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Thermoacoustic-wave equations for gas in a channel and a tube subject to temperature gradient

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This paper develops a general theory for linear propagation of acoustic waves in a gas enclosed in a two-dimensional channel and in a circular tube subject to temperature gradient axially and extending infinitely. A 'narrow-tube approximation' is employed by assuming that a typical axial length is much longer than a span length, but no restriction on a thickness of thermoviscous diffusion layer is made. For each case, basic equations in this approximation are reduced to a spatially one-dimensional equation in terms of an excess pressure by making use of a method of Fourier transform. This equation, called a thermoacoustic-wave equation, is given in the form of an integro-differential equation due to memory by thermoviscous effects. Approximations of the equations for a short-time and a long-time behaviour from an initial state are discussed based on the Deborah number and the Reynolds number. It is shown that the short-time behaviour is well approximated by the equation derived previously by the boundary-layer theory, while the long-time behaviour is described by new diffusion equations. It is revealed that if the diffusion layer is thicker than the span length, the thermoviscous effects give rise to not only diffusion but also wave propagation by combined action with temperature gradient, and that negative diffusion may occur if the gradient is steep.

1. Introduction

Thermoacoustics has attracted much attention over the past few decades from the viewpoint of potential in application to novel heat engines. Central phenomena are due to thermoviscous effects of a gas in contact with a solid wall subject to temperature gradient. If the gradient is steep, it happens that the effects cause instability of the gas and eventually give rise to its spontaneous oscillations. Although such phenomena have been known empirically or experimentally since 19th century, no analysis of stability was made until the pioneering work initiated by Kramers (1949) and Rott (1969). They attempted to seek a marginal condition for the onset of the Taconis oscillations (Taconis *et al.* 1949) in a helium-filled tube in cryogenics. Although Kramers (1949) failed, Rott (1969, 1973) succeeded in deriving the conditions, which are confirmed experimentally by Yazaki, Tominaga & Narahara (1980). Since then, Rott's linear theory has played a fundamental role and has been applied to design experimental devices (Swift 1988, 2002).

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When temperature gradient is absent, the thermoviscous effects appear normally to give rise to damping of acoustic waves. Many theoretical works have been made since Kirchhoff (1868) derived a dispersion relation rigorously by taking both effects due to viscous and thermal diffusion. Kirchhoff's theory was fully reproduced by Rayleigh (1945). In discussing the wave propagation there are two important parameters: the ratio of a tube radius to a typical axial wavelength and the ratio of a typical thickness of diffusion layer to the radius. While the former ratio is usually much smaller than unity, there are various cases depending on the latter ratio. If it is much smaller than unity, the thermoviscous effects are confined only in a thin boundary layer on the tube wall, outside of which lossless and one-dimensional propagation may be assumed. Such a case is called a 'wide tube' and is discussed by Kirchhoff (1868). On the contrary, the other limit where the diffusion layer is extremely thick is discussed by Rayleigh (1945) and is called a 'narrow tube.'

While Weston (1953) classified various cases based on Kirchhoff's dispersion relation, it is shown by Zwikker & Kosten (1949) and Tijdeman (1975) that the approximation based on the small ratio of the tube radius to the wavelength is practically useful. Recently Yazaki, Tashiro & Biwa (2007) have made experiments to check the approximations by Kirchhoff and Rayleigh and to confirm the validity of the results by Tijdeman (1975) over a very wide range of frequency normalized by a characteristic thermal diffusion time. It is then unveiled that the wide tube provides a good approximation especially for a propagation velocity even when the thickness of the diffusion layer becomes comparable with a tube radius.

The author has developed a boundary-layer theory for acoustic waves in a wide tube subject to temperature gradient. Although the theory, once used by Kramers (1949), was later regarded as being incapable of describing the Taconis oscillations by Rott (1969), it has now turned out to be applicable to derive a marginal condition of the Taconis oscillations (Sugimoto & Yoshida 2007). The failure by Kramers and Rott results from the use of a regular asymptotic expansion in terms of a thickness of boundary layer. Quite recently, it has also been applied to derivation of marginal conditions for the Sondhauss tube and resonators (Sugimoto & Takeuchi 2009). Because the theory is now extended to non-harmonic disturbances by taking account of memory due to the thermoviscous effects, it can be incorporated into a framework of weakly nonlinear theory (Sugimoto & Shimizu 2008). Using this theory, initial instability and ensuing emergence of self-excited Taconis oscillations are simulated numerically (Shimizu & Sugimoto 2009, 2010). Such a success has motivated us to clarify not only a reason as to why the boundary-layer theory is applicable but also a condition under which the theory is valid. Further, it suggests us to seek a wave equation in such a general case that the diffusion layer is not restricted to be thin.

To this end, this paper develops a general theory for linear propagation of acoustic waves in a gas confined in a two-dimensional channel and in a circular tube subject to temperature gradient and extending infinitely. While the thickness of thermoviscous diffusion layer is kept arbitrary, use is made of a 'narrow-tube approximation' in the sense that a typical axial length is assumed to be much longer than a span length. An adjective 'narrow' in this context is used differently from that by Rayleigh. This approximation is simply the one used by Zwikker & Kosten (1949) and is called 'low-reduced frequency' by Tijdeman (1975) and Rott (1969) for the analysis of the Taconis oscillations. But no other assumptions are made, for example as to a form of temporal variations. Basic equations in the narrow-tube approximation are reduced rigorously to a one-dimensional integro-differential equation in terms of an

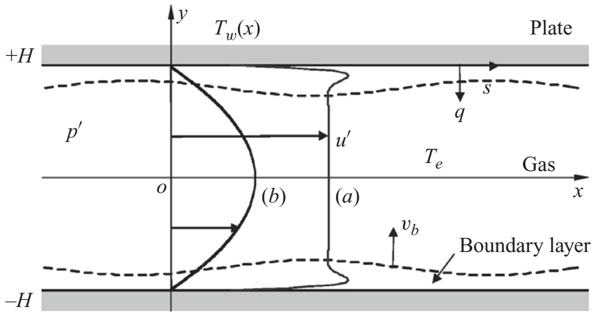

FIGURE 1. Illustration of a two-dimensional channel of width $2H$ filled with a thermoviscous gas and subject to non-uniform temperature distribution axially. The temperature of the gas in a quiescent equilibrium state T_e is assumed to be equal to the temperature of the plates T_w imposed externally. An excess pressure p' over p_0 in the equilibrium is uniform over a cross-section, while profiles labelled (*a*) and (*b*) are of the axial velocity of gas u' with respect to y for typical cases of a thin diffusion layer and of a thick layer, respectively. In the former case, the thermoviscous effects are confined in a thin layer, i.e. a boundary layer, designated by a region between the plate surface and the broken curve. In the latter case, the diffusion layer fills fully the channel width. Here v_b represents velocity directed normal to the plate surface and away from it at the edge of the boundary layer, while s and q represent, respectively, shear stress acting on the gas at the plate surface and heat flux flowing into the gas through it.

excess pressure. This is called a thermoacoustic-wave equation. Approximations of this equation for a short-time and a long-time behaviour from an initial state are discussed based on the Deborah number and the Reynolds number.

In what follows, starting from the linearized theory for an ideal gas of Newtonian fluid, basic equations in the narrow-tube approximation are presented in the case of a two-dimensional channel, from which a thermoacoustic-wave equation is derived by applying a method of Fourier transform in § 2. No account of a thermal coupling between the gas and the solid wall is taken into account. Approximations of the equation are made, respectively, for a short-time behaviour and a long-time behaviour in § 3. In § 4, modifications in the case of a circular tube are briefly described and the thermoacoustic-wave equation is derived. Section 5 is devoted to discussions of the thermoacoustic-wave equations in both cases and their approximations.

2. Formulation of a two-dimensional problem

2.1. *Linearized basic equations*

Suppose a two-dimensional propagation of acoustic waves occurs in a gas enclosed in a channel of infinite length between two parallel plates subject to temperature gradient along them. Taking the x-coordinate along the wave propagation, the y-coordinate is taken normal to the plates with its origin at a midpoint between them separated by distance $2H$ (see figure 1). The temperature of both plates at a position x is assumed to be equal with each other.

In a quiescent equilibrium state where no gravity is assumed, a pressure of gas takes a uniform value p_0 throughout. A temperature of gas T_e satisfies $\nabla \cdot (k \nabla T_e) = 0$, k being a thermal conductivity and T_e must be equal to the temperature of the plates $T_w(x)$ at the plate surfaces. If k is assumed to be a constant independent of T_e and the channel width is much narrower than a typical axial length in T_w, T_e is solved asymptotically as $T_e = T_w + (\mathrm{d}^2 T_w/\mathrm{d}x^2)(H^2 - y^2)/2 + \dots$. As long as the axial distribution of T_w is gentle enough for a contribution from $\mathrm{d}^2T_w/\mathrm{d}x^2$ to be negligible, the temperature of gas may be set equal to that of the plates. This approximation is stipulated quantitatively by a following requirement on T_w

$$\frac{H^2}{T_w}\left|\frac{\mathrm{d}^2T_w}{\mathrm{d}x^2}\right| \ll \frac{H}{T_w}\left|\frac{\mathrm{d}T_w}{\mathrm{d}x}\right| \ll 1. \tag{2.1}$$

The present theory neglects the leftmost quantity and takes account of non-uniformity of temperature distribution up to its first-order derivative $\mathrm{d}T_w/\mathrm{d}x$. Since T_e is now identified as T_w, no distinction will be made between them and $T_e(x)$ will be used in the following. With the pressure and temperature thus specified, a density of gas is determined by an equation of state. If the ideal gas is assumed, the density $\rho_e(x)$ satisfies Charles' law with $\rho_e T_e$ held constant. Here and hereafter a quantity with the subscript e is understood to be dependent on x through T_e.

Assuming infinitesimally small disturbances from the above equilibrium state, the equation of continuity, Navier–Stokes equation and equations of energy and of state are linearized around the state as follows:

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot(\rho_e \boldsymbol{v}') = 0, \tag{2.2}$$

$$\rho_e\frac{\partial \boldsymbol{v}'}{\partial t} = -\nabla p' + \nabla\left[2\mu\left(\boldsymbol{e}' - \frac{1}{3}\boldsymbol{I}\nabla\cdot\boldsymbol{v}'\right)\right] + \nabla(\mu_{\mathrm{v}}\nabla\cdot\boldsymbol{v}'), \tag{2.3}$$

$$\rho_e c_p\left(\frac{\partial T'}{\partial t} + \boldsymbol{v}'\cdot\nabla T_e\right) = \frac{\partial p'}{\partial t} + \nabla\cdot(k\nabla T'), \tag{2.4}$$

$$\frac{p'}{p_0} = \frac{\rho'}{\rho_e} + \frac{T'}{T_e}, \tag{2.5}$$

in $-\infty < x < \infty$ and $-H < y < H$, with $\nabla T_e = (\mathrm{d}T_e/\mathrm{d}x, 0)$ where ρ, p, T, $\boldsymbol{v}$ [$= (u, v)$], $\boldsymbol{e}$ [$= (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$ with $(x_1, x_2) = (x, y)$ and $(v_1, v_2) = (u, v)$ for $i, j = 1$ or 2] and $\boldsymbol{I}$ denote, respectively, density, pressure, absolute temperature, velocity vector, rate of strain tensor and unit tensor, the prime attached implying a disturbance from the respective equilibrium values, and all variables are assumed to depend on x, y and t, t being the time measuring from a remote past $t \to -\infty$; μ, μ_{v} and c_p denote, respectively, a shear viscosity, a bulk viscosity and a specific heat at constant pressure.

The viscosities and thermal conductivity depend on the temperature and pressure in general. (But the kinetic theory of gas suggests no dependence on the pressure and the exponent β in (2.8) takes 1/2 (see, for example Sone 2002).) On the right-hand side of (2.3), the pressure dependence gives rise to $(\partial\mu/\partial p)\nabla p'$ and $(\partial\mu_{\mathrm{v}}/\partial p)\nabla p'$ and therefore yields nonlinear terms in the disturbances. In contrast, the temperature dependence gives rise to $(\partial\mu/\partial T)\nabla T$ and $(\partial\mu_{\mathrm{v}}/\partial T)\nabla T$, and linear terms remain because $\nabla T = \nabla T_e + \nabla T' \approx \nabla T_e$. Expanding μ, μ_{v} and their derivatives around the local equilibrium values, (2.3) may be rewritten in the linearized approximation as

$$\rho_e\frac{\partial \boldsymbol{v}'}{\partial t} = -\nabla p' + \mu_e\Delta\boldsymbol{v}' + \left(\frac{1}{3}\mu_e + \mu_{\mathrm{v}e}\right)\nabla\nabla\cdot\boldsymbol{v}' + 2\left.\frac{\partial\mu}{\partial T}\right|_e \nabla T_e\left(\boldsymbol{e}' - \frac{1}{3}\boldsymbol{I}\nabla\cdot\boldsymbol{v}'\right) + \left.\frac{\partial\mu_{\mathrm{v}}}{\partial T}\right|_e \nabla T_e(\nabla\cdot\boldsymbol{v}'), \tag{2.6}$$

where the symbol $...|_e$ designates evaluation at the equilibrium state. In (2.4) as well, the term $\nabla\cdot(k\nabla T)$ is treated in a similar manner. Thus (2.4) is rewritten as

$$\rho_e c_p\left(\frac{\partial T'}{\partial t}+\boldsymbol{v}'\cdot\nabla T_e\right)=\frac{\partial p'}{\partial t}+k_e\Delta T'+\left.\frac{\partial k}{\partial T}\right|_e \nabla T_e\cdot\nabla T', \tag{2.7}$$

where k_e denotes the value of k in the equilibrium state.

Experimental data suggest that the shear viscosity and heat conductivity may be fitted by a power law of the temperature given by

$$\frac{\mu}{\mu_0}=\left(\frac{T}{T_0}\right)^{\beta} \quad \text{and} \quad \frac{k}{k_0}=\left(\frac{T}{T_0}\right)^{\beta}, \tag{2.8}$$

where quantities with the suffix 0 imply respective values in a reference state and β takes a positive constant between 0.5 and 0.6 for air. No relation for μ_v is given because of its few data, and also because the term with μ_v does not appear in the following analysis. In contrast to the viscosity and heat conductivity, the specific heat c_p and the Prandtl number Pr defined by $\mu c_p/k$ change little so that their values may be regarded as being constant.

Boundary conditions at the plate surfaces are imposed as follows:

$$u'=v'=T'=0 \quad \text{at} \quad y=\pm H. \tag{2.9}$$

Here, the last condition needs a remark. It assumes that both plates have a large heat capacity. Thus the temperature of the plates is always kept at T_w so that the temperature of gas adjacent to them is equal to T_w. No account of a thermal coupling with the solid is taken. Boundary conditions at both infinity of x are not specified.

2.2. *Narrow-tube approximation*

At first, it is noticed that there are three typical length-scales involved in this problem besides H. Letting a typical angular frequency be ω, one is a wavelength a/ω where a is a typical speed of wave propagation. This speed is taken to be an adiabatic sound speed a_0 $(=\sqrt{\gamma p_0/\rho_0})$ at a reference state, γ being the ratio of specific heats.

Another is a typical length of the temperature gradient, l, estimated by

$$l\sim\left|\frac{1}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\right|^{-1}, \tag{2.10}$$

where the symbol $\sim$ implies equality in order of magnitude. Using l, the leftmost quantity in (2.1) is estimated to be of order $(H/l)^2$ $(\ll 1)$. Since this is to be neglected, the temperature gradient should not be steep enough to violate this assumption. The other is a thickness of a viscous diffusion layer $\sqrt{\nu/\omega}$, ν being a typical kinematic viscosity. Of course, a thickness of a thermal diffusion layer $\sqrt{\kappa/\omega}$ may be taken, κ $(=k/\rho c_p)$ being a typical thermal diffusivity. But because the Prandtl number Pr $(=\mu c_p/k=\nu/\kappa)$ takes a value of order unity for usual gases, the thermal diffusion layer is represented by the viscous one.

Among the three lengths, the wavelength a/ω may be assumed to be much longer than H in usual thermoacoustic devices operating at an audible frequency, while l should be much longer than H, as stipulated by (2.1). The thickness $\sqrt{\nu/\omega}$ is much smaller than a/ω and l. The present theory assumes that l is shorter than a/ω or comparable with it, while $\sqrt{\nu/\omega}$ is arbitrary in comparison with H. To include a special case without temperature gradient $(l\to\infty)$ for comparison, a typical axial length is denoted by L rather than l. When temperature gradient is absent, L is taken to be a/ω, while when the gradient is present, L is taken to be a smaller one between l and a/ω. In any case, L should be much longer than H.

Thus three dimensionless parameters are introduced as follows:

$$\frac{H}{L} \equiv \lambda \ll 1, \quad \frac{\omega L}{a} \equiv \frac{1}{\chi} \leqslant O(1) \quad \text{and} \quad \frac{\sqrt{\nu/\omega}}{H} \equiv \delta = O(1), \tag{2.11}$$

where χ takes a larger value than unity, but it is equal to unity if temperature gradient is absent. A reduction to be guided by the first two assumptions of (2.11), i.e. $\omega H/a = \lambda/\chi \ll 1$, may be called a low-frequency approximation or a long-wave approximation or a narrow-tube approximation. On the other hand, if $\delta \ll 1$ where the diffusion layer may be treated as a boundary layer, this case is called a 'wide tube,' while if $\delta \gg 1$, it is called a 'narrow tube,' as mentioned in § 1.

If $\lambda/\chi \ll 1$ and $\lambda/\chi \ll \delta^{-1}$ ($\sqrt{\nu\omega}/a \ll 1$), this is called a low-reduced frequency by Tijdeman (1975) because he claimed that use of 'wide' or 'narrow' is misleading. But this paper uses 'narrow' from a standpoint that the 'narrowness' of the tube should be measured in comparison with the axial length L, not with the thickness of the diffusion layer. The approximation by the first two conditions of (2.11) is thus called a 'narrow-tube approximation' here. As the third condition specifies the thickness of diffusion layer, the terms 'thin' and 'thick' are used for the diffusion layer as $\delta \ll 1$ and $\delta \gg 1$, respectively. The former is a case to which the boundary-layer approximation is applicable. For a circular tube, δ is defined by a tube radius R instead of H. In passing, $Pr/2\delta^2$ is often called $\omega\tau$ by experimentalists (Yazaki *et al.* 2007), τ being a typical thermal diffusion time $R^2/2\kappa$.

Using this narrow-tube approximation, the magnitude of each term in (2.2), (2.6) and (2.7) is estimated. The x- and y-coordinates are measured by L and H, respectively, while the time t is measured by ω^{-1}. In (2.2), the order of ρ' and the one of v' are estimated in terms of u' as follows:

$$\frac{\rho'}{\rho_0} \sim \chi \frac{u'}{a_0} \quad \text{and} \quad v' \sim \lambda u'. \tag{2.12}$$

It is to be noted that if $\chi \gg 1$, the relative change in density is much greater than the one in the axial speed. When the gradient is absent, of course, both changes are comparable. The spanwise velocity is much smaller than the axial one.

From the x-component of (2.6), a balance between the inertia term and the pressure gradient estimates the order of variation of p' to be

$$\frac{p'}{p_0} \sim \frac{1}{\chi} \frac{u'}{a_0}. \tag{2.13}$$

This suggests that the relative change in the pressure is smaller than the one in the axial speed if χ is large. In the viscous terms, the spanwise curvature $\partial^2 u'/\partial y^2$ dominates over the other terms to balance with the inertia term as far as δ is $O(1)$. The terms due to temperature dependence of μ and μ_v remain secondary because $\partial\mu/\partial T|_e = \beta\mu_e/T_e$ and $\partial\mu_v/\partial T|_e = \beta\mu_{ve}/T_e$. Using the first assumption of (2.11) in the y-component of (2.6), it follows, to the lowest order, that the spanwise gradient of the pressure should vanish so that the pressure is uniform over a cross-section of the channel. In (2.7) as well, the spanwise curvature of the temperature dominates over the axial one. Changes in density, pressure and temperature are linked with each other through the equation of state. If χ is large, a relative variation of density is estimated to be much larger than the one of pressure. It is found from (2.5) that a relative change in density should balance with the one in temperature with opposite sign as

$$\frac{\rho'}{\rho_e} \approx -\frac{T'}{T_e} \sim \chi^2 \frac{p'}{p_0}. \tag{2.14}$$

This is nearly an isobaric process.

Summarizing the above results, the basic equations are simplified by the narrow-tube approximation as follows:

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}(\rho_e u') + \frac{\partial}{\partial y}(\rho_e v') = 0, \tag{2.15}$$

$$\rho_e \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} + \mu_e \frac{\partial^2 u'}{\partial y^2}, \tag{2.16}$$

$$0 = -\frac{\partial p'}{\partial y}, \tag{2.17}$$

$$\rho_e c_p \left(\frac{\partial T'}{\partial t} + u' \frac{\mathrm{d}T_e}{\mathrm{d}x} \right) = \frac{\partial p'}{\partial t} + k_e \frac{\partial^2 T'}{\partial y^2}, \tag{2.18}$$

together with the equation of state (2.5). Here it should be remarked that this system of equations is the same as the one used by Rott (1969). If no temperature gradient is assumed, it is equivalent to that used by Tijdeman (1975).

2.3. *Solutions by the method of Fourier transform*

Thanks to (2.17), the pressure may be regarded as being uniform over a cross-section of the channel. This consequence stems from the first assumption of (2.11) and holds also for the wide tube. Since p' is written as $p'(x, t)$, the other physical variables are expressed in term of p'. Solutions to (2.15)–(2.18) with (2.5) are sought by applying a method of Fourier transform defined by

$$\mathscr{F}\{u'\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u'(x, y, t) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \equiv \hat{u}'(x, y, \omega), \tag{2.19}$$

with its inverse transform given by

$$\mathscr{F}^{-1}\{\hat{u}'\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}'(x, y, \omega) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega = u'(x, y, t). \tag{2.20}$$

Applying the method to (2.16), a solution $\hat{u}'$ is sought so as to satisfy the boundary conditions. It is expressed in terms of p' as

$$\hat{u}' = -\frac{1}{\rho_e} \sigma^{-1} \frac{\partial \hat{p}'}{\partial x} f, \tag{2.21}$$

with $\sigma = -\mathrm{i}\omega$ where f is defined as

$$f(x, y) = 1 - \frac{\cosh(y/H\delta_e)}{\cosh(1/\delta_e)}, \tag{2.22}$$

and δ_e is defined by

$$\delta_e(x) = \frac{1}{H} \left(\frac{\nu_e}{\sigma} \right)^{1/2}, \tag{2.23}$$

with $\nu_e(x) = \mu_e/\rho_e$. Here $\sigma^{-1/2}$ is defined to take a positive real part for a positive value of ω.

Substituting this solution into (2.18) and using the boundary conditions, $\hat{T}'$ is obtained as follows:

$$\hat{T}' = \frac{1}{\rho_e c_p}\hat{p}' f_P + \frac{1}{\rho_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\sigma^{-2}\frac{\partial \hat{p}'}{\partial x}\left(-\frac{Pr}{1-Pr}f + \frac{1}{1-Pr}f_P\right), \tag{2.24}$$

where f_P is defined as

$$f_P(x,y) = 1 - \frac{\cosh(y\sqrt{Pr}/H\delta_e)}{\cosh(\sqrt{Pr}/\delta_e)}. \tag{2.25}$$

The subscript P is used to indicate dependence of f_P on Pr and therefore to imply thermal effects because $\delta_e/\sqrt{Pr}$ $[= (\kappa_e/\sigma)^{1/2}/H]$ measures the thickness of thermal diffusion layer relative to H, $\kappa_e(x)$ being $k_e/\rho_e c_p$. Unless P is attached, f designates the function f_P in which Pr is set equal to unity formally. From f, conversely, f_P is recovered by substituting $\delta_e/\sqrt{Pr}$ in place of δ_e. Such a use of the subscript P will also be made in the following.

Now that $\hat{T}'$ is available, $\hat{\rho}'$ is expressed in terms of $\hat{p}'$ by using a following relation derived from (2.5) transformed:

$$\hat{\rho}' = \frac{\rho_e}{p_0}\hat{p}' - \frac{\rho_e}{T_e}\hat{T}' = \frac{\gamma}{a_e^2}\hat{p}' - \frac{(\gamma-1)\rho_e c_p}{a_e^2}\hat{T}', \tag{2.26}$$

where $a_e(x)$ is a local adiabatic sound speed $\sqrt{\gamma p_0/\rho_e}$, and a_e^2 is alternatively written as $a_e^2 = (\gamma-1)c_p T_e$. Substituting (2.26) into (2.15) and integrating it with respect to y, $\rho_e\hat{v}'$ is obtained as follows:

$$\begin{aligned}\rho_e\hat{v}' = &-\frac{1}{a_e^2}\left[\sigma\hat{p}' - \frac{\partial}{\partial x}\left(a_e^2\sigma^{-1}\frac{\partial \hat{p}'}{\partial x}\right)\right]y \\ &-\frac{\partial}{\partial x}\left(\sqrt{\nu_e}\sigma^{-3/2}\frac{\partial \hat{p}'}{\partial x}g\right) - \frac{\gamma-1}{\sqrt{Pr}}\frac{\sqrt{\nu_e}}{a_e^2}\sigma^{1/2}\hat{p}' g_P \\ &+\frac{1}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\sqrt{\nu_e}\sigma^{-3/2}\frac{\partial \hat{p}'}{\partial x}\left[\frac{Pr}{1-Pr}g - \frac{1}{(1-Pr)\sqrt{Pr}}g_P\right] + \text{const.},\end{aligned} \tag{2.27}$$

where g and g_P are defined, respectively, by

$$g(x,y) = \frac{\sinh(y/H\delta_e)}{\cosh(1/\delta_e)}, \tag{2.28}$$

and

$$g_P(x,y) = \frac{\sinh(y\sqrt{Pr}/H\delta_e)}{\cosh(\sqrt{Pr}/\delta_e)}, \tag{2.29}$$

the const. being an arbitrary integration constant.

Imposing the boundary conditions $\hat{v}' = 0$ at $y = \pm H$, the constant must vanish and a relation for $\hat{p}'$ is available as follows:

$$\begin{aligned}\sigma\hat{p}' &- \frac{\partial}{\partial x}\left(a_e^2\sigma^{-1}\frac{\partial \hat{p}'}{\partial x}\right) + \frac{\partial}{\partial x}\left[\frac{a_e^2\sqrt{\nu_e}}{H}\sigma^{-3/2}\frac{\partial \hat{p}'}{\partial x}g(x,H)\right] \\ &+ \frac{\gamma-1}{\sqrt{Pr}}\frac{\sqrt{\nu_e}}{H}\sigma^{1/2}\hat{p}' g_P(x,H) - \frac{a_e^2}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\sqrt{\nu_e}}{H}\sigma^{-3/2}\frac{\partial \hat{p}'}{\partial x} \\ &\times\left[\frac{1}{1-Pr}g(x,H) - \frac{1}{(1-Pr)\sqrt{Pr}}g_P(x,H)\right] = 0,\end{aligned} \tag{2.30}$$

with $g(x,H) = \tanh(\delta_e^{-1})$ and $g_P(x,H) = \tanh(\sqrt{Pr}\delta_e^{-1})$, where note that $\mathrm{d}a_e^2/\mathrm{d}x = a_e^2 T_e^{-1}\mathrm{d}T_e/\mathrm{d}x$. Equation (2.30) governs a behaviour of the pressure transformed. Before proceeding to the inverse transform, we look at the problem in terms of mean quantities over the cross-section.

2.4. *Equations averaged over the cross-section and thermoacoustic-wave equation*

Defining the mean $\bar{\phi}$ of a quantity $\phi(x, y, t)$ averaged over the cross-section by

$$\frac{1}{2H}\int_{-H}^{+H} \phi(x, y, t)\mathrm{d}y \equiv \bar{\phi}(x, t), \tag{2.31}$$

(2.15), (2.16) and (2.18) may be rewritten as follows:

$$\frac{\partial \bar{\rho}'}{\partial t} + \frac{\partial}{\partial x}\left(\rho_e \bar{u}'\right) = 0, \tag{2.32}$$

$$\rho_e \frac{\partial \bar{u}'}{\partial t} = -\frac{\partial p'}{\partial x} + \frac{s}{2H}, \tag{2.33}$$

$$\rho_e c_p \left(\frac{\partial \bar{T}'}{\partial t} + \bar{u}'\frac{\mathrm{d}T_e}{\mathrm{d}x}\right) = \frac{\partial p'}{\partial t} + \frac{q}{2H}, \tag{2.34}$$

where s and q denote, respectively, shear stress acting on the gas at the plate surfaces and heat flux flowing into the gas through them, which are given respectively by

$$s = \mu_e \frac{\partial u'}{\partial y}\bigg|_{y=+H} - \mu_e \frac{\partial u'}{\partial y}\bigg|_{y=-H}, \tag{2.35}$$

and

$$q = k_e \frac{\partial T'}{\partial y}\bigg|_{y=+H} - k_e \frac{\partial T'}{\partial y}\bigg|_{y=-H}. \tag{2.36}$$

Equations (2.32)–(2.34) are combined to eliminate ρ', u' and T' into a single equation for p':

$$\frac{\partial^2 p'}{\partial t^2} - \frac{\partial}{\partial x}\left(a_e^2 \frac{\partial p'}{\partial x}\right) = \frac{a_e^2}{c_p T_e}\frac{\partial}{\partial t}\left(\frac{q}{2H}\right) - \frac{\partial}{\partial x}\left(a_e^2 \frac{s}{2H}\right). \tag{2.37}$$

While the left-hand side represents lossless propagation in the gas non-uniform in temperature, the first and second terms on the right-hand side represent, respectively, the effects due to the heat flux in the form of a monopole and the shear stress in the form of a dipole (see, for example Howe 1998). In the present case, s and q are determined as a part of the solution.

Using the solution (2.21) obtained, the shear stress is given in the transformed form as

$$\hat{s} = 2\sqrt{\nu_e}(\sigma^{-1/2}\tanh\delta_e^{-1})\frac{\partial \hat{p}'}{\partial x}. \tag{2.38}$$

To make an inverse transform, it is useful to note the following formulas:

$$\frac{1}{\sqrt{2\pi}}\mathscr{F}^{-1}\left\{\sigma^{-1/2}\tanh\delta_e^{-1}\right\} = \frac{1}{\sqrt{\pi t}}G\left(\frac{\nu_e t}{H^2}\right)h(t), \tag{2.39}$$

where G is defined by

$$G(t) = 1 + 2\sum_{n=1}^{\infty}(-1)^n \exp\left(-\frac{n^2}{t}\right), \tag{2.40}$$

and $h(t)$ is a unit step function. Note that t in (2.40) and $h(t)$ implies a dimensionless argument. In the present context, no confusion would occur with the dimensional time. The transform (2.39) is obtained by expanding the hyperbolic tangent function $\tanh X$ for $|X| \gg 1$ as

$$\tanh X = 1 + 2\sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{-2nX}, \tag{2.41}$$

X being δ_e^{-1}, and using the transform

$$\mathscr{F}\left\{\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{b}{t}\right)h(t)\right\} = \frac{1}{\sqrt{2\pi}}\sigma^{-1/2}\exp\left[-2(b\sigma)^{1/2}\right], \tag{2.42}$$

b being a positive constant (p. 12 & p. 123 in Oberhettinger 1957). It is also noted that

$$\int_0^{\infty}\frac{1}{\sqrt{\pi t}}G(t)\mathrm{d}t = 1. \tag{2.43}$$

Using these formulas, the inverse transform of $\hat{s}$ is expressed in the form of a convolution integral given by

$$s = 2\sqrt{\nu_e}\mathscr{M}\left(\frac{\partial p'}{\partial x}\right), \tag{2.44}$$

where $\mathscr{M}(\phi)$ designates a functional of a function $\phi(x,t)$, which is defined as a special case of a following functional $\mathscr{M}_P(\phi)$ by setting Pr to be equal to unity formally:

$$\mathscr{M}_P\left[\phi(x,t)\right] \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{G[\nu_e(t-\tau)/Pr H^2]}{\sqrt{t-\tau}}\phi(x,\tau)\mathrm{d}\tau. \tag{2.45}$$

In a similar way, the inverse transform of $\sigma\hat{q}$ is expressed in the following form:

$$\begin{aligned}\frac{\partial q}{\partial t} = 2c_pT_e\sqrt{\nu_e}\Bigg\{&-\frac{\gamma-1}{\sqrt{Pr}}\mathscr{M}_P\left(\frac{1}{a_e^2}\frac{\partial^2 p'}{\partial t^2}\right)\\ &+\frac{1}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\left[\frac{1}{1-Pr}\mathscr{M}\left(\frac{\partial p'}{\partial x}\right)-\frac{1}{(1-Pr)\sqrt{Pr}}\mathscr{M}_P\left(\frac{\partial p'}{\partial x}\right)\right]\Bigg\}.\end{aligned} \tag{2.46}$$

Here it is to be noted that

$$\mathscr{M}_P\left[\frac{\partial\phi}{\partial t}(x,t)\right] = \frac{\partial}{\partial t}\mathscr{M}_P[\phi(x,t)]. \tag{2.47}$$

Now that the shear stress and the heat flux are expressed in terms of p', (2.37) is given as follows:

$$\begin{aligned}&\frac{\partial^2 p'}{\partial t^2}-\frac{\partial}{\partial x}\left(a_e^2\frac{\partial p'}{\partial x}\right)+\frac{\partial}{\partial x}\left[\frac{a_e^2\sqrt{\nu_e}}{H}\mathscr{M}\left(\frac{\partial p'}{\partial x}\right)\right]+\frac{\gamma-1}{\sqrt{Pr}}\frac{\sqrt{\nu_e}}{H}\mathscr{M}_P\left(\frac{\partial^2 p'}{\partial t^2}\right)\\ &-\frac{a_e^2}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\sqrt{\nu_e}}{H}\left[\frac{1}{1-Pr}\mathscr{M}\left(\frac{\partial p'}{\partial x}\right)-\frac{1}{(1-Pr)\sqrt{Pr}}\mathscr{M}_P\left(\frac{\partial p'}{\partial x}\right)\right]=0.\end{aligned} \tag{2.48}$$

This equation is called a thermoacoustic-wave equation for the excess pressure p' of the gas in the channel. It is also obtainable directly without averaging over the cross-section by making an inverse transform of (2.30) after σ is multiplied. The physical meaning of each term is clear. While the first two terms represent lossless propagation in the presence of temperature gradient, the third term stem from the shear stress and the remaining ones stem from the heat flux. In particular, note that the fourth term is present even when no temperature gradient is present and the last term represents the thermoviscous effects combined with the temperature gradient. The last term would become pronounced for a gas with Pr close to unity but vanish in the limit of no-heat-conducting gas as $Pr \to \infty$.

Equation (2.48) is no longer a differential equation but an integro-differential equation. The thermoviscous effects introduce memory (hereditary) effects even in the Newtonian fluids in such a geometrical configuration as diffusion layers adjacent to a solid surface. The memory effects are the essence of the thermoacoustic phenomena.

If an initial-value problem is considered with $p' = 0$ for $t < 0$, then p' is set equal to $p'h(t)$ and the lower bounds of the integral in $\mathscr{M}$ and $\mathscr{M}_P$ are set equal to zero. Because (2.48) is free from a choice of time origin, the lower bound of the integrals may set, without any loss of generality, at either zero or minus infinity. It is remarked that the coefficient $\sqrt{\nu_e}/H$ in (2.48) may be incorporated into the integrals by noting a dimensionless variable $\nu_e t/H^2$ instead of t. This suggests that a short-time behaviour corresponds to a case with large H, roughly speaking, whereas a long-time behaviour corresponds to a case with small H.

Finally, the mean values of the physical variables other than the pressure are presented. It is convenient to express them in the form of temporal derivatives as follows:

$$\rho_e \frac{\partial \bar{u}'}{\partial t} = -\frac{\partial p'}{\partial x} + \frac{\sqrt{\nu_e}}{H}\mathscr{M}\left(\frac{\partial p'}{\partial x}\right), \tag{2.49}$$

$$\frac{\partial^2 \bar{\rho}'}{\partial t^2} = \frac{\partial^2 p'}{\partial x^2} - \frac{\partial}{\partial x}\left[\frac{\sqrt{\nu_e}}{H}\mathscr{M}\left(\frac{\partial p'}{\partial x}\right)\right], \tag{2.50}$$

$$\rho_e c_p \frac{\partial^2 \bar{T}'}{\partial t^2} = \frac{a_e^2}{\gamma - 1}\left\{\frac{\gamma}{a_e^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} + \frac{\partial}{\partial x}\left[\frac{\sqrt{\nu_e}}{H}\mathscr{M}\left(\frac{\partial p'}{\partial x}\right)\right]\right\}, \tag{2.51}$$

where (2.30) and therefore (2.48) have been used. The mean value of v' vanish of course because of the odd function in y. These are the alternative forms of (2.32)–(2.34) expressed in terms of p' alone and the functional involved is due to the shear stress.

3. Approximations of the thermoacoustic-wave equation

3.1. *Properties of the relaxation function*

It is revealed from the explicit forms of the shear stress and the heat flux that the viscous and thermal diffusive effects give rise to memory. For example, a current value of the shear stress at t is determined by a whole past history of the pressure gradient in τ $(< t)$. The function $G[\nu_e(t-\tau)/H^2]/\sqrt{\pi(t-\tau)}$ weighs the magnitude of dependence of the stress on the value of $\partial p'/\partial x$ in the past tracked back by a time $t-\tau$. In analogy with viscoelasticity (see, for example Christensen 1982), this function may be called a relaxation function here, although that relates a current stress to a past history of a strain rate and the pressure gradient has no dimension of it.

The arguments of G, i.e. $\nu_e t/H^2$ and $\nu_e t/PrH^2$ are also to be remarked. These dimensionless quantities correspond to the inverse of a Deborah number De in rheology (Reiner 1964; Tanner 1985), which is here defined by

$$De \equiv \frac{H^2/\nu_e}{t}. \tag{3.1}$$

This number measures a viscous diffusion time H^2/ν_e relative to a time concerned. Besides H^2/ν_e, a thermal diffusion time H^2/κ_e may be taken. Then the Deborah number becomes $H^2/\kappa_e t$, which is equal to $PrH^2/\nu_e t$. As was mentioned earlier, however, the viscous diffusion time is taken here in defining the Deborah number.

If an initial-value problem is concerned and De is large enough, i.e. a short-time behaviour is concerned compared with the diffusion time, the relaxation function may be approximated as

$$\frac{1}{\sqrt{\pi t}} G\left(\frac{\nu_e t}{H^2}\right) = \frac{1}{\sqrt{\pi t}}\left[1 - 2\exp\left(-\frac{H^2}{\nu_e t}\right) + \cdots\right]. \tag{3.2}$$

Thus, G may be set equal to unity as $\nu_e t/H^2 \to 0$.

But as the value of De becomes small, the sum in G expressed by (2.40) becomes delicate. To see a limiting value of G as $t \to \infty$, it is useful to invoke the following formula of the theta function $\vartheta_4(0, q)$, which is sometimes written as $\vartheta_0(0)$ (p. 579 in Abramowitz & Stegun 1972):

$$1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \prod_{n=1}^{\infty}(1 - q^{2n}) \prod_{n=1}^{\infty}(1 - q^{2n-1})^2 = \sqrt{\frac{2}{\pi} k' K(k)}, \tag{3.3}$$

with $\log(1/q) = \pi K(k')/K(k)$ where k and k' are the modulus of the complete elliptic integral $K(k)$ and the complimentary modulus defined by $k'^2 = 1 - k^2$. Although q and k have been defined, respectively, as the heat flux and the thermal conductivity, no confusion would occur in this context. In the limit as $q \to 1$, the sum $\sum_{n=1}^{\infty}(-1)^n$ is found to be equal to $-1/2$. Thus $G(t)$ tends to vanish as $t \to \infty$. Note that as $q \to 1$, k tends to unity and $K(k)$ tends to diverge. Using the asymptotic expressions $K(k) \approx (1/2)\log[8/(1-k)]$ and $K(k') \approx \pi/2$ as $k \to 1$ (p. 591 in Abramowitz & Stegun 1972), the expression on the rightmost term in (3.3) is approximated as

$$\sqrt{\frac{2}{\pi} k' K(k)} \approx [2(1-k)]^{1/4} \sqrt{\frac{1}{\pi}\log\left(\frac{8}{1-k}\right)}, \tag{3.4}$$

with $(1-k)/8 \approx \exp(-\pi^2 \nu_e t/H^2)$. Using this expression, an asymptotic behaviour of $G/\sqrt{\pi t}$ as $t \to \infty$ is evaluated as

$$\frac{1}{\sqrt{\pi t}} G\left(\frac{\nu_e t}{H^2}\right) \approx \frac{2\sqrt{\nu_e}}{H} \exp\left(-\frac{\pi^2}{4}\frac{\nu_e t}{H^2}\right). \tag{3.5}$$

But this is available by executing the inverse transform of (2.39) directly. It is rewritten in the following form of the inverse Laplace transform:

$$\frac{1}{\sqrt{2\pi}} \mathscr{F}^{-1}\left\{\sigma^{-1/2}\tanh \delta_e^{-1}\right\} = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \sigma^{-1/2} \tanh\left[\left(\frac{\sigma}{\nu_e}\right)^{1/2} H\right] \mathrm{e}^{\sigma t}\,\mathrm{d}\sigma. \tag{3.6}$$

Note that $\sigma^{1/2}$ has a branch point at the origin, but it disappears in the integrand. Because simple poles are located only in the left half-plane of σ at $\sigma = -(2n-1)^2\pi^2\nu_e/4H^2$ ($n = 1, 2, 3, \cdots$), the integral vanishes for $t < 0$. For $t > 0$, it is evaluated

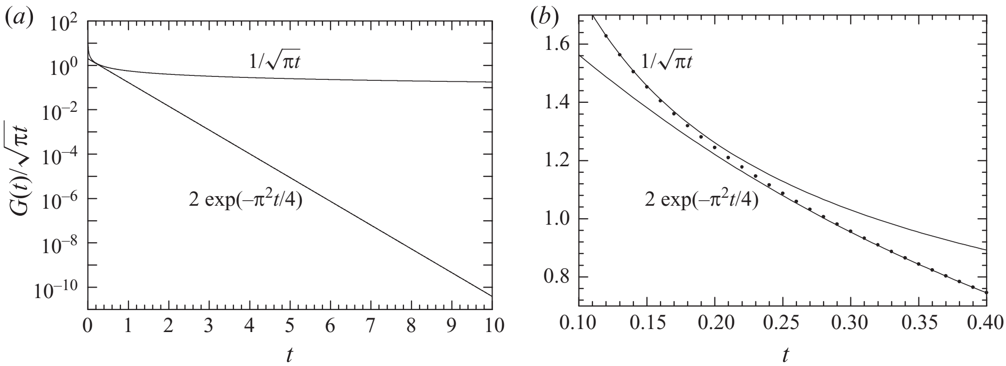

FIGURE 2. Graph of the relaxation function $G(t)/\sqrt{\pi t}$ against t in the intervals $0 < t \leqslant 10$ in (*a*) and $0.1 \leqslant t \leqslant 0.4$ in (*b*), respectively. In (*a*), the profile of $G(t)/\sqrt{\pi t}$ appears to coincide with the asymptotic expressions: $1/\sqrt{\pi t}$ for $t < 0.2$ and $2\exp(-\pi^2 t/4)$ for $t > 0.2$. In (*b*), the dots indicate the values of $G(t)/\sqrt{\pi t}$ numerically calculated, and they are seen to move on from one asymptotic branch to the other over a narrow transition interval as t varies.

by the residue theorem. Thus it follows that

$$\frac{1}{\sqrt{\pi t}} G\left(\frac{\nu_e t}{H^2}\right) = \frac{2\sqrt{\nu_e}}{H} \sum_{n=1}^{\infty} \exp\left[-\frac{(2n-1)^2\pi^2}{4}\frac{\nu_e t}{H^2}\right] h(t). \tag{3.7}$$

The asymptotic expression (3.5) agrees with the leading term of (3.7).

Figure 2 shows the graph of $G(t)/\sqrt{\pi t}$ together with the two leading asymptotic expressions of (3.2) and (3.5) with ν_e/H^2 suppressed. In the scale of t shown in figure 2(*a*), no visible difference from the asymptotic expressions is seen over the interval $0 < t \leqslant 10$. The function $G(t)/\sqrt{\pi t}$ tends to diverge as $t \to 0$ and tends to vanish as $t \to \infty$. It appears to coincide with (3.2) for $t < 0.2$, while with (3.5) for $t > 0.2$. Figure 2(*b*) blows up the graph in the interval $0.1 \leqslant t \leqslant 0.4$, where the dots indicate the values of $G(t)/\sqrt{\pi t}$ numerically calculated by using (2.40). It is surprising to find that the dots are located just on the asymptotic curves except for a narrow transition interval in $0.15 < t < 0.3$. This finding is very useful because $G(t)/\sqrt{\pi t}$ may be approximated substantially by either one of the two asymptotic expressions so that the treatment of the functionals will be facilitated.

3.2. *Approximation of the thermoacoustic-wave equation based on a Deborah number*

Making use of the properties of the relaxation function, the thermoacoustic-wave equation (2.48) is approximated as $De \gg 1$ or $De \ll 1$. When an initial-value problem at $t = 0$ is considered with $p' = 0$ for $t < 0$, and while $De \gg 1$, (3.2) may be truncated at the first term as

$$\frac{1}{\sqrt{\pi t}} G\left(\frac{\nu_e t}{Pr H^2}\right) \approx \frac{1}{\sqrt{\pi t}}. \tag{3.8}$$

This approximation may also be regarded as a case of large H formally. For $De \gg 1$ in (2.48), the contributions due to the memory remain small so that the first two terms may balance mainly with each other. Substituting (3.8) into (2.48), replacing $\partial^2 p'/\partial t^2$ in the fourth term by $(\partial/\partial x)(a_e^2 \partial p'/\partial x)$, and using $(\mathrm{d}/\mathrm{d}x)(a_e^2\sqrt{\nu_e}) = (3/2 +$

$\beta/2)a_e^2\sqrt{\nu_e}T_e^{-1}\mathrm{d}T_e/\mathrm{d}x$, it follows that

$$\frac{\partial^2 p'}{\partial t^2}-\frac{\partial}{\partial x}\left(a_e^2\frac{\partial p'}{\partial x}\right)+\frac{a_e^2\sqrt{\nu_e}}{H}\left[C\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left(\frac{\partial^2 p'}{\partial x^2}\right)+\frac{(C+C_T)}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left(\frac{\partial p'}{\partial x}\right)\right]=0, \tag{3.9}$$

with

$$C=1+\frac{\gamma-1}{\sqrt{Pr}} \quad \text{and} \quad C_T=\frac{1}{2}+\frac{\beta}{2}+\frac{1}{\sqrt{Pr}+Pr}, \tag{3.10}$$

where the derivative of minus half-order is defined over a semi-infinite interval by (Gel'fand & Shilov 1964)

$$\frac{\partial^{-1/2}\phi}{\partial t^{-1/2}}\equiv\frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{\phi(x,\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau, \tag{3.11}$$

and the lower bound may be set equal to zero for the initial-value problem. Equation (3.9) is of the same form as the one derived previously by using the boundary-layer approximation (Sugimoto & Yoshida 2007). But note that the temperature dependence of the viscosity and heat conduction is included here. It is remarked that (3.9) is always valid as long as De is much greater than unity, even if no idea of the boundary layer is introduced.

Next is a case with $De \ll 1$ for a long-time behaviour from an initial state. This case corresponds formally to a case with small H. In this case, the initial state is almost forgotten and the lower bound of the integrals may be set to be minus infinity. If $G(t)/\sqrt{\pi t}$ is viewed over such a long time, it may be approximated to be a delta function in light of the property (2.43) as

$$\frac{1}{\sqrt{\pi t}}G\left(\frac{\nu_e t}{PrH^2}\right)\approx\sqrt{\frac{Pr}{\nu_e}}H\delta(t), \tag{3.12}$$

where note that $\delta(bt)=\delta(t)/b$, b being a positive constant. If this approximation is used in the functional (2.45), it is reduced to

$$\mathscr{M}_P(\phi)\approx\sqrt{\frac{Pr}{\nu_e}}H\phi(x,t). \tag{3.13}$$

Substituting this relation in (2.48), it follows that the wave equation is reduced to an equation $\gamma\partial^2 p'/\partial t^2=0$. This trivial case is also derived from the transformed equation (2.30) by taking the limit $|\delta_e^{-1}|\to 0$ to set $g(x,H)\to\delta_e^{-1}$ and $g_P\to\sqrt{Pr}\delta_e^{-1}$. Then (2.30) is reduced to $\gamma\sigma\hat{p}'=0$.

Thus, corrections due to deviation of the relaxation function from the delta function are desired. They are systematically derived by expanding $g(x,H)$ and $g_P(x,H)$ with respect to δ_e^{-1}, i.e. by using the expansion $\tanh X = X - X^3/3+2X^5/15+\cdots$ for $|X|\ll 1$, X being either δ_e^{-1} or $\delta_e^{-1}/\sqrt{Pr}$. It then follows from the lowest (apart from the above trivial case) relation that

$$\gamma\sigma\hat{p}'-\frac{\partial}{\partial x}\left(\frac{a_e^2H^2}{3\nu_e}\frac{\partial\hat{p}'}{\partial x}\right)+\frac{a_e^2H^2}{3\nu_e T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial\hat{p}'}{\partial x}=0. \tag{3.14}$$

As the magnitude of the second and third terms relative to the first one is of order $|\delta_e|^{-2}\chi^2$, it follows from the balance among them that $|\delta_e|\sim\chi$. While (3.14) is valid for an extremely long-time behaviour, note that this is also applicable to a case that the diffusion layer is extremely thick.

Let a new diffusivity α be defined by

$$\alpha = \frac{a_e^2 H^2}{3\gamma\nu_e}, \tag{3.15}$$

where $a_e/\sqrt{\gamma}$ is a local, isothermal sound speed $\sqrt{p_0/\rho_e}$. If no temperature dependence of viscosity is assumed, then α is a constant. But because α is proportional to $T_e^{-\beta}$, it decreases as T_e increases unlike usual diffusions. Using α, (3.14) is transformed inversely to yield

$$\frac{\partial p'}{\partial t} - \frac{\partial}{\partial x}\left(\alpha\frac{\partial p'}{\partial x}\right) + \frac{\alpha}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial p'}{\partial x} = 0. \tag{3.16}$$

The memory integral now disappears in the limit as $t \to \infty$ and the diffusion equation revives. But it is revealed that the temperature gradient gives rise to the third term responsible for wave propagation.

If further higher-order terms are included, (3.14) is modified as

$$\sigma\hat{p}' - \frac{\partial}{\partial x}\left(\alpha\frac{\partial\hat{p}'}{\partial x}\right) + \frac{\alpha}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial\hat{p}'}{\partial x} - \frac{(\gamma-1)Pr\alpha}{a_e^2}\sigma^2\hat{p}' - \frac{2}{5}(1+Pr)\frac{\alpha H^2}{\nu_e T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\sigma\frac{\partial\hat{p}'}{\partial x} + \frac{2}{5}\frac{\partial}{\partial x}\left(\frac{\alpha H^2}{\nu_e}\sigma\frac{\partial\hat{p}'}{\partial x}\right) = 0. \tag{3.17}$$

The higher-order terms are of order $|\delta_e|^{-4}\chi^2$ ($\sim|\delta_e|^{-2}$) in comparison with the first three terms. Using the lowest relation (3.14) in (3.17), the last term may be approximated as

$$\frac{2}{5}\frac{\partial}{\partial x}\left(\frac{\alpha H^2}{\nu_e}\sigma\frac{\partial\hat{p}'}{\partial x}\right) \approx \frac{2}{5}\frac{H^2}{\nu_e}\left(\sigma^2\hat{p}' - \frac{\alpha\beta}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\sigma\frac{\partial\hat{p}'}{\partial x}\right). \tag{3.18}$$

Thus the higher-order equation is written as

$$\frac{\partial p'}{\partial t} - \frac{\partial}{\partial x}\left(\alpha\frac{\partial p'}{\partial x}\right) + \frac{\alpha}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial p'}{\partial x} + \left[\frac{6}{5}\gamma - (\gamma-1)Pr\right]\frac{\alpha}{a_e^2}\frac{\partial^2 p'}{\partial t^2} - \frac{2}{5}(1+\beta+Pr)\frac{\alpha H^2}{\nu_e T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial^2 p'}{\partial t\partial x} = 0. \tag{3.19}$$

It is to be remarked that the fourth term given by the second-order derivative of t is so small in order that it cannot govern the temporal evolution, though it is higher in order of differentiation than the first one. Discussions on the equations (3.9), (3.16) and (3.19) will be given in § 5.4.

4. Modifications in the case of a circular tube

4.1. *Changes in derivation process of thermoacoustic-wave equation*

The results obtained in the preceding section are qualitatively valid for an axisymmetric wave propagation in a circular tube of radius R and of infinite length, but need to be modified quantitatively. Because the mathematical procedures are the same, their description is kept minimum and only changes are given.

Instead of y, a radial coordinate r is taken and a radial velocity component is designated by v. The condition (2.1) is modified by replacing H with R. Equations (2.15)–(2.18) are replaced by the following ones:

$$\frac{\partial\rho'}{\partial t} + \frac{\partial}{\partial x}(\rho_e u') + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_e v') = 0, \tag{4.1}$$

$$\rho_e \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} + \frac{\mu_e}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u'}{\partial r}\right), \tag{4.2}$$

$$0 = -\frac{\partial p'}{\partial r}, \tag{4.3}$$

$$\rho_e c_p \left(\frac{\partial T'}{\partial t} + u'\frac{\mathrm{d}T_e}{\mathrm{d}x}\right) = \frac{\partial p'}{\partial t} + \frac{k_e}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T'}{\partial r}\right), \tag{4.4}$$

in $-\infty < x < \infty$ and $0 \leqslant r < R$. The boundary conditions require that $u' = v' = T' = 0$ at $r = R$ and they are to be finite at $r = 0$.

Defining the mean value of a variable $\phi(x, r, t)$ over the cross-section by

$$\frac{1}{\pi R^2}\int_0^R 2\pi r \phi(x, r, t)\mathrm{d}r \equiv \bar{\phi}(x, t), \tag{4.5}$$

(2.37) remains valid but $s/2H$ and $q/2H$ are replaced, respectively, by $2s/R$ and $2q/R$ with the definitions of the shear stress and heat flux modified as

$$s = \mu_e \left.\frac{\partial u'}{\partial r}\right|_{r=R} \quad \text{and} \quad q = k_e \left.\frac{\partial T'}{\partial r}\right|_{r=R}. \tag{4.6}$$

The transformed solutions $\hat{u}'$ and $\hat{T}'$ are given by (2.21) and (2.24), respectively, with f and f_P replaced by

$$f = 1 - \frac{I_0(r/R\delta_e)}{I_0(1/\delta_e)}, \tag{4.7}$$

and

$$f_P = 1 - \frac{I_0(r\sqrt{Pr}/R\delta_e)}{I_0(\sqrt{Pr}/\delta_e)}, \tag{4.8}$$

respectively, with

$$\delta_e = \frac{1}{R}\left(\frac{\nu_e}{\sigma}\right)^{1/2}, \tag{4.9}$$

where I_0 and I_1 below denote the modified Bessel functions of zeroth and first order, respectively. While (2.26) for $\hat{\rho}'$ remains valid, (2.27) for $\rho_e \hat{v}'$ still holds with const. $= 0$ and the proviso that y is replaced by $r/2$, and g and g_P are replaced, respectively, by

$$g(x, r) = \frac{I_1(r/R\delta_e)}{I_0(1/\delta_e)}, \tag{4.10}$$

and

$$g_P(x, r) = \frac{I_1(r\sqrt{Pr}/R\delta_e)}{I_0(\sqrt{Pr}/\delta_e)}. \tag{4.11}$$

A functional $\mathscr{N}_P(\phi)$ corresponding to $\mathscr{M}_P(\phi)$ is introduced as

$$\mathscr{N}_P[\phi(x, t)] \equiv \int_{-\infty}^{t} \Theta\left[\frac{\nu_e(t-\tau)}{PrR^2}\right]\phi(x, \tau)\mathrm{d}\tau, \tag{4.12}$$

where Θ is defined in terms of the inverse transform as

$$\frac{1}{\sqrt{2\pi}}\mathscr{F}^{-1}\left\{\sigma^{-1/2}\frac{I_1(\sqrt{Pr}/\delta_e)}{I_0(\sqrt{Pr}/\delta_e)}\right\} = \Theta\left(\frac{\nu_e t}{PrR^2}\right), \tag{4.13}$$

and $\mathscr{N}(\phi)$ denotes $\mathscr{N}_P(\phi)$ with $Pr=1$ formally. As will be shown later, Θ vanishes for $t<0$. The shear stress s and the heat flux q are expressed as follows:

$$s=\sqrt{\nu_e}\mathscr{N}\left(\frac{\partial p'}{\partial x}\right), \tag{4.14}$$

$$\frac{\partial q}{\partial t}=c_pT_e\sqrt{\nu_e}\left\{-\frac{\gamma-1}{\sqrt{Pr}}\mathscr{N}_P\left(\frac{1}{a_e^2}\frac{\partial^2 p'}{\partial t^2}\right)+\frac{1}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\left[\frac{1}{1-Pr}\mathscr{N}\left(\frac{\partial p'}{\partial x}\right)-\frac{1}{(1-Pr)\sqrt{Pr}}\mathscr{N}_P\left(\frac{\partial p'}{\partial x}\right)\right]\right\}. \tag{4.15}$$

The thermoacoustic-wave equation for the gas in the circular tube is given by

$$\frac{\partial^2 p'}{\partial t^2}-\frac{\partial}{\partial x}\left(a_e^2\frac{\partial p'}{\partial x}\right)+\frac{\partial}{\partial x}\left[\frac{2a_e^2\sqrt{\nu_e}}{R}\mathscr{N}\left(\frac{\partial p'}{\partial x}\right)\right]+\frac{2(\gamma-1)}{\sqrt{Pr}}\frac{\sqrt{\nu_e}}{R}\mathscr{N}_P\left(\frac{\partial^2 p'}{\partial t^2}\right)-\frac{2a_e^2}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\sqrt{\nu_e}}{R}\left[\frac{1}{1-Pr}\mathscr{N}\left(\frac{\partial p'}{\partial x}\right)-\frac{1}{(1-Pr)\sqrt{Pr}}\mathscr{N}_P\left(\frac{\partial p'}{\partial x}\right)\right]=0. \tag{4.16}$$

Although the relaxation function Θ is different from $G/\sqrt{\pi t}$, (4.16) is of the same form as (2.48) if $1/H$ and $\mathscr{M}$ are identified, respectively, as $2/R$ and $\mathscr{N}$. The coefficient $2/R$ comes from the ratio of the wetted perimeter to the cross-sectional area. For the channel of unit thickness in the direction normal to a sheet of paper, the wetted perimeter is 2 with no account of the lateral sides, while the area is $2H$. In this context, the mean values of the physical variables over the cross-section of the circular tube are given by the same expressions as (2.49)–(2.51), provided that $1/H$ is replaced with $2/R$ and $\mathscr{M}$ with $\mathscr{N}$.

4.2. *Relaxation function for the circular tube*

As Θ has a dimension of inverse square root of time, it is made dimensionless by setting $\Theta(t)=(\nu_e/PrR^2)^{1/2}\theta(t)$ in (4.13), t being dimensionless in this context. Then θ is expressed in the form of the inverse Laplace transform as

$$\theta(t)=\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}z^{-1/2}\frac{I_1(z^{1/2})}{I_0(z^{1/2})}\,\mathrm{e}^{zt}\mathrm{d}z. \tag{4.17}$$

Simple poles of the integrand are located in the left half-plane of z at $z=-j_n^2$, where j_n $(n=1,2,3,\cdots)$ denote roots $z=j_n$ of $J_0(z)=0$ $(0<j_1<j_2<j_3\cdots)$ with $j_1\approx 2.40$, $j_2\approx 5.52$ and $j_3\approx 8.65$. Thus θ is obtained as

$$\theta(t)=2\sum_{n=1}^{\infty}\mathrm{e}^{-j_n^2 t}h(t). \tag{4.18}$$

Incidentally the integral of θ satisfies

$$\int_0^{\infty}\theta(t)\mathrm{d}t=2\sum_{n=1}^{\infty}j_n^{-2}=\frac{1}{2}. \tag{4.19}$$

Although (4.18) is valid except for $t=0$, it is inappropriate for a small value of t. An alternative expression for $t\ll 1$ is available by expanding the integrand of (4.17)

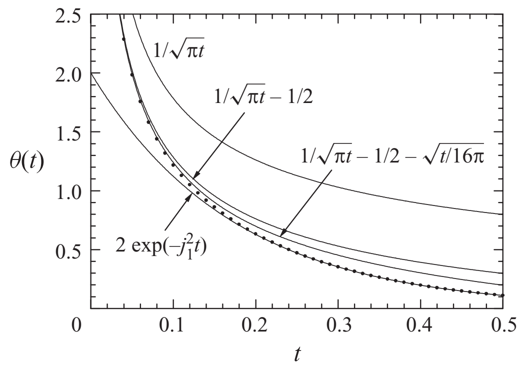

FIGURE 3. Graphs of the relaxation function $\theta(t)$ against t and the leading term of (4.18) and the higher-order approximation given by (4.21) where the profile of $\theta(t)$ appears to coincide with the asymptotic expression $1/\sqrt{\pi t} - 1/2 - \sqrt{t/16\pi}$ for $t < 0.05$ and with the expression $2\exp(-j_1^2 t/4)$ with $j_1 \approx 2.40$ for $t > 0.2$.

around infinity of z. Executing the expansion as

$$z^{-1/2}\frac{I_1(z^{1/2})}{I_0(z^{1/2})} = z^{-1/2} - \frac{1}{2}z^{-1} - \frac{1}{8}z^{-3/2}\cdots, \tag{4.20}$$

it is found from the formulas of the inverse Laplace transform that

$$\theta = \frac{1}{\sqrt{\pi t}} - \frac{1}{2} - \sqrt{\frac{t}{16\pi}} + \cdots, \tag{4.21}$$

for $0 < t \ll 1$. Using these expressions, it follows that

$$\Theta\left(\frac{\nu_e t}{R^2}\right) \approx \frac{1}{\sqrt{\pi t}} - \sqrt{\frac{\nu_e}{4R^2}} - \frac{\nu_e}{R^2}\sqrt{\frac{t}{16\pi}} + \cdots, \tag{4.22}$$

for $0 < \nu_e t/R^2 \ll 1$, while

$$\Theta\left(\frac{\nu_e t}{R^2}\right) \approx \frac{2\sqrt{\nu_e}}{R}\exp\left(-j_1^2\frac{\nu_e t}{R^2}\right), \tag{4.23}$$

for $\nu_e t/R^2 \gg 1$.

Comparing $\theta(t)$ with $G(t)/\sqrt{\pi t}$, both behaviours are quantitatively different from each other. For $t \ll 1$, the leading behaviour is the same but higher-order terms in θ contribute to non-exponential decay. For $t \gg 1$, on the other hand, θ decays faster than $G(t)/\sqrt{\pi t}$ because $\pi/2 < j_1$. But if $R/2$ is identified as H, then (4.23) decays slower than (3.5). This interpretation seems to be appropriate because the gas in the circular tube is surrounded by the tube wall in all angle so that the thermoviscous effects appear more pronouncedly than those in the channel.

Figure 3 plots the value of $\theta(t)$ by dots together with the asymptotic expression (4.21) taken up the third order and the leading expression of (4.18) drawn by curves. The integral (4.17) is evaluated numerically by using the double exponential formula due to Ooura & Mori (1991). It is seen in this case as well that θ switches from lying on (4.21) to (4.18) over a narrow transition interval $0.05 < t < 0.2$, though two more terms than $1/\sqrt{\pi t}$ are necessary in (4.21).

4.3. *Approximation of the thermoacoustic-wave equation*

Equation (4.16) is also approximated just as in the case of the channel. For a large value of De, the leading terms of Θ agrees with that of (3.2). Thus it is found that (3.9) holds provided H is replaced by $R/2$ as

$$\frac{\partial^2 p'}{\partial t^2} - \frac{\partial}{\partial x}\left(a_e^2 \frac{\partial p'}{\partial x}\right) + \frac{2a_e^2\sqrt{\nu_e}}{R}\left[C\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left(\frac{\partial^2 p'}{\partial x^2}\right) + \frac{(C+C_T)}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left(\frac{\partial p'}{\partial x}\right)\right] = 0. \tag{4.24}$$

For a small value of De as well, use is made of the expansion $I_1(X)/I_0(X) = X/2 - X^3/16 + X^5/96 + \cdots$ for $|X| \ll 1$. The higher-order equation is derived by the same way as the previous one. The definition of the diffusivity α is modified to be

$$\alpha = \frac{a_e^2 R^2}{8\gamma\nu_e}. \tag{4.25}$$

If $R/2$ is identified as H, α is larger than that for the channel. Thus the approximate equation for a long-time behaviour is given by

$$\frac{\partial p'}{\partial t} - \frac{\partial}{\partial x}\left(\alpha\frac{\partial p'}{\partial x}\right) + \frac{\alpha}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial p'}{\partial x} + \left[\frac{4}{3}\gamma - (\gamma-1)Pr\right]\frac{\alpha}{a_e^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{6}(1+\beta+Pr)\frac{\alpha R^2}{\nu_e T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial^2 p'}{\partial t \partial x} = 0. \tag{4.26}$$

5. Discussions of the results

This section is devoted to discussions on the thermoacoustic-wave equations (2.48) and (4.16) and the approximation of them. It is pointed out firstly that they are rigorous and equivalent to the system of equations (2.15)–(2.18) and (4.1)–(4.4) supplemented by (2.5) in the narrow-tube approximation, since no approximations have been made at all. It is a great merit that the spatial dimension of the original system is reduced to one dimension, though the price must be paid by having to treat the memory integrals instead. Provided a solution to p' were available, the other variables are determined by tracking back to the formulas expressed in terms of p'.

5.1. *Accuracy of uniformity in pressure over the cross-section*

That the system is reduced to the one dimension owes to uniformity in pressure over each cross-section derived as the outcome of the narrow-tube approximation. Therefore, this approximation may alternatively be called a ‘uniform-pressure approximation.’

As the uniformity results from neglect of higher-order terms, its accuracy is discussed. Unless this approximation was made, (2.2), (2.6), (2.7) and (2.5) must be solved. Let a deviation of p' from a uniform value p'_{uni} in each cross-section be denoted by p'_{dev} and the pressure be the sum $p'_{uni} + p'_{dev}$. Because the pressure must satisfy (2.7), it follows that

$$\frac{\partial p'_{dev}}{\partial t} = -\frac{\partial p'_{uni}}{\partial t} + \rho_e T_e\left(\frac{\partial T'}{\partial t} + u'\frac{\mathrm{d}T_e}{\mathrm{d}x}\right) - k_e\left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2}\right) - \left.\frac{\partial k}{\partial T}\right|_e \frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial T'}{\partial x} \approx -k_e\frac{\partial^2 T'}{\partial x^2} - \left.\frac{\partial k}{\partial T}\right|_e \frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial T'}{\partial x}, \tag{5.1}$$

where T' is approximated by the one obtained by the narrow-tube approximation and (2.18) has been used. The deviatoric pressure is found to be related to the axial gradient and curvature of T' as well as the temperature gradient $\mathrm{d}T_e/\mathrm{d}x$. The magnitude of p'_{dev}/p_0 is estimated to be of $|\delta_e|^2\lambda^2 T'/T_0$. If (2.14) is assumed, the order is of $|\delta_e|^2\lambda^2\chi^2 p'_{uni}/p_0$. In order for the deviatoric pressure to be negligible, $|\delta_e|\lambda\chi$ [$=(\sqrt{\nu_e\omega}/a_e)\chi^2$] should be much smaller than unity. This condition is usually met at an audible frequency because $\sqrt{\nu_e\omega}/a_e=1$ for the atmospheric air at $\omega/2\pi\approx 10^9$ Hz.

5.2. *Reduction to equation of pressure amplitude for a time-harmonic oscillation*

The thermoacoustic-wave equations are capable of describing a spatio-temporal behaviour of any disturbance in p' in the framework of the linear theory. If p' is assumed to be time-harmonic in the form of $P(x)\exp(\mathrm{i}\omega t)$, $P(x)$ being a complex pressure amplitude, then it is verified that (2.48) and (4.16) are reduced to the ordinary differential equations for $P(x)$ derived by Rott (1969). Noting, for example, that

$$\mathscr{M}_P[\exp(\mathrm{i}\omega t)] = (\mathrm{i}\omega)^{-1/2}\tanh\left[\sqrt{Pr}\,H\left(\frac{\mathrm{i}\omega}{\nu_e}\right)^{1/2}\right]\exp(\mathrm{i}\omega t), \tag{5.2}$$

equations for P are readily available by reversing the sign of ω in the expressions so far derived. Following the notations used by Rott (1969), the pressure equations for P are derived from (2.48) and (4.16) as follows:

$$\omega^2[1+(\gamma-1)f_j^*]P + \frac{\mathrm{d}}{\mathrm{d}x}\left[a_e^2(1-f_j)\frac{\mathrm{d}P}{\mathrm{d}x}\right] - \frac{a_e^2}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\left(\frac{f_j^*-f_j}{1-Pr}\right)\frac{\mathrm{d}P}{\mathrm{d}x} = 0, \tag{5.3}$$

where the subscript j distinguishes the cases of the channel and of the circular tube, respectively, by $j=0$ and $j=1$, and $f_j(\eta_0)$ are defined as

$$f_0(\eta_0) = \frac{\tanh\eta_0}{\eta_0} \quad \text{and} \quad f_1(\eta_0) = \frac{2J_1(\mathrm{i}\eta_0)}{\mathrm{i}\eta_0 J_0(\mathrm{i}\eta_0)} = \frac{2I_1(\eta_0)}{\eta_0 I_0(\eta_0)}, \tag{5.4}$$

and $f_j^* = f_j(\sqrt{Pr}\,\eta_0)$, η_0 being the inverse of δ_e with the sign of ω reversed as $(\mathrm{i}\omega/\nu_e)^{1/2}H$ for the channel and $(\mathrm{i}\omega/\nu_e)^{1/2}R$ for the circular tube, respectively. Thus it is found that Rott's equations are included in (2.48) and (4.16) as a special case.

5.3. *Deborah number and Reynolds number*

As was defined in § 3.1, the Deborah number is a dimensionless parameter of a temporal scale in comparison with a diffusion time. By contract, the well-known Reynolds number Re defined by using a typical speed U as

$$Re = \frac{UH}{\nu_e}, \tag{5.5}$$

H being replaced by R for the circular tube, may be regarded as a parameter of a spatial scale which measures a thickness of the viscous diffusion layer ν_e/U in comparison with H (or R). If the thermal diffusion layer κ_e/U is taken on the right-hand side of (5.5), then it is the Péclet number Pe ($\equiv RePr$). As the Deborah number and Reynolds number illuminate, respectively, the temporal and spatial aspects of the diffusion process, they are independent of each other in general.

For a short-time behaviour from the initial state, a time elapsed is chosen to be typical, and then the approximate equations (3.9) and (4.24) are well applicable. In the course of time, the Deborah number decreases, while the diffusion layer is not yet fully established so that the Reynolds number is not well defined. As the value of De

becomes smaller than unity, (3.9) and (4.24) tend to be invalid. Then it appears that no approximations are available so that the full equations (2.48) and (4.16) must be employed.

Thanks to the properties of the relaxation functions, however, one crude approximation is to truncate them at a finite time, for example $G/\sqrt{\pi t}$ for the channel at $t = 0.2$, and to neglect the rapid exponential decay that follows. But this function does not satisfy (2.43). Another choice is to approximate it by the two asymptotic functions as

$$\frac{1}{\sqrt{\pi t}}G(t) = \begin{cases} \dfrac{1}{\sqrt{\pi t}} & \text{for} \quad t < M, \\ 2\exp\left(-\dfrac{\pi^2 t}{4}\right) & \text{for} \quad t > M, \end{cases} \tag{5.6}$$

where a value of M is chosen for (5.6) to satisfy (2.43). This condition gives $M \approx 0.213$ but the discontinuity occurs at $t = M$ with jump of 0.04. For $\theta(t)$ in the case of circular tube, however, inclusion of the higher-order terms given by (4.21) is necessary to make a jump small, whereas only the leading term is taken in (4.18) because inclusion of the higher-order terms makes the jump larger. Then $M \approx 0.123$ with the jump of 0.077. The point of switch at $t = M$ is located in the middle of the transition interval in both cases (see figures 2 and 3).

Using (5.6) into (2.45), it consists of the integral in terms of the derivative of minus half-order and a new integral of $(G-1)/\sqrt{\pi t}$ over the interval $0 \leqslant \tau \leqslant t-(PrH^2/\nu_e)M$ for $\nu_e t/PrH^2 > M$. Substitution of this into (2.48) yields (3.9) supplemented by the new integrals. Here, note that the replacement of $\partial^2 p'/\partial t^2$ in the fourth term of (2.48) with $(\partial/\partial x)(a_e^2 \partial p'/\partial x)$ previously made is not allowable now. But it is found that if the approximation (5.6) is used, (3.9) may be revived with these modifications.

If a further long time has passed so that the initial state may be forgotten virtually, this case corresponds to a situation in which the lower bound in the memory integrals may be regarded as minus infinity. Then the Deborah number vanishes and loses its significance. What is then taken to be a typical time? If an acoustic field is oscillating or fluctuating, its period or inverse of a peak frequency in Fourier spectrum should be taken as a typical time, for example ω^{-1}. Then the Deborah number may be modified by using this time as

$$De = \frac{H^2/\nu_e}{\omega^{-1}}. \tag{5.7}$$

This is nothing but $|\delta_e^{-2}|$ and substantially the Reynolds number because $H\omega$ is regarded as U. Thus, it is the Reynolds number that is the only parameter in this case.

To summarize the approximations of the thermoacoustic-wave equations so far made based on the Deborah number and the Reynolds number, figure 4 illustrates a diagram in the case of the channel indicating qualitatively in which temporal and spatial domains the approximate equations are valid. For the case of the circular tube as well, of course, the same diagram is drawn. Here the horizontal axis represents a typical length ν/U normal to the plate surface, while the vertical axis represents a time t. A state prior to $t = 0$ is assumed to be undisturbed. The time may be taken unlimited toward future, whereas the length is limited to be smaller than H by the plate surface. But this length should be understood as a thickness to be determined by the diffusion only in the absence of the other plate. The horizontal axis corresponds to an initial state.

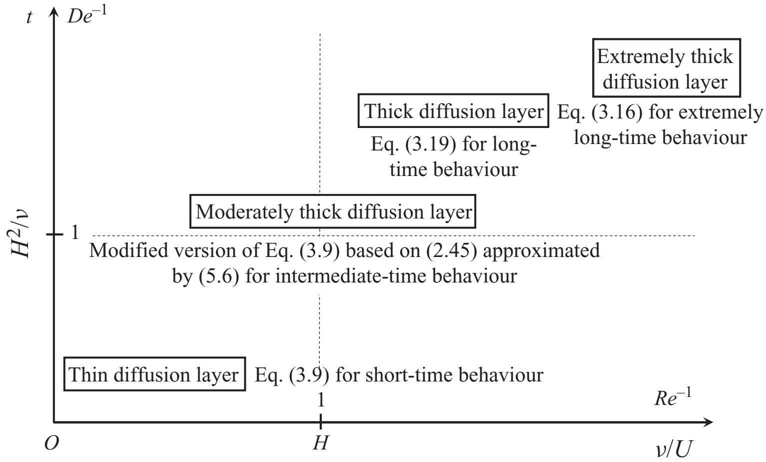

FIGURE 4. Diagram of space and time domains for the approximate equations of the thermoviscous-wave equation (2.48) for the channel to be valid where the horizontal and vertical axes measure, respectively, a typical length ν/U normal to the plate surface, U being a typical axial speed of gas, and a time t, which correspond to Re^{-1} and De^{-1}, respectively.

The horizontal and vertical axes also measure the inverse of the Reynolds number and the inverse of the Deborah number, respectively. The unit in the horizontal and vertical axes corresponds, respectively, to a length for ν/U to be equal to the thickness H and to a time for t to be equal to the diffusion time H^2/ν. The scales are not linear but rather logarithmic. Equation (2.48) is valid everywhere in the whole domain.

A domain for $De^{-1} \ll 1$ corresponds to a situation where only a short time has passed from the initial state. Remember that this time is measured by the diffusion time. Even if short, it happens in the case of a thin diffusion layer, i.e. a wide tube that a corresponding dimensional time becomes very long compared with a period of oscillations. Equation (3.9) is valid in a domain for De^{-1} less than 0.2 approximately. But it should be emphasized that (3.9) holds irrespective of the value of Re^{-1}. In the domain around $De^{-1} \approx 1$ and $Re^{-1} \approx 1$ corresponding to an intermediate time and a moderately thick diffusion layer, the modified version of (3.9) may be applicable if $G/\sqrt{\pi t}$ in (2.45) is approximated by using (5.6). Right above this, there spreads a domain for $De^{-1} \gg 1$ and $Re^{-1} \gg 1$ in which (3.19) holds for a long-time behaviour and a thick diffusion layer. At the right top corner, where $De^{-1} \to \infty$ and $Re^{-1} \to \infty$, (3.16) may be applicable. For a long-time behaviour, the meaning of the vertical axis is lost and the approximations should be referred to the horizontal axis, i.e. to the thickness of diffusion layer only.

5.4. *Properties of the approximate wave equations*

General properties of the respective approximate equations are discussed. It has been unveiled that (3.9) and (4.24) in the case of $De \gg 1$ are the same as the equation derived by the boundary-layer approximation. What is to be emphasized again is that they are always valid as long as De is large, whatsoever the thickness of the thermoviscous layer may be. This may be a reason why the initial instability is described by the boundary-layer theory (Sugimoto & Shimizu 2008). Equations (3.9) and (4.24) are also obtained if a formal limit as $H \to \infty$ or $R \to \infty$ is taken in the memory integrals. Thus they may be regarded as the approximation for a wide tube. Even if a long time has passed, they are applicable as long as $\nu_e t/H^2$ is much smaller than unity, and then the lower bound of the integral may be set to be minus infinity by a shift of time origin. But it is a merit to find that (3.9) and (4.24) may be employed beyond the range of validity, if appropriate modifications are included.

In this connection, reference is made to the equations in a case without temperature gradient. With $\mathrm{d}T_e/\mathrm{d}x = 0$, (4.24) is simply (3) in Sugimoto & Horioka (1995) (if the diffusivity of sound ν_d is negligible) derived on the basis of the boundary-layer theory developed by Chester (1964) (see also pp. 519–525 in Blackstock 2000). Chester gave the memory integral but did not use the derivative of minus half-order defined by (3.11). For use of fractional calculus in this context, see Sugimoto (1989). The constant C should be attributed to Kirchhoff. It consists of two factors, 1 and $(\gamma - 1)/\sqrt{Pr}$, which stem from the shear stress and the heat flux at the plate surfaces, respectively. This is found from (2.37) with (4.14) and (4.15). When temperature gradient is present, the other constant C_T appears, which is the coefficient of $T_e^{-1}\mathrm{d}T_e/\mathrm{d}x$ involved in the velocity v at the edge of the boundary layer to be shown in (5.17).

For $De \ll 1$, on the other hand, the equation in the form of (3.16) is derived, to the lowest approximation, for the circular tube as well, though the value of the diffusivity α is different. If no temperature gradient is present, then the third term vanishes and (3.16) is reduced simply to the diffusion equation. Its time-harmonic solution propagating toward the positive direction of x while decaying is easily obtained in the form of $\exp(\mathrm{i}\omega t - mx)$ with $m > 0$, where m^2 is given by $m^2 = \mathrm{i}\omega/\alpha$. The value of m agrees with those obtained by Rayleigh (1945) for the channel and the circular tube in his 'narrow tube' ((29) and (24) on pp. 327–328) corresponding to a case of thick diffusion layer in the present context.

Rewriting m as $\mathrm{i}\omega/c$ in the form of $\exp[\mathrm{i}\omega(t - x/c)]$, the complex phase velocity c is given by $(1+\mathrm{i})\sqrt{\alpha\omega/2}$. For the circular tube, in particular, this velocity is given by

$$c = \left(\frac{1+\mathrm{i}}{4}\right) \frac{R}{\sqrt{\nu_0/\omega}} \frac{a_0}{\sqrt{\gamma}}. \tag{5.8}$$

For the channel, $R/4$ is replaced by $H/\sqrt{6}$. Note that, $a_0/\sqrt{\gamma}$ is the isothermal sound speed, while $R/\sqrt{\nu_0/\omega}$ is the ratio of the thickness of the diffusion layer to the radius, which is much smaller than unity, i.e. $|\delta_e|^{-1} \ll 1$. Thus, c becomes even slower than the isothermal sound speed by $|\delta|^{-1}$. For $R = 0.1$ mm, the real part of (5.8) takes 47.2 m/s for $\omega/2\pi = 100$ Hz in the atmospheric air at 15°C, for which $|\delta_e|^{-1} \approx 0.658$.

Now the effect of the temperature gradient is discussed. The coefficient of $\partial p'/\partial x$ in the third term of (3.16) is denoted by V as

$$V(x) = \frac{\alpha}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x} = \frac{a_e^2 H^2}{3\gamma\nu_e T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}, \tag{5.9}$$

where $H^2/3$ is replaced by $R^2/8$ for the circular tube. Note that V has a dimension of speed and the sign is determined by the temperature gradient. This velocity is brought about by the action of diffusion in the presence of the temperature gradient. If the temperature varies according to $(T_e/T_0)^\beta = 1/(1 - \beta V_0 x/\alpha_0)$, T_0, α_0 and V_0 implying, respectively, the values of T_e, α and V at $x = 0$, then V takes a constant equal to V_0 everywhere. Supposing that $T_e^{-1}\mathrm{d}T_e/\mathrm{d}x$ is 10 m^{-1} at $x = 0$ and 15°C for the atmospheric air enclosed in a circular tube with $R = 0.1$ mm, it is found that V_0 takes 71.1 m/s.

For the sake of simplicity, the temperature dependence of the viscosity and thermal conductivity is ignored in the following. Then β is set to be zero so that α is a constant, independent of the temperature. Further if the temperature distribution is assumed to be exponential in the form of $\exp(x/l)$, then $T_e^{-1}\mathrm{d}T_e/\mathrm{d}x$ becomes constant and so does V. Then the solution of (3.16) is easily anticipated. If a spatial frame moving with V is introduced, (3.16) may be written as the diffusion equation in this frame. While the diffusion in the presence of the temperature gradient yields the wave propagation, it is found that no active action occurs in the case that the diffusion layer is extremely thick.

This prompts us to examine effects by the higher-order equations (3.19) and (4.26). Suppose that the temperature gradient is so steep that the third term may surpass the second term and therefore the first term for evolution balances mainly with the third one, i.e. $\partial p'/\partial t + V\partial p'/\partial x \approx 0$ to the lowest order. Using this approximation, the time derivatives in the higher-order equation are expressed in terms of the spatial derivative. Then (3.19) and (4.26) are expressed in the following form:

$$\frac{\partial p'}{\partial t} + V\frac{\partial p'}{\partial x} = D\frac{\partial^2 p'}{\partial x^2}, \tag{5.10}$$

where the diffusion coefficient D is given by

$$D = \alpha\left\{1 - [n\gamma(2+Pr) - (\gamma-1)Pr]\, V^2/a_e^2\right\}, \tag{5.11}$$

where n is 6/5 for the channel and 4/3 for the circular tube, and β has already been set equal to zero. If the temperature gradient is assumed to be steep enough to make D negative for plausible values of γ and Pr, the higher-order terms will contribute to a negative diffusion. Hence, it is expected that the temperature gradient will give rise to instability convected with the velocity V.

5.5. *Approximation of the acoustic field*

Finally the acoustic field is discussed in two approximations. It has already been noted that the equations (2.15)–(2.18) and (2.5) are the same as those derived for the boundary layer ((4.1)–(4.4) in Sugimoto & Tsujimoto 2002). If the second-order derivatives with respect to y for the viscosity and heat condition are dropped, they are simply the equations for lossless and one-dimensional propagation. Keeping this in mind, approximation of the acoustic field is considered.

If a thin diffusion layer is concerned, i.e. $|\delta_e| \ll 1$, then the acoustic field outside of the diffusion layer is almost uniform over the cross-section. In fact, f and f_P given by (2.22) and (2.25) for the channel are approximated to be unity because, for example

$$\frac{\cosh(y\sqrt{Pr}/H\delta_e)}{\cosh(\sqrt{Pr}/\delta_e)} \approx \exp\left[-\frac{\sqrt{Pr}}{\delta_e}\left(1-\frac{|y|}{H}\right)\right]. \tag{5.12}$$

For the circular tube, f_P given by (4.8) is approximated by (5.12) with $|y|/H$ replaced by r/R. In the outside of the diffusion layer where $1-|y|/H \gg |\delta_e|$, the contribution by (5.12) is exponentially small. Then (2.21) and (2.24) are simply the transformed relations derived from (2.16) and (2.18) with the y-dependence dropped. The neglect of the diffusion terms is thus justified by the above approximation, and the neglect of $\partial v'/\partial y$ will be shown to be appropriate below.

Among the physical variables, above all v' is interesting to be examined. In (2.27), for example, g_P is approximated as

$$\frac{\sinh(y\sqrt{Pr}/H\delta_e)}{\cosh(\sqrt{Pr}/\delta_e)} \approx \operatorname{sgn} y\, \exp\left[-\frac{\sqrt{Pr}}{\delta_e}\left(1-\frac{|y|}{H}\right)\right], \tag{5.13}$$

so that g and g_P are negligible in the outside of the diffusion layer. Thus v' changes linearly with y (or r) in each cross-section. At the edge of the boundary layer where $|y| \approx H$ but $(1-|y|/H)/|\delta_e| \gg 1$, $\hat{v}'$ in (2.27) may be approximated as

$$\hat{v}' \approx -\frac{1}{\rho_e a_e^2}\left[\sigma \hat{p}' - \frac{\partial}{\partial x}\left(a_e^2 \sigma^{-1} \frac{\partial \hat{p}'}{\partial x}\right)\right] H, \tag{5.14}$$

where the boundary layer adjacent to the plate at $y = H$ is concerned. For the circular tube, H is replaced by $R/2$. The factor bracketed in (5.14) corresponds to the first two terms in (2.30). When they are replaced by the remaining terms in (2.30), $\hat{v}'$ is found to be proportional to $\sqrt{\nu_e}$ and the magnitude of $\hat{v}'$ is estimated to be smaller than the one of $\hat{u}'$ by the factor $|\delta_e|\lambda$. Thus the neglect of v' is justified.

By executing the above replacement and setting $g = g_{\mathrm{P}} = 1$, $\hat{v}'$ is written as

$$\hat{v}' = \frac{\sqrt{\nu_e}}{\rho_e a_e^2}\left(C\sigma^{1/2}\hat{p}' + C_T \frac{a_e^2}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\sigma^{-3/2}\frac{\partial \hat{p}'}{\partial x}\right), \tag{5.15}$$

where the lowest relation has been used for the lossless propagation $\sigma^2 \hat{p}' = (\partial/\partial x)(a_e^2 \partial \hat{p}'/\partial x)$ to remove the second-order derivative with respect to x. To make a comparison with the velocity derived previously by the boundary-layer theory, (5.15) is rewritten in terms of $\hat{u}'$. Using the transformed relations for the lossless propagation: $\sigma \hat{p}'/\rho_e a_e^2 + \partial \hat{u}'/\partial x = 0$ and $\rho_e \sigma \hat{u}' = -\partial \hat{p}'/\partial x$, the first being simply the equation of continuity (2.15) and rewritten by expressing $\hat{\rho}'$ in terms of $\hat{p}'$ ((4.8) in Sugimoto & Tsujimoto 2002), it follows that $\hat{v}'$ is expressed as

$$\hat{v}' = -\sqrt{\nu_e}\left(C\sigma^{-1/2}\frac{\partial \hat{u}'}{\partial x} + \frac{C_T}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\sigma^{-1/2}\hat{u}'\right). \tag{5.16}$$

The inverse transform of (5.16) yields

$$v' = -\sqrt{\nu_e}\left[C\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left(\frac{\partial u'}{\partial x}\right) + \frac{C_T}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial^{-1/2}u'}{\partial t^{-1/2}}\right]. \tag{5.17}$$

This relation agrees perfectly with $-v_b$ derived by the boundary-layer theory ((4.22) in Sugimoto & Tsujimoto 2002) where v_b is taken positive when directed away from the wall and β in C_T is set equal to zero. The relation (5.17) is also valid in the case of the circular tube as it is.

In a similar fashion, the shear stress s acting on the gas at both plate surfaces is given by

$$s = 2\sqrt{\nu_e}\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left(\frac{\partial p'}{\partial x}\right), \tag{5.18}$$

while the heat flux q flowing into the gas through them is given by

$$q = 2\rho_e c_p T_e \sqrt{\nu_e}\left[(C-1)\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left(\frac{\partial u'}{\partial x}\right) + \left(C_T - \frac{1}{2} - \frac{1}{2}\beta\right)\frac{1}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial^{-1/2}u'}{\partial t^{-1/2}}\right]. \tag{5.19}$$

These may also be available directly from (2.44) and (2.46). The heat flux through one plate $q/2$ agrees with the result previously obtained ((4.24) in Sugimoto & Tsujimoto 2002) and holds for the circular tube.

Next, the opposite case for the thick diffusion layer with $|\delta_e| \sim \chi \gg 1$ is concerned. The functions f and g without and with the subscript P are now expanded into the power series of δ_e^{-1}. For the case of the channel, for example f_P in (2.25) and g_P in

(2.29), are approximated, respectively, as

$$f_P = \frac{Pr\sigma}{2\nu_e}(H^2 - y^2), \tag{5.20}$$

and

$$g_P = \frac{\sqrt{Pr}\,\sigma^{1/2}}{\sqrt{\nu_e}}\left[1 + \frac{Pr\sigma}{\nu_e}\left(\frac{y^2}{6} - \frac{H^2}{2}\right)\right]. \tag{5.21}$$

Thus the axial velocity is given by

$$u' = -\frac{1}{2\mu_e}\frac{\partial p'}{\partial x}(H^2 - y^2). \tag{5.22}$$

This distribution is simply that of the Poiseulle flow of incompressible fluids driven by a pressure gradient. But there appears v' in this case, which is given by

$$v' = \frac{\gamma}{2\rho_e a_e^2 H^2}\frac{\partial p'}{\partial t}\left(H^2 - y^2\right) y, \tag{5.23}$$

where the lowest approximation (3.16) has been used. To this approximation, the temperature disturbance T' is taken to vanish everywhere since the channel is too narrow for the temperature to change from the one imposed by the boundary condition.

Using (5.22) and (5.23), the order of magnitudes of u' and v' is estimated in terms of p'/p_0 as

$$\frac{u'}{a_e} \sim \frac{1}{|\delta_e|}\frac{p'}{p_0} \quad \text{and} \quad v' \sim \lambda u'. \tag{5.24}$$

Since $|\delta_e| \gg 1$ so that the axial pressure gradient balances with the viscous term, the first estimate is different from (2.13). By the isothermal approximation with $T' = 0$, ρ' may be set equal to $\rho_e p'/p_0$ $(=\gamma p'/a_e^2)$. Using this in (2.32) and also $\rho_e \bar{u}' = -(H^2/3\nu_e)\partial p'/\partial x$ calculated by (5.22), it is found that (3.16) is recovered. Thus (3.16) may be regarded as the equation of continuity averaged and expressed in terms of p'. Of course it is also verified that (5.22) and (5.23) satisfy (2.15) locally, as far as the lowest approximation (3.16) is concerned. Using (5.23), in passing, the order of terms neglected in (2.17) is checked. Because $\rho_e \partial v'/\partial t$ is smaller in magnitude than $\mu_e \partial^2 v'/\partial y^2$ by $|\delta_e|^{-2}$, $\partial p'_{dev}/\partial y$ balances with the latter. Thus p'_{dev} is found to be of order $\lambda^2 p'_{uni}$.

If the higher-order approximation is concerned, then a small change in T' is obtained by (2.24) as

$$T' = \frac{1}{2k_e}\frac{\partial p'}{\partial t}(H^2 - y^2) + \frac{Pr}{24\rho_e \nu_e^2}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial p'}{\partial x}(5H^4 - 6H^2y^2 + y^4), \tag{5.25}$$

where both terms are of the same order as long as $|\delta_e| \sim \chi$ and the order of magnitude of T'/T_e is found to be smaller than that of p'/p_0 by $|\delta_e|^{-2}$. The temperature gradient gives rise to a quartic distribution in y in addition to the quadratic one in the case of no gradient. Using the distribution (5.25) and higher-order terms in (5.22), (3.17) follows from (2.32). The first and second terms in (5.25) yield parts of the fourth and fifth terms in (3.19), respectively. For the circular tube, the factor $(H^2 - y^2)$ in (5.22) and (5.25) is replaced by $(R^2 - r^2)/2$, while $(1 - y^2/H^2)y$ in (5.23) is replaced by $(1 - r^2/R^2)r$. The quartic distribution $(5H^4 - 6H^2y^2 + y^4)/24$ in (5.25) is replaced by $(3R^4 - 4R^2r^2 + r^4)/64$.

Finally, the shear stress and the heat flux from the plate surfaces are given, respectively, by

$$s = 2\frac{\partial p'}{\partial x}H, \tag{5.26}$$

and

$$q = -2\frac{\partial p'}{\partial t}H, \tag{5.27}$$

where H is replaced by $R/4$ for the circular tube. Note that no viscosity and heat conduction appear. These relations result from the right-hand sides of (2.33) and (2.34) because the axial velocity and the temperature change are so little that the left-hand side may be negligible.

6. Conclusions

This paper has developed a general theory for linear propagation of acoustic waves in the gas enclosed in the two-dimensional channel and in the circular tube subject to temperature gradient axially. In the framework of the narrow-tube approximation, the respective systems of basic equations for the gas are reduced rigorously to the one-dimensional, integro-differential equations for the excess pressure. Essence of the thermoacoustic effects lies in the memory integrals. The thermoacoustic-wave equations thus derived are general and always valid.

But the approximations of the equations are useful in discussing a short-time and a long-time behaviour, which correspond, respectively, to the case with large H (or R) and the one with small H (or R). For a short-time behaviour, the equations are reduced to simply the one derived previously by the boundary-layer theory. It now turns out that the boundary-layer theory is valid in any case as long as the Deborah number remains large. This is the reason why the boundary-layer theory is applicable to derive the marginal conditions of initial instability. But as this number becomes smaller than unity, it is limited to a case in which the diffusion layer is thin. Nevertheless, it is expected to be applicable to some problems just as Kirchhoff's theory for a wide tube is useful even when the thickness of a diffusion layer becomes comparable with the tube radius. This is endorsed by the finding that the equations valid for a large value of the Deborah number are still applicable if some modifications are introduced.

For a long-time behaviour as the Deborah number tends to vanish, the memory due to the thermoviscous effects disappears and the simple diffusion revives. It is revealed that the temperature gradient gives rise to wave propagation but no instability occurs after an extremely long time. This also suggests no instability if the diffusion layer is extremely thick. As temperature gradient becomes steeper, however, it may happen that the higher-order terms give rise to negative diffusion and convective instability over a long time even if the diffusion layer is thick. To expect the instability, therefore, the span should not be taken too narrow. Hence, the thermoacoustic-wave equations cover any situation which will occur in the channel and the tube, and they also provide the basis of the approximations depending on situations of interest.

REFERENCES

Abramowitz, M. & Stegun, I. A. 1972 *Handbook of Mathematical Functions*. Dover.
Blackstock, D. T. 2000 *Fundamentals of Physical Acoustics*. John Wiley & Sons.
Chester, W. 1964 Resonant oscillations in closed tubes. *J. Fluid Mech.* **18**, 44–64.
Christensen, R. M. 1982 *Theory of Viscoelasticity: An Introduction*, 2nd edn. Academic Press.

Gel'fand, I. M. & Shilov, G. E. 1964 *Generalized Functions*, vol. 1. pp. 115–122. Academic Press.
Howe, M. S. 1998 *Acoustics of Fluid–Structure Interactions*. Cambridge University Press.
Kramers, H. A. 1949 Vibrations of a gas column. *Physica.* **15**, 971–984.
Kirchhoff, G. 1868 Ueber den Einfluß der Wärmeleitung in einem Gase auf die Schallbewegung. *Ann. Phys. Chem.* **134**, 177–193.
Oberhettinger, F. 1957 *Tabellen zur Fourier Transformation*. Springer.
Ooura, T. & Mori, M. 1991 The double exponential formula for oscillatory functions over the half infinite interval. *J. Comput. Appl. Math.* **38**, 353–360.
Rayleigh, Lord 1945 *The Theory of Sound* vol. 2, pp. 319–328. Dover.
Reiner, M. 1964 The Deborah number. *Phys. Today* **17**, 62.
Rott, N. 1969 Damped and thermally driven acoustic oscillations in wide and narrow tubes. *Z. Angew. Math. Phys.* **20**, 230–243.
Rott, N. 1973 Thermally driven acoustic oscillations. Part II. Stability limit for helium. *Z. Angew. Math. Phys.* **24**, 54–72.
Shimizu, D. & Sugimoto, N. 2009 Physical mechanisms of thermoacoustic Taconis oscillations. *J. Phys. Soc. Japan.* **78**, 094401 (1–6).
Shimizu, D. & Sugimoto, N. 2010 Numerical study of thermoacoustic Taconis oscillations. *J. Appl. Phys.* **107**, 034910 (1–11).
Sone, Y. 2002 *Kinetic Theory and Fluid Mechanics*. Birkhäuser.
Sugimoto, N. 1989 'Generalized' Burgers equations and fractional calculus. In *Nonlinear Wave Motion* (ed. A. Jeffrey), pp. 162–179. Longman Scientific & Technical.
Sugimoto, N. & Horioka, T. 1995 Dispersion characteristics of sound waves in a tunnel with an array of Helmholtz resonators. *J. Acoust. Soc. Am.* **97**, 1446–1459.
Sugimoto, N. & Shimizu, D. 2008 Boundary-layer theory for Taconis oscillations in a helium-filled tube. *Phys. Fluids* **20**, 104102 (1–11).
Sugimoto, N. & Takeuchi, R. 2009 Marginal conditions for thermoacoustic oscillations in resonators. *Proc. R. Soc.* A **465**, 3531–3552.
Sugimoto, N. & Tsujimoto, K. 2002 Amplification of energy flux of nonlinear acoustic waves in a gas-filled tube under an axial temperature gradient. *J. Fluid Mech.* **456**, 377–409.
Sugimoto, N. & Yoshida, M. 2007 Marginal condition for the onset of thermoacoustic oscillations of a gas in a tube. *Phys. Fluids* **19**, 074101 (1–13).
Swift, G. W. 1988 Thermoacoustic engines. *J. Acoust. Soc. Am.* **84**, 1145–1180.
Swift, G. W. 2002 *Thermoacoustics: A Unifying Perspective for Some Engines and Refrigerators*. Acoustical Society of America.
Taconis, K. W., Beenakker, J. J. M., Nier, A. O. C. & Aldrich, L. T. 1949 Measurements concerning the vapour–liquid equilibrium of solutions of He^3 in He^4 below 2.19°K. *Physica.* **15**, 733–739.
Tanner, R. I. 1985 *Engineering Rheology*. Oxford University Press.
Tijdeman, H. 1975 On the propagation of sound waves in cylindrical tubes. *J. Sound Vib.* **39**, 1–33.
Yazaki, T., Tashiro, Y. & Biwa, T. 2007 Measurements of sound propagation in narrow tubes. *Proc. R. Soc.* A **463**, 2855–2862.
Yazaki, T., Tominaga, A. & Narahara, Y. 1980 Experiments on thermally driven acoustic oscillations of gaseous helium. *J. Low Temp. Phys.* **41**, 45–60.
Weston, D. E. 1953 The theory of the propagation of plane sound waves in tubes. *Proc. Phys. Soc. Lond.* B **66**, 695–709.
Zwikker, C. & Kosten, C. W. 1949 *Sound Absorbing Materials*, pp. 34–40. Elsevier.