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SUBSIDIES IN AN ECONOMY WITH ENDOGENOUS CYCLES OVER INVESTMENT AND INNOVATION REGIMES

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We explore the roles of subsidies in the Matsuyama model [K. Matsuyama, *Econometrica* 67 (1999), 335–347] of growth through cycles with a Solow investment phase and a Romer innovation phase when innovation and intermediate goods production rely on existing capital. We show that subsidies to R&D investment or to the purchase of new intermediate goods can arbitrarily reduce the threshold level of capital per type of intermediate good beyond which the economy moves to the innovation phase. Sufficient subsidization can eventually eliminate cycles. For plausible parameterizations, optimal subsidies can achieve significant welfare gains equivalent to as much as 10% rises in consumption at all times.

Keywords: Subsidization, Innovation, Capital Accumulation, Cycles, Growth

1. INTRODUCTION

One theme of the macroeconomic aspect of public policy analysis is exploration of whether and how government policies can promote output growth and mitigate output fluctuations for welfare gains. In this regard, different macroeconomic models have different answers. The neoclassical growth model pioneered by Solow (1956) may need no government intervention so long as consumers choose their consumption path optimally, as in Cass (1965) and Koopmans (1965), according to the welfare theorems. This policy implication remains valid even when exogenous shocks are introduced into the neoclassical growth model for the creation of cycles,

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as in the real business cycle models led by Kydland and Prescott (1982) and Long and Plosser (1983).

The R&D-based growth models developed in the last two decades generate sustainable growth through costly innovations that create new varieties of intermediate goods or improve the quality of existing intermediate goods [see, e.g., Romer (1990); Aghion and Howitt (1992)]. In these models, R&D activities intended for a new variety or a quality improvement incur a fixed cost, whereas the production of each intermediate good incurs a constant marginal cost, and both the innovation and the intermediate goods production use labor or current final output. Patents are granted to innovators to allow them to recoup their innovational costs. The consequent monopoly pricing reduces the demand for new intermediate goods, causing lower final output and slower growth than the socially optimal levels in the Romer model. Such efficiency losses justify government subsidization either of R&D investment or of the purchase of new intermediate goods in the Romer model, as shown in Barro and Sala-i-Martin (1995) with inelastic labor and in Zeng and Zhang (2007) with elastic labor, among others. In Aghion and Howitt (1992), with vertical expansion arising from innovations, there is a business-stealing effect or creative destruction, possibly leading to too much R&D activity and very different implications of subsidization from those in the Romer model. In contrast to the neoclassical growth models, however, the economy is always on the balanced growth path in such innovation-driven growth models, which are known as the AK model in essence.

Matsuyama (1996, 1999) unifies the Solow and Romer models by assuming that R&D activities and intermediate goods production must use capital from previous savings. Under this neoclassical-style assumption, innovation can break even to recover the fixed cost only when capital per variety exceeds a critical level for a profitable scale of the demand for newly invented intermediates. Once innovation occurs, however, part of the capital stock must be used for the fixed innovation cost and the amount of capital remaining for manufacturing intermediate goods declines. Consequently, current innovation, if responding more elastically than capital investment to initial abundance in capital per variety, can reduce capital per variety to the extent such that future innovation becomes unprofitable until enough capital is formed again through a neoclassical investment phase. It is argued that this is an empirically plausible scenario: The balanced growth path with innovation is unstable and the economy fluctuates perpetually between a Solow investment phase and a Romer innovation phase.

In a different (new Keynesian) type of R&D growth model proposed by Francois and Shi (1999) and Haruyama (2009), labor is the sole input for innovations and for intermediate goods production, as assumed in Grossman and Helpman (1991), Segerstrom (1991), Aghion and Howitt (1992), Young (1993), Parente (1994), and Fan (1995). Endogenous cycles can also arise in Francois and Shi (1999) because of contemporaneous complementarities between investors devoting labor to innovation for temporary profits. Haruyama (2009) shows that optimal steady-state R&D subsidization fails to eliminate cycles and should be

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state-dependent when labor is the only input for innovation and intermediate goods production.

Using R&D for general purpose technologies (GPTs), Helpman and Trajtenberg (1998), Freeman et al. (1999), Francois and Lloyd-Ellis (2003), Maliar and Maliar (2004), and Walde (2002, 2005) also obtain sustainable growth through endogenous cycles. Cycles arise from the reallocation of resources between production and innovation as in Matsuyama (1999). However, the GPT cycle is often driven by creative destruction and thus appears more suitable to characterize long-wave fluctuations caused by drastic technology breakthroughs such as the "steam engine age," "electricity age," and "information technology age." In Bental and Peled (1996), firms producing final goods engage in costly search for better technology from a known and fixed pool of technologies. By contrast, the Matsuyama cycle captures industry-level innovations without creative destruction and thus is generally shorter than the GPT cycle. The exact length of the Matsuyama cycle depends on the nature of the industry, the applied terms of patents, and the length of time-to-build for capital accumulation, likely in the range of medium-term cycles (between 32 and 200 quarters), shown empirically and theoretically in Comin and Gertler (2006).

The integration of factor accumulation and innovation in Matsuyama (1996, 1999) provides a new framework for the study of government policies with growth through cycles, given the following source of market failures occurring at the heart of the model. First, the fixed capital requirement to conduct innovation distorts the relative prices of different types of intermediate goods used in production. The subsequent uneven use of symmetric intermediates leads to static and dynamic efficiency losses. Second, private innovators and investors fail to internalize intertemporal spillovers of innovation and investment, leading to too little R&D, too little investment, and hence slower growth. Third, the endogenous cyclical fluctuations in consumption, investment, innovation, and growth represent another type of efficiency loss, given diminishing marginal utility and diminishing marginal products. This differs from the socially optimal fluctuation driven by exogenous shocks in Long and Plosser (1983). An important macroeconomic question arises: Can government policies mitigate such frictions and enhance social welfare?

To answer this question, we focus on subsidies to R&D activities or to the purchase of new R&D products. Existing studies have focused on how such subsidies mitigate efficiency losses on a stable balanced growth path. The only work on subsidization in the unified model of Matsuyama (1999) is that of Aloi and Lasselle (2007) using a lump-sum subsidy to innovators financed by a lump-sum tax and finding that it can promote growth, stabilize innovation cycles, and increase welfare. However, their subsidy adds directly to existing capital and thus relaxes the capital constraint on innovation and on intermediate goods production.

In this paper, we explore whether flat-rate subsidization financed by consumption taxation can promote growth and mitigate or eliminate cycles for welfare gains using the Matsuyama model. The subsidies provide additional awards to innovators without relaxing the constraint on available capital for innovation and intermediate goods production. We find that proportional subsidies to R&D investment or to the purchase of new intermediate goods can arbitrarily reduce the threshold level of capital per variety, beyond which the economy moves from the investment phase to the innovation phase. Moreover, sufficient subsidization can stabilize the balanced growth path and thus eliminate cycles. In numerical examples for plausible parameterizations, optimal subsidy rates can achieve substantial welfare gains (equivalent to about a 10% increase in consumption in every period) and lie in the range that leads to convergence toward the stable balanced growth path.

Our results appear consistent with postwar experiences in some industrial nations such as the United States, where substantial subsidies are provided to R&D activities and to the purchase of new equipment: for example, a 50% immediate writing-off of equipment investment, expensing of R&D expenditures, and accelerated depreciation allowances. [For these subsidies in the U.S. tax system, see Gordon et al. (2004a, 2004b) and others.] Worldwide, 26 out of 34 OECD countries and a number of non-OECD economies have R&D tax incentives in place [OECD (2011)]. At the same time, these countries observe many more innovations but dampened recessions compared to previous times on average. As found by Comin and Mulani (2009) using cross-country data between 1979 and 1997, there exists a large and statistically significant relationship between increments in R&D tax credits and declines in aggregate volatility; however, the relationship between R&D tax credits and growth is less pronounced (insignificant though positive), perhaps because of the limitation of sample size or of not including subsidies on the purchase of new products.

The rest of the paper proceeds as follows. Section 2 introduces the model and characterizes the equilibrium. Section 3 analyzes the steady state in different regimes as well as the global dynamics for different levels of subsidization. Section 4 presents quantitative implications and optimal subsidy rates using numerical simulations. Section 5 concludes the paper.

2. THE MODEL

The model is based on Matsuyama (1999, 2001), with discrete time extending from period 1 to infinity ($t = 1, 2, \ldots, \infty$), a constant population of identical, infinitely lived agents of mass L, and a single consumption–investment good taken as a numeraire. The economy comprises sectors of production, innovation, households, and the government. To deal with the various efficiency losses mentioned earlier in that model, we consider subsidies to R&D spending and to the purchase of newly invented intermediate goods, financed by a consumption tax.

We describe each sector and determine the equilibrium path in turn, together with the details of subsidies.

2.1. Production

Final-good production uses capital and labor. Labor supply is entirely inelastic and normalized to one unit per worker (L in aggregate) in each period. Let K_{t-1}

denote the aggregate capital stock available for production and innovation in period t, given an initial capital stock $K_0 > 0$. Capital must be converted into a CES composite of intermediates. Let $x_t(z)$ denote the zth type of intermediate good available in the range $[0, N_t]$ in period t. Labor and the composite of intermediates are combined for final-good production through a Cobb–Douglas technology,

$$Y_t = A\,(L)^{1/\sigma}\left\{\int_0^{N_t}[x_t\,(z)]^{1-1/\sigma}\,dz\right\}, \tag{1}$$

where $A > 0$ is total factor productivity and $\sigma > 1$ the direct partial elasticity of substitution between every pair of intermediate goods.

In each period t, old intermediates in the range $z \in [0, N_{t-1}]$ are sold competitively, starting with $N_0 > 0$ in period 1. New intermediates in the range $z \in [N_{t-1}, N_t]$ may be introduced and sold exclusively in period t under one-period patent protection. By symmetry, let $x_t(z) \equiv x_t^c$ for competitively supplied old intermediate goods at a price p_t^c and $x_t(z) \equiv x_t^m$ for monopolistically supplied new intermediate goods at a price p_t^m.

The profit for a producer in the final-good sector is given by

$$\Pi_t = A\,(L)^{1/\sigma}\left[N_{t-1}\left(x_t^c\right)^{1-1/\sigma} + (N_t - N_{t-1})\left(x_t^m\right)^{1-1/\sigma}\right] \\ - N_{t-1}p_t^c x_t^c - (1-s_x)\,(N_t - N_{t-1})\,p_t^m x_t^m - w_t L, \quad 0 \le s_x < 1, \tag{2}$$

where s_x is a time-invariant subsidy rate to the purchase of new intermediate goods and w_t is the wage rate. This subsidy, which is not considered in Aloi and Lasselle (2007), reduces the user cost and thus has the potential to encourage demand for new products in such a way as to mitigate the efficiency loss of unequal use of symmetric intermediates in production.

The final-good sector is perfectly competitive, and factors are paid by their marginal products:

$$p_t^c = (1-1/\sigma)\,A\,(L)^{1/\sigma}\left(x_t^c\right)^{-1/\sigma}, \tag{3}$$

$$p_t^m\,(1-s_x) = (1-1/\sigma)A\,(L)^{1/\sigma}\left(x_t^m\right)^{-1/\sigma}, \tag{4}$$

$$w_t = (1/\sigma)\,Y_t/L. \tag{5}$$

Equations (3) and (4) determine the demand for old and new intermediate goods as functions of the prices, respectively, leading to the relative demand $x_t^c/x_t^m = \{p_t^c/[p_t^m(1-s_x)]\}^{-\sigma}$.

2.2. Innovation

One unit of an intermediate good is manufactured from one unit of capital. The creation of a new intermediate good costs F (fixed) units of capital. Let r_t denote the price of capital. Thus, the one-period monopoly profit for innovation is equal to $\pi_t = p_t^m x_t^m - r_t[x_t^m + (1-s_n)F]$, where s_n is a time-invariant subsidy rate to the R&D cost $r_t F$. A lump-sum subsidy is used instead by Aloi and Lasselle (2007), based on the Matsuyama model.

With a one-period patent, a new intermediate is priced to maximize profit, knowing the demand function in (4) from final producers. This profit-maximization problem yields the monopolistic price $p_t^m = [\sigma/(\sigma - 1)]r_t$. In contrast, the competitive price of each old intermediate is just equal to the marginal cost $p_t^c = r_t$. Obviously, the monopolistic price of new intermediates exceeds the competitive price of old intermediates, $p_t^m > p_t^c$, by the markup factor $\sigma/(\sigma - 1) > 1$ on the marginal cost r_t. The markup $1/(\sigma - 1)$ is independent of subsidies and decreasing with the elasticity of substitution between any pair of intermediates. Substituting the pricing rules into $x_t^c/x_t^m = \{p_t^c/[p_t^m(1 - s_x)]\}^{-\sigma}$ yields the ratio of an old to a new intermediate in terms of equilibrium levels, as a decreasing function of the subsidy on the purchase of a new intermediate:

$$\frac{x_t^c}{x_t^m} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma} (1 - s_x)^{\sigma}. \tag{6}$$

Absent subsidies, the higher price of new intermediates than of old intermediates engenders a smaller equilibrium quantity of each new intermediate than of each old intermediate: $x_t^m < x_t^c$. The lower equilibrium quantity for new than for old intermediates must lead to static and dynamic efficiency losses. The dynamic efficiency loss takes the form of decelerating the rate of innovation because it reduces the profitability for innovators to recover the fixed R&D cost. The static efficiency loss takes the form of decreasing final output because all intermediates enter final goods production symmetrically and make diminishing marginal contributions to final output.

Subsidizing the purchase of new intermediate goods strengthens the demand for new relative to that for old intermediate goods by reducing the user cost of new intermediate goods. From (6), when $s_x < 1/\sigma$, $x_c > x_m$; when $s_x = 1/\sigma$, $x_c = x_m$; when $s_x > 1/\sigma$, $x_c < x_m$. Thus, the level of s_x may affect the dynamic system significantly in this model.

There is free entry into R&D activities, implying nonpositive profit for innovators. However, innovation cannot occur unless it can break even, meaning nonnegative profit. We thus have

$$\begin{aligned} &x_t^m \leq (\sigma - 1)(1 - s_n) F, \quad N_t \geq N_{t-1}, \\ &\left[x_t^m - (\sigma - 1)(1 - s_n) F\right](N_t - N_{t-1}) = 0. \end{aligned} \tag{7}$$

By lowering the cost of innovation to virtually any level, the subsidy on the R&D cost can reduce this break-even level of demand for a new intermediate to virtually any level and may thus have significant effects on the dynamic system of the model.

Regardless of the value of the subsidy rates, intermediate goods production and innovation in period t are constrained by available capital carried over from the

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previous period:

$$K_{t-1} = N_{t-1}x_t^c + (N_t - N_{t-1})\left(x_t^m + F\right). \tag{8}$$

Without this constraint, the economy would always be on a unique balanced growth path, as in the Romer model. In Aloi and Lasselle (2007), however, subsidies relax this constraint.

Substituting (6) and (7) into (8) for x_t^c and x_t^m determines the rate of innovation,

$$\frac{N_t - N_{t-1}}{N_{t-1}} = \max\left\{0, \quad \frac{K_{t-1}/N_{t-1} - \theta\sigma F\left(1-s_x\right)^{\sigma}\left(1-s_n\right)}{\left[\sigma - s_n\left(\sigma-1\right)\right]F}\right\}, \tag{9}$$

where

$$\theta \equiv \left(1 - \frac{1}{\sigma}\right)^{1-\sigma}, \quad \theta \in [1, e], \quad e = 2.71828\ldots$$

Here, θ is increasing with σ. Intuitively, for innovators to break even in period t, the amount of available capital K_{t-1} must be abundant enough relative to available variety N_{t-1}. According to (9), the ratio of capital to variety must exceed a critical level, $\theta\sigma F(1-s_x)^{\sigma}(1-s_n)$, to induce new innovations. Clearly, increasing each subsidy reduces the critical level for innovators to break even, given any initial state K_{t-1}/N_{t-1}. Once innovators can break even, raising each subsidy also increases the rate of innovation in (9) because a higher subsidy either strengthens the relative demand for new products in (6) or reduces the cost of innovation in (7).

From (6), (8), and (9), the equilibrium level of each type of intermediate is determined by

$$x_t^c = \left(1 - \frac{1}{\sigma}\right)^{-\sigma}\left(1-s_x\right)^{\sigma} x_t^m = \min\left\{\frac{K_{t-1}}{N_{t-1}}, \ \theta\sigma F\left(1-s_x\right)^{\sigma}\left(1-s_n\right)\right\}. \tag{10}$$

In essence, all capital has to be converted into old intermediates until capital per variety exceeds the critical level for innovators to break even. More clearly, from (10), sufficient subsidies can reduce the break-even level of the capital/variety ratio to induce innovations given any initial level of capital and variety. Also, once innovators can break even, increasing the subsidy rate on the R&D cost reduces the level of each old and each new intermediate because the increase in variety makes capital thinner over all varieties. Given the same initial state that allows innovation, however, increasing the subsidy rate on the purchase of new intermediates only reduces the level of each old intermediate without changing the level of each new intermediate according to (10).

We can now rewrite final output in (1) as

$$Y_t = A\left(L\right)^{1/\sigma}\left[N_{t-1}\left(x_t^c\right)^{1-1/\sigma} + \left(N_t - N_{t-1}\right)\left(x_t^m\right)^{1-1/\sigma}\right]. \tag{11}$$

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Given any initial state that allows innovators to break even, the subsidies can increase final output by promoting innovation, on one hand, but may reduce final output by reducing the level of each intermediate, on the other. Also, the subsidy on the purchase of new intermediates can increase final output by narrowing the gap in the levels of new and old intermediates. To give more details in this regard, let $k_t \equiv K_t/N_t$ be the capital–variety ratio and $k_c \equiv \theta\sigma F(1-s_x)^{\sigma}(1-s_n)$ be its critical level, above which innovation occurs. Then substituting (7)–(10) into (11) gives

$$Y_t = A\,(LN_{t-1})^{1/\sigma}\,(K_{t-1})^{1-1/\sigma}\,, \quad \text{if } k_{t-1} \le k_c; \tag{12}$$

$$Y_t = A(L)^{1/\sigma}\left\{K_{t-1}\frac{[(\sigma-1)\,F\,(1-s_n)]^{1-1/\sigma}}{[\sigma - s_n\,(\sigma-1)]\,F}\right.$$
$$\left. + N_{t-1}\,(1-s_x)^{\sigma-1}\,\frac{[s_n + s_x\sigma\,(1-s_n)]}{[\sigma - s_n\,(\sigma-1)]}\,[\theta\sigma F\,(1-s_n)]^{1-1/\sigma}\right\}, \quad \text{if } k_{t-1} \ge k_c.$$

The critical value of the capital–variety ratio, k_c, divides government action in this model into *policy-dormant* and *policy-active* regions, respectively. As noted earlier, increasing the rate of either subsidy can reduce the threshold level of the capital–variety ratio, k_c, virtually to anywhere above zero, enhancing the chance for the economy to stay in the policy-active region with R&D activities. According to (12), given an initial state (N_{t-1}, K_{t-1}) such that $k_{t-1} \ge k_c$, subsidizing either the R&D cost or the purchase of new intermediates can increase final output if the subsidy rates are sufficiently low so that their positive impact on variety expansion dominates. The opposite occurs for further increases in the subsidy rates if the subsidy rates are already high enough so that their negative impact on the demand for intermediates dominates. To see this more clearly, we differentiate final output with respect to one subsidy at a time for any initial state (N_{t-1}, K_{t-1}) such that $k_{t-1} \ge k_c$. Focusing first on how s_n affects Y_t at $s_x = 0$ and $k_{t-1} \ge k_c$, dY_t/ds_n is signed by two parts additively. One part, containing the derivative of $(1-s_n)^{1-1/\sigma}/[\sigma - s_n(\sigma-1)]$ with respect to s_n, is signed by $-s_n$, and the other part, containing the derivative of $s_n(1-s_n)^{1-1/\sigma}/[\sigma - s_n(\sigma-1)]$, is signed by $1 - s_n(2-1/\sigma)$. Overall, dY_t/ds_n must be positive for very small s_n (say zero), but it becomes negative for large s_n, at least when $s_n > 1/(2-1/\sigma)$.

Focusing on s_x at $s_n = 0$, dY_t/ds_x is signed by $1 - \sigma s_x$ through signing the derivative of $(1-s_x)^{\sigma-1}s_x$ with respect to s_x. Thus, final output increases with s_x when $s_x < 1/\sigma$, under which $x_c > x_m$; final output peaks at $s_x = 1/\sigma$, at which $x_c = x_m$; any further increase in s_x leads to $x_c < x_m$ and thus reduces final output. Therefore, a rise in subsidies may improve (worsen) static efficiency when it starts from a low (high) enough subsidy level.

To economize on notation, we normalize $L = 1$ and $F = 1/\theta\sigma$ without affecting the essence of the results. Then, the critical value of k becomes $k_c \equiv (1-s_x)^{\sigma}\,(1-s_n)$. Given any initial state of capital and variety, the determination

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of innovation and output in (9) and (12) is simplified to

$$\frac{N_t}{N_{t-1}} \equiv \psi(k_{t-1}, s_x, s_n) = \max\left\{1,\quad 1 + \frac{\theta\sigma}{\sigma - s_n(\sigma-1)}(k_{t-1} - k_c)\right\}, \tag{13}$$

$$\frac{Y_t}{K_{t-1}} \equiv \phi(k_{t-1}, s_x, s_n)$$
$$= \begin{cases} A(k_{t-1})^{-1/\sigma}, & \text{if } k_{t-1} \le k_c; \\ \dfrac{A(1-s_n)^{1-1/\sigma}\left\{1 + \dfrac{(1-s_x)^{\sigma-1}}{k_{t-1}}\left[\dfrac{s_n}{\sigma} + s_x(1-s_n)\right]\right\}}{\left[1 - s_n\left(1 - \dfrac{1}{\sigma}\right)\right]}, & \text{if } k_{t-1} \ge k_c. \end{cases} \tag{14}$$

We now turn to capital accumulation through household savings.

2.3. Households

The infinitely lived representative agent derives utility from consumption according to

$$U = \sum_{t=1}^{\infty} \beta^t \ln(C_t), \quad 0 < \beta < 1, \tag{15}$$

where β is the discount factor. The logarithmic utility provides tractability.

In period t, the agent receives capital income, $r_t K_{t-1}$, and earns labor income, $w_t L$. He consumes C_t, faces a proportional consumption tax $\tau_{c,t}$, and carries over K_t to the next period. The flow budget constraint for the agent is

$$K_t = r_t K_{t-1} + w_t L - (1 + \tau_{c,t})C_t. \tag{16}$$

Also, the consumer faces an intertemporal solvency restriction:

$$\lim_{t\to\infty} \frac{K_t}{\prod_{s=1}^{t} r_s} \ge 0. \tag{17}$$

Relegating the derivation to Appendix A, we give the solution to the consumer problem as follows:

$$K_t = \beta\left(1 - \frac{1}{\sigma}\right) Y_t \equiv s^E Y_t, \tag{18}$$

$$C_t = \frac{(1 - s^E)Y_t}{1 + \tau_{c,t}}. \tag{19}$$

From (18) and (19), the agent carries a constant fraction of income, $s^E = \beta(1 - 1/\sigma)$, as capital into the next period and spends the remaining fraction on consumption. The higher is the elasticity of substitution between intermediate inputs σ, the greater is the saving rate s^E. A higher tax rate on consumption spending reduces consumption (for greater subsidies).

2.4. The Government

The government runs a balanced budget in every period between taxes and subsidies:

$$\tau_{c,t} C_t = s_n F r_t (N_t - N_{t-1}) + s_x a r_t \sigma/(\sigma - 1)(N_t - N_{t-1}) x_t^m. \tag{20}$$

Without innovation occurring ($N_t = N_{t-1}$), the tax/subsidy equals zero (the policy-dormant regime).

Combining (18), (19), and (20), together with (7) and (10), determines the tax rate as a function of the subsidy rates and the state:

$$\tau_{c,t} = \max \left\{ 0, \frac{(k_{t-1} - k_c)[s_n + \sigma s_x (1 - s_n)]}{\left(\frac{\sigma}{\sigma - 1} - \beta\right)[\sigma - s_n(\sigma - 1)] k_{t-1} - (k_{t-1} - k_c)[s_n + \sigma s_x (1 - s_n)]} \right\}. \tag{21}$$

2.5. The Equilibrium Path

From (13), (14), and (18), the dynamic path of the economy can be uniquely determined by the following system of first-order difference equations in K and N:

$$K_t = s^E \phi(k_{t-1}, s_x, s_n) K_{t-1}, \tag{22}$$

$$N_t = N_{t-1} + \max \left\{ 0, \frac{\theta}{1 - s_n(1 - 1/\sigma)} \left[K_{t-1} - (1 - s_x)^\sigma (1 - s_n) N_{t-1} \right] \right\}, \tag{23}$$

starting from an initial state $(K_0, \ N_0)$, given time-invariant subsidy rates, s_x and s_n. Here, $\phi(k_{t-1}, s_x, s_n)$ is the ratio of output to available capital stock, Y_t/K_{t-1}, in (14).

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From (21), (22), and (23), the law of motion for the capital–variety ratio, k_t, can be further converted into the following one-dimensional mapping, $\Phi : R_+ \to R_+$,

$$k_t = \Phi\left(k_{t-1}\right) \equiv \begin{cases} s^E A\left(k_{t-1}\right)^{1-1/\sigma}, & \text{if } k_{t-1} \le k_c; \\ \dfrac{s^E A\left(1-s_n\right)^{1-1/\sigma}\left\{k_{t-1} + \left(1-s_x\right)^{\sigma-1}\left[s_n/\sigma + s_x\left(1-s_n\right)\right]\right\}}{1 - s_n\left(1-1/\sigma\right) + \theta\left[k_{t-1} - \left(1-s_x\right)^{\sigma}\left(1-s_n\right)\right]}, & \text{if } k_{t-1} \ge k_c, \end{cases} \quad (24)$$

where $k_c = (1-s_x)^{\sigma}(1-s_n)$. The equilibrium path for an initial state k_0 is given by $\{\Phi^t(k_0)\}$, where $\Phi^t(k)$ is defined iteratively by $\Phi^1(k) \equiv \Phi(k)$ and $\Phi^t(k) \equiv \Phi(\Phi^{t-1}(k))$.

3. THE STEADY STATE AND DYNAMICS WITH SUBSIDIES

This section explores how subsidization affects the steady state and the dynamics of the equilibrium path in turn. A thorough analysis of a similar equilibrium path is made in Matsuyama (1999, 2001) and Gardini et al. (2008) in the absence of subsidization.

3.1. The Steady State

The dynamic system in (24) has a unique steady state where the ratio of capital to variety, $k_t = K_t/N_t$, stays constant over time for any time-invariant subsidy rates, s_x and s_n. Even though the mapping $k_t = \Phi(k_{t-1})$ in (24) has two fixed points, the first fixed point, $k = 0$, is repelling because $\Phi'(0) > 1$ (hence trivial). The second fixed point corresponds to the unique steady state, which can be either k^*, satisfying $k^* = \Phi(k^*)$, if $k^* \le k_c$ (the Solow regime), as shown in Figure 1 or k^{**}, satisfying $k^{**} = \Phi(k^{**})$, if $k^{**} > k_c$ (the Romer regime), as shown in Figure 2.

If $k = k^* \le k_c$ in a steady state, then $N_t = N_{t-1}$ and $K_t = K_{t-1}$, from (13) and from the definition of k_t. In other words, in this steady state, there is no innovation; all the intermediates are competitively supplied; and the economy does not grow in the long run. From (24), on this neoclassical stationary path, $k^* \equiv (s^E A)^{\sigma}$. The existence of such a stationary path requires that $s^E A \le (k_c)^{1/\sigma}$.

If $k = k^{**} > k_c$ holds in a steady state, then from (22) and (23) the balanced growth path satisfies the following:

$$\frac{K_t}{K_{t-1}} = \frac{N_t}{N_{t-1}} = s^E \phi\left(k^{**}, s_x, s_n\right) = 1 + \frac{\theta\sigma}{\sigma - s_n\left(\sigma-1\right)}\left(k^{**} - k_c\right) > 1.$$

In this steady state, the available capital stock of the economy is large enough relative to the number of existing intermediates so that new intermediates are introduced. The existence of such a balanced growth path requires that $s^E \phi(k^{**}, s_x, s_n) > 1$.

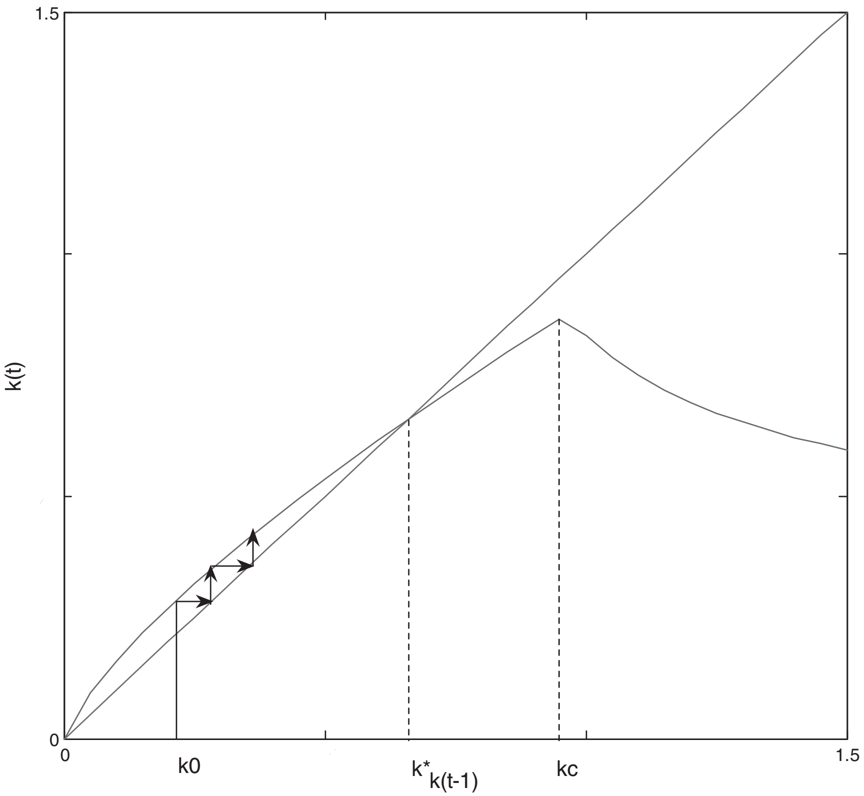

FIGURE 1. $G < 1$.

Relegating the proof to Appendix B, we give the results about the steady state as follows.

PROPOSITION 1. *Let* $G \equiv s^E A/(k_c^{1/\sigma})$ *with* $k_c = (1 - s_x)^\sigma (1 - s_n)$, $0 \leq s_x < 1$, *and* $0 \leq s_n < 1$. *The dynamic system has a unique steady state:*

(i) *If* $G \leq 1$, *the steady state is a neoclassical stationary path* $k^* = \Phi(k^*)$ *with* $k^* = (s^E A)^\sigma \leq k_c$.
(ii) *If* $G > 1$, *the steady state is a balanced growth path* $k^{**} = \Phi(k^{**})$ *with*

$$k^{**} = [\theta k_c + s_n (1 - 1/\sigma) - 1 + \beta (1 - 1/\sigma) A (1 - s_n)^{1-1/\sigma} + \Delta^{1/2}] / (2\theta) > k_c,$$

$$\Delta = \left[1 - s_n (1 - 1/\sigma) - \theta k_c - \beta (1 - 1/\sigma) A (1 - s_n)^{1-1/\sigma}\right]^2 + 4\theta\beta (1 - 1/\sigma) A (1 - s_n)^{1-1/\sigma} (1 - s_x)^{\sigma-1} [s_n/\sigma + s_x (1 - s_n)] .$$

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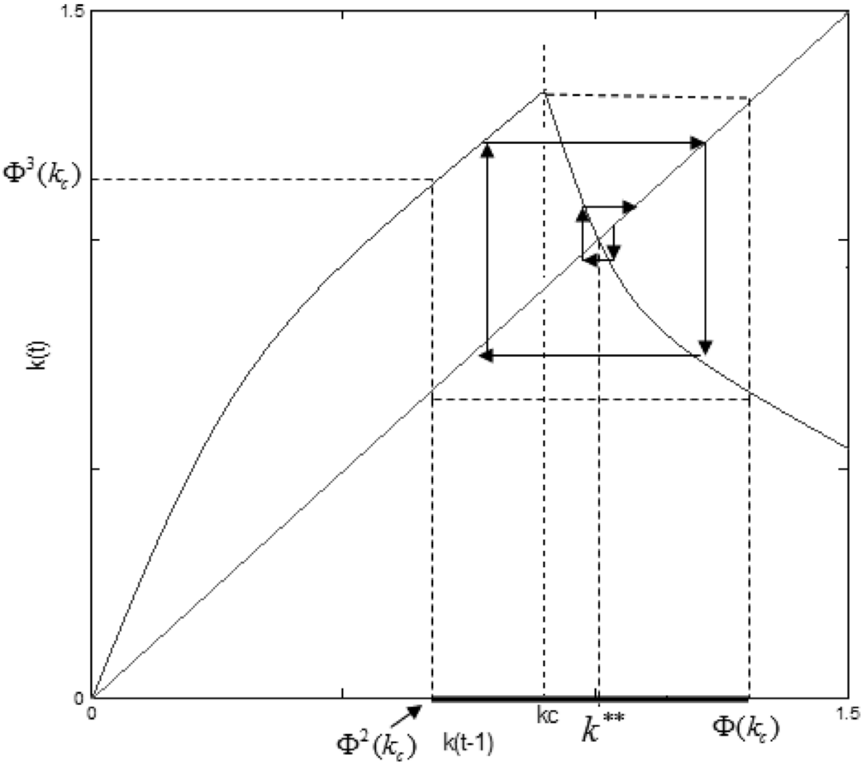

FIGURE 2. $G > 1$, $s_x < 1/\sigma$, and $\Phi'(k^{**}) < -1$.

The balanced growth rate for (N_t, K_t) *is given by*

$$g = \beta(1 - 1/\sigma)\,\phi\left(k^{**}, s_x, s_n\right) = 1 + \frac{\theta\sigma}{\sigma - s_n(\sigma - 1)}\left(k^{**} - k_c\right).$$

Whether the economy grows or not in the steady state depends on the fundamentals (such as the discount factor, the total factor productivity, and the degree of substitution between intermediates) as well as on the subsidy rates. Given the fundamentals, the higher the rates of subsidies, the more likely the economy is to move beyond the critical k_c toward the balanced growth path.

For example, suppose that $s^E A = \beta(1 - 1/\sigma)A < 1$. Then, starting from a case without subsidies, $s_x = 0$ and $s_n = 0$, such that $k_c = 1$, the low saving and low total factor productivity yield no growth potential in the long run, $G \leq 1$, and the steady state is in the Solow regime.

Now, we use one type of subsidy at a time. Once the subsidy rates are set in the range $\{s_x, s_f | s_x = 0 \cap s_n \in (1 - (s^E A)^\sigma, 1)\}$ or in the range $\{s_x, s_n | s_x \in (1 - s^E A, 1) \cap s_n = 0\}$, the condition for long-run growth, $G = s^E A/(k_c)^{1/\sigma} > 1$, is satisfied. That is, sufficient subsidization can rule out the

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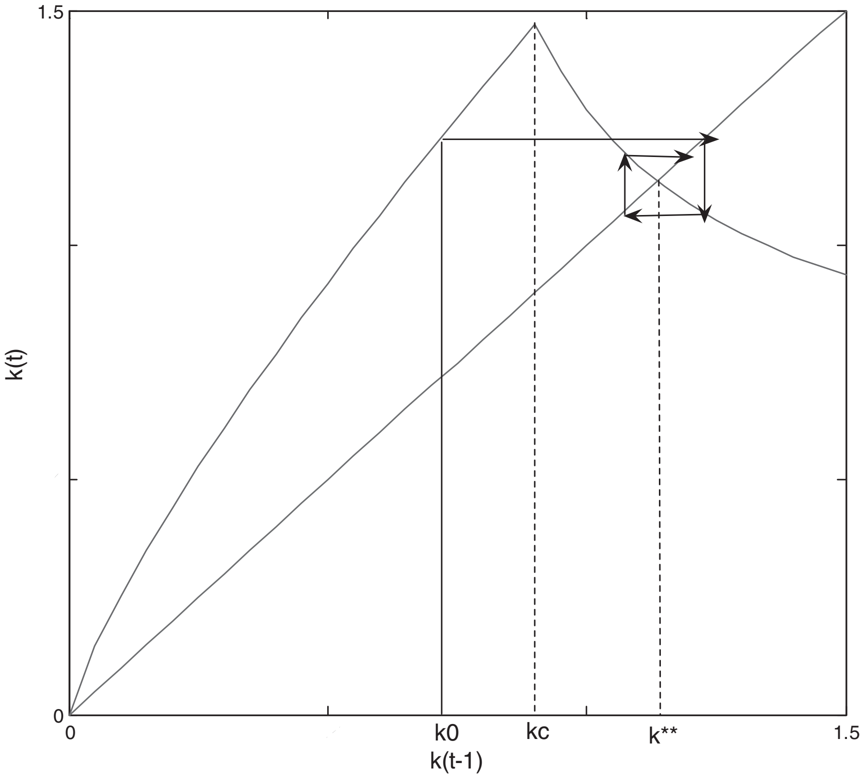

FIGURE 3. $G > 1$, $s_x < 1/\sigma$, and $-1 < \Phi'(k^{**}) < 0$.

neoclassical steady state in the long run and replace it by the steady state with balanced growth in capital and in the variety of intermediates.

3.2. The Dynamics

The asymptotic behavior of $k_t = K_t/N_t$, from any initial state $k_0 = K_0/N_0 > 0$, is characterized by a continuous mapping $k_t = \Phi(k_{t-1})$ in equation (24). This mapping is increasing in the range of $(0, k_c)$ and may be increasing or decreasing in the range of (k_c, ∞), as illustrated in Figures 1–4.

When $k_{t-1} \leq k_c$, there is no innovation in period t; that is, $N_t = N_{t-1}$. Consequently, the subsidies are nonoperative. When $k_{t-1} > k_c$, new intermediates are introduced and the subsidies become operative. Economic growth in this policy-active region is driven by both investment and innovation. Intuitively, when the rate of capital accumulation exceeds the rate of variety expansion, $K_t/K_{t-1} > N_t/N_{t-1}$, the resultant ratio of capital per variety, $k_t = K_t/N_t$, will increase; conversely, it will decrease. The slope of the transition curve of

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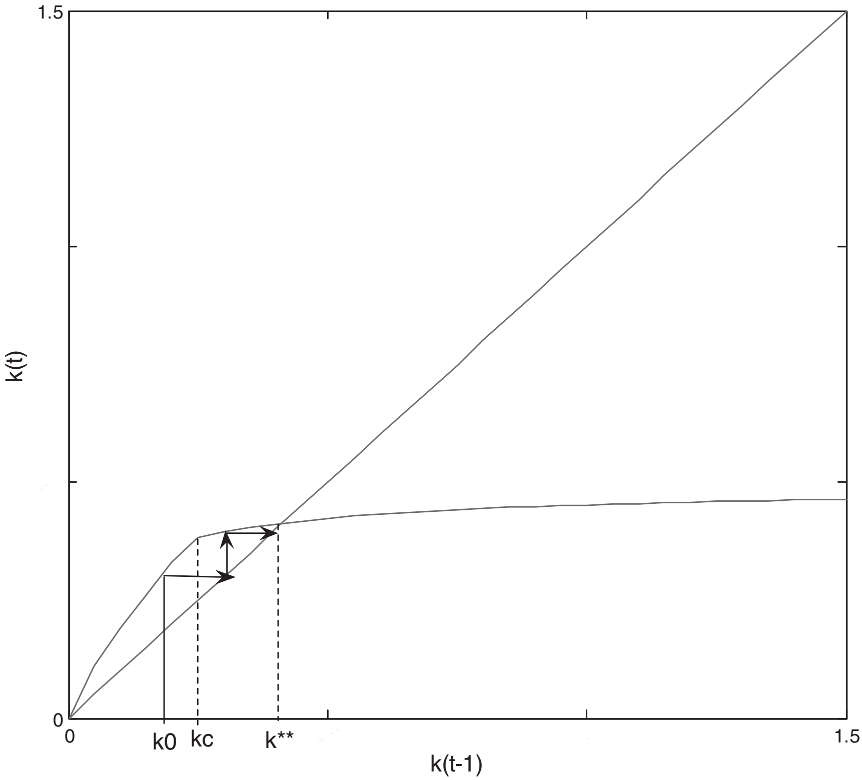

FIGURE 4. $G > 1$ and $s_x > 1/\sigma$.

$k_t = \Phi(k_{t-1})$ in this Romer regime varies with subsidies and plays a crucial role in determining the asymptotic behavior of k_t. Thus, it deserves careful investigation.

Without the use of subsidies at $s_x = s_n = 0$, the dynamics of $k_t = \Phi(k_{t-1})$ in (24) will become exactly the same as that in Matsuyama (1999), where $k_c = 1$and the mapping for k_t is unimodal and always decreasing for $k_{t-1} > k_c$, as in Figures 1–3. Under the empirically plausible conditions $1 < G < \theta - 1$ in the Matsuyama model, period-2 cycles prevail, with $k_t$alternating between the Solow and Romer regimes forever, as in Figure 2. It is important to ask how the subsidies can change the slope of the transition equation $k_t = \Phi(k_{t-1})$ to mitigate or even eliminate the cycles. Four scenarios of the asymptotic path of k_t are summarized as follows (the proof is relegated to Appendix C).

PROPOSITION 2. *Suppose* $\theta > 2$. *Define* $G_0 \equiv \beta(1 - 1/\sigma)A$ *and* $G \equiv \beta(1 - 1/\sigma)A/(k_c^{1/\sigma})$ *with* $k_c = (1 - s_x)^\sigma(1 - s_n)$ *and* $0 \leq s_n,\ s_x < 1$. *For any given* $k_0 > 0$, *the distinct asymptotic dynamic behaviors are as follows:*

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(i) *If* $G \leq 1$, *then the economy converges toward a stable neoclassical stationary path with* $\lim_{t\to\infty} \Phi^t(k) = k^*$ *in the policy-dormant region.*
(ii) *If* $G > 1$, $s_x \leq 1/\sigma$, *and the subsidy rates are low enough so that* $\Phi'(k^{**}) < -1$, *then period-2 cycles may exist forever.*
(iii) *If* $G > 1$, $s_x \leq 1/\sigma$, *and the subsidy rates are high enough, such as* $s_n > \sigma(\theta - 2)/[1 + \sigma(\theta - 2)]$ at $s_x = 0$ *or* $s_x \to 1/\sigma$ at $s_n = 0$, *then* $-1 < \Phi'(k^{**}) < 0$. *The economy fluctuates around and eventually converges toward a stable balanced growth path,* $\lim_{t\to\infty} \Phi^t(k) = k^{**}$.
(iv) *If* $G > 1$ *and* $s_x \geq 1/\sigma$, *then* $0 < \Phi'(k^{**}) < 1$. *The economy converges monotonically toward a stable balanced growth path,* $\lim_{t\to\infty} \Phi^t(k) = k^{**}$.

At the heart of period-2 cycles in the empirically plausible situation in Matsuyama (1999), the period of intense innovation can be associated with productivity and growth slowdowns. It is only after the period of innovation that the innovated good releases its full potential for faster growth of output. According to Proposition 2 and illustrations in Figures 1–4, sufficiently high rates of subsidies can eventually eliminate cycles by stabilizing the balanced growth path with innovation. This is achieved either by strengthening the demand for new intermediates (via a higher s_x) or by reducing the innovation cost (via a higher s_n) so that R&D activities are profitable even at a low capital–variety ratio. By increasing varieties, such subsidization can exert different impacts on final output and thus on capital investment with a constant saving rate. First, it can directly increase final output by increasing the number of varieties according to (11). Second, it can indirectly reduce final output by reducing the equilibrium quantity of each type of intermediate input, as the subsequent increase in the total fixed innovation cost competes for the given amount of existing capital. For $0 \leq s_x < 1/\sigma$, a higher s_x reduces the gap in the equilibrium quantity of old and new intermediates, thereby mitigating the efficiency loss of unequal use of symmetric intermediates in production.

It is convenient to look at the effects of subsidies on the stability of the dynamic system by using a general expression for the slope of the transition curve $k_t = \Phi(k_{t-1})$:

$$\Phi'(k_{t-1}) = d(K_t/N_t)/dk_{t-1} = [N_t(dK_t/dk_{t-1}) - K_t(dN_t/dk_{t-1})]/N_t^2.$$

Rewriting it in the following ways may help our interpretation:

$$\begin{aligned}\Phi'(k_{t-1}) &= \left(\frac{dK_t}{dk_{t-1}} - \frac{dN_t}{dk_{t-1}}k_t\right)\frac{1}{N_t} \\ &= \left(\frac{dK_t}{dk_{t-1}}\frac{1}{K_t} - \frac{dN_t}{dk_{t-1}}\frac{1}{N_t}\right)\frac{K_t}{N_t} \\ &= \left(\frac{dK_t}{dk_{t-1}}\frac{k_{t-1}}{K_t} - \frac{dN_t}{dk_{t-1}}\frac{k_{t-1}}{N_t}\right)\frac{k_t}{k_{t-1}}.\end{aligned}$$

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The sign of $\Phi'(k_{t-1})$ is positive (negative) if the ratio of the derivative of capital investment to the derivative of variety expansion with respect to the initial abundance of capital, $(dK_t/dk_{t-1})/(dN_t/dk_{t-1})$, is greater (smaller) than the resultant capital–variety ratio, k_t. The absolute value of $\Phi'(k_{t-1})$ depends positively on the difference between the two responses, as fractions of their new stocks, $[(dK_t/dk_{t-1})/K_t - (dN_t/dk_{t-1})/N_t]$, as well as on the new capital–variety ratio, K_t/N_t. In the Romer steady state with $k_{t-1} = k_t > k_c$, both the sign and the magnitude of $\Phi'(k_{t-1})$ will depend solely on the gap in the respective elasticity of investment K_t and innovation N_t with respect to k_{t-1}.

Note that $G_0 = s^E A$ is the growth factor absent subsidization. Holding N_{t-1} constant, if $k_{t-1} > k_c$, then (14) and (22) lead to $dK_t/dk_{t-1} = G_0(1-s_n)^{1-1/\sigma}N_{t-1}/[1-s_n(1-1/\sigma)] > 0$ and (23) leads to $dN_t/dk_{t-1} = \theta N_{t-1}/[1-s_n(1-1/\sigma)] > 0$. That is, in the Romer regime, both capital investment and variety expansion respond positively to the initial abundance of capital.

Under the assumption that $1 < G_0 < \theta - 1$, however, in the absence of subsidization, variety's response to the initial abundance of capital is more elastic than investment's response, causing instability of the balanced growth path in the original Matsuyama model. At the steady state k^{**} on the balanced growth path without subsidization, Proposition 1 and (23) lead to

$$k^{**} = \frac{G_0 - 1 + \theta}{\theta} > 1 \quad \text{for } G_0 > 1 \text{ and } s_x = s_n = 0; \qquad N_t/N_{t-1} = G_0.$$

So $\Phi'(k_{t-1}) = [(dK_t/dk_{t-1}) - k_t(dN_t/dk_{t-1})]/N_t$ at the steady state k^{**} on the balanced growth path becomes $\Phi'(k^{**}) = (G_0 - k^{**}\theta)N_{t-1}/N_t = (1-\theta)/G_0$, which is negative under $1 < \theta$ and less than -1 under $G_0 < \theta - 1$ and $2 < \theta$. In this case, period-2 cycles may exist, as in Matsuyama (1999).

Starting from stable period-2 cycles with $\Phi'(k^{**}) = (1-\theta)/G < -1$, a decrease in G toward 1 may eventually trigger a bifurcation and lead to chaotic dynamics, as discussed in Matsuyama (1999) and shown in Gardini et al. (2008). Following one numerical example in Gardini et al. (2008), given the parameterization of $\sigma = 5$ and $s_x = s_n = 0$, period-2 cycles lose stability when G enters the chaotic region $G \in (1, 1.0725)$, where the trajectory enters four chaotic intervals in a few iterations.[1] Such chaotic situations can be avoided if subsidies lift G above the upper bound of the chaotic region.

Subsidizing the innovation cost strengthens the response of variety expansion to the initial abundance of capital, $dN_t/dk_{t-1} = \theta N_{t-1}/[1-s_n(1-1/\sigma)] > 0$, but weakens the response of investment, $dK_t/dk_{t-1} = G_0(1-s_n)^{1-1/\sigma}N_{t-1}/[1-s_n(1-1/\sigma)] > 0$, setting $s_x = 0$. From (24), if the subsidy s_n is large enough, at least for $s_n > 1/(2-1/\sigma) \in (0,1)$, a further increase in s_n will lead to lower capital per variety k_t as long as $k_t > k_c$ and $k_{t-1} > k_c$. This arises because beyond the level $s_n = 1/(2-1/\sigma) \in (0,1)$, a further rise in s_n (at $s_x = 0$) reduces output (hence investment for a constant saving rate) on one hand, as mentioned earlier, and increases innovation, on the other hand. The sign of $\Phi'(k_{t-1}) = [G_0(1 - s_n)^{1-1/\sigma} - k_t\theta](N_{t-1}/N_t)/[1 - s_n(1 - 1/\sigma)]$ is only determined by the factor $[G_0(1 - s_n)^{1-1/\sigma} - k_t\theta]$, where both terms, $G_0(1 - s_n)^{1-1/\sigma}$ and $k_t\theta$, eventually decline with the subsidy rate on the innovation cost when the subsidy rate becomes large enough. This helps to explain why the sign of $\Phi'(k_{t-1})$ remains negative for all levels of the subsidy rate in the Romer regime. The remaining factors, which determine only the magnitude, not the sign, of $\Phi'(k_{t-1})$ are decreasing with s_n as well:

$$(N_{t-1}/N_t)/[1-s_n(1-1/\sigma)] = \{1-s_n(1-1/\sigma) + \theta[k_{t-1} - (1-s_n)]\}^{-1} \quad \text{for } \theta > 1.$$

This explains why the absolute value of $\Phi'(k_{t-1})$ depends inversely on the subsidy rate s_n.

On the other hand, consider how the subsidy on the purchase of new intermediates affects the dynamic feature of the ratio of capital to variety. At $s_n = 0$, the sign of $\Phi'(k_{t-1}) = (G_0 - \theta k_t)\, N_{t-1}/N_t$ is determined only by $(G_0 - k_t\theta)$, which is initially negative under $1 < G_0 < \theta - 1$ in the Romer regime when the subsidy rate on the purchase of new intermediates is equal to zero. From (14) and (22), a higher subsidy on the purchase of new intermediates exerts a positive (zero, negative) effect on investment, K_t, if the subsidy rate is lower than (equal to, higher than) $1/\sigma$. From (23), a higher subsidy for the purchase of new intermediates promotes innovation in the Romer regime and thus reduces the reciprocal of the rate of innovation, $N_{t-1}/N_t = \{1 + \theta[k_{t-1} - (1 - s_x)^\sigma]\}^{-1}$, tending to reduce the absolute value of $\Phi'(k_{t-1})$. From (24), a higher subsidy rate for the purchase of new intermediates reduces the amount of capital per variety k_t as long as $k_t > k_c$ and $k_{t-1} > k_c$ in the Romer regime: dk_t/ds_x is signed by $[1 - \sigma s_x - \theta(1 - s_x)^\sigma - \theta k_{t-1}(\sigma - 1)] < 0$, in which $1 - \sigma s_x - \theta(1 - s_x)^\sigma < 0$ attains a maximum $1/\sigma - 1 < 0$ at $s_x = 1/\sigma$. By reducing k_t, a higher subsidy rate for the purchase of new intermediates increases the value of $(G_0 - \theta k_t)$. Overall, a higher s_x reduces the absolute value of $\Phi'(k_{t-1})$ when $G_0 - k_t\theta < 0$. At $s_x = 1/\sigma$ and $s_n = 0$, $G_0 = k^{**}\theta$ on the balanced growth path, implied by case (ii) of Proposition 1, leading to $\Phi'(k_{t-1}) = 0$ at the steady state. When s_x is increased further for $s_x > 1/\sigma$, $(G_0 - k^{**}\theta) > 0$ must hold, leading to $\Phi'(k_{t-1}) > 0$ on the balanced growth path. Recall that subsidizing the purchase of new intermediates at a rate $s_x > 1/\sigma$ leads to $x_c < x_m$, thereby creating a loss in final output. The loss in final output, due to a higher s_x beyond $s_x = 1/\sigma$, will in turn lead to a decline in investment for a constant saving rate given any initial k_{t-1}, whereas variety expansion is increasing with s_x. Consequently, a higher s_x with $s_x > 1/\sigma$ will reduce k^{**} to a level at which $(G_0 - k^{**}\theta) > 0$ and thus $\Phi'(k_{t-1}) > 0$ in the steady state. For $s_n = 0$, the absolute value of $\Phi'(k_{t-1})$ on the balanced growth path is derived as

$$\Phi'(k^{**}) = \frac{G_0 + 1 - \theta(1 - s_x)^\sigma - \Delta^{1/2}}{G_0 + 1 - \theta(1 - s_x)^\sigma + \Delta^{1/2}} < 1.$$

Therefore, the balanced growth path becomes stable once the subsidy rate on the purchase of new intermediates is set high enough so that $|\Phi'(k^{**})| < 1$.

This combination of growth and cycles makes it difficult to derive a reduced-form expression for welfare in this model. As a consequence, it is difficult to derive optimal subsidy rates to maximize social welfare analytically. Intuitively, however, there are several sources of efficiency losses in laissez-faire, which can be mitigated by subsidies for welfare improvements.

First, the emergence of cycles or even chaos when the steady state in the Romer regime loses stability is likely inefficient, given the desire for consumption smoothing from decreasing marginal utility. From Proposition 2, sufficiently high subsidies can change the slope of the mapping (24) for stability in the steady state in the Romer regime to mitigate and eventually eliminate cycles. This implies a possible welfare improvement arising from the subsidies.

Second, private innovators fail to attach any value to the benefits from current innovation that accrue beyond the temporary patent protection. However, from a social perspective, the gains from any invention will last forever, making future invention less costly when based on more advanced current technology. Therefore, the private rate of return to R&D is likely below the social rate. Consequently, too little capital is allocated to R&D and too much capital is allocated to manufacturing intermediates. The resultant quantity of each intermediate should be too high. This intertemporal externality bears similarity to the results of Romer (1990) and Aghion and Howitt (1992). Because in the Romer model there is no business-stealing effect, there is always too little R&D and slower growth in the Romer regime. By promoting innovation, both subsidies can be welfare-improving.

Further, the assumption of a fixed amount of capital necessary for innovation distorts the relative price of intermediates, leading to the uneven use of old and new intermediates in final production. This drives down the already lower marginal rate of contribution, making it further from the social rate. The two types of subsidies work to internalize the intertemporal spillover from innovation and mitigate the static and dynamic inefficiencies, respectively. For example, the subsidy to R&D investment, s_n, induces more resources to the R&D sector; the subsidy to the purchase of new inputs, s_x, narrows the quantity gap between new and old varieties.

Third, as capital is accumulated in a neoclassical fashion in the Solow regime, there is a friction in the model, because innovators take past accumulation as given rather than using that information to optimally plan the amount of resources to devote to R&D activity. Imposing a consumption tax to finance the subsidies can reduce consumption for investment and innovation and can thus mitigate this type of efficiency loss.

4. NUMERICAL SIMULATION RESULTS

We now gauge the quantitative implications and the welfare gains from the subsidies and explore optimal subsidy rates by numerical simulations for plausible parameterizations. Concerning the method, given any initial state (N_{t-1}, K_{t-1}),

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one can update it one period to obtain (N_t, K_t) by using (22) and (23). One can then find the levels of intermediates in (10), final output in (11), consumption in (19), and welfare in (15) in each period.

We set a benchmark parameterization as $A = 2.5$, $\beta = 0.63$, $\theta = 2.4414$, and $\sigma = 5$. The value of $\sigma = 5$ is in line with that in Matsuyama (1999), where it plays dual roles: $1 - 1/\sigma = 0.8$ is the share of capital (interpreted broadly as both physical and human capital); $1/(\sigma - 1) = 0.25$ is the markup enjoyed by the innovator. When one period (the length of patent protection) corresponds to 15 years, an annual discount factor of 0.97 leads to $\beta = (0.97)^{15} = 0.63$. In the WTO's Agreement on Trade-Related Aspects of Intellectual Property Rights, the term of patents is 20 years from the filing date of the application. However, many countries have patent laws for shorter terms of 6 to10 years. Such patent lengths are in the range for medium-term business cycles, as in Comin and Gertler (2006). In our simulation, we take the average term of patents as 15 years. In fact, the shorter the term of patents, the larger the discount factor, and thus the stronger the effects of subsidization.

The value of $A = 2.5$ is chosen so that the benchmark parameterization without subsidies ($s_x = 0$,$s_n = 0$) satisfies $1 < G_0 < \theta - 1$ for period-2 cycles. This is an empirically plausible assumption, as argued in Matsuyama (1999). Also, we set $K_0 = 0.4$ and $N_0 = 1$. The benchmark case can be seen in Figure 5a, which depicts the values of k for 100 periods, starting from $k_0 = 0.4$. The dotted line denotes the separation between the Solow and Romer regimes. The economy fluctuates in the Romer regime for quite a while before eventually settling down to period-2 cycles across the two regimes.

To better comprehend the change in welfare, we measure its equivalent variation in consumption in every period. Define the equivalent variation in consumption in each period by Δ, which allows the benchmark case without subsidization to reach the same welfare level as that in a case with subsidization type i:

$$U_{\text{no-subsidization}} + \frac{\beta}{1-\beta}\ln(1+\Delta) = U_{s_i}.$$

This corresponds to adding $\sum_{t=1}^{\infty} \beta^t \ln(1+\Delta) = \beta[\ln(1+\Delta)]/(1-\beta)$ to welfare in the case without subsidization. We use a large number of periods in each case (say 1,000) so that any further increase in the number of periods has no impact on welfare within ten decimal digits.

In Tables 1 and 2, we report simulation results for increasing s_x and s_n from zero to reach and go beyond a peak of the welfare level, one at a time, respectively. In Figure 5, we plot the value of the capital–variety ratio for 100 periods from the initial state and compare the benchmark case with three selective cases of different types and levels of subsidization. In Figures 6 and 7, we also depict the asymptotic capital–variety ratios in the Solow and Romer regimes (k^s, k^r) in the scenario of period-2 cycles or the steady state capital variety ratio, k^{**}, on the balanced growth path for increasing s_x and s_n from zero to sufficiently high rates. The subsequent

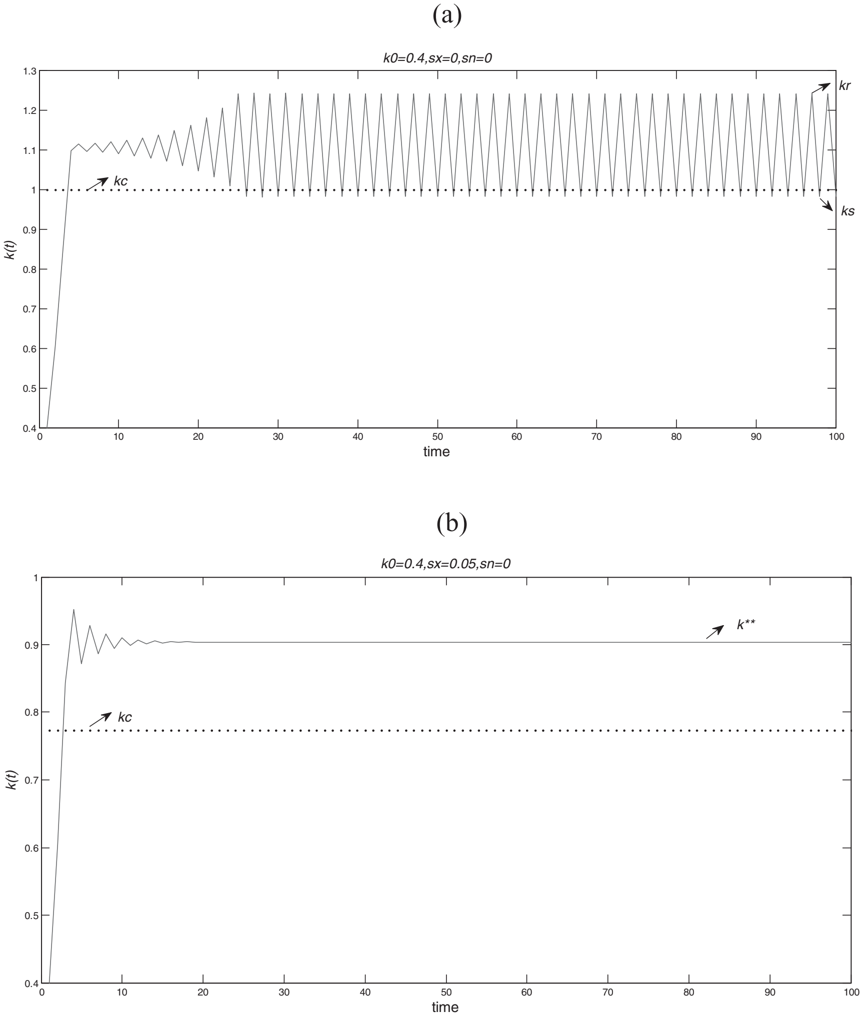

FIGURE 5. The transitional paths of the capital–variety ratio k_t.

welfare levels are plotted in the same figure to better illustrate the optimal rates of subsidies that maximize welfare.

In Table 1, we set $s_n = 0$ and examine the dynamic behavior, the asymptotic capital–variety ratio, the balanced growth rate, the consumption tax rate, the welfare level, and the consumption equivalent variation when s_x is varied gradually from 0 in the benchmark case to 40%. When s_x is sufficiently small (e.g., $s_x = 0.01$), the economy still alternates between $k^s = 0.945$ and $k^r = 1.204$ in the Solow and Romer regimes, where both the growth rate and social welfare are higher than in the benchmark case. When s_x is increased further but still below

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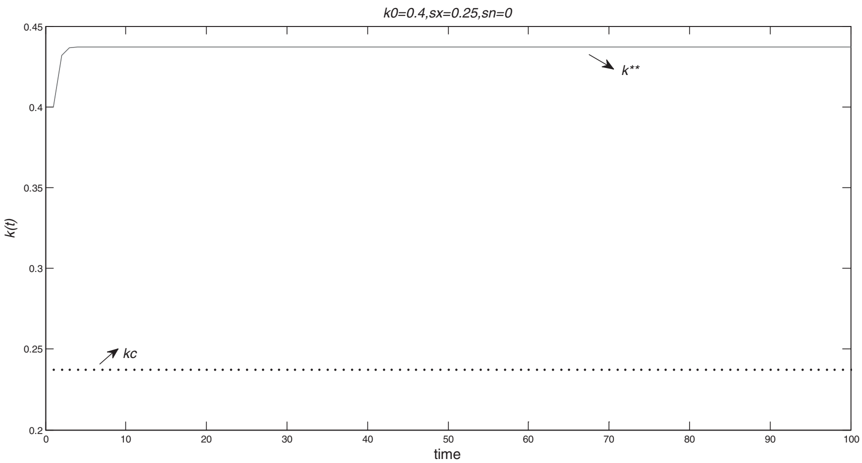
(c)



(d)

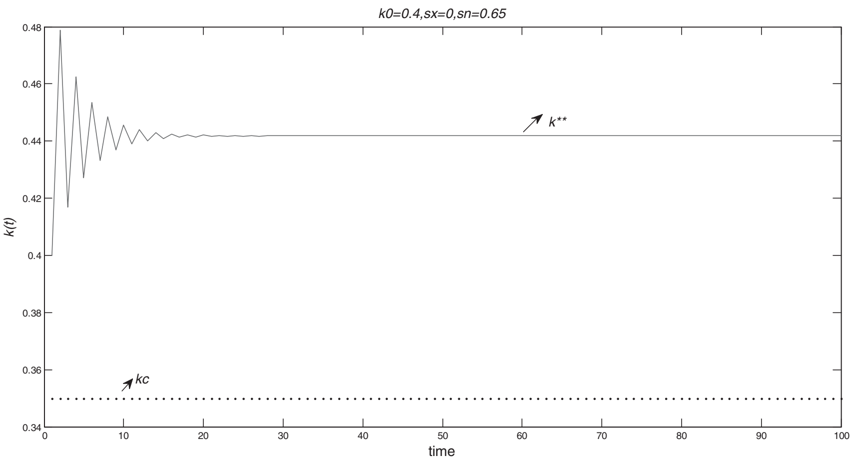


FIGURE 5. Continued.

$1/\sigma = 0.2$, the steady state k^{**} becomes stable and the economy converges toward k^{**}, with fluctuations for some periods, as predicted in Proposition 2. As shown in Figure 5b, when $s_x = 0.05$, the fluctuations subside in magnitude and die off within 20 periods before reaching k^{**}. As s_x is increased beyond $1/\sigma = 0.2$, the convergence becomes monotonic, as shown by Figure 5c with $s_x = 0.25$.

For a better view of the welfare effect of the subsidy on the purchase of new products, we vary s_x from 0 to 0.6 ($s_n = 0$) in Figure 6 and find the subsequent

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TABLE 1. Results of changing the subsidy on the purchase of new intermediate goods (parameterization: $A = 2.5$, $\beta = 0.63$, $\theta = 2.4414$, $\sigma = 5$, $K_0 = 0.4$, $N_0 = 1$, $s_n = 0$)

Subsidy rate	Mode of dynamics	k^{**} or (k^s, k^r)	Growth rate gross (annual %)	Tax rate (%)	Welfare level	Cequivalent variation (%)
$s_x = 0$	Period-2 cycle	(0.983, 1.243)	1.262 (1.56)	0.0, 0.0	−0.0811	0.0
$s_x = 0.01$	Period-2 cycle	(0.945, 1.204)	1.272 (1.62)	0.0, 0.34	−0.0716	0.56
$s_x = 0.02$	O-convergence	1.020	1.283 (1.67)	0.37	−0.0626	1.09
$s_x = 0.05$	O-convergence	0.904	1.317 (1.85)	1.17	−0.0292	3.10
$s_x = 0.10$	O-convergence	0.743	1.371 (2.13)	3.42	0.0226	6.28
$s_x = 0.15$	O-convergence	0.616	1.420 (2.37)	7.25	0.0631	8.84
$s_x^* = 0.16$	O-convergence	0.594	1.429 (2.41)	8.27	0.0643	8.91
$s_x = 0.20$	M-convergence	0.516	1.460 (2.56)	13.35	0.0482	7.89
$s_x = 0.25$	M-convergence	0.437	1.488 (2.68)	22.60	−0.0624	1.10
$s_x = 0.30$	M-convergence	0.374	1.503 (2.75)	36.32	−0.2762	−12.14
$s_x = 0.35$	M-convergence	0.323	1.504 (2.76)	56.60	−0.5908	−34.90
$s_x = 0.40$	M-convergence	0.280	1.493 (2.71)	87.23	−1.0077	−72.32

Notes: (1) O-convergence and M-convergence refer to convergence with fluctuations and monotonic convergence to steady states, respectively. (2) The growth rate in the period-2 cycle economy is calculated as the geometric average of the corresponding growth rates in the two regions following Matsuyama (1999). (3) The values in the brackets beside growth rates indicate the discounted annual rates. (4) The subsidy rates with $*$ indicate the optimal rate maximizing the social welfare.

welfare levels. It is worth noting that the welfare level is a concave and smooth curve, peaking at a unique point with $s_x = 0.16$. This optimal rate lies in the range where the subsidy eliminates the cycles. The resultant welfare level at the optimal subsidy rate is 0.0643, compared to −0.0811 in the benchmark case without subsidization. The corresponding consumption tax rate is equal to 8.27%. The maximum welfare gain with this type of subsidy on the purchase of new products

TABLE 2. Results of changing the subsidy to the R&D investment (parameterization: $A = 2.5$, $\beta = 0.63$, $\theta = 2.4414$, $\sigma = 5$, $K_0 = 0.4$, $N_0 = 1$, $s_x = 0$)

Subsidy rate	Mode of dynamics	k^{**} or (k^s, k^r)	Growth rate gross (annual %)	Tax rate (%)	Welfare level	Cequivalent variation (%)
$s_n = 0$	Period-2 cycle	(0.983, 1.243)	1.262 (1.56)	0.0, 0.0	−0.0811	0.0
$s_n = 0.1$	Period-2 cycle	(0.888, 1.146)	1.286 (1.69)	0,0, 0.76	−0.0616	1.15
$s_n = 0.2$	Period-2 cycle	(0.794, 1.047)	1.311 (1.82)	0.0, 1.85	−0.0370	2.62
$s_n = 0.3$	Period-2 cycle	(0.700, 0.947)	1.339 (1.97)	0.0, 3.44	−0.0063	4.49
$s_n = 0.31$	O-convergence	0.795	1.342 (1.98)	1.79	−0.0034	4.67
$s_n = 0.4$	O-convergence	0.703	1.371 (2.13)	2.87	0.0210	6.18
$s_n = 0.5$	O-convergence	0.600	1.407 (2.30)	4.69	0.0578	8.50
$s_n = 0.6$	O-convergence	0.495	1.446 (2.49)	7.70	0.0665	9.06
$s_n^* = 0.61$	O-convergence	0.485	1.450 (2.51)	8.10	0.0668	9.07
$s_n = 0.7$	O-convergence	0.388	1.486 (2.68)	13.15	0.0141	5.75
$s_n = 0.8$	O-convergence	0.277	1.523 (2.85)	24.94	−0.2787	−12.31
$s_n = 0.9$	O-convergence	0.159	1.518 (2.82)	63.02	−1.5241	−133.37

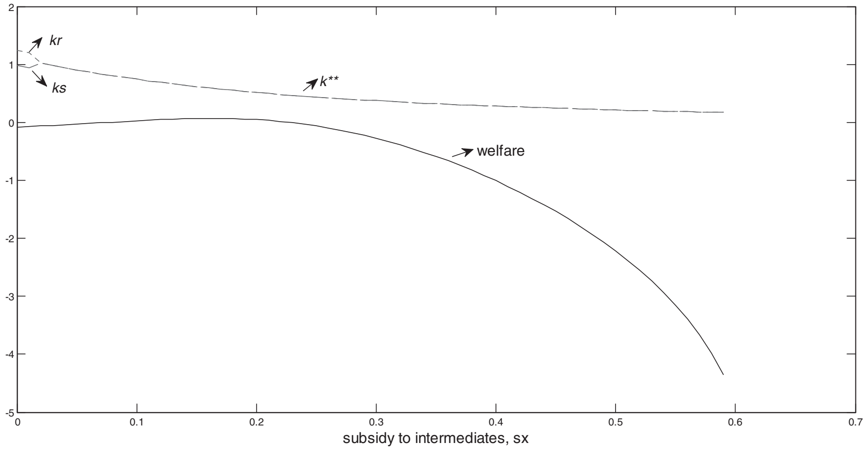

FIGURE 6. The asymptotic capital–variety ratio and welfare levels with changing s_x.

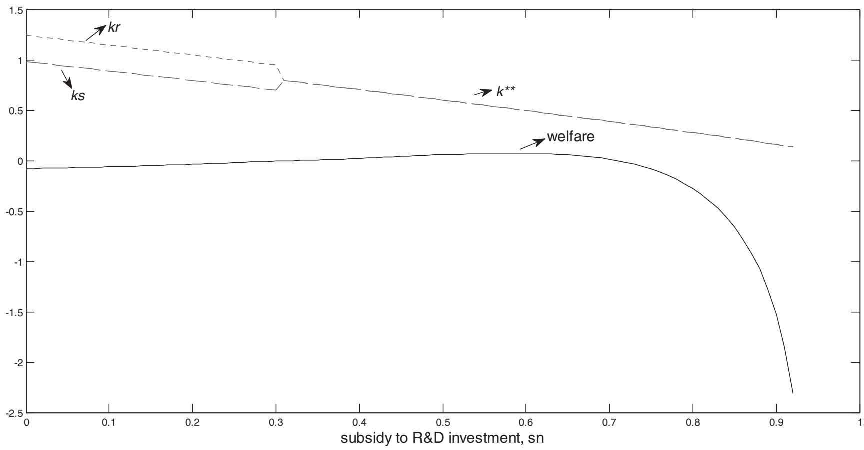

FIGURE 7. The asymptotic capital–variety ratio and welfare levels with changing s_n.

is equivalent to a significant 8.91% rise in consumption in every period from the benchmark case.

In Table 2, we set $s_x = 0$ and focus on the effects of the subsidy on the R&D investment, s_n. Period-2 cycles persist for a relatively wide range of s_n. After that, when it is high enough to satisfy the stability condition in Proposition 2, period-2 cycles are replaced by convergence to k^{**} in the policy-active region. Figure 5d illustrates this process of convergence when $s_n = 0.65$.

In Figure 7, we vary s_n from 0 to 90% ($s_x = 0$) and find the subsequent welfare levels. It is worth noting that the welfare curve is concave and smooth as well, peaking at a unique point when $s_n^* = 0.61$ that achieves convergence toward the

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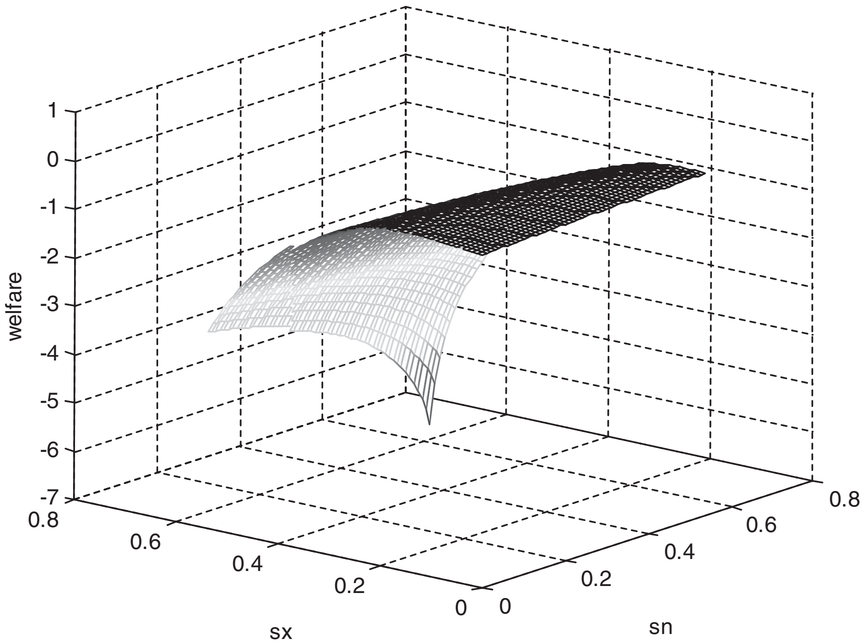

FIGURE 8. The welfare surface using both s_x and s_n.

steady state with balanced growth in capital and variety. The welfare curve is flatter before peaking and takes longer to reach the optimal level of the subsidy rate than in Figure 6, because s_n does not change the price gap and therefore does not create an additional gain or loss in static efficiency like those created by s_x on either side of $s_x = 1/\sigma$. So the efficiency gain from faster variety expansion and from convergence to the steady state at a higher subsidy rate to the R&D cost is gradually offset by a loss from the subsequent decline in the equilibrium quantity of each intermediate. Beyond this optimal subsidy rate, the welfare level declines rapidly, because the efficiency loss from the declined use of each intermediate is increasing at the margin. The maximized welfare level is 0.0668, as opposed to -0.0811 in the benchmark case. The welfare change is equivalent to a significant 9.07% rise in consumption in every period. The corresponding consumption tax rate of 8.1% also falls into the reasonable range.

In Figure 8, we jointly use the two subsidies at the same time and plot the welfare surface when varying both s_x and s_n from zero to sufficiently high rates. It turns out that the welfare surface is concave and smooth, peaking at a unique point when $s_x = 0.1$ and $s_n = 0.38$. The capital–variety ratio will converge to the steady state, $k^{**} = 0.4972$, in the policy-active regime when the subsidies are financed by a consumption tax at a rate 9.21%. The maximized welfare level is 0.0866, higher than those in cases with one subsidy at a time, as expected. The welfare change from the benchmark case is now equivalent to a substantial 10.35% rise in consumption in each period.

The respective optimal rates of subsidies mitigate and eventually eliminate the cycles in all the reported cases. So part of the welfare gain comes from smoothing consumption through subsidization.

5. CONCLUSION

We have examined the implications of two types of subsidies, one to the purchase of new intermediate products and the other to R&D investment, in the model of Matsuyama (1999, 2001) with growth through endogenous cycles. One contribution of doing so is that the subsidization can reduce the critical level of the capital–variety ratio substantially, enhancing the possibility for the economy to stay in the policy-active region with sustainable innovation and growth. Sufficient subsidization can rule out the neoclassical regime without innovation from the steady state in the long run.

Another contribution is that we have characterized several possible scenarios for the asymptotic paths of the representative agent economy from any initial state, depending on the values of the economic fundamentals and subsidy rates. In a novel yet empirically plausible scenario, sufficient subsidization stabilizes the balanced growth path with nonmonotonic or even monotonic convergence. Also, we have discussed the sources of market failures that give roles to subsidization in enhancing social welfare.

In the end, we have used numerical examples to gauge the welfare gains from the two types of subsidies, starting from an empirically plausible parameterization for a period-2 cycle economy in the absence of subsidization. It turns out that both types of subsidies can help enhance welfare significantly in terms of a 9–10% rise in consumption for every period; and the optimal subsidy rates maximizing social welfare are calculated in their plausible ranges, which eventually eliminate cycles for consumption smoothing.

Our results in this paper appear consistent not only with substantial subsidization to new investment and R&D spending but also with the combination of intensified innovation and dampened cyclical fluctuations in many industrial nations in the postwar era. This is consistent with the evidence in Comin and Gertler (2009).

Our results about R&D subsidies are different from those in Haruyama (2009), which uses a model with cycles but without accumulation: optimal R&D subsidies cannot eliminate cycles in Haruyama's model but can do so in our model. As mentioned earlier, the models are very different: intermediates production and innovation must use accumulated capital in the present model, but only use labor in Haruyama's model. So our results concerning the R&D subsidies are complementary to Haruyama's results. Comin and Mulani (2009) find a positive effect of the R&D subsidy on R&D innovations but a negative effect of the R&D subsidy on general innovations in a different model focusing on growth and volatility without cycles. Finally, our results about the subsidies to the purchase of new intermediates are new in models with endogenous cycles and sustainable growth.

NOTE

1. For a thorough derivation of the sufficient and necessary conditions of the chaotic intervals for the type of mapping in (24), see Devaney (1989) and Gardini et al. (2008).

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APPENDIX A: DERIVATION OF SOLUTION TO THE CONSUMER PROBLEM

Taking the initial capital stock $K_0 > 0$ and the sequences of prices and tax rates $(\tau_{c,t},\ r_t,\ w_t)_{t=1}^{\infty}$ as given, the agent chooses the sequence of allocations $(C_t,\ K_t)_{t=1}^{\infty}$ to maximize utility in (15) subject to (16) and (17). The optimal intertemporal condition is

$$\frac{1}{(1+\tau_{c,t})C_t} = \frac{\beta r_{t+1}}{(1+\tau_{c,t+1})C_{t+1}},$$

and the binding solvency (transversality) condition can be written as

$$\lim_{t\to\infty} \frac{K_t}{\prod_{s=1}^{t} r_s} = \lim_{t\to\infty} \beta^t \frac{K_t}{(1+\tau_{c,t})C_t} = 0.$$

Production factors are compensated competitively according to $w_t L = (1/\sigma)\,Y_t$ and $r_t K_{t-1} = (1 - 1/\sigma)\,Y_t$. Combining them with (16) and the intertemporal condition yields the optimal solution in (18) and (19). Q.E.D.

APPENDIX B: PROOF OF PROPOSITION 1

The solutions for the steady state k^*or k^{**} follow the respective scenarios in (24). What remains to show is that $k^{**} > k_c$ for $G > 1$. First, let us rewrite the expression for Δ in

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case (ii) as

$$\begin{aligned}\Delta &= [1 - s_n(1 - 1/\sigma) - \theta k_c - \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}]^2 \\ &+ 4\theta\beta\,(1 - 1/\sigma)\,Ak_c(1 - s_n)^{-1/\sigma}(1 - s_x)^{-1}[s_x(1 - s_n) + s_n/\sigma] \\ &= \theta^2k_c^2 + [1 - s_n(1 - 1/\sigma) - \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}]^2 \\ &- 2\theta k_c[1 - s_n(1 - 1/\sigma) - \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}] \\ &+ 4\theta\beta\,(1 - 1/\sigma)\,Ak_c(1 - s_n)^{-1/\sigma}(1 - s_x)^{-1}[s_x(1 - s_n) + s_n/\sigma].\end{aligned}$$

Here, $1 - s_n(1 - 1/\sigma) - (1 - s_x)(1 - s_n) = s_x(1 - s_n) + s_n/\sigma \geq 0$ and $G > 1$ implies that $\beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma} > (1 - s_x)(1 - s_n)$. We can now rewrite the expression of Δ as

$$\begin{aligned}\Delta &= \theta^2k_c^2 + [1 - s_n(1 - 1/\sigma) - \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}]^2 \\ &+ 2\theta k_c\{2\beta\,(1 - 1/\sigma)\,A(1 - s_n)^{-1/\sigma}(1 - s_x)^{-1}[s_x(1 - s_n) + s_n/\sigma] \\ &- 1 + s_n(1 - 1/\sigma) + \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}\} \\ &> \theta^2k_c^2 + [1 - s_n(1 - 1/\sigma) - \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}]^2 \\ &+ 2\theta k_c[1 - s_n(1 - 1/\sigma) - (1 - s_x)(1 - s_n)] \\ &\times [2\beta\,(1 - 1/\sigma)\,A(1 - s_n)^{-1/\sigma}(1 - s_x)^{-1} - 1] \\ &> \theta^2k_c^2 + [1 - s_n(1 - 1/\sigma) - \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}]^2 \\ &+ 2\theta k_c[1 - s_n(1 - 1/\sigma) - (1 - s_x)(1 - s_n)] \\ &> \theta^2k_c^2 + [1 - s_n(1 - 1/\sigma) - \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}]^2 \\ &+ 2\theta k_c[1 - s_n(1 - 1/\sigma) - \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}] \\ &= [\theta k_c + 1 - s_n(1 - 1/\sigma) - \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma}]^2.\end{aligned}$$

So $k^{**} = [\theta k_c - 1 + s_n(1 - 1/\sigma) + \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma} + \Delta^{1/2}]/(2\theta) > 2\theta k_c/(2\theta) = k_c$. The other root,

$$k^{**} = [\theta(1 - s_x)^\sigma(1 - s_n) + s_n(1 - 1/\sigma) - 1 + \beta\,(1 - 1/\sigma)\,A(1 - s_n)^{1-1/\sigma} - \Delta^{1/2}]/(2\theta),$$

is dropped for being inconsistent with $k^{**} > k_c$. Q.E.D.

APPENDIX C: PROOF OF PROPOSITION 2

In case (i), with $k^* = [\beta(1 - 1/\sigma)A]^\sigma < k_c = (1 - s_x)^\sigma(1 - s_n)$ and $k_{t-1} < k_c$, the slope of $k_t = \Phi(k_{t-1}) = \beta(1 - 1/\sigma)A(k_{t-1})^{1-1/\sigma}$ in (24) is always positive, exceeding 1 at the origin ($k_{t-1} \to 0$) and falling below 1 in the steady state ($k^* = [\beta(1 - 1/\sigma)A]^\sigma$) according to $\Phi'(k^*) = 1 - 1/\sigma$, because $\sigma > 1$. The steady state $k^*$is thus globally stable and the sequence $\{k_t\}_{t=0}^{\infty}$ converges toward k^* for any $k_0 > 0$ as in the standard neoclassical growth model. We illustrate case (i) in Figure 1.

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In cases (ii)–(iv) with $k^* = [\beta(1-1/\sigma)A]^\sigma > k_c = (1-s_x)^\sigma(1-s_n)$, the slope of the transition equation $k_t = \Phi(k_{t-1})$ in (24) for $k_{t-1} > k_c$is derived as

$$\Phi'(k_{t-1}) = \frac{\beta\,(1-1/\sigma)\,A(1-s_n)^{1-1/\sigma}[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}]}{\{1-s_n(1-1/\sigma)+\theta[k_{t-1}-(1-s_x)^\sigma(1-s_n)]\}^2}.$$

Here, $1-s_n(1-1/\sigma) > 0$ because $s_n \in [0, 1)$ and $\sigma > 1$. Also, $k_{t-1}-(1-s_x)^\sigma(1-s_n) > 0$ for $k_{t-1} > k_c$. So the sign of $\Phi'(k_{t-1})$ is the same as the sign of $1-\theta(1-s_x)^{\sigma-1}$. Recalling that $\theta = (1-1/\sigma)^{1-\sigma} > 1$ under $\sigma > 1$, we have $\Phi'(k_{t-1}) \geq 0$ if and only if $1 > s_x \geq 1/\sigma$, because with $s_x \in [0, 1)$, $1-\theta(1-s_x)^{\sigma-1} \geq 0$ corresponds to $1 > s_x \geq 1/\sigma$. Accordingly, $\Phi'(k_{t-1}) < 0$ if and only if $0 \leq s_x < 1/\sigma$, under which $1-\theta(1-s_x)^{\sigma-1} < 0$. Also, the absolute value of $\Phi'(k_{t-1})$ is monotonically decreasing in $k_{t-1}$and approaches zero when k_{t-1}approaches infinity, implying that the dynamic system in (24) cannot diverge toward infinity.

For $k_{t-1} > k_c$, there are thus two possibilities, with either $\Phi'(k_{t-1}) < 0$ or $\Phi'(k_{t-1}) \geq 0$. If $0 \leq s_x < 1/\sigma$ and thus $\Phi'(k_{t-1}) < 0$ beyond k_c, then the economy may either fluctuate forever with cycles or eventually converge toward the steady state of the balanced growth path so that $\Phi^t\,(k) \to k^{**}$ as $t \to \infty$, depending on whether $|\Phi'\,(k^{**})| \geq 1$or < 1. The proof of cases (ii) and (iii) is the same as that in Matsuyama (1999). For the purposes of our paper, we focus on how subsidies change the slopes of the transition equation in (24) and switch the economy across cases.

Specifically, if the subsidy rates are low enough (say zero), then $|\Phi'\,(k^{**})| > 1$ prevails under $\theta > 2$ and the economy behaves in the same way as in the original model of Matsuyama (1999) with endogenous cycles forever once it enters the trapping region denoted by the interval $[\Phi^2(k_c), \Phi(k_c)]$ in Figure 2. If the subsidy rates are high enough, then we show that $|\Phi'\,(k^{**})| < 1$ for $\Phi'(k_{t-1}) \leq 0$, as follows. First, the slope of $k_t = \Phi(k_{t-1})$ at $k_{t-1} = k^{**}$ can be rewritten as

$$\begin{aligned}\Phi'\,(k^{**}) &= \frac{\beta\,(1-1/\sigma)\,A(1-s_n)^{1-1/\sigma}[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}]}{\{1-s_n(1-1/\sigma)+\theta[k^{**}-(1-s_x)^\sigma(1-s_n)]\}^2}\\ &= \frac{4\beta\,(1-1/\sigma)\,A(1-s_n)^{1-1/\sigma}[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}]}{\left\{1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta\,(1-1/\sigma)\,A(1-s_n)^{1-1/\sigma}+\Delta^{1/2}\right\}^2},\end{aligned}$$

using the expression for k^{**} given in Proposition 1 for substitution.

For the special case without subsidies, we have $G_0 = \beta\,(1-1/\sigma)\,A$ and

$$\Phi'\,(k^{**}) = \frac{1-\theta}{G} < 0, \quad \text{at}\, s_x = s_n = 0.$$

The absolute value of this slope exceeds one (unstable k^{**}) if and only if $1 < G < \theta-1$as in the original model of Matsuyama (1999). This condition applies under $\theta > 2$.

For $0 \leq s_x \leq 1/\sigma$, showing that $|\Phi'\,(k^{**})| < 1$ is equivalent to showing that

$$\begin{aligned}&F(s_n, s_x)\\ &\equiv \left[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta\,(1-1/\sigma)\,A(1-s_n)^{1-1/\sigma}+\Delta^{1/2}\right]^2\\ &\quad+4\beta\,(1-1/\sigma)\,A(1-s_n)^{1-1/\sigma}[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}] > 0,\end{aligned}$$

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where $1-\theta(1-s_x)^{\sigma-1} \leq 0$. Substituting the expression for Δin Proposition 1 into$F(s_n, s_x)$leads to

$$F(s_n,s_x) = [1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}]^2$$
$$+\Delta+2\Delta^{1/2}[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}]$$
$$+4\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}]$$
$$=2[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)]^2+2[\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}]^2$$
$$+4\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}\{\theta(1-s_x)^{\sigma-1}[s_x(1-s_n)+s_n/\sigma]$$
$$+[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}]\}$$
$$+2\Delta^{1/2}[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)$$
$$+\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}].$$

Here, $\{\theta(1-s_x)^{\sigma-1}[s_x(1-s_n)+s_n/\sigma]+[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}]\}$ can be shown to be equal to $[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)]$. Thus, we have

$$F(s_n,s_x) = 2[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)]^2$$
$$+2[\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}]^2$$
$$+4\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)]$$
$$+2\Delta^{1/2}[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}]$$
$$=2[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}]^2$$
$$+2\Delta^{1/2}[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}].$$

A sufficient yet unnecessary condition for $F(s_n, s_x) > 0$ is

$$[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}]$$
$$> [1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+(1-s_x)(1-s_n)]\,(\text{under}\, G>1) > 0.$$

This condition is satisfied by the stated conditions on the subsidy rates:

$$s_n > \sigma(\theta-2)/[1+\sigma(\theta-2)] \in (0,1)\ \text{under}\ \theta > 2\ \text{at}\ s_x = 0; \quad \text{or} \quad s_x \to 1/\sigma\ \text{at}\ s_n = 0.$$

That is, if the subsidy rates are sufficiently high so that $|\Phi'(k^{**})| < 1$, then the economy will eventually converge toward the stable steady state, or the stable balanced growth path. We depict cases (ii) and (iii) in Figures 2 and 3, respectively.

Recall that $1-\theta(1-s_x)^{\sigma-1}$ and $\Phi'(k_{t-1})$ must have the same sign. For $s_x \geq 1/\sigma$ in (iv), we have $1-\theta(1-s_x)^{\sigma-1} \geq 0$ and thus $\Phi'(k_{t-1}) \geq 0$. This is a new case compared to those in Matsuyama (1999, 2001). Now, showing that $\Phi'(k^{**}) < 1$ for a steady state k^{**} is equivalent to showing the following:

$$\left[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}+\Delta^{1/2}\right]^2$$
$$-4\beta(1-1/\sigma)A(1-s_n)^{1-1/\sigma}[1-s_n(1-1/\sigma)][1-\theta(1-s_x)^{\sigma-1}] \geq 0$$

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under $1-\theta(1-s_x)^{\sigma-1} \geq 0$. The left-hand side of this inequality can be decomposed into

$$\begin{aligned}
&\left[1-s_n(1-1/\sigma)-\beta\,(1-1/\sigma)\,A(1-s_n)^{1-1/\sigma}\right]^2+[\theta(1-s_x)^\sigma(1-s_n)]^2+\Delta \\
&\quad+2\Delta^{1/2}\left[1-s_n(1-1/\sigma)-\theta(1-s_x)^\sigma(1-s_n)+\beta\,(1-1/\sigma)\,A(1-s_n)^{1-1/\sigma}\right] \\
&\quad+2\theta\,[1-s_n(1-1/\sigma)]\,(1-s_n)^{1-1/\sigma}(1-s_x)^{\sigma-1} \\
&\quad\times\left[\beta\,(1-1/\sigma)\,A-(1-s_x)(1-s_n)^{1/\sigma}\right] \\
&\quad+2\theta\beta\,(1-1/\sigma)\,A(1-s_x)^{\sigma-1}(1-s_n)^{1-1/\sigma}\,[s_n/\sigma+s_x(1-s_n)]\,,
\end{aligned}$$

which is nonnegative under $G>1$, $0\leq s_n<1$, and $1/\sigma\leq s_x<1$. As in case (i), it suffices to establish stability and convergence for case (iv) graphically in Figure 4. Q.E.D.

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