tag{14}$$

for some positive constant k_4. Also, from (13) and (1),

$$D_n^{(\alpha)} \leqslant \sum_{s=0}^{p} \phi_{s,p}. \tag{15}$$

By (11), for $q = p + 2 - s \leqslant p_1$, the terms in (13) making up $\phi_{s,p}$ are each less than $\dfrac{\lambda_1}{n(n-1)}$; and so

$$\sum_{s=p-p_1+2}^{p} \phi_{s,p} < \lambda_1.$$

By (14) we have

$$\sum_{s=0}^{p-p_1+1} \phi_{s,p} < k_4 \sum_{q=p_1+1}^{p+2} q^{-2}$$

$$< k_4 \sum_{q=1}^{\infty} q^{-2}$$

$$= \lambda_2, \quad \text{a finite positive constant.}$$

Thus, by (15)

$$D_n^{(\alpha)} < \lambda_1 + \lambda_2,$$

for $n = 2^p$, $p > p_1$. Hence by Theorem C, the capacity of the part of $S(\omega)$ between 0 and $T(1, \omega)$ is positive. This completes the proof of Theorem 2.

† Here $[x]$ denotes the integer part of x.

Corollary. *The absolute constant γ defined by Theorem 1 satisfies $\gamma \geqslant \frac{1}{2}$.*

This follows immediately from the above theorem, Lemma A and Theorem A.

3. *The graph of a Brownian path.* For fixed $\omega \in \Omega$, the set of values of $X(t, \omega)$ for $0 \leqslant t \leqslant 1$ is clearly a linear closed interval. The end-points of this interval satisfy known probability laws. However, each interior point of the interval has a multiplicity $2^{\aleph_0}$, the power of the continuum. Thus we need some device to examine the set of values of $x(t, \omega)$.

For ω in Ω, let $C(\omega)$ be the set of points $(t, x(t, \omega))$ $(t \geqslant 0)$. Then, with probability 1, $C(\omega)$ is a continuous curve in the plane passing through $(0, 0)$ and it is non-rectifiable. We now think of $C(\omega)$ as a set of points in 2-space defined by the stochastic process ω. If $0 \leqslant t_1 \leqslant t_2$ we denote by $C(t_1, t_2, \omega)$ the set of points $(t, x(t, \omega))$ $(t_1 \leqslant t \leqslant t_2)$.

Theorem 3. *For almost all ω of Ω, the curve $C(\omega)$ has dimensional number $\frac{3}{2}$.*

For any $\lambda < \frac{1}{2}$, there is probability 1 that the function $x(t, \omega)$ satisfies a uniform Lipschitz condition of order λ in the range $0 \leqslant t \leqslant 1$ (this is proved as Theorem XLVII in (10)). Hence, by the main theorem of (2), $\dim C(0, 1, \omega) \leqslant 2 - \lambda$, with probability 1. It follows that, for amost all ω of Ω,

$$\dim \mathrm{C}(\omega) \leqslant \tfrac{3}{2}.$$

To obtain the opposite inequality we again consider the capacity of the set $C(0, 1, \omega)$. Put

$$r(t, \omega) = [|x(t, \omega)|^2 + t^2]^{\frac{1}{2}}; \tag{16}$$

$$f(t, r) = \mu\{\omega : r(t, \omega) < r\}. \tag{17}$$

Then, if $r \geqslant t > 0$,

$$f(t, r) = \mu\{\omega : |x(t, \omega)|^2 < r^2 - t^2\}$$
$$= 2(2\pi t)^{-\frac{1}{2}} \int_0^{(r^2 - t^2)^{\frac{1}{2}}} \exp(-u^2/2t)\, du;$$

while $f(t, r) = 0$ if $0 \leqslant r \leqslant t$. Thus

$$\frac{\partial f(t, r)}{\partial r} = 2(2\pi t)^{-\frac{1}{2}} r (r^2 - t^2)^{-\frac{1}{2}} \exp\{-(r^2 - t^2)/2t\} \quad \text{if} \quad r > t > 0;$$
$$= 0 \quad \text{if} \quad 0 \leqslant r < t.$$

For $0 < \alpha < \frac{3}{2}$, put

$$P_\alpha(t) = \int_\Omega \frac{d\omega}{|r(t, \omega)|^\alpha} = \int_0^\infty r^{-\alpha} \frac{\partial f}{\partial r}\, dr, \quad \text{by (17)}.$$

Thus

$$P_\alpha(t) = 2(2\pi t)^{-\frac{1}{2}} \int_t^\infty r^{1-\alpha} (r^2 - t^2)^{-\frac{1}{2}} \exp\{-(r^2 - t^2)/2t\}\, dr$$
$$= 2(2\pi)^{-\frac{1}{2}} t^{\frac{1}{2} - \frac{1}{2}\alpha} \int_0^\infty x^{\frac{1}{2}} (x^2 + t)^{-\frac{1}{2}\alpha} \exp(-\tfrac{1}{2} x^2)\, dx,$$

by the change of variable $r = [t(x^2 + t)]^{\frac{1}{2}}$. Now $x^2 + t \geqslant t$ for all x, and so

$$P_\alpha(t) \leqslant 2(2\pi)^{-\frac{1}{2}} t^{\frac{1}{2} - \alpha} \int_0^\infty x^{\frac{1}{2}} \exp(-x^2/2)\, dx.$$

That is,

$$P_\alpha(t) \leqslant k_5 t^{\frac{1}{2} - \alpha}, \tag{18}$$

where k_5 is a finite positive constant. Let $\psi(s, t)$ be the distance from $C(s, s, \omega)$ to $C(t, t, \omega)$; that is,

$$\psi(s, t) = \{|x(t, \omega) - x(s, \omega)|^2 + |t - s|^2\}^{\frac{1}{2}}.$$

Now, by Fubini's theorem

$$\int_\Omega d\omega \int_0^1\int_0^1 \frac{ds\,dt}{\{\psi(s,t)\}^\alpha} = \int_0^1\int_0^1 ds\,dt \int_\Omega \frac{d\omega}{\{\psi(s,t)\}^\alpha} \leqslant k_5 \int_0^1\int_0^1 \frac{ds\,dt}{|t-s|^{\alpha-\frac{1}{2}}}$$

by (18). This is finite for $\alpha < \frac{3}{2}$. Hence for almost all ω of Ω, $\int_0^1\int_0^1 \frac{ds\,dt}{\{\psi(s,t)\}^\alpha}$ is finite; and, by Theorem B, the α-capacity of $C(0,1,\omega)$ is positive. It follows by Theorem A that $\dim C(0,1,\omega) \geqslant \alpha$. Since α can be arbitrarily near $\frac{3}{2}$, we have

$$\dim C(\omega) \geqslant \tfrac{3}{2},$$

with probability 1. This completes the proof of the theorem.

To obtain the probable value of $\Lambda^{\frac{3}{2}} C(\omega)$ we need

LEMMA C. *Given $\eta > 0$, and the space Ω of paths ω, there exists an increasing sequence of integers $k_1, k_2, \ldots$ such that if*

$$D_r = \{\omega\colon \sup_{0\leqslant k\leqslant 1/k_r} |x(t,\omega)| < \eta k_r^{-\frac{1}{2}}\},$$

then

(i) $\mu(D_r) = p > 0 \quad (r = 1, 2, \ldots)$,

(ii) $\mu(D_{r+1}/D_1' D_2' \ldots D_r') > \frac{1}{2}p$,

(iii) *for any positive integer N*, $\mu\left(\bigcup_{r=N}^{\infty} D_r\right) = 1$.

This is a restatement of Lemmas 5, 6 and 7 of (11).

THEOREM 4. *For almost all Brownian paths ω in Ω,*

$$\Lambda^{\frac{3}{2}} C(\omega) = 0.$$

Proof. Let ω be a fixed element of Ω. If δ is any small positive number, put $\eta = \frac{1}{12}\delta$ and apply Lemma C. Let p_1 be the smallest of the integers $k_1, k_2, \ldots$ such that

$$p_1 > \eta^{-2} \tag{19}$$

and $\omega \in D_{p_1}$: p_1 exists with probability 1. The rectangle with vertices $(0, \pm\eta p_1^{-\frac{1}{2}})$, $(1/p_1, \pm\eta p_1^{-\frac{1}{2}})$, will contain all of $C(0, 1/p_1, \omega)$. This rectangle can be covered by n_1 squares of side $1/p_1$, where $n_1 \leqslant 2\eta p_1^{\frac{1}{2}} + 1 \leqslant 3\eta p_1^{\frac{1}{2}}$, by (19). Each of these squares is contained in a circle of diameter $\sqrt{2}/p_1$. Hence there is a set S_1 of n_1 circles, each of diameter $\sqrt{2}/p_1$ which covers $C(0, 1/p_1, \omega)$; thus

$$\sum_{S_1} d^{\frac{3}{2}} = n_1(\sqrt{2}/p_1)^{\frac{3}{2}} < 6\eta(1/p_1) = \tfrac{1}{2}\delta(1/p_1).$$

Now repeat the argument for $C(1/p_1, \infty, \omega)$; this gives an integer p_2 such that $C(1/p_1, 1/p_1 + 1/p_2, \omega)$ is contained in a set S_2 of n_2 circles each of diameter $\sqrt{2}/p_2$, and $n_2 < 3\eta p_2^{\frac{1}{2}}$. As each step can be carried out with probability 1, it is possible to obtain a sequence $0 = t_0 < t_1 < t_2 \ldots$ such that $C(t_{r-1}, t_r, \omega)$ is contained in a set S_r of circles with

$$\sum_{S_r} d^{\frac{3}{2}} < \tfrac{1}{2}\delta(t_r - t_{r-1}) \quad (r = 1, 2, \ldots). \tag{20}$$

Now, by (iii) of Lemma C, there exists an integer M such that

$$\mu\{\omega\colon p_i < M\} > \tfrac{1}{2} \quad (i = 1, 2, \ldots).$$

These events are independent; hence, by the Borel-Cantelli lemma, the sequence $p_1, p_2, \ldots$ will, with probability 1, contain an infinite number of terms less than M; and so the sequence $t_1, t_2, \ldots$ cannot have a limit point before 2. Since $t_r - t_{r-1} \leqslant 1$, there must exist an integer m with $1 \leqslant t_m < 2$. The set $V = \bigcup_{r=1}^{m} S_r$ of circles then covers $C(0, t_m, \omega)$ and therefore $C(0, 1, \omega)$. Further, by (20),

$$\sum_V d^{\frac{3}{2}} < \sum_{r=1}^{m} \tfrac{1}{2}\delta(t_r - t_{r-1}) < \delta.$$

Since δ is arbitrary, $\Lambda^{\frac{3}{2}} C(0, 1, \omega) = 0$ with probability 1. Hence $\Lambda^{\frac{3}{2}} C(\omega) = 0$ for almost all ω of Ω.

4. *The $\Lambda^{\frac{1}{2}}$-measure of $R(\omega)$.* Let $R_\lambda(0, 1, \omega)$ be the set of values of t for which $x(t, \omega) = \lambda$, $0 \leqslant t \leqslant 1$. Clearly it is a linear set given by the intersection of $C(0, 1, \omega)$ with $y = \lambda$. We need the following

Lemma 2. *Let $y = f(x)$ be a single-valued continuous function of x defined for $0 \leqslant x \leqslant 1$, and let E_y be the set $f^{-1}(y)$, $0 \leqslant x \leqslant 1$. Then, if $\alpha > 0$, $\eta > 0$, $\Lambda_\eta^\alpha E_y$ is an upper semi-continuous function of y.*

Proof. We require to show that, for any y_0,

$$\Lambda_\eta^\alpha E_{y_0} \geqslant \limsup_{y \to y_0} \Lambda_\eta^\alpha E_y.$$

Suppose $\Lambda_\eta^\alpha E_{y_0} = p$; then $0 \leqslant p < \infty$. For any $\epsilon > 0$, let $Q = U(E_{y_0}, \eta)$ be a covering of E_{y_0} by intervals of length not greater than η such that

$$\sum_Q d^\alpha < \Lambda_\eta^\alpha E_{y_0} + \epsilon.$$

Since $f(x)$ is continuous, E_{y_0} is closed, and we may assume that Q consists of a finite number of open intervals. The plane set T of infinite strips with base Q', the complement of Q, is closed. Also the plane set C of points $(x, f(x))$ $(0 \leqslant x \leqslant 1)$ is closed. Hence the set $T \cap C$ and the line $y = y_0$, are two closed sets with no common points. These sets must be at a positive distance δ apart. Thus, for $|y - y_0| < \delta$, E_y is covered by a set of open intervals congruent to Q. Hence

$$\Lambda_\eta^\alpha E_y \leqslant \sum_Q d^\alpha \quad \text{for} \quad |y - y_0| < \delta;$$

and so

$$\Lambda_\eta^\alpha E_{y_0} \geqslant \limsup_{y \to y_0} \Lambda_\eta^\alpha E_y - \epsilon.$$

Since ϵ is arbitrary, the lemma is proved.

The above lemma shows that $\Lambda^\alpha E_y = \lim_{\eta \to 0} \Lambda_\eta^\alpha E_y$ is a measurable function of y. It is this fact that we need in the sequel.

Theorem 5. *For almost all Brownian paths ω in Ω,*

$$\Lambda^{\frac{1}{2}} R(\omega) = 0.$$

Proof. Suppose the theorem is false. Then Lemma 1 shows that, for all $\tau > 0$,

$$\mu\{\omega: \Lambda^{\frac{1}{2}} R(0, \tau, \omega) > 0\} = 1.$$

Let δ be a small positive number, and Ω^* the subset of Ω for which

$$\sup_{0 \leqslant t \leqslant \frac{1}{2}} x(t, \omega) > 3\delta \quad \text{and} \quad \inf_{0 \leqslant t \leqslant \frac{1}{2}} x(t, \omega) < -3\delta.$$

As $\delta \to 0$, $\mu(\Omega^*) \to 1$. Define a probability measure μ^* in Ω^* by

$$\mu^*(E) = \frac{\mu(E)}{\mu(\Omega^*)} \quad \text{when} \quad E \subset \Omega^*.$$

Then
$$\mu^*\{\omega: \Lambda^{\frac{1}{2}} R(0, \tau, \omega) > 0\} = 1.$$

Define an enumerable sequence $\{\lambda_i\}$ as follows. Let ξ be any positive number such that ξ/δ is irrational. For $n = 1, 2, \ldots$, let λ_n be the unique number such that $|\lambda_n| < 2\delta$, and $|n\xi - \lambda_n| = 4q\delta$ for some integer q. Then the sequence $\{\lambda_i\}_{i=1,2,\ldots}$ is equidistributed on $(-2\delta, 2\delta)$ in the following sense. If E is any set contained in $[-\delta, \delta]$, define

$$\begin{aligned}\phi_i(z, E) &= 1 \quad \text{if} \quad \lambda_i + z \in E, \\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Then, for almost all z in the range $|z| < \delta$ (in the Lebesgue sense),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \phi_i(z, E) = \frac{|E|}{4\delta}. \tag{21}$$

(This is an immediate application of the famous result of Weyl, see pp. 315, 316 of (12).) For a fixed z satisfying $|z| < \delta$, put

$$y_i = \lambda_i + z \quad (i = 1, 2, \ldots).$$

Now suppose ω is in Ω^*. Then

$$\mu^*\{\omega: \Lambda^{\frac{1}{2}} R_{y_1}(0, 1, \omega) > 0\} = 1,$$

for there is a first point in $R_{y_1}(0, 1, \omega)$, and behaviour to the right of this point is independent of behaviour before. Further, no matter what is known about $\Lambda^{\frac{1}{2}} R_{y_1}(0, 1, \omega)$ for $i = 1, 2, \ldots, m-1$, the conditional probability for

$$\{\omega: \Lambda^{\frac{1}{2}} R_{y_m}(0, 1, \omega) > 0\}$$

must still be 1 since $\min_{1 \leqslant i \leqslant m-1} |y_i - y_m| > 0$. Thus, for almost all ω in Ω^*,

$$\Lambda^{\frac{1}{2}} R_{y_i}(0, 1, \omega) > 0 \quad \text{for} \quad i = 1, 2, \ldots. \tag{22}$$

Let $E(\omega)$ be the set of values of y for which

$$\Lambda^{\frac{1}{2}} R_y(0, 1, \omega) = 0 \quad \text{and} \quad |y| \leqslant \delta.$$

By Lemma 2, for almost all ω of Ω^*, $E(\omega)$ is a G_δ set and therefore measurable. Hence, by (21), for almost all z such that $|z| < \delta$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \phi_i(z, E(\omega)) = \frac{|E(\omega)|}{4\delta}. \tag{23}$$

Suppose, for the moment, that with probability $q > 0$

$$|E(\omega)| > 0.$$

Define a product measure in $\Omega^* \times Z$, where Z is the set of real numbers $|z| \leqslant \delta$, as the product of the measure μ^* in Ω^* and Lebesgue measure in Z. Then, by (23), for almost all $z \in Z$, there is probability q that at least one of the sets $R_{z+\lambda_i}(0, 1, \omega)$ $(i = 1, 2, \ldots)$ has zero $\Lambda^{\frac{1}{2}}$-measure. This contradicts (22), which states that for any $z \in Z$ the probability of this event is zero. Hence we must have, with probability 1,

$$|E(\omega)| = 0.$$

There must exist $p > 0$ such that

$$E(p, \omega) = \{y \colon |y| \leqslant \delta, \Lambda^{\frac{1}{2}} R_y(0, 1, \ldots) \geqslant p\}$$

satisfies
$$\mu^*\{\omega \colon |E(p, \omega)| > \delta\} > \tfrac{1}{2}. \tag{24}$$

The main theorem of (9) now shows that, if $\omega \in \Omega^*$ is such that $|E(p, \omega)| > \delta$, there exists a constant $k > 0$ with
$$\Lambda^{\frac{3}{2}} C(0, 1, \omega) > kp\delta.$$

Thus, by (24),
$$\mu^*\{\omega \colon \Lambda^{\frac{3}{2}} C(0, 1, \omega) \geqslant kp\delta\} > \tfrac{1}{2};$$

and so, in Ω,
$$\mu\{\omega \colon \Lambda^{\frac{3}{2}} C(0, 1, \omega) \geqslant kp\delta\} > \tfrac{1}{2}\mu(\Omega^*) > 0.$$

Since $kp\delta > 0$, this contradicts Theorem 4. Our original assumption is contradicted and the theorem is proved.

Remark. In the present paper we have restricted ourselves to considering measure with respect to the measure functions x^α, $a > 0$. More general Hausdorff measure functions might be considered; this was done by Lévy in (8) for Brownian paths in n-space. He succeeds in obtaining an improved upper bound for the measure of sets considered, but it seems more difficult to show that the result is 'best possible'. In the present case, analogous improved upper bounds can be obtained, but I omit to state the results as I have been unable to prove them 'best possible'.

REFERENCES

(1) Besicovitch, A. S. and Taylor, S. J. On the complementary intervals of a linear closed set of zero Lebesgue measure. *J. Lond. math. Soc.* 29 (1954), 449–59.

(2) Besicovitch, A. S. and Ursell, H. D. Sets of fractional dimensions, V: On dimensional numbers of some continuous curves. *J. Lond. math. Soc.* 12 (1937), 18–25.

(3) Fekete, M. Über den transfiniten Durchmesser ebener Punktmengen. I. *Math. Z.* 32 (1930), 108–14.

(4) Frostman, O. Potentiel d'équilibre et capacité des ensembles, avec quelques applications à la théorie des fonctions. *Medd. Lunds Univ. mat. Semin.* 3 (1935).

(5) Kametani, S. On Hausdorff's measures and generalised capacities with some of their applications to the theory of functions. *Jap. J. Math.* 19 (1946), 217–57.

(6) Lévy, P. Sur certains processus stochastiques homogènes. *Compos. math.* 7 (1940), 283–339.

(7) Lévy, P. *Processus stochastiques et mouvements browniens* (Paris, 1948).

(8) Lévy, P. La mesure de Hausdorff de la courbe du mouvement brownien. *G. Inst. ital. Attuari*, 16 (1953), 1–37.

(9) Marstrand, J. M. The dimension of Cartesian product sets. *Proc. Camb. phil. Soc.* 50 (1954), 198–202.

(10) Paley, R. E. A. C. and Wiener, N. *Fourier transforms in the complex domain* (Colloq. Publ. Amer. math. Soc. no. 19, 1934).

(11) Taylor, S. J. The Hausdorff α-dimensional measure of Brownian paths in n-space. *Proc. Camb. phil. Soc.* 49 (1953), 31–9.

(12) Weyl, H. Über die Gleichwerteilung von Zahlen mod. Eins. *Math. Ann.* 77 (1916), 313–54.

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