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Phase portraits of the quadratic polynomial Liénard differential systems

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We classify the global phase portraits in the Poincaré disc of the quadratic polynomial Liénard differential systems

$$\dot{x} = y, \quad \dot{y} = (ax + b)y + cx^2 + dx + e,$$

where $(x, y) \in \mathbb{R}^2$ are the variables and a, b, c, d, e are real parameters.

Keywords: Quadratic system; Liénard system; Poincaré compactification; Global phase portraits

2010 *Mathematics subject classification:* 37C29; 37G15

1. Introduction

A *quadratic polynomial differential system* is a system of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

where P and Q are polynomials in the variables x and y, and the maximum of the degrees of P and Q is two.

The quadratic polynomial differential systems and their applications have been studied intensively these last 30 years, see for instance the exhaustive bibliography about these systems in the books of Reyn [40] and Ye Yanqian [48]. More concretely, classes of quadratic systems that have been studied are: homogeneous (see [13, 34, 36]), bounded (see [11, 16, 19]), having a star nodal point (see [6]), chordal (see [22, 23]), with a weak focus of second or third order (see [4, 5, 29, 32]), with four infinite critical points and one invariant straight line (see [42]), Hamiltonian (see [2]), gradient (see [10]), having a focus and one antisaddle (see [3]), integrable quadratic systems using Carleman and Painlevé tools (see [**24**]), having a centre (see [**32**, **45**]), ...

On the other hand a *polynomial Liénard differential system* is a system of the form

$$\dot{x} = y, \quad \dot{y} = f(x)y + g(x),$$

where f and g are polynomials in the variable x.

The polynomial Liénard differential systems and their applications also have been analysed by many authors these recent years. Thus some authors studied their limit cycles (see for instance [**14**, **15**, **17**, **21**, **26**, **27**, **39**, **41**]), or their algebraic limit cycles (see [**28**, **31**, **37**]), or their invariant algebraic curves (see [**7**, **8**, **49**]), or their canard limit cycles (see [**43**]), or the shape of their limit cycles (see [**46**]), or the period function of their centres (see [**47**]), or their integrability (see [**9**, **30**]), or a kind of a generalized Liénard system (see [**18**]).

Roughly speaking the *Poincaré disc* $\mathbb{D}^2$ is the closed unit disc centred at the origin of coordinates of $\mathbb{R}^2$, where its interior is identified with $\mathbb{R}^2$ and its boundary $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$, i.e. in the plane we can go to or come from infinity in as many directions as points for the circle $\mathbb{S}^1$. So a polynomial differential system in $\mathbb{R}^2$ (i.e. in the interior of $\mathbb{D}^2$) can be extended analytically to the whole $\mathbb{D}^2$. In this way we can study the dynamics of the differential system in a neighbourhood of infinity. For details on the Poincaré disc see § 3, and chapter 5 of [**20**].

Up to now the phase portraits in the Poincaré disc of the quadratic polynomial Liénard differential systems have not been studied, their study is the goal of this paper. More precisely, our objective is to classify the different topological phase portraits in the Poincaré disc of the systems

$$\dot{x} = y, \quad \dot{y} = (ax + b)y + cx^2 + dx + e, \tag{1.1}$$

where $(x, y) \in \mathbb{R}^2$ are the variables and a, b, c, d, e are real parameters.

We denote by $\mathcal{X} = (y, (ax + b)y + cx^2 + dx + e)$ the vector field defined by system (1.1). We observe that since we are interested in the quadratic polynomial Liénard differential systems we must assume that the parameters satisfy $a^2 + c^2 \neq 0$ and $a^2 + b^2 \neq 0$ in order to avoid the non-quadratic systems and in order to have systems of Liénard type respectively. Moreover we need that $c^2 + d^2 + e^2 \neq 0$, otherwise $y = 0$ is a straight line filled of equilibria and the system can be reduced to a linear one.

Two phase portraits in the Poincaré disc $\mathbb{D}^2$ are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{D}^2 \to \mathbb{D}^2$ which sends orbits of one of the phase portraits into orbits of the other phase portrait, preserving or reversing the orientation of all the orbits.

Our main result is the following one.

Theorem 1. *A quadratic polynomial Liénard differential system* (1.1) *has a phase portrait in the Poincaré disc topologically equivalent to one of the phase portraits of figures* 3, 4, 7, 9, 10, 11 and 12. That is, there are 17 different topological phase portraits in the Poincaré disc for system (1.1).

Table 1. *Normal forms for system* (1.1).

	System (1.1)	Normal form
(i)	$a \neq 0$ $c \neq 0$	$\dot{x} = y$, $\dot{y} = x^2 + xy + Ax + By + C$
(ii)	$a = 0$, $bc \neq 0$	$\dot{x} = y$, $\dot{y} = x^2 + y + Dx + E$
(iii)	$ab \neq 0$, $c = 0$	$\dot{x} = y$, $\dot{y} = xy + y + Gx + H$
(iv)	$a \neq 0$, $b = c = 0$, $d > 0$	$\dot{x} = y$, $\dot{y} = xy + x + I$
(v)	$a \neq 0$, $b = c = 0$, $d < 0$	$\dot{x} = y$, $\dot{y} = xy - x + I$
(vi)	$a \neq 0$, $b = c = d = 0$, $e \neq 0$	$\dot{x} = y$, $\dot{y} = xy + 1$

In order to prove theorem 1 we will make use of the normal forms of system (1.1) in § 2, which simplify in somehow the envolved calculations. Afterthat in § 3 we analyse the local phase portraits of the infinite singular points. Finally in § 4 we study the local phase portraits of the finite singular points, and the proof of theorem 1 is given in § 4.

For studying the local phase portraits at the finite and infinite singular points of the compactified quadratic polynomial Liénard differential systems we use notations and results presented in chapters 2, 3 and 5 of [**20**]. For classifying the global phase portraits of the quadratic polynomial Liénard differential systems in the plane $\mathbb{R}^2$ extended to infinity we follow the notations and results on Poincaré disc in chapter 5 in [**20**], and with the result due to Markus [**33**], Neumann [**35**] and Peixoto [**38**], which guarantees that we only need to classify all the different configurations of separatrices of the compactified quadratic polynomial Liénard differential systems, in order to obtain their topologically different phase portraits in the Poincaré disc.

2. Normal forms

The next result will simplify the study of the phase portraits of system (1.1) in the Poincaré disc.

PROPOSITION 2. *All systems* (1.1) *are topologically equivalent to one of the normal forms* (i)–(vi) *in table* 1, *where* $A, B, C, D, E, G, H,$ and I *are parameters.*

Proof. Fixed $\alpha, \beta, \gamma \neq 0$, after the linear change of coordinate $(x, y) \mapsto (\alpha X, \beta Y)$ and the time rescaling $t \mapsto \gamma T$, system (1.1) becomes

$$\frac{\mathrm{d}X}{\mathrm{d}T} = \frac{\gamma\beta}{\alpha}Y, \quad \frac{\mathrm{d}Y}{\mathrm{d}T} = c\frac{\gamma\alpha^2}{\beta}X^2 + a\alpha\gamma XY + d\frac{\alpha\gamma}{\beta}X + b\gamma Y + e\frac{\gamma}{\beta}. \tag{2.1}$$

We study six cases separately. In each case, we assume

$$\frac{\gamma\beta}{\alpha} = 1. \tag{2.2}$$

Case (i): $ac \neq 0$. After the above change of coordinate we can take (2.2), $c\gamma\alpha^2/\beta = 1$ and $a\alpha\gamma = 1$. These conditions are satisfied if $\alpha = c/a^2$, $\beta = c^2/a^3$ and

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$\gamma = a/c$. System (2.1) becomes

$$\frac{dX}{dT} = Y, \quad \frac{dY}{dT} = X^2 + XY + AX + BY + C,$$

where $A = d/c^2$, $B = ab/c$ and $C = a^4e/c^3$.

Case (ii): $a = 0$ and $bc \neq 0$. Then we assume (2.2), $c\gamma\alpha^2/\beta = 1$ and $b\gamma = 1$, and the solution is $\alpha = b^2/c$, $\beta = b^3/c$ and $\gamma = 1/b$. System (2.1) becomes

$$\frac{dX}{dT} = Y, \quad \frac{dY}{dT} = X^2 + DX + Y + E,$$

where $D = dc/b^5$ and $E = ec/b^4$.

Case (iii): $ab \neq 0$ and $c = 0$. Then we assume (2.2), $a\alpha\gamma = 1$ and $b\gamma = 1$. System (2.1) becomes

$$\frac{dX}{dT} = Y, \quad \frac{dY}{dT} = XY + Y + GX + H,$$

where $G = d/b^2$ and $H = eb/a$.

Case (iv): $a \neq 0$, $b = c = 0$ and $d > 0$. Then we assume (2.2), $a\alpha\gamma = 1$ and $d\alpha\gamma/\beta = 1$. System (2.1) becomes

$$\frac{dX}{dT} = Y, \quad \frac{dY}{dT} = XY + X + I,$$

where $I = \pm ae/d^{3/2}$.

Case (v): $a \neq 0$, $b = c = 0$ and $d < 0$. Then we assume (2.2), $a\alpha\gamma = 1$ and $d\alpha\gamma/\beta = -1$. System (2.1) becomes

$$\frac{dX}{dT} = Y, \quad \frac{dY}{dT} = XY - X + I,$$

where $I = \pm ae/|d|^{3/2}$.

Case (vi): $a \neq 0$, $b = c = d = 0$ and $e \neq 0$. Then we assume (2.2), $a\alpha\gamma = 1$ and $e\gamma/\beta = 1$. System (2.1) becomes

$$\frac{dX}{dT} = Y, \quad \frac{dY}{dT} = XY + 1.$$

This complete the proof of the proposition. □

3. Infinite singular points

In this section we study the infinite singular points of the quadratic polynomial Liénard differential systems using the notation and results of chapter 5 in [**20**].

3.1. Infinite singular points in the local charts U_1 and V_1

From equation (5.2) in [**20**], we obtain that the expression of the Poincaré compactification $p(\mathcal{X})$ of system (1.1) in the local chart U_1 is

$$\begin{aligned} \dot{u} &= c + au + dv + buv + ev^2 - u^2v, \\ \dot{v} &= -uv^2. \end{aligned} \tag{3.1}$$

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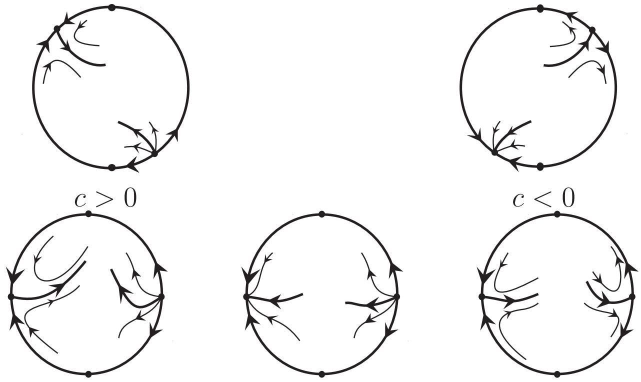

Figure 1. Local phase portraits of the infinite singular points of the chart U_1 and V_1 in the Poincaré disc.

Proposition 3. *If $a = 0$ there is no infinite singular points in U_1. If $a \neq 0$ there is a unique equilibrium point in U_1 and another unique equilibrium point at V_1 as described in figure 1.*

Proof. Taking $v = 0$ in (3.1), which correspond to the points of boundary $\mathbb{S}^1$ of the Poincaré disc, the infinite singular points are the solutions of the system

$$\dot{u} = au + c = 0, \quad \dot{v} = 0. \tag{3.2}$$

So, if $a \neq 0$ at infinity there is a unique singular point, namely $(u, 0) = (-c/a, 0)$. When $a = 0$ there are no infinite singular points in the local chart U_1 because $a^2 + c^2 \neq 0$.

At the singular point $(-c/a, 0)$ the Jacobian matrix has trace a and determinant 0. From § 1.5 in [**20**] we know that $(-c/a, 0)$ is a semi-hyperbolic singular point, and using theorem 2.19 in [**20**] we obtain that it is a saddle-node such that when $c > 0$ it has in $v > 0$ the unstable parabolic sector, and in $v < 0$ there are two hyperbolic sectors, recall that the infinity $v = 0$ is invariant. When $c < 0$ the sectors of the saddle-node interchange their localization with respect the line of the infinity.

Since the local phase portraits in the infinite singular points of the local chart V_1 of the Poincaré sphere are the symmetric phase portrait with respect to the centre of the sphere reversing the orientation of the orbits, we have that the infinite singular point $(-c/a, 0)$ in V_1 when $c > 0$ has in $v > 0$ the two hyperbolic sectors, and in $v < 0$ the parabolic sector. When $c < 0$ the sectors of the saddle-node interchange their localization with respect the line of the infinity.

When $c = 0$ the origin is the unique infinite singular point and it is a semi-hyperbolic singular point. Again, by theorem 2.19 of [**20**] we obtain that $(0, 0)$ is either a saddle if $ad < 0$, or an unstable node if $ad > 0$, or a saddle-node if $d = 0$ and $e \neq 0$. □

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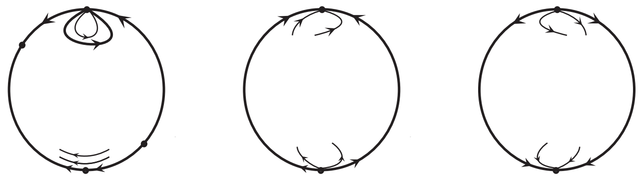

Figure 2. Local phase portraits of the infinite singular points of the origins of the chart U_2 and V_2 in the Poincaré disc.

3.2. The origin of the local charts U_2 and V_2

Once we have studied the infinite singular points in the local charts U_1 and V_1, it only remains to study if the origins of the local charts U_2 and V_2 are infinite singular points.

Again using the results stated in [20], chapter 5, we obtain the expression of the Poincaré compactification $p(\mathcal{X})$ of system (1.1) in the local chart U_2, i.e.

$$\begin{aligned} \dot{u} &= v - au^2 - buv - cu^3 - du^2v - euv^2, \\ \dot{v} &= -auv - bv^2 - cu^2v - duv^2 - ev^3. \end{aligned} \tag{3.3}$$

Clearly the origin $(u, v) = (0, 0)$ is an infinite singular point of this system.

Proposition 4. *The local phase portrait at the origin of U_2 and V_2 is described in figure 2.*

Proof. The Jacobian matrix of system (3.3) at the singular point $(0, 0)$ has trace and determinant equal to zero, but it is not the zero matrix. So according to the definitions of § 1.5 in [20] the origin of system (3.3) is a nilpotent singular point, and we can study its local phase portrait using theorem 3.5 of [20]. Doing that we get that if $a \neq 0$ then the local phase portrait at the origin is formed by two sectors one elliptic and one hyperbolic, of course separating these two sectors we can consider two parabolic sectors. When $a = 0$ we have that $c \neq 0$, and then the origin is a stable node if $c > 0$, and an unstable node if $c < 0$.

In order to know the position of the invariant straight line of the infinity $v = 0$ with respect to the elliptic and hyperbolic sectors when $a \neq 0$, we need to do the changes of variable known as blow up's, see for more details chapter 3 of [20] or [1]. Doing such changes we get that the elliptic sector is in $v > 0$ and the hyperbolic sector is in $v < 0$. Moreover there are no parabolic sectors in $v < 0$. □

In short we have completed the description of the local phase portraits at the infinite singular points.

4. Finite singular points for each normal form

In this section we study the local behaviour of the finite singular points for each normal form of system (1.1) presented in table 1. To analyse the local phase portraits

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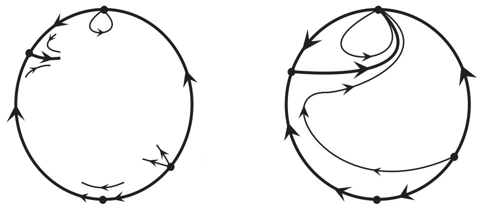
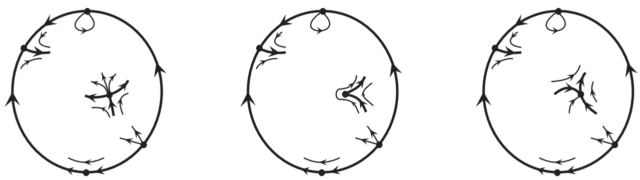


(a) Local phase portrait (b) Global phase portrait

Figure 3. Local and global phase portraits in the Poincaré disc of Case (i) when $A^2 - 4C < 0$.



$B - A/2 > 0$ (a) $B - A/2 = 0$ (b) $B - A/2 < 0$ (c)

Figure 4. Local phase portraits in the Poincaré disc of Case (i) when $A^2 - 4C = 0$.

of every finite singular point we use the results of [**20**]. Furthermore, we aim to obtain all the possible global phase portraits for each Poincaré disc with a different configuration of all local phase portraits of finite and infinite singular points. In order to get this we use the following important remark for all the six cases that we have to analyse in this section.

Remark 5. *For determining the possible global phase portraits from a local phase portrait it is sufficient to consider all the possible α- and ω-limits of the separatrices of the hyperbolic sectors of the correspondent local phase portrait.*

Case (i). The system (1.1) is

$$\dot{x} = y, \quad \dot{y} = x^2 + xy + Ax + By + C. \tag{4.1}$$

If $A^2 - 4C < 0$ this system has no finite singular points. The local phase portrait and the correspondent global phase portrait are shown in figure 3.

Assuming $A^2 - 4C \geqslant 0$ the finite singular points of system (4.1) are $p_\pm = ((-A \pm \sqrt{A^2 - 4C})/2, 0)$, and the Jacobian matrix in each singular point has determinant $\Delta_\pm = \mp\sqrt{A^2 - 4C}$ and trace $T_\pm = B + (-A \pm \sqrt{A^2 - 4C})/2$.

When $A^2 - 4C = 0$ we get $\Delta = 0$ and there is a unique singular point, that is, $p = (-A/2, 0)$ which is nilpotent if $T = B - A/2 = 0$, or semi-hyperbolic if $T \neq 0$. Applying theorem 2.19 of [**20**] in the semi-hyperbolic case we get that p is a saddle-node. For the case nilpotent we apply theorem 3.5 of [**20**] and conclude that p is a cusp. These local phase portraits are shown in figure 5. Using remark 5 the correspondent global phase portraits are shown in figure 5.

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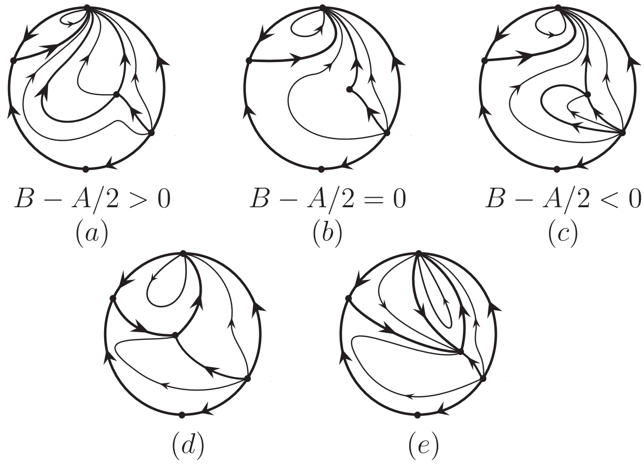

Figure 5. Global phase portraits in the Poincaré disc of Case (i) when $A^2 - 4C = 0$.

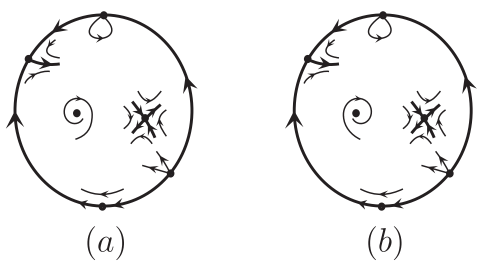

Figure 6. Local Phase portraits in the Poincaré disc of Case (i) when $A^2 - 4C > 0$.

When $A^2 - 4C > 0$ there are two singular points, that is, $p_{\pm}$. Since $\Delta_+ < 0$ we have that p_+ is a saddle, and p_- is either a focus, or a node. If $T_-^2 - 4\Delta_- < 0$ then p_- is either a stable focus if $T_- < 0$, or an unstable focus if $T_- > 0$. If $T_-^2 - 4\Delta_- > 0$ then p_- is either a stable node if $T_- < 0$, or an unstable node if $T_- > 0$. See these local phase portraits in figure 6. Using remark 5 it is easy to check that there are only six possible global phase portraits of figure 7. And in table 2 we check that the six phase portraits of figure 7 are realizable.

In all the above phase portraits there exists at most one limit cycle for the Liénard differential system that we are analysing, this is a known result which can be found in [**12**].

Hopf Bifurcation: In Case (i) with $A^2 - 4C > 0$ and $T_- = 0$ as the singular point p_- is a weak focus it, eventually, could be a centre, but this is not the case. In order to prove this we compute the eigenvalues of the Jacobian matrix of system (4.1) at the singular point p_-. The real part of the eigenvalues is $-A + 2B - \alpha$, where $\alpha^2 = A^2 - 4C$, and we assume $\alpha > 0$ to facilitate the calculations. The possible situation where the weak focus p_- could be a centre is when the real part of its eigenvalues is zero, i.e. $\alpha = -A + 2B$. With some effort we can check that the singular point p_- is always an unstable focus, and never a centre, because

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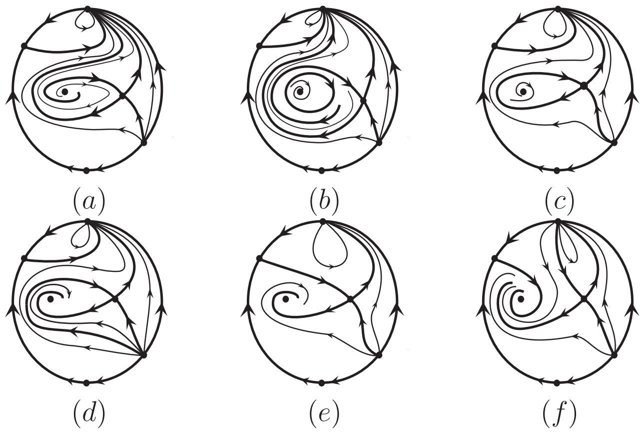

Figure 7. Global phase portraits in the Poincaré disc of Case (i) when $A^2 - 4C > 0$. In (a) $T_- \geqslant 0$, in (b), and in (c), (d), (e) and (f) $T_- < 0$.

Table 2. *Parameters of Case (i) realizing all the possible phase portraits in the Poincaré disc. Here* $B_1 \in (2.3, 2.4)$ *and* $B_2 \in (1.6, 1.7)$.

$A^2 - 4C > 0$		$A = 3, B \geqslant (3+\sqrt{5})/2, C = 1$	Figure 7(a)
		$A = 0, B_1 < B < (3+\sqrt{5})/2, C = -3/2$	Figure 7(b)
		$B = B_1$	Figure 7(c)
		$A = 3, B_2 < B < B_1, C = 1$	Figure 7(d)
		$B = B_2$	Figure 7(e)
		$A = 3, B < B_2, C = 1$	Figure 7(f)
$A^2 - 4C = 0$	$B - A/2 > 0$	$A = -1, B = 1/2, C = 1/4$	Figure 5(a)
	$B - A/2 = 0$	$A = 2, B = 1, C = 1$	Figure 5(b)
	$B - A/2 < 0$	$A = 3, B = 1, C = 9/4$	Figure 5(c)
		$A^* \in (2, 5), B = 1, C = A^*/4$	Figure 5(d)
		$A = 5, B = 1, C = 25/4$	Figure 5(e)
$A^2 - 4C < 0$		$A = 2, B = 1, C = 2$	Figure 3

the first non-zero Lyapunov constant is equal $\pi/4\beta^5 > 0$, where $\beta > 0$ and is given by $\beta^2 = 2B - A$. The result proved in [**12**] ensures that the Liénard differential system that we are analysing has at most one periodic orbit. Using a result proved in [**25**] (theorem 3.3), when $B_1 = (A + \sqrt{A^2 - 4C})/2$ we have a Hopf bifurcation, and an unstable limit cycle bifurcates from p_- when $B < B_1$ (see figure 7(b)). This limit cycle ends in a loop formed with the sadle p_+ and a homoclinic orbit to it (see figure 7(c)).

Case (ii). Then system (1.1) becomes

$$\dot{x} = y, \quad \dot{y} = x^2 + y + Dx + E. \tag{4.2}$$

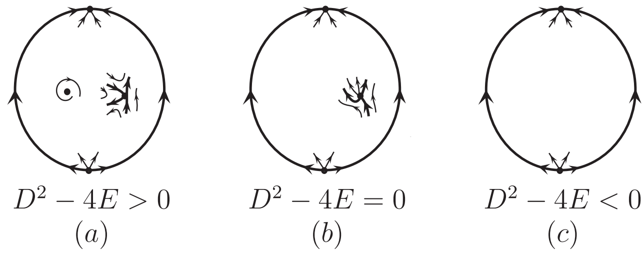

Figure 8. Local phase portraits in the Poincaré disc of Case (ii).

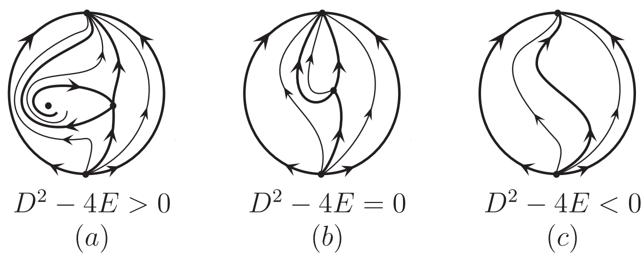

Figure 9. Global phase portraits in the Poincaré disc of Case (ii).

The finite singular points of system (4.2) are $p_{\pm} = ((-D \pm \sqrt{D^2 - 4E})/2, 0)$, and the Jacobian matrix at $p_{\pm}$ is

$$J|_{p_{\pm}} = \begin{pmatrix} 0 & 1 \\ \pm\sqrt{D^2 - 4E} & 1 \end{pmatrix},$$

which has trace $T = 1$ and determinant $\Delta = \sqrt{D^2 - 4E}$ for p_-, and $\Delta = -\sqrt{D^2 - 4E}$ for p_+. First we observe the necessary condition $D^2 - 4E > 0$ for the existence of the equilibrium points $p_{\pm}$. If $D^2 - 4E = 0$ we get a unique singular point which is semi-hyperbolic. Applying theorem 2.19 of [**20**] at this semi-hyperbolic singular point we get that this singularity is a saddle-node.

If $D^2 - 4E > 0$ we obtain two singular points, i.e. $p_{\pm}$. In this case p_+ is a saddle because $T = 1$ and $\Delta = -\sqrt{D^2 - 4E} < 0$, and the other equilibrium point p_- is an unstable node if $0 < D^2 - 4E \leqslant 1/16$, or an unstable focus if $D^2 - 4E > 1/16$. Since the divergence of system (4.2) is constant equal to 1, this system cannot have periodic orbits or homoclinic loops (by Bendixson theorem, see for instance theorem 7.10 of [**17**]). Then from the local phase portraits of figure 8 we obtain the correspondent global phase portraits as shown in figure 9.

All these global phase portraits in figure 9 are achievable with the respective parameters shown in table 3.

Case (iii). Then the correspondent normal form is

$$\dot{x} = y, \quad \dot{y} = xy + y + Gx + H. \tag{4.3}$$

If $G = 0$ there is no finite singular point for system (4.3). And, if $G \neq 0$, this system has a unique finite singular point $(-H/G, 0)$.

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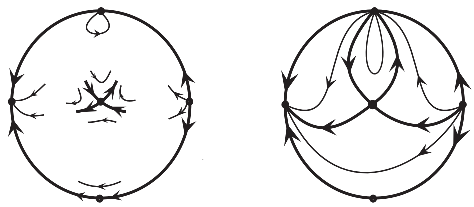
Table 3. *Parameters of Case (ii).*

$D^2 - 4E > 0$	$D = 3, E = 2$	Figure 9(a)
$D^2 - 4E = 0$	$D = 3, E = 9/4$	Figure 9(b)
$D^2 - 4E < 0$	$D = 3, E = 3$	Figure 9(c)



(a) Local phase portrait. (b) Global phase portrait.

Figure 10. Local and global phase portraits in the Poincaré disc of Case (iii) when $G > 0$.

Consider $G \neq 0$, then the Jacobian matrix at this singular point $(-H/G, 0)$ is

$$J|_{(-H/G,0)} = \begin{pmatrix} 0 & 1 \\ G & 1 - H/G \end{pmatrix}.$$

Then $J|_{(-H/G,0)}$ has trace $T = 1 - H/G$ and determinant $\Delta = -G$. Hence, if $G > 0$ then $\Delta < 0$ and the singular point $(-H/G, 0)$ is a saddle. When $G < 0$ we get $\Delta > 0$ and in this case we have to analyse two possibilities. More precisely we can have

(i) $T^2 \geqslant 4\Delta \Leftrightarrow H^2 - 2GH + G^2 + 4G^3 \geqslant 0$,

(ii) $4\Delta > T^2 > 0 \Leftrightarrow H^2 - 2GH + G^2 + 4G^3 < 0$.

In Case (i) the singular point is a node and in Case (ii) the singular point is a focus. In both cases the singular point is stable if $H > G$ and unstable if $H < G$.

For all these local phase portraits and the correspondent global phase portraits see figure 10 when $G > 0$, figure 11 when $G < 0$ and when $G = 0$ the local and the correspondent global phase portrait are topologically equivalent to those ones of figure 3. It is easy to check that the quadratic differential system (4.3) has no invariant straight lines. So by a result of Sotomayor and Paterlini [**44**] it follows that the two finite separatrices of the saddles at the origins of the local charts U_1 and V_1 of figure 11(a) cannot connect, otherwise the connection must be through an invariant straight line. Therefore the global phase portrait of figure 11(b) is the unique possible. All these global phase portraits are achievable with the respective parameters shown in table 4.

Case (iv). The correspondent normal form is

$$\dot{x} = y, \quad \dot{y} = xy + x + I, \tag{4.4}$$

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Figure 11. Local and global phase portraits in the Poincaré disc of Case (iii) when $G < 0$ and $G < H$. If $G > H$ then the phase portraits are the same but reversing the orientation of all their orbits.

Table 4. *Parameters of Case (iii).*

$G > 0$	$G = 1, H = 1$	Figure 10
$G = 0$	$G = 0, H = 1$	Figure 3(b)
$G < 0$	$G = -1, H = 0$	Figure 11

which has a unique finite singular point $(-I, 0)$. The Jacobian matrix at this singular point is

$$J|_{(-I,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -I \end{pmatrix}.$$

Since $J|_{(-I,0)}$ has trace $T = -I$ and determinant $\Delta = -1$ we conclude that the singular point $(-I, 0)$ is a saddle. This local phase portrait and the correspondent global phase portrait are topologically equivalent to those ones presented in figure 10.

Case (v). The correspondent normal form is

$$\dot{x} = y, \quad \dot{y} = xy - x + I, \tag{4.5}$$

which has a unique finite singular point $(I, 0)$. The Jacobian matrix at this singular point is

$$J|_{(I,0)} = \begin{pmatrix} 0 & 1 \\ -1 & I \end{pmatrix}.$$

which has trace $T = I$ and determinant $\Delta = 1$. If $I = 0$ then the singular point $(0, 0)$ is a centre non-hyperbolic because in this case the system is reversible with involution given by $(x, y, t) \mapsto (-x, y, -t)$. The local and global phase portraits are shown in figure 12. If $I \neq 0$ there exist two possibilities

(i) $T^2 \geqslant 4\Delta > 0 \Leftrightarrow I^2 \geqslant 4 > 0 \Leftrightarrow I \leqslant -2$ or $I \geqslant 2$,

(ii) $4\Delta > T^2 > 0 \Leftrightarrow 4 > I^2 \Leftrightarrow -2 < I < 2$.

In Case (i) the singular point $(I, 0)$ is a stable node if $I \leqslant -2$, or an unstable node if $I \geqslant 2$. In Case (ii) the singular point $(I, 0)$ is a stable focus if $-2 < I < 0$, or an

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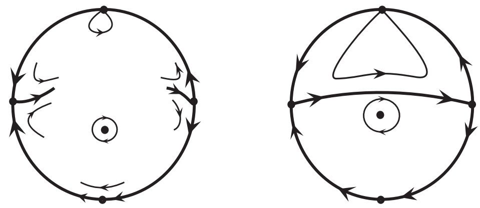

Figure 12. Local and global phase portraits in the Poincaré disc of Case (v) when $I = 0$.

unstable focus if $0 < I < 2$. These local phase portraits and the correspondent global phase portraits are topologically equivalent to those ones presented at figure 11.

Case (vi). The correspondent normal form is

$$\dot{x} = y, \quad \dot{y} = xy + 1, \tag{4.6}$$

which has no finite singular point. This local phase portrait and the correspondent global phase portrait are topologically equivalent to those ones of figure 3.

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