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LEBESGUE DENSITY AND Π_1^0 CLASSES

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Abstract. Analyzing the effective content of the Lebesgue density theorem played a crucial role in some recent developments in algorithmic randomness, namely, the solutions of the ML-covering and ML-cupping problems. Two new classes of reals emerged from this inquiry: the *positive density points* with respect to *effectively closed* (or Π_1^0) sets of reals, and a proper subclass, the *density-one points*. Bienvenu, Hölzl, Miller, and Nies have shown that the Martin-Löf random positive density points are exactly the ones that do not compute the halting problem. Treating this theorem as our starting point, we present several new results that shed light on how density, randomness, and computational strength interact.

§1. A generalization of 1-genericity. The Lebesgue density theorem says that if A is any Lebesgue measurable set of reals, for almost every point x of A, the *density of A at x is* 1. Roughly speaking, the more we "zoom in" on x by looking at a smaller and smaller interval containing it, the closer to 1 is the fractional measure of A within that interval.

Suppose that $\mathcal{C}$ is a countable collection of Lebesgue measurable subsets of the unit interval. We say $x \in [0, 1]$ is a *positive density point for $\mathcal{C}$* if for every $P \in \mathcal{C}$ that contains x, the density of P at x is positive. We say x is a *density-one point for $\mathcal{C}$* if for every $P \in \mathcal{C}$ that contains x, the density of P at x is 1. It follows from the Lebesgue density theorem that almost every point in the unit interval is a density-one point for $\mathcal{C}$. Of particular interest are the positive density and density-one points we obtain when $\mathcal{C}$ is the collection of *effectively closed* (or Π_1^0) subsets of the unit interval. These have been at the heart of several recent developments in algorithmic randomness, such as the solutions of the ML-covering and ML-cupping problems [1, 4, 5]. An interesting fact that emerged from this line of research is a new characterization of the Turing incomplete[1] Martin-Löf random reals:

THEOREM 1.1 (Bienvenu, Hölzl, Miller, and Nies [3]). *A Martin-Löf random real is a positive density point if and only if it is incomplete.*

The positive density points are properly contained within the class of Kurtz random reals, but not within the Martin-Löf random reals. So Theorem 1.1 leads us to ask: Are positive density points computationally weak in general? In the

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[1] A real is *Turing incomplete* (henceforth, simply *incomplete*) if it does not Turing compute the halting problem.

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other direction, are the Kurtz random reals that are not positive density points computationally powerful?

The 1-generics, which have been widely studied in computability theory, are closely connected to the density-one points of Π^0_1 classes. In fact, every 1-generic is a density-one point. If the former is a member of a Π^0_1 class P, then P contains an open interval around it. A general density-one point can then be viewed as a *more tolerant 1-generic*. It permits P to have gaps in the interval, as long as the gaps are not too big in fractional measure, and this measure goes down as we shrink the interval. A natural question is, how *unlike* 1-generics can these points be?

Bienvenu, Greenberg, Kučera, Nies, and Turetsky [2] distinguish between *dyadic density* and *full density*. The former is the natural notion of density in Cantor space, while the latter is the natural one on the unit interval. In Section 3, we strongly separate the two by constructing a dyadic density-one point that is not a full positive density point (Theorem 3.5). We also show (with Joseph S. Miller) that when we restrict our attention to the Martin-Löf random reals, being dyadic density-one is equivalent to being full density-one (Theorem 3.12).

In Section 4, we turn to the computational power of dyadic positive density points, showing that one direction of Theorem 1.1 fails when the assumption of Martin-Löf randomness is removed: There is a dyadic density-one point Turing above any degree (Theorem 4.1). In Section 5, we lift Theorem 4.1 to full density on the unit interval (Theorem 5.3).

In Section 6, we probe the connection between 1-generics and density-one points further. We find that the "van Lambalgen property" fails for dyadic density-one points. However, no dyadic positive density point can be of minimal Turing degree: Every such point is either Martin-Löf random, or computes a 1-generic (Theorem 6.2).

In Section 7, we explore the relationship between randomness and various notions of computability-theoretic strength within the class of reals that are not positive density. We observe (Proposition 7.1) that there is a computably random real that is incomplete and not positive density. On the other hand, the property of being not positive density does imply a weaker form of computational strength on the class of Schnorr random reals. In Proposition 7.3, we show that every such real is high.

§2. Preliminaries. We assume familiarity with basic concepts in computability theory, on the level of the first chapters of Downey and Hirschfeldt [6], and Nies [11].

A *string* is a finite binary sequence, i.e., an element of $2^{<\omega}$. Cantor space, denoted by 2^ω, is the space of all infinite binary sequences, with the topology generated by basic clopen sets of the form $[\sigma] = \{X \in 2^\omega : \sigma \prec X\}$, where σ is a string. If S is a set of strings, then $[S]^\prec$ denotes the open set $\bigcup_{\sigma\in S}[\sigma]$. If S is c.e., $[S]^\prec$ is a Σ^0_1 *class* in Cantor space, while its complement is a Π^0_1 *class*. We will refer more often to Σ^0_1 classes than the c.e. sets of strings that generate them, so we depart slightly from convention and let $\langle W_e\rangle_{e\in\omega}$ denote a uniform enumeration of Σ^0_1 classes.

The empty string is denoted by $\langle\rangle$. If σ is a string, $|\sigma|$ denotes its length. If it is not the empty string, σ^- denotes the string obtained from σ by removing the last bit. If ρ is another string, we write $\sigma \preceq \rho$ to indicate that σ is a prefix of ρ. If $X \in 2^\omega$, $\sigma \prec X$ means that σ is an initial segment of X, while $X \upharpoonright n$ denotes the initial segment of X of length n. If $i \in \{0,1\}$, $\bar{i}$ denotes the *other* binary digit, namely, $1-i$.

Any irrational $x \in [0,1]$ can be identified uniquely with an infinite binary sequence, namely, its binary expansion. Since we are seldom concerned with rationals, we use the term *real* to refer both to infinite binary sequences and elements of $[0,1]$. By associating the open set $(0.\sigma, 0.\sigma + 2^{-|\sigma|})$ to the string σ, we can speak of Σ^0_1 and Π^0_1 classes on the unit interval.

The symbol μ refers both to the uniform measure on Cantor space and to Lebesgue measure on the unit interval, which are measure-theoretically isomorphic via the correspondence just described. Given $\sigma \in 2^{<\omega}$ and a measurable set $C \subseteq 2^\omega$, the shorthand $\mu_\sigma(C)$ denotes the relative measure of C in $[\sigma]$, i.e.,

$$\mu_\sigma(C) = \frac{\mu([\sigma] \cap C)}{\mu([\sigma])}.$$

If I and C are measurable subsets of $[0,1]$, and I is not null, then $\mu_I(C)$ denotes the relative measure of C in I, i.e.,

$$\mu_I(C) = \frac{\mu(I \cap C)}{\mu(I)}.$$

Finally, the concept of Martin-Löf randomness is an important one in this paper. A *Martin-Löf test* is a uniform sequence $\langle U_n \rangle_{n\in\omega}$ of Σ^0_1 classes such that $\mu(U_n) \leq 2^{-n}$. A real is *Martin-Löf random* if it is not contained in $\bigcap_n U_n$ for any Martin-Löf test $\langle U_n \rangle_{n\in\omega}$.

§3. Dyadic density vs. full density.

Definition 3.1. Let C be a measurable subset of 2^ω and $X \in 2^\omega$. The *(lower) dyadic density* of C at X, written $\varrho_2(C \mid X)$, is

$$\liminf_n \mu_{X \restriction n}(C).$$

Definition 3.2. A real $X \in 2^\omega$ is a *dyadic positive density point* if for every Π^0_1 class C containing X, $\varrho_2(C \mid X) > 0$. It is a *dyadic density-one point* if for every Π^0_1 class C containing X, $\varrho_2(C \mid X) = 1$.

Even though dyadic density seems like the natural notion of density in Cantor space, it is a simplification of the version of density that appears in the classical Lebesgue Density Theorem:

Definition 3.3. Let C be a measurable subset of $\mathbb{R}$ and $x \in \mathbb{R}$. The (*lower*) *full density* of C at x, written $\varrho(C \mid x)$, is

$$\liminf_{\gamma,\delta\to 0^+} \frac{\mu((x-\gamma, x+\delta) \cap C)}{\gamma+\delta}.$$

Definition 3.4. We say $x \in [0,1]$ is a *full positive density point* if for every Π^0_1 class $C \subseteq [0,1]$ containing x, $\varrho(C \mid x) > 0$. It is a *full density-one point* if for every Π^0_1 class $C \subseteq [0,1]$ containing x, $\varrho(C \mid x) = 1$.

As pointed out earlier, if x is irrational, we can identify it uniquely with a binary sequence. So it makes sense to ask if x is a dyadic density-one point. Likewise, it

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makes sense to ask if a sequence $X \in 2^\omega$ is a full density-one point. Clearly, every full density-one point is dyadic density-one. That the converse fails is our main result in this section:

THEOREM 3.5. *There is a dyadic density-one point that is not a full positive density point.*

The real described by this theorem is not 1-generic, and as we will see shortly, not Martin-Löf random. Its construction illustrates a method by which we can break out of those classes, and serves as the basic template for the constructions in Sections 4 and 5. We begin with a lemma that is a restatement of the well-known "Kolmogorov inequality for martingales" (see, for example, [11], 7.1.9):

LEMMA 3.6. *Suppose $W \subseteq 2^\omega$ is open. For any ε such that $\mu(W) \leq \varepsilon \leq 1$, let $U_\varepsilon(W)$ denote the set $\{X \in 2^\omega : \mu_\rho(W) \geq \varepsilon$ for some $\rho \prec X\}$. Then $\mu(U_\varepsilon(W)) \leq \mu(W)/\varepsilon$.*

PROOF. For each $X \in U_\varepsilon$, let ρ_X denote the least initial segment ρ of X such that $\mu_\rho(W) > \varepsilon$. Let $V = \{\rho_X : X \in U_\varepsilon\}$. Note that V is prefix-free and $[V] = U_\varepsilon$. Since W is open, for every $Y \in W$, some initial segment of Y is in V and so $[V]$ covers W. Now, for each $\rho \in V$,

$$\mu_\rho(W) = \frac{\mu(W \cap [\rho])}{2^{-|\rho|}} \geq \varepsilon.$$

So $2^{-|\rho|} \leq \mu(W \cap [\rho])/\varepsilon$ and

$$\mu([V]) = \sum_{\rho \in V} 2^{-|\rho|} \leq \sum_{\rho \in V} \frac{\mu(W \cap [\rho])}{\varepsilon} = \frac{\mu(W)}{\varepsilon}.$$

⊣

PROOF OF THEOREM 3.5. We build the desired real Y by computable approximation. At each stage s of the construction, we have a sequence of finite strings $\sigma_{0,s} \prec \sigma_{1,s} \prec \cdots$ approximating Y. At the same time, we build a Σ^0_1 class B, the complement of which witnesses the fact that Y is not a full positive density point. The main idea for accomplishing this is depicted in Figure 1, where σ is the longest initial segment of Y that "sees" the measure that we enumerate into B. This measure is small inside $[\sigma]$, but there is an interval containing Y, namely, the closure of $[\sigma 01^j] \cup [\sigma 10^j]$, in which the measure is quite large.

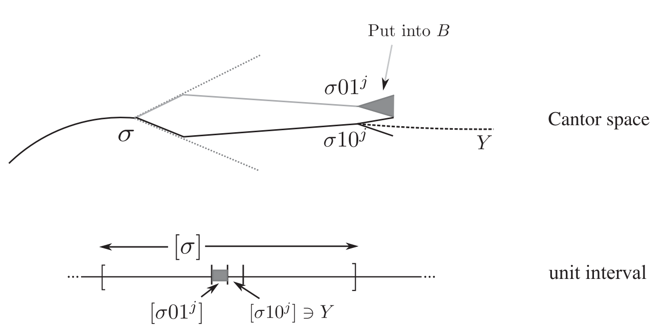

FIGURE 1. Separating dyadic and full density.

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Recall that W_e denotes the e-th Σ^0_1 class. Each such class represents a requirement that needs to be met by Y. In other words, for each e, if Y is not in W_e, we require that $\lim_{\rho \prec Y} \mu_\rho(W_e) = 0$. Priorities are assigned to Σ^0_1 classes in the usual manner, with W_j being of higher priority than W_i for any $i > j$. We make use of the following shorthand: Let C be a measurable set and τ and τ' two strings such that $\tau \prec \tau'$. If for every ρ such that $\tau \preceq \rho \prec \tau'$, $\mu_\rho(C) < \alpha$, then we say that *between* τ *and* τ', $\mu(C) < \alpha$.

At any stage s, for each $k \leq s$, we will be working above $\sigma_{k,s}$ to define $\sigma_{k+1,s}$. We have two goals in mind. First, for any $e < k$ such that $[\sigma_{k,s}]$ is not already contained in W_e, we must keep the measure of W_e between $\sigma_{k,s}$ and $\sigma_{k+1,s}$ below a certain threshold. If the threshold is exceeded, say above a string ρ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$, we will "move σ_{k+1}": $\sigma_{k+1,s+1}$ will be a string extending ρ so that the cone above it is contained entirely in W_e. In doing so, we must ensure that the measure of B remains small between $\sigma_{k,s}$ and $\sigma_{k+1,s+1}$. It is here that Lemma 3.6 plays a key part: it allows us to bound the measure of reals that have an initial segment above which the measure of B is too large.

Second, we must ensure that there is an interval $I \subseteq [\sigma_{k,s}]$ such that $[\sigma_{k+1,s}] \subseteq I$ and $\mu_I(B)$ is large. Both goals must be satisfied while keeping Y from entering B. Globally, we must maintain the fact that between $\sigma_{k,s}$ and $\sigma_{k+1,s}$, the measure of B remains *strictly below* a threshold $\beta_s(k)$, which is updated each time we act above $\sigma_{k,s}$ by moving σ_{k+1}. We begin the construction by setting $\sigma_{0,0} = \langle\rangle$.

Procedure above $\sigma_{k,s}$. When we first start working above $\sigma_{k,s}$, say at stage s_0, we set $\beta_{s_0}(k)$ to an initial value $\beta^*(k)$, which will have to be chosen small enough to accommodate the actions of this procedure (see below for how $\beta^*(k)$ is defined). If $k > 0$, then we start by choosing a $\nu \succ \sigma_{k,s_0}$ long enough so that between σ_{k-1,s_0} and σ_{k,s_0}, $\mu(B_{s_0} \cup [\nu]) < \beta_{s_0}(k-1)$. We let $\sigma_{k+1,s_0} = \nu 10^j$ and enumerate $[\nu 01^j]$ into B, where j is chosen large enough so that the measure of B between σ_{k,s_0} and σ_{k+1,s_0} remains below $\beta^*(k)$. If $k = 0$, ν can be chosen to be $\langle\rangle$.

In a subsequent stage s, suppose that $C_0, \ldots, C_l$ are those among the first k Σ^0_1 classes in which $[\sigma_{k+1,s}]$ is not already contained, in order of descending priority. Now if for some ρ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$ and some $j \leq l$, $\mu_\rho(C_j)$ exceeds $\sqrt{\beta_s(k)}$ and no action has yet been taken for a higher priority $C_{j'}$, then we *act* by moving σ_{k+1} to a string extending ρ. Let $\nu \succeq \rho$ be a string such that $[\nu] \subseteq C_j$ and let it be long enough so that:

(1) Between ρ and ν, $\mu(B_s) < \sqrt{\beta_s(k)}$.
(2) $B_s \cap [\nu] = \emptyset$.
(3) If $k > 0$, then between $\sigma_{k-1,s}$ and $\sigma_{k,s}$, $\mu(B_s \cup [\nu])$ must be strictly less than $\beta_s(k-1)$.

Let j be large enough so that between $\sigma_{k,s}$ and ν, $\mu(B_s \cup [\nu 01^j])$ remains strictly below $\sqrt{\beta_s(k)}$. We set $\sigma_{k+1,s+1} = \nu 10^{j+k}$ and enumerate $[\nu 01^j]$ into B. Finally, we set $\beta_{s+1}(k) = \sqrt{\beta_s(k)}$.

This describes the construction, save for the choice of the initial values of the thresholds, which we now address.

Choosing $\beta^*(k)$. We move $[\sigma_{k+1}]$ into C_j when the following is seen to occur at some stage s: For some ρ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$, $\mu_\rho(C_j)$ exceeds the threshold

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$\sqrt{\beta_s(k)}$. If this does not occur, we wish to limit the measure of C_j to 2^{-k} between $\sigma_{k,s}$ and $\sigma_{k+1,s}$. Each action above $\sigma_{k,s}$ raises the threshold by a power of $1/2$, and there are at most k actions (i.e., at most one for each W_e with $e < k$), so it suffices to ensure that the initial value of the threshold $\beta^*(k)$ satisfies

$$(\beta^*(k))^{1/2^{k+1}} \leq 2^{-k}.$$

Verification.

Claim 3.7. *Unless the procedure above $\sigma_{k,s}$ acts, the measure of B remains strictly below $\beta_s(k)$ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$.*

Proof. Condition (2) above ensures that if σ_k is moved at stage s due to an action above $\sigma_{l,s}$ for some $l < k$, then $\mu(B_s \cap [\sigma_{k,s}]) = 0$. If we act above $\sigma_{k+1,s}$, then condition (3) ensures that $\mu(B_s)$ remains below $\beta_s(k)$ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$. Note that there is a string v such that $\sigma_{k+1,s} \prec v \prec \sigma_{k+2,s}$ and $\mu(B_s \cup [v]) < \beta_s(k)$ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$. So if we act above $\sigma_{l,s}$ for some $l > k+1$, then we add some measure to B, but this measure is contained entirely in $[v]$. ⊣

Claim 3.8. *We can act above $\sigma_{k,s}$ while satisfying requirements* (1) *through* (3) *above.*

Proof. By Claim 3.7, $\mu(B_s) < \beta_s(k)$ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$. If for some ρ between $\sigma_{k,s}$ and $\sigma_{k+1,s}$, $\mu(C_j)$ exceeds $\sqrt{\beta_s(k)}$ then by Lemma 3.6, $\mu_\rho(U_{\sqrt{\beta_s(k)}}(B_s)) < \beta_s(k)/\sqrt{\beta_s(k)} < \mu_\rho(C_j)$. So there is an $X \in C_j$ extending ρ such that for every α such that $\rho \preceq \alpha \prec X$, $\mu_\alpha(B) < \sqrt{\beta_s(k)}$. Thus there are arbitrarily long strings extending ρ satisfying condition (1). Conditions (2) and (3) are met by simply choosing a long enough such string. ⊣

Claim 3.9. *For each $k \in \omega$, $\sigma_k = \lim_s \sigma_{k,s}$ exists, and $Y = \bigcup_k \sigma_k$ is total.*

Proof. Assume that $\sigma_{k,s}$ has stabilized by stage s. Then σ_{k+1} is moved after stage s only by the procedure above $\sigma_{k,s}$, hence at most once for each W_e where $e < k$. ⊣

Claim 3.10. *Y is a dyadic density-one point.*

Proof. Suppose that $Y \notin W_e$. Let k be large enough so that $k > e$ and for all $e' < e$, if $Y \in W_{e'}$, then $[\sigma_k] \subseteq W_{e'}$. Fixing a $k' > k$, let s be large enough so that $\sigma_{k',s}$ has stabilized. By our choice of k, we never act above $\sigma_{k',s}$ for the sake of $W_{e'}$ for any $e' < e$, and by the assumption that $Y \notin W_e$, we never act for the sake of W_e. Let $t > s$ be such that $\sigma_{k'+1,t}$ has stabilized. For all $t' > t$, between $\sigma_{k',t'}$ and $\sigma_{k'+1,t'}$, $\mu(W_e)$ does not exceed $\sqrt{\beta_{t'}(k')}$, which is always bounded by $2^{-k'}$. ⊣

Claim 3.11. *Y is not a full positive density point.*

Proof. Let σ_k and σ_{k+1} be the final values of $\sigma_{k,s}$ and $\sigma_{k+1,s}$ respectively. Then by construction there is a string v such that $\sigma_k \prec v \prec \sigma_{k+1} \prec Y$, and $\sigma_{k+1} = v10^{j+k}$ for some j and $[v01^j] \subseteq B$. Let $l = |v| + j + 1$ and let I be the interval $(0.v1-2^{-l}, 0.v1+2^{-(l+k)})$. Since Y is a dyadic density-one point, Y is not a rational and so $Y \in (0.v1, 0.v1 + 2^{-(l+k)}) \subset I$, and $\mu_I(B) \geq 1/(1+2^{-k})$. ⊣

This completes the proof of Theorem 3.5. ⊣

Bienvenu, et al. have observed (see [3], Remark 3.4) that Theorem 1.1 remains true if full density is replaced by dyadic density. It follows that a Martin-Löf random real is dyadic positive density if and only if it is full positive density. We now show

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that the notions of dyadic density-one and full density-one also coincide on the class of Martin-Löf random reals.

THEOREM 3.12 (Khan, J. Miller). *Suppose X is Martin-Löf random. Then X is a dyadic density-one point if and only if X is a density-one point.*

In order to prove Theorem 3.12, we need to introduce *nonporosity points*.

DEFINITION 3.13. We say that a Π^0_1 class C is *porous* at $X \in 2^\omega$ if there is an $\varepsilon > 0$ such that for every $\alpha > 0$, there is a $0 < \beta < \alpha$ such that $(X - \beta, X + \beta)$ contains an open interval of length $\varepsilon\beta$ that is disjoint from C.

We say $Y \in 2^\omega$ is a *nonporosity point* if every Π^0_1 class to which Y belongs is nonporous at Y.

LEMMA 3.14 (Khan, J. Miller). *If $X \in 2^\omega$ is dyadic density-one but not full density-one, then there is a Π^0_1 class that is porous at X.*

PROOF. Suppose that the Σ^0_1 class W witnesses that X is not a full density-one point, i.e., there is an $\varepsilon > 0$ such that for all $\delta > 0$, there is an open interval $I \subseteq (X - \delta, X + \delta)$ containing X such that $\mu_I(W) > \varepsilon$. Since X is a dyadic density-one point, there is an initial segment σ of X such that for all $\tau \succeq \sigma$, $\mu_\tau(W) \leq \varepsilon/6$, and σ is not all zeros or all ones[2].

If ρ is a string of length k that is not all zeros or all ones, let ρ^- and ρ^+ denote the lexicographically preceding and succeeding strings of length k. The intervals $[\rho^-]$, $[\rho]$ and $[\rho^+]$ are all of the same length and adjacent.

Now let $I \subseteq [\sigma]$ be any open interval containing X, and let ρ be the longest initial segment of X extending σ such that the closure of $[\rho^-] \cup [\rho] \cup [\rho^+]$ covers I and denote this closure by I'. Then $\mu(I) \geq \mu(I')/6$. To see this, assume without loss of generality, that $X \succ \rho 0$. By the maximality of ρ, it cannot be the case that I is contained in the closure of $[\rho 0^-] \cup [\rho 0] \cup [\rho 0^+]$. So I must overlap either half of the interval $[\rho^-]$ or half of the interval $[\rho]$, which means that $\mu(I) > \mu([\rho])/2 = \mu(I')/6$.

Next, assume that $\mu_{\rho^-}(W) \leq \varepsilon/6$ and $\mu_{\rho^+}(W) \leq \varepsilon/6$. Since ρ extends σ, $\mu_\rho(W) \leq \varepsilon/6$. It follows that $\mu_{I'}(W) \leq \varepsilon/6$. Then

$$\mu_I(W) = \frac{\mu(W \cap I)}{\mu(I)} \leq \frac{6\mu(W \cap I)}{\mu(I')} \leq \frac{6\mu(W \cap I')}{\mu(I')} = 6\mu_{I'}(W) \leq \varepsilon.$$

We have shown that if $\mu_I(W) > \varepsilon$, then either $\mu_{\rho^-}(W)$ or $\mu_{\rho^+}(W)$ must exceed $\varepsilon/6$. We build C as follows: whenever we see a $\rho \succeq \sigma$ such that $\mu_\rho(W) > \varepsilon/6$, enumerate $[\rho]$ into the complement of C. Note that we never enumerate an initial segment of X into the complement of C, so C contains X. Moreover, C is porous at X: Given an $\alpha > 0$, there is an open interval $I \subseteq [\sigma]$ containing X such that $I \subseteq (X - \alpha/24, X + \alpha/24)$ and $\mu_I(W) > \varepsilon$. Let ρ be chosen as above, and I' accordingly. Then $I' \subseteq (X - \alpha/4, X + \alpha/4)$. Finally, let $\beta = 2 \cdot 2^{-|\rho|}$. Then $(X - \beta, X + \beta) \subseteq (X - \alpha/2, X + \alpha/2)$, and there is a subinterval of $(X - \beta, X + \beta)$ of size $\beta/2$, namely, one of $[\rho^-]$ or $[\rho^+]$, that lies in the complement of C. ⊣

[2]Note that X is not computable.

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Theorem 3.12 now follows from two facts. By Theorem 1.1 (and the fact that it holds for dyadic density), X in Lemma 3.14 is incomplete, while the Π^0_1 class C is porous at X. But then X cannot be Martin-Löf random:

THEOREM 3.15 (Bienvenu, et al. [3]). *Every incomplete Martin-Löf random real is a nonporosity point.*

Nies [10] has extended Lemma 3.14 to show that if X is a nonporosity point, then for each Π^0_1 class C, $\varrho(C \mid X) = \varrho_2(C \mid X)$.

§4. A dyadic density-one point above any degree.

We have seen that the Martin-Löf random positive density points are incomplete. Every 1-generic G satisfies $G \oplus 0' \equiv_T G'$ and is therefore also incomplete. However, the proof of Theorem 3.5 suggests a way of constructing dyadic density-one points outside of those classes. In this section, we use this framework to show that general dyadic density-one points can be arbitrarily powerful as oracles. Our ultimate goal is Theorem 5.3 which shows this to be true of full density-one points, but working on the unit interval presents complications that obscure the idea behind the proof of that theorem. For this reason, we first present the dyadic version.

THEOREM 4.1. *For every $X \in 2^\omega$, there is a dyadic density-one point $Y \in 2^\omega$ such that $X \leq_T Y \leq_T X \oplus \emptyset'$.*

PROOF. We build a Δ^0_2 perfect tree $F : 2^{<\omega} \to 2^{<\omega}$ and a functional Γ such that for every $X \in 2^\omega$, $F(X)$ is a density-one point and $\Gamma^{F(X)} = X$. F will be obtained as the limit of partial computable functions $F_s : 2^{<\omega} \to 2^{<\omega}$. For each s, we will ensure that if $F_s(\sigma)$ is defined, then $\Gamma^{F_s(\sigma)} = \sigma$. If, at any stage s, we set $F_s(\sigma)$ to a new value, it should be assumed that for any σ' properly extending σ, we undefine $F_s(\sigma')$. Each Σ^0_1 class now represents a requirement that needs to be met by each path on the tree. In other words, for each e and for each $X \in 2^\omega$, if $F(X)$ is not in W_e, we require that $\lim_{\rho \prec F(X)} \mu_\rho(W_e) = 0$. Priorities are assigned as before.

Above $F_s(\sigma)$, we work to define $F_s(\sigma i)$ for $i \in \{0, 1\}$. We want to ensure that for each $e < |\sigma|$, if $[F_s(\sigma i)]$ is not already contained in W_e, then between $F_s(\sigma)$ and $F_s(\sigma i)$, $\mu(W_e)$ remains below a certain threshold. If the threshold is exceeded above some ρ between $F_s(\sigma)$ and $F_s(\sigma i)$, we will "move $F(\sigma i)$": $F_{s+1}(\sigma i)$ will be chosen to be a string ν extending ρ such that $[\nu]$ is contained in W_e. Complications arise because ν cannot be such that Γ^ν properly extends σi or is incompatible with σi. In the proof of Theorem 3.5, we built a single forbidden Σ^0_1 class B, the measure of which we had to keep small along the approximation. Here, we maintain a Σ^0_1 class B_σ for every nonempty string σ: if $\sigma = \alpha i$, then B_σ consists of the union of the set of current or previous values of $[F(\sigma 0)]$, $[F(\sigma 1)]$ and $[F(\alpha \bar{i})]$. We also maintain thresholds $\beta_s(\sigma)$, and the fact that at every stage s, for every nonempty string σ, between $F_s(\sigma^-)$ and $F_s(\sigma)$, the measure of $B_{\sigma,s}$ is strictly below $\beta_s(\sigma)$.

We begin the construction by setting $F_0(\langle\rangle) = \langle\rangle$.

Procedure for $F_s(\sigma i)$. Let t be the stage at which $F_s(\sigma)$ is first set to its current value. Both $\beta_t(\sigma 0)$ and $\beta_t(\sigma 1)$ are set to the same initial value $\beta^*(|\sigma|)$. The strings $F_t(\sigma i)$ for $i \in \{0, 1\}$ are chosen initially so that:

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- The measure of $[F_t(\sigma i)]$ between $F_t(\sigma)$ and $F_t(\sigma\bar{i})$ is strictly below $\beta^*(|\sigma|)$.
- If σ is not the empty string, $F_t(\sigma 0)$ and $F_t(\sigma 1)$ must be long enough so that between $F_t(\sigma^-)$ and $F_t(\sigma)$, $\mu(B_{\sigma,t}) < \beta_t(\sigma)$.

Suppose that $C_0, \dots, C_l$ are those among the first $|\sigma|$ many Σ^0_1 classes in which $[F_s(\sigma i)]$ is not already contained, in order of descending priority. Now if for some ρ between $F_s(\sigma)$ and $F_s(\sigma i)$ and some $j \leq l$, $\mu_\rho(C_j)$ exceeds $\sqrt{\beta_s(\sigma i)}$ and no action has yet been taken for a higher priority $C_{j'}$, then we *act*: Let ν be a string extending ρ such that $[\nu] \subseteq C_j$ and

(1) between ρ and ν, $\mu(B_{\sigma i,s}) < \sqrt{\beta_s(\sigma i)}$.
(2) ν is long enough so that $\mu(B_{\sigma\bar{i},s} \cup [\nu]) < \beta_s(\sigma\bar{i})$ between $F_s(\sigma)$ and $F_s(\sigma\bar{i})$, and $\mu(B_{\sigma,s} \cup [\nu]) < \beta_s(\sigma)$ between $F_s(\sigma^-)$ and $F_s(\sigma)$.
(3) $B_{\sigma i,s} \cap [\nu] = \emptyset$.

We set $F_{s+1}(\sigma i) = \nu$ and $\beta_{s+1}(\sigma i) = \sqrt{\beta_s(\sigma i)}$.

Choosing $\beta^*(|\sigma|)$. We move $[F(\sigma i)]$ into C_j when the following is seen to occur at some stage s: For some ρ between $F_s(\sigma)$ and $F_s(\sigma i)$, $\mu_\rho(C_j)$ exceeds the threshold $\sqrt{\beta_s(\sigma i)}$. If this does not occur, we wish to limit the measure of C_j to $2^{-|\sigma|}$ between $F_s(\sigma)$ and $F_s(\sigma i)$. Each action raises the threshold by a power of $1/2$, and there are at most $|\sigma|$ actions, so we require that $\beta^*(|\sigma|)$, the initial value of $\beta(\sigma i)$, satisfy

$$(\beta^*(|\sigma|))^{1/2^{|\sigma|+1}} \leq 2^{-|\sigma|}.$$

Verification.

CLAIM 4.2. *For every $\sigma \in 2^{<\omega}$, $\lim_s F_s(\sigma)$ exists.*

PROOF. Assume that $F_s(\sigma)$ has stabilized by stage s_0. For each $i \in \{0, 1\}$, $F(\sigma i)$ is moved after stage s_0 at most $|\sigma|$ times. ⊣

CLAIM 4.3. *The procedure for $F_s(\sigma i)$ can act while satisfying requirements* (1) *through* (3) *above.*

PROOF. Suppose at stage s, we move $F(\sigma i)$ for the sake of C_j, i.e., for some ρ between $F_s(\sigma)$ and $F_s(\sigma i)$, $\mu_\rho(C_{j,s}) > \sqrt{\beta_s(\sigma i)}$. By Lemma 3.6, there is a $Y \in 2^\omega$ extending ρ such that for each α such that $\rho \preceq \alpha \prec Y$, $\mu_\alpha(B_{\sigma i,s}) < \sqrt{\beta_s(\sigma i)}$. Thus there are arbitrarily long strings α extending ρ satisfying condition (1). To satisfy (2) and (3), we simply choose an α long enough and designate it $F_{s+1}(\sigma i)$. ⊣

CLAIM 4.4. *Suppose at stage $s+1$, we set $F_{s+1}(\sigma i) = \tau$ and set $\Gamma^\tau_{s+1} = \sigma i$. Then $\Gamma^\tau_s \preceq \sigma i$. In other words, setting $\Gamma^\tau = \sigma i$ keeps Γ consistent.*

PROOF. We first show by induction that if t is the stage when $F_s(\sigma)$ is first set to its current value, then for all $\rho \succeq F_t(\sigma)$, $\Gamma^\rho_t = \sigma$. The base case is trivial since $F_s(\langle\rangle) = \langle\rangle$ for all s. Suppose $\sigma = \alpha j$ for some $j \in \{0, 1\}$. When $F_s(\alpha)$ is first set to its current value, say at stage t_0, then for all $\nu \succeq F_s(\alpha)$, $\Gamma^\nu_{t_0} = \alpha$. Now suppose at some subsequent stage t_1, we set $F_{t_1}(\alpha j) = \tau$, then because of requirement (3), $B_{\alpha j,t_1} \cap [\tau]$ is empty, and hence for all $\rho \succeq \tau$, $\Gamma^\rho_{t_1} = \sigma$.

Subsequent to initialization, $[F_s(\sigma i)]$ is always disjoint from $B_{\sigma i,s}$, hence $\Gamma^{F_s(\sigma i)}$ never properly extends σi or becomes incompatible with σi. ⊣

CLAIM 4.5. *For each $X \in 2^\omega$, $F(X) = \bigcup_{k\in\omega} \lim_{s\to\infty} F_s(X \upharpoonright k)$ is a dyadic density-one point.*

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PROOF. Suppose that $F(X) \notin W_e$. Let $\sigma \prec X$ be long enough so that $|\sigma| > e$ and for all $e' < e$, if $F(X) \in W_{e'}$, then $F(\sigma) \in W_{e'}$. Let ρ be any initial segment of X that properly extends σ and let t be large enough so that $F_t(\rho)$ has stabilized. By our choice of σ, we never move $F(\rho)$ after stage t for the sake of $W_{e'}$ for any $e' < e$, and by the assumption that $F(X) \notin W_e$, we never act for the sake of W_e. Hence, for all $t' \geq t$, between $F_{t'}(\rho^-)$ and $F_{t'}(\rho)$, $\mu(W_e)$ never exceeds $\sqrt{\beta_{t'}(\rho)}$, which is always bounded by $2^{-(|\rho|-1)}$. ⊣

Now, for every $X \in 2^\omega$, $\Gamma^{F(X)} = X$, and because F is Δ^0_2, $F(X) \leq_T X \oplus 0'$. This concludes the proof of Theorem 4.1. ⊣

§5. A full density-one point above any degree.

We can adapt the previous construction to produce a full density-one point above any degree. We will need a version of Lemma 3.6 for the unit interval:

LEMMA 5.1 (Bienvenu, et al. [3]). *Suppose $W \subseteq [0,1]$ is open. For any ε such that $\mu(W) \leq \varepsilon \leq 1$, let $\mathcal{U}_\varepsilon(W)$ denote the set*

$$\{X \in [0,1] : \exists \text{ an interval } I,\ X \in I, \text{ and } \mu_I(W) \geq \varepsilon\}.$$

Then $\mu(\mathcal{U}_\varepsilon(W)) \leq 2\mu(W)/\varepsilon$.

Lemma 5.1 has a subtle shortcoming. When relativizing it to an interval $J \subset [0,1]$, we obtain a bound on $\mu_J(\mathcal{U}_\varepsilon(W \cap J))$, but in our construction we will be concerned about $\mu_J(\mathcal{U}_\varepsilon(W))$. Fortunately, this is easily remedied:

LEMMA 5.2. *Let $W \subseteq [0,1]$ be open, and let K be an open interval such that for all open intervals L containing K, $\mu_L(W) < \delta$. Then for any interval I containing K, and any ε such that $\mu(W) \leq \varepsilon \leq 1$, $\mu_I(\mathcal{U}_\varepsilon(W)) < 6\delta/\varepsilon$.*

PROOF. By Lemma 5.1, $\mu_I(\mathcal{U}_\varepsilon(W \cap I)) \leq 2\delta/\varepsilon$. Let $S = \mathcal{U}_\varepsilon(W) \setminus \mathcal{U}_\varepsilon(W \cap I)$ and $c = \mu_I(S)$. If $c = 0$, then $\mu_I(\mathcal{U}_\varepsilon(W)) = \mu_I(\mathcal{U}_\varepsilon(W \cap I))$, and we are done. If $c > 0$, there exists an $X \in S \cap I$ such that X is at least $\mu(I)c/4$ away from the nearest endpoint of I. Let J be an interval containing X such that $\mu_J(W) \geq \varepsilon$. Since $X \notin \mathcal{U}_\varepsilon(W \cap I)$, J cannot be contained in I, so $\mu(J \cap I) > \mu(I)c/4$. We now have $\mu(J)/\mu(J \cup I) \geq \mu(J \cap I)/\mu(I) > c/4$, and so:

$$\mu_{I\cup J}(W \cap J) = \frac{\mu(W \cap J)}{\mu(I \cup J)} = \frac{\mu(W \cap J)}{\mu(J)}.\frac{\mu(J)}{\mu(I \cup J)} > \mu_J(W)\frac{c}{4} \geq \frac{\varepsilon c}{4}.$$

On the other hand, $\mu_{I\cup J}(W \cap J) < \delta$ by assumption, so $c < 4\delta/\varepsilon$. ⊣

The following shorthand is convenient: Let C be a measurable set and I and I' intervals such that $I' \subseteq I$. If for every interval J such that $I' \subseteq J \subseteq I$, $\mu_J(C) < \alpha$, then we say that *between I and I', $\mu(C) < \alpha$*.

We briefly outline the obstacles to lifting Theorem 4.1 to the unit interval. The first is that what was an advantage in the proof of Theorem 3.5 now works against us. In building a full density-one point X, we can no longer restrict our attention to relative measures of Σ^0_1 classes within dyadic cones of the form $[X_s \upharpoonright n]$. As an example, consider the intervals we enumerate into B in the proof of Theorem 3.5, which appear small in dyadic cones along the approximation, but big when we consider their fractional measure within arbitrary intervals around X_s.

The second obstacle is subtler. In the proof of Theorem 4.1 we decompose a density requirement with respect to a single Σ^0_1 class into countably many subrequirements. At each level of the construction, we attempt to satisfy stronger and stronger subrequirements with respect to W_e that, when taken together, ensure that the limiting density requirement is satisfied. The key is that if $\nu_0 \preceq \nu_1 \preceq \nu_2$ are strings, and the measure of the set W is below ε between ν_0 and ν_1, and also between ν_1 and ν_2, then the measure of W is below ε between ν_0 and ν_2. However, if $I_0 \subseteq I_1 \subseteq I_2$ are intervals, it may be the case that the measure of W is below ε between I_0 and I_1, and also between I_1 and I_2, but not between I_0 and I_2.

THEOREM 5.3. *For every $X \in 2^\omega$, there is a density-one point $Y \in 2^\omega$ such that $X \leq_T Y \leq_T X \oplus \emptyset'$.*

PROOF. Let $\mathcal{I}$ denote the collection of closed subintervals of the unit interval with dyadic rational endpoints. By computable approximation, we build a tree $F : 2^{<\omega} \to \mathcal{I}$ of intervals and a functional Γ such that for all every σ in $2^{<\omega}$, and every $Y \in F(\sigma)$, $\Gamma^Y \upharpoonright |\sigma| = \sigma$. F is obtained as a limit of partial computable functions F_s such that if $\sigma \preceq \sigma'$ and $F_s(\sigma)$ and $F_s(\sigma')$ are both defined, then $F_s(\sigma') \subseteq F_s(\sigma)$. If at any stage s, we "move $F(\sigma)$", i.e., we define $F_{s+1}(\sigma)$ to be something other than $F_s(\sigma)$, it should be assumed that we set $\Gamma^X_{s+1} = \sigma$ for all X in $F_{s+1}(\sigma)$ and we undefine $F_{s+1}(\sigma')$ for any σ' properly extending σ.

As in the previous construction, we will be working within $F_s(\sigma)$ to define $F_s(\sigma i)$ for $i \in \{0, 1\}$. A key difference is that we now maintain a proper subinterval $J_s(\sigma)$ of $F_s(\sigma)$ within which $F_s(\sigma 0)$ and $F_s(\sigma 1)$ reside. If we act at stage s by setting $F_{s+1}(\sigma i)$ to a new value, we are allowed to move it outside $J_s(\sigma)$, in which case we expand $J_s(\sigma)$ to a larger interval $J_s(\sigma)^+ = J_{s+1}(\sigma)$ that contains $F_{s+1}(\sigma i)$. We postpone explaining how the initial value of $J_s(\sigma)$ is chosen and how $J_s(\sigma)^+$ is defined.

Let $B_{\sigma i,s}$ denote the union over all $t \leq s$ of $F_t(\sigma i 0) \cup F_t(\sigma i 1) \cup F_t(\sigma \bar{i})$. By moving $F(\sigma i 0)$, say, we contribute measure to $B_{\sigma i}$. We shall have to ensure that we can do this without violating the measure constraint $\beta(\sigma)$ for $B_{\sigma i}$.

We begin the construction by setting $F_0(\langle\rangle) = [0, 1]$.

Procedure for $F_s(\sigma i)$. Let t be the stage at which $F_s(\sigma)$ is first set to its current value. We set $\beta_t(\sigma 0)$ and $\beta_t(\sigma 1)$ to the same initial value $\beta^*(|\sigma|)$. We set $J_t(\sigma) = Int(F_t(\sigma), |\sigma|)$ (we define Int later) and choose $F_t(\sigma 0)$ and $F_t(\sigma 1)$ to satisfy the following conditions:

- Both are contained in $J_t(\sigma)$.
- Between $J_t(\sigma)^+$ and $J_t(\sigma 0)$, $\mu(F_t(\sigma 1)) < \beta^*(|\sigma|)$.
- Between $J_t(\sigma)^+$ and $J_t(\sigma 1)$, $\mu(F_t(\sigma 0)) < \beta^*(|\sigma|)$.
- If σ is not the empty string, let $\alpha = \sigma^-$. Then between $J_t(\alpha)^+$ and $J_t(\sigma)$, $\mu([B_{\sigma,t}]) < \beta_t(\sigma)$.

It is not hard to see that these conditions can be met by ensuring that the intervals are small enough and far enough apart relative to their width.

In a subsequent stage s, let $C_0, \dots, C_l$ be those among the first $|\sigma|$ many Σ^0_1 classes that $F_s(\sigma i)$ has not already entered, in order of descending priority. Suppose that for some interval I such that $J_s(\sigma)^+ \supseteq I \supseteq J_s(\sigma i)$, $\mu_I(C_j)$ exceeds $6\sqrt{\beta_s(\sigma i)}$, and no action has yet been taken within $F_s(\sigma)$ for a higher priority $C_{j'}$. Then there is an interval $L \subseteq C_j$ such that:

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(1) For every $Z \in L$, and every interval $K \subseteq J_s(\sigma)^+$ such that $Z \in K$, $\mu_K(B_{\sigma i,s}) < \sqrt{\beta_s(\sigma i)}$. To see that such an interval exists, note that we inductively maintain the property that between $J_s(\sigma)^+$ and $J_s(\sigma i)$, $\mu(B_{\sigma i,s}) < \beta_s(\sigma i)$. By Lemma 5.2,

$$\mu_I(\mathcal{U}_{\sqrt{\beta_s(\sigma i)}}(B_{\sigma i,s} \cap J_s(\sigma)^+)) < 6\sqrt{\beta_s(\sigma i)}.$$

Now let L be an interval contained in $C_j \cap I$ that is disjoint from $\mathcal{U}_{\sqrt{\beta_s(\sigma i)}}(B_{\sigma i,s} \cap J_s(\sigma)^+)$.

(2) L is small enough so that between $J_s(\sigma)^+$ and $J_s(\sigma\bar{i})$, $\mu(B_{\sigma\bar{i},s} \cup L) < \beta_s(\sigma\bar{i})$.

(3) If σ is not the empty string, let $\alpha = \sigma^-$. Then between $J_s(\alpha)^+$ and $J_s(\sigma)$, $\mu(B_{\sigma,s} \cup L) < \beta_s(\sigma)$.

(4) $L \cap B_{\sigma i,s} = \emptyset$.

In this case, we let $F_{s+1}(\sigma i) = L$, $J_{s+1}(\sigma) = J_s(\sigma)^+$, and $\beta_{s+1}(\sigma i) = \sqrt{\beta_s(\sigma i)}$.

Choosing $\beta^*(|\sigma|)$. If W_j is a Σ_1^0 class that $F_s(\sigma)$ has not already entered, then if $F(\sigma i)$ never enters W_j, we wish to limit the measure of W_j to 2^{-k} between $J_s(\sigma)^+$ and $J_s(\sigma i)$. The idea is the same as in the proof of Theorem 4.1, the only difference being the factor of 6 in the statement of Lemma 5.2. It suffices to pick $\beta^*(|\sigma|)$ small enough so that

$$6(\beta^*(|\sigma|))^{1/2^{k+1}} \leq 2^{-k}.$$

Defining *Int* and $^+$. For an interval I, let I^+ be obtained by padding I on either side with intervals of the same length. For an interval J, let $Int(J,k)$ be a subinterval I of J small enough so that $2k$ applications of the $^+$ operation to I result in an interval still contained in J.

Verification.

CLAIM 5.4. *The procedure for $F_s(\sigma i)$ can act without violating any measure constraints.*

PROOF. Given a string v, there are two ways in which the measure constraint for $B_{v,s}$ could be affected: by the direct addition of measure to $B_{v,s}$, or by the expansion of $J_s(\sigma)$. It is easy to see that the only such v are σ and $\sigma\bar{i}$.

Property (2) of L above ensures that $\mu(B_{\sigma\bar{i},s} \cup L) < \beta_s(\sigma\bar{i})$ between $J_s(\sigma)^+$ and $J_s(\sigma\bar{i})$, and since $F_s(\sigma) \cap (B_{\sigma\bar{i},s} \cup L)$ is contained entirely in $J_s(\sigma)^+ = J_{s+1}(\sigma)$, also between $J_{s+1}(\sigma)^+$ and $J_{s+1}(\sigma\bar{i})$.

Property (3) of L ensures that $\mu(B_{\sigma,s} \cup L) < \beta_s(\sigma)$ between $J_s(\sigma^-)^+$ and $J_s(\sigma)$, and hence between $J_{s+1}(\sigma^-)^+$ and $J_{s+1}(\sigma) = J_s(\sigma)^+$. ⊣

The argument for the following claim is virtually the same as for Claim 4.4.

CLAIM 5.5. *Suppose at stage $s+1$, we set $F_{s+1}(\sigma i) = I$. Then for any $X \in I$, $\Gamma_s^X \preceq \sigma i$. In other words, setting $\Gamma_{s+1}^X = \sigma i$ for all $X \in I$ keeps Γ consistent.*

CLAIM 5.6. *Let X be any real, and let Y be a real in $\bigcap_{\sigma \prec X} F(\sigma)$. Then Y is a density-one point.*

PROOF. For $\sigma \in 2^{<\omega}$, let $F(\sigma)$, $J(\sigma)$, and $\beta(\sigma)$ denote the limiting values of $F_s(\sigma)$, $J_s(\sigma)$, and $\beta_s(\sigma)$, respectively. Suppose that Y is not in W_j. Let σ be an initial segment of X such that $|\sigma| > j$ and if $Y \in W_l$ for any $l < j$, then $F(\sigma) \subseteq W_l$. We claim that for any $I \subseteq J(\sigma)^+$ such that $Y \in I$, $\mu_I(W_j) \leq 2^{-|\sigma|+1}$.

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Let $\rho \succeq \sigma$ be the longest initial segment of X such that I is entirely contained in $J(\rho)^+$. Suppose $X \succ \rho i$. Let $I' = I \cup J(\rho i)$. Now,

$$\frac{\mu(I')}{\mu(I)} \leq \frac{\mu(I \cup J(\rho i))}{\mu(I)} \leq 1 + \frac{\mu(J(\rho i))}{\mu(I)}.$$

By the maximality of ρ, $I \not\subseteq J(\rho i)^+$. Since $J(\rho i)^+$ is obtained by pasting a copy of $J(\rho i)$ on either side of $J(\rho i)$, $\mu(I) \geq \mu(J(\rho i))$. So the ratio above is bounded by 2. Since I' is an interval between $J(\rho)^+$ and $J(\rho i)$, $\mu_{I'}(W_j)$ never exceeds $\beta(\rho i) \leq 2^{-|\rho|}$. Therefore, $\mu_I(W_j)$ never exceeds $2^{-|\rho|+1}$. ⊣

Now, for any $X \in 2^\omega$, and any $Y \in \bigcap_{\sigma \prec X} F(\sigma)$, $\Gamma^Y = X$, and Y is a full density-one point. Since F is Δ^0_2, $Y \leq_T X \oplus 0'$. This concludes the proof of Theorem 5.3. ⊣

§6. **Nonminimality.** It is easy to see that if $A \oplus B$ is dyadic positive density, then so are A and B (and if $A \oplus B$ is dyadic density-one, so are A and B). The similarities with 1-generics seem to end here. It can be shown using the techniques of the constructions above that the "van Lambalgen property" fails badly for density-one points:

Proposition 6.1 (Khan [8]). *There is a dyadic density-one point $A \oplus B$ such that $A \equiv_T B$.*

It is nevertheless a consequence of the main result of this section that no positive density point can be of minimal Turing degree.

Theorem 6.2. *Every dyadic positive density point is either Martin-Löf random or computes a 1-generic.*

Proof. Let $\langle U_n \rangle_{n\in\omega}$ be a Martin-Löf test. For each n, let S_n be a prefix-free c.e. set of strings such that $U_n = [S_n]^\prec$. We can assume that $S_{0,s} = \{\langle\rangle\}$ for all s, and if $\tau \in S_{j+1,s}$, then there is some $\sigma \preceq \tau$ such that $\sigma \in S_{j,s}$. Let V_e denote the e-th c.e. set of strings.

We define a functional Γ such that for each $Y \in \bigcap_n U_n$,

(1) Γ^Y is total, and
(2) if Y is a dyadic positive density point, Γ^Y is 1-generic.

We define Γ inductively on a sequence $\langle R_n \rangle_{n\in\omega}$ of c.e. sets of strings. Let $R_0 = S_0$, and let $\Gamma^{\langle\rangle} = \langle\rangle$. When a string τ enters R_n, we choose m large enough so that $2^{-m} \leq 2^{-|\tau|-n}$, and so $\mu_\tau(U_m) \leq 2^{-n}$. Then, whenever a string ν extending τ enters S_m at stage s, we extend the definition of Γ as follows: If there exists an $e \leq s$ such that $[\Gamma^\tau]$ is not already contained in $[V_{e,s}]^\prec$ and there is an extension of Γ^τ in $V_{e,s}$, then let e' be the least such index and let σ be an extension of Γ^τ in $V_{e',s}$. We set $\Gamma^\nu = \sigma 0$. On the other hand, if no such e exists, we set $\Gamma^\nu = \Gamma^\tau 0$. In either case, we enumerate ν into R_{n+1}. This completes the definition of Γ.

Consider a $Y \in \bigcap_n U_n$. To see that (1) holds, note that for each n, Y has a unique initial segment σ_n in R_n, and $\Gamma^{\sigma_{n+1}}$ properly extends Γ^{σ_n}. It remains to verify (2). If Γ^Y is not 1-generic, then let e be the least index such that V_e is dense along it, but $Y \notin [V_e]^\prec$. Let M be large enough so that for each $e' < e$, if $\Gamma^Y \in [V_{e'}]^\prec$, then $[\Gamma^{\sigma_M}] \subseteq [V_{e'}]^\prec$, otherwise $[\Gamma^{\sigma_M}] \cap [V_{e'}]^\prec = \emptyset$. We exhibit a Σ^0_1 class B such that $Y \in \overline{B}$ and $\varrho_2(\overline{B} \mid Y) = 0$. For each $n \geq M$ and for each $\tau \in R_n$, we wait for a stage $s \geq e$ such that an extension of Γ^τ is in $V_{e,s}$. If this occurs, we enumerate the open set $[\tau] \setminus [R_{n+1,s}]^\prec$ into B.

If τ is an initial segment of Y, then since V_e is dense along Γ^Y, such a stage s must occur. Let ν be the initial segment of Y in R_{n+1}, and let $t \geq e$ be the least stage such that $\nu \in R_{n+1,t}$. By our choice of M, if an extension σ of Γ^τ occurs in $V_{e,t}$, we would have set $\Gamma^\nu = \sigma 0$. Therefore, $t < s$, which implies that $Y \in \overline{B}$. Moreover, $\mu_\tau([R_{n+1}]^{\prec}) \leq 2^{-n}$, and so $\varrho_2(\overline{B} \mid Y) = 0$. ⊣

COROLLARY 6.3. *No dyadic positive density point is of minimal degree.*

Theorem 6.2 has an interesting consequence. Bienvenu, et al. [2] introduce *Oberwolfach randomness* and show that every Oberwolfach random real is a full density-one point. Based on earlier work by Figueira, Hirschfeldt, Miller, Ng, and Nies [7], they observe that one "half" of every Martin-Löf random real is always Oberwolfach random, hence full density-one[3]:

PROPOSITION 6.4 (Bienvenu, et al. [2]; Figueira, et al. [7]). *If $A \oplus B$ is Martin-Löf random, then either A or B is a full density-one point.*

Thus, every Martin-Löf random real computes a full density-one point, which, together with Theorem 6.2, implies:

COROLLARY 6.5. *Every dyadic positive density point computes a full density-one point.*

§7. **Randomness and computational strength.** We have already mentioned that Theorem 1.1 holds regardless of whether we use dyadic density or full density, so one direction of that theorem can be rephrased as follows: Every Martin-Löf random point that is not dyadic positive density computes $0'$. Theorem 6.2 implies that we cannot weaken the hypothesis from Martin-Löf randomness to computable randomness. To see this, note that there is a computably random real of minimal degree. By Corollary 6.3, it cannot be dyadic positive density.

PROPOSITION 7.1. *There is a computably random real that is not dyadic positive density and is incomplete.*

In this section, we will see that the property of not being positive density does imply some form of computational strength, namely, being high, on the computably random reals, and in fact, on a more general randomness class, the *Schnorr random* reals. While this fact is a straightforward consequence of Theorem 1.1 and the result by Nies, Stephan, and Terwijn [12] that every nonhigh Schnorr random real is Martin-Löf random, we give a direct proof here that does not appeal to Theorem 1.1, and which highlights the extent to which the result is uniform.

DEFINITION 7.2. A *Schnorr test* is a Martin-Löf test $\langle G_n \rangle_{n\in\omega}$ where $\mu(G_n)$ is uniformly computable in n. A real X is *Schnorr random* if there is no Schnorr test $\langle G_n \rangle_{n\in\omega}$ such that X is contained in G_n for infinitely many n.

PROPOSITION 7.3 (Nies, et al. [12]; Bienvenu, et al. [3]). *Every Schnorr random real that is not full positive density is high.*

We will need the following lemma:

LEMMA 7.4 (Bienvenu, et al. [3]). *Let $W \subseteq [0,1]$ be open. Fixing an $\varepsilon \in (0,1)$, let*

$$\mathcal{U}_\varepsilon(W) = \{z : \exists \text{ an open interval } I,\ z \in I, \text{ and } \mu_I(W) > 1-\varepsilon\}.$$

Then $\mu(\mathcal{U}_\varepsilon(W) \setminus W) < 2\varepsilon$.

[3]The author thanks A. Kučera for bringing this fact to his attention.

Proof of Proposition 7.3. Fix $z \in 2^\omega$ such that it is Schnorr random, and let B be a Σ^0_1 class such that $z \in \overline{B}$ and $\varrho(\overline{B} \mid z) = 0$. Let $f \leq_T z$ be the function such that $f(n)$ is the least stage s such that there is an open interval I containing z with $\mu_I(B_s) > 1 - 2^{-n}$. Note that f is total, and that it is uniformly computable in z and an index for B. We claim that f is a dominating function.

Suppose that g is a computable function that it fails to dominate. Then for each $n \in \omega$, let

$$G_n = U_{2^{-n-1}}(B_{g(n)}) \setminus B_{g(n)}.$$

Each G_n is a Σ^0_1 class modulo the rationals. By Lemma 7.4, $\mu(G_n) < 2^{-n}$. It is not hard to see that $\mu(G_n)$ is uniformly computable in n, and in fact, that $\mathcal{U}_{2^{-n-1}}(B_{g(n)})$ is the union of a finite collection of open intervals with rational endpoints that can be computed from $B_{g(n)}$. Moreover, there are infinitely many n such that $z \in G_n$, which contradicts the assumption that z is Schnorr random. $\dashv$

Figure 2 shows the relationship between three important randomness classes and three forms of classical computability-theoretic strength within the class of reals that are not positive density[4]. Every computably random real is Schnorr random, and so Proposition 7.1 yields nonimplication (a).

To see nonimplication (b), let X be a minimal degree below $0'$. Every minimal degree is GL_2, and so X satisfies $(X \oplus 0')' \equiv_T X''$, which implies that it is not high. Because it is minimal, X cannot compute a 1-generic, so by Theorem 6.2, it is not dyadic positive density. However, every hyperimmune degree contains a Kurtz random real [9], and so $X \equiv_T Y$, where Y is Kurtz random, not dyadic positive density (because it is minimal), and not high.

For implication (c), we appeal to a result by L. Yu (see, for example, [6], Theorem 8.11.12) that every hyperimmune-free Kurtz random is weakly 2-random. For each Π^0_1 class C and each rational $\varepsilon > 0$, the set of points $\{X \in C : \varrho(C \mid X) < 1 - \varepsilon\}$ is a null Π^0_2 set. The weakly 2-random reals are exactly those which avoid every null Π^0_2 set. Therefore, every hyperimmune-free Kurtz random real is, in fact, full density-one.

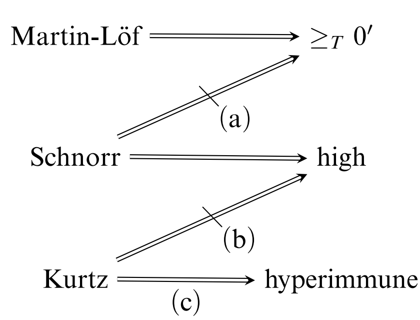

Figure 2. Relationships between randomness and notions of computability-theoretic strength within the reals that are not positive density.

[4] It does not matter whether we use dyadic or full density.

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We conclude with a question. In Theorems 4.1 and 5.3, we saw that general positive density points (in fact, density-one points) can be arbitrarily powerful as oracles. It is unknown whether this remains true under the assumption of any form of randomness intermediate between Kurtz and Martin-Löf randomness.

QUESTION 7.5. *Is there a positive density real which is Schnorr random and complete*?

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