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Faltings extension and Hodge-Tate filtration for abelian varieties over p-adic local fields with imperfect residue fields

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Abstract. Let K be a complete discrete valuation field of characteristic 0, with not necessarily perfect residue field of characteristic $p > 0$. We define a Faltings extension of $\mathcal{O}_K$ over $\mathbb{Z}_p$, and we construct a Hodge-Tate filtration for abelian varieties over K by generalizing Fontaine's construction [Fon82] where he treated the perfect residue field case.

1 Introduction

1.1 Let K be a complete discrete valuation field of characteristic 0, with residue field k of characteristic $p > 0$. Let $\overline{K}$ be an algebraic closure of K, let G_K be the Galois group of $\overline{K}$ over K, let C be the p-adic completion of $\overline{K}$. We denote by $C(r)$ the r-th Tate twist. For an abelian variety X over K, we denote its Tate module by $T_p(X)$. When k is perfect and X has good reduction, Tate [Tat67] constructed a canonical G_K-equivariant exact sequence

(1.1.1)
$$0 \longrightarrow H^1(X, \mathcal{O}_X) \otimes_K C(1) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(X), C(1)\right) \longrightarrow H^0(X, \Omega^1_{X/K}) \otimes_K C \longrightarrow 0.$$

In the same paper, Tate also computed the Galois cohomology groups of $C(r)$. He proved in particular that $H^1(G_K, C(r)) = 0$ for any $r \neq 0$, which implies that the sequence (1.1.1) has a G_K-equivariant splitting, and that $H^0(G_K, C(r)) = 0$ for any $r \neq 0$, which implies that the splitting is unique. Tate conjectured that for any proper smooth scheme X over K, there is a canonical G_K-equivariant decomposition (called the *Hodge–Tate decomposition*)

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C = \bigoplus_{i=0}^{n} H^i(X, \Omega^{n-i}_{X/K}) \otimes_K C(i-n).$$

Then subsequently, Raynaud used the semistable reduction theorem to show that any abelian variety over K admits a Hodge–Tate decomposition ([sga72, IX 3.6, 5.6]). Afterwards, Fontaine [Fon82] gave a new proof for general abelian varieties. He constructed a canonical map $H^0(X, \Omega^1_{X/K}) \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(T_p(X), C(1))$, by computing

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$\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ and pulling back differentials. The conjecture of Tate was finally settled by Faltings [Fal88, Fal02] and Tsuji [Tsu99, Tsu02] independently.

When k is not necessarily perfect, Hyodo proved that there is still an exact sequence (1.1.1) for abelian varieties with good reduction, following the same argument as in [Tat67] ([Hyo86, Remark 1]). But the sequence does not split in general ([Hyo86, Theorem 3]). In this paper, we will construct the exact sequence (1.1.1) for general abelian varieties by generalizing Fontaine's method to the imperfect residue field case.

We remark that Scholze [Sch13] has generalized the conjecture of Tate to any proper smooth rigid-analytic variety X over C. He proved that there is a canonical filtration (called the *Hodge–Tate filtration*) $\mathrm{Fil}^\bullet$ on $H^n_{\text{ét}}(X,\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} C$, such that

$$\mathrm{Fil}^i(H^n_{\text{ét}}(X,\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} C)/\,\mathrm{Fil}^{i+1}(H^n_{\text{ét}}(X,\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} C) = H^i(X,\Omega^{n-i}_{X/C})\otimes_C C(i-n).$$

1.2 For any abelian group M, we set

$$T_p(M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p]/\mathbb{Z}, M) \quad \text{and} \quad V_p(M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], M).$$

In Section 4, we construct a Faltings extension of $\mathcal{O}_K$ over $\mathbb{Z}_p$. It is a canonical exact sequence of C- G_K-modules that splits as a sequence of C-modules (*cf.* Theorem 4.4),

$$0 \longrightarrow C(1) \xrightarrow{\iota} V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \xrightarrow{\nu} C\otimes_{\mathcal{O}_C}(\mathcal{O}_{\overline{K}}\otimes_{\mathcal{O}_K}\Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^\wedge \longrightarrow 0, \tag{1.2.1}$$

where $(-)^\wedge$ denotes the p-adic completion. Based on Hyodo's computation of Galois cohomology (*cf.* Theorem 3.8), we will show that the connecting map of the above sequence

$$\delta : \big(C\otimes_{\mathcal{O}_C}(\mathcal{O}_{\overline{K}}\otimes_{\mathcal{O}_K}\Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^\wedge\big)^{G_K} \longrightarrow H^1(G_K, C(1)) \tag{1.2.2}$$

is an isomorphism (*cf.* Corollary 4.5).

Following Fontaine, we deduce from the above sequence and its cohomological properties a canonical K-linear injective homomorphism (*cf.* Theorem 5.6)

$$\rho : H^0(X,\Omega^1_{X/K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(T_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})). \tag{1.2.3}$$

The arguments are essentially the same as in [Fon82].

Our main result can be stated as follows (*cf.* Theorem 7.4 and Paragraphs 7.5, 7.6).

Theorem 1.3 *For any abelian variety X over K, there is a canonical exact sequence of C- G_K-modules*

$$\begin{aligned} 0 \longrightarrow H^1(X,\mathcal{O}_X)\otimes_K C(1) &\xrightarrow{\psi} \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) \\ &\xrightarrow{\phi} H^0(X,\Omega^1_{X/K})\otimes_K C \longrightarrow 0 \end{aligned} \tag{1.3.1}$$

satisfying the following properties:

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(i) *Any C-linear retraction of ι in (1.2.1) induces a C-linear section of ϕ. More precisely, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) & \xrightarrow{\ \phi\ } & H^0(X, \Omega^1_{X/K}) \otimes_K C \\ & \nwarrow^{\pi} & \downarrow{\scriptstyle\rho} \\ & & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})), \end{array}$$

where ρ is induced by the map (1.2.3), and π is induced by any retraction of ι in (1.2.1).

(ii) *The connecting map δ' associated with (1.3.1) fits into a commutative diagram*

$$\begin{array}{ccc} H^0(X, \Omega^1_{X/K}) & \xrightarrow{\ \delta'\ } & H^1(G_K, H^1(X, \mathcal{O}_X) \otimes_K C(1)) \\ \downarrow{\scriptstyle\rho} & & \downarrow \\ \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})) & \xrightarrow{\ \pi'\ } & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C \otimes_{\mathcal{O}_C} (\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^\wedge), \end{array}$$

where ρ is the map (1.2.3), π' is induced by $-\nu$ of (1.2.1), and the unlabeled arrow is induced by δ^{-1} (1.2.2) and ψ of (1.3.1).

Corollary 1.4 *For any abelian variety X over K, sequence (1.3.1) splits if and only if the image of ρ (1.2.3) lies in $\mathrm{Hom}_{\mathbb{Z}_p[G_K]}(T_p(X), C(1))$. In fact, when it splits, the splitting is unique.*

Remark 1.5 Caraiani and Scholze [CS17] constructed a relative version of Hodge–Tate filtration for proper smooth morphisms of adic spaces. And recently, Abbes and Gros [AG20] constructed a relative version of Hodge–Tate spectral sequence for projective smooth morphisms of logarithmic schemes. Unlike these works that rely on advanced theories and results, our proof for abelian varieties uses only basic algebraic geometry and p-adic Galois cohomology computation of Tate and Hyodo. For instance, we do not use Faltings' almost purity theorem.

2 Notation

2.1 Let K be a complete discrete valuation field of characteristic 0, with residue field k of characteristic $p > 0$. Let $\overline{K}$ be an algebraic closure of K, let G_K be the Galois group of $\overline{K}$ over K. Let C be the p-adic completion of $\overline{K}$, v_p the valuation on C such that $v_p(p) = 1$, $|\ |_p$ the absolute value on C such that $|p|_p = 1/p$. We fix a complete discrete valuation subfield K_0 of K such that $\mathcal{O}_{K_0}/p\mathcal{O}_{K_0} = k$ (by Cohen structure theorem, *cf.* [Gro64, 0_{IV} 19.8.6]). We remark that K/K_0 is a totally ramified finite extension. We fix elements $(u_i)_{i\in I}$ of $\mathcal{O}_{K_0}$ such that the reductions $(\overline{u_i})_{i\in I}$ form a p-base of k. For each $i \in I$, we fix elements $(w_{im})_{m\geq 0}$ of $\mathcal{O}_{\overline{K}}$ such that $w_{i,m+1}^p = w_{i,m}$ and $w_{i,0} = u_i$. We denote by $(e_i)_{i\in I}$ the standard basis of $\oplus_{i\in I}\mathbb{Z}$.

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2.2 For any discrete valuation field L of characteristic 0, with residue field of characteristic p, we denote by

$$\hat{\Omega}^1_{\mathcal{O}_L} = (\Omega^1_{\mathcal{O}_L/\mathbb{Z}_p})^\wedge$$

the p-adic completion of the module of differentials of $\mathcal{O}_L$ over $\mathbb{Z}_p$.

For any algebraic extension L' over L, we set

$$\hat{\Omega}^1_{\mathcal{O}_L}(\mathcal{O}_{L'}) = \operatorname*{colim}_{L_1/L} \hat{\Omega}^1_{\mathcal{O}_{L_1}},$$

where L_1 runs through all finite subextensions of L'/L. We remark that $\hat{\Omega}^1_{\mathcal{O}_L}(\mathcal{O}_{L'}) = \hat{\Omega}^1_{\mathcal{O}_{L_1}}(\mathcal{O}_{L'})$ for any finite subextension L_1 of L'/L, and that $\hat{\Omega}^1_{\mathcal{O}_L}(\mathcal{O}_L) = \hat{\Omega}^1_{\mathcal{O}_L}$.

2.3 For any abelian group M, we define

$$T_p(M) = \varprojlim_{x\mapsto px} M[p^n] = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p]/\mathbb{Z}, M),$$
$$V_p(M) = \varprojlim_{x\mapsto px} M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], M).$$

Being an inverse limit of $\mathbb{Z}$-modules each killed by some power of p, $T_p(M)$ is a p-adically complete $\mathbb{Z}_p$-module ([Jan88, 4.4]). If M is p-primary torsion, then $V_p(M) = T_p(M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and thus it has a natural $\mathbb{Q}_p$-module structure. If M is a $\mathbb{Z}_p$-module, then $T_p(M) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, M)$, $V_p(M) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, M)$. We set $\mathbb{Z}_p(1) = T_p(\mathcal{O}_{\overline{K}}^\times)$, a free $\mathbb{Z}_p$-module of rank 1 with continuous G_K-action. For any $\mathbb{Z}_p$-module M and $r \in \mathbb{Z}$, we set $M(r) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes r}$, the r-th Tate twist of M. Let X be an abelian variety over K. We set $T_p(X) = T_p(X(\overline{K}))$ and $V_p(X) = V_p(X(\overline{K}))$.

3 Review of Hyodo's Computation of Galois Cohomology Groups of $C(r)$

Lemma 3.1 *Let B/A be a finite extension of discrete valuation rings, whose fraction field extension and residue field extension are both separable. We assume that A is henselian, or that B/A is totally ramified. Let R be a subring of A. Then the canonical map $B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/R}$ is injective.*

Proof After replacing A by its maximal unramified extension in B, we can assume that B is totally ramified over A. Hence, B is of the form $A[X]/(f(X))$ for some irreducible polynomial $f \in A[X]$. Let x be the image of X in B. Then we have

$$\Omega^1_{B/R} = (B \otimes_A \Omega^1_{A/R} \oplus B\mathrm{d}X)/B(\mathrm{d}_A f(x) + f'(x)\mathrm{d}X),$$

where $\mathrm{d}_A f \in A[X] \otimes_A \Omega^1_{A/R}$ is obtained by differentiating the coefficients of f. Since $f'(x) \neq 0$, the canonical map $B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/R}$ is injective. ■

Lemma 3.2 ([Hyo86, 4-4]) *There is an isomorphism of $\mathcal{O}_{K_0}$-modules*

$$(\oplus_{i\in I}\mathcal{O}_{K_0})^\wedge \xrightarrow{\sim} \hat{\Omega}^1_{\mathcal{O}_{K_0}}, \quad e_i \longmapsto \mathrm{d}\log u_i, \quad \forall i \in I.$$

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Proof As $(\overline{u_i})_{i\in I}$ form a p-base of the residue field of $\mathcal{O}_{K_0}$, we have $\Omega^1_{\mathcal{O}_{K_0}/\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\mathbb{Z}_p/p\mathbb{Z}_p = \Omega^1_{k/\mathbb{F}_p} = \oplus_{i\in I}k$, where e_i corresponds to $\mathrm{d}\log\overline{u_i}$. Since $\mathcal{O}_{K_0}$ is flat over $\mathbb{Z}_p$ and k is formally smooth over $\mathbb{F}_p$, $\mathcal{O}_{K_0}/p^n\mathcal{O}_{K_0}$ is formally smooth over $\mathbb{Z}_p/p^n\mathbb{Z}_p$ for each $n\geq 1$ ([Gro64, 0_{IV} 19.7.1], [Sta20, 031L]). In particular, $\Omega^1_{\mathcal{O}_{K_0}/\mathbb{Z}_p}\otimes\mathbb{Z}_p/p^n\mathbb{Z}_p$ is a projective $\mathcal{O}_{K_0}/p^n\mathcal{O}_{K_0}$-module. Hence, we have an exact sequence

$$0\longrightarrow \Omega^1_{\mathcal{O}_{K_0}/\mathbb{Z}_p}\otimes\mathbb{Z}_p/p\mathbb{Z}_p \xrightarrow{\cdot p^{n-1}} \Omega^1_{\mathcal{O}_{K_0}/\mathbb{Z}_p}\otimes\mathbb{Z}_p/p^n\mathbb{Z}_p \longrightarrow \Omega^1_{\mathcal{O}_{K_0}/\mathbb{Z}_p}\otimes\mathbb{Z}_p/p^{n-1}\mathbb{Z}_p\longrightarrow 0,$$

from which we get isomorphisms $\oplus_{i\in I}\mathcal{O}_{K_0}/p^n\mathcal{O}_{K_0}\xrightarrow{\sim}\Omega^1_{\mathcal{O}_{K_0}/\mathbb{Z}_p}\otimes\mathbb{Z}_p/p^n\mathbb{Z}_p$ by induction. The conclusion follows by taking limit over n. ∎

Proposition 3.3 ([Hyo86, 4-2-1]) *There is an exact sequence of $\mathcal{O}_K$-modules*

$$0\longrightarrow(\oplus_{i\in I}\mathcal{O}_K)^\wedge\xrightarrow{\theta}\hat{\Omega}^1_{\mathcal{O}_K}\longrightarrow\Omega^1_{\mathcal{O}_K/\mathcal{O}_{K_0}}\longrightarrow 0,$$

where $\theta(e_i)=\mathrm{d}\log u_i$ for any $i\in I$.

Proof The sequence of modules of differentials of $\mathcal{O}_K/\mathcal{O}_{K_0}/\mathbb{Z}_p$,

$$0\longrightarrow\mathcal{O}_K\otimes_{\mathcal{O}_{K_0}}\Omega^1_{\mathcal{O}_{K_0}/\mathbb{Z}_p}\longrightarrow\Omega^1_{\mathcal{O}_K/\mathbb{Z}_p}\longrightarrow\Omega^1_{\mathcal{O}_K/\mathcal{O}_{K_0}}\longrightarrow 0,$$

is exact by Lemma 3.1. Passing to p-adic completions, as $\Omega^1_{\mathcal{O}_K/\mathcal{O}_{K_0}}$ is killed by a power of p, we still get an exact sequence [Sta20, 0BNG]. The conclusion follows from Lemma 3.2, and the isomorphism $\mathcal{O}_K\otimes_{\mathcal{O}_{K_0}}(\oplus_{i\in I}\mathcal{O}_{K_0})^\wedge\xrightarrow{\sim}(\oplus_{i\in I}\mathcal{O}_K)^\wedge$ as $\mathcal{O}_K$ is finite free over $\mathcal{O}_{K_0}$. ∎

Lemma 3.4 ([Hyo86, 4-4]) *Let $M_0=\bigcup_{i\in I,m\geq 0}K_0(w_{im})\subseteq\overline{K}$. Then there is an isomorphism of $\mathcal{O}_{M_0}$-modules*

$$M_0\otimes_{\mathcal{O}_{K_0}}(\oplus_{i\in I}\mathcal{O}_{K_0})^\wedge\xrightarrow{\sim}\hat{\Omega}^1_{\mathcal{O}_{K_0}}(\mathcal{O}_{M_0}),\quad p^{-m}\otimes e_i\longmapsto\mathrm{d}\log w_{im},\ \forall i\in I, m\in\mathbb{N}.$$

Proof For an integer $N>0$ and a finite subset $J\subseteq I$, let $L_0=\bigcup_{i\in J}K_0(w_{iN})$. Then by Lemma 3.2, $(\oplus_{i\in I}\mathcal{O}_{L_0})^\wedge$ is isomorphic to $\hat{\Omega}^1_{\mathcal{O}_{L_0}}$ by sending e_i to $\mathrm{d}\log w_{iN}$ if $i\in J$, and to $\mathrm{d}\log u_i$ if $i\notin J$. The conclusion follows by taking colimit over J and N. ∎

Lemma 3.5 ([Hyo86, 4-7]) *With the same notation as in Lemma 3.4, let M be a finite extension of M_0. Then there is a canonical exact sequence of $\mathcal{O}_M$-modules*

$$0\longrightarrow\mathcal{O}_M\otimes_{\mathcal{O}_{M_0}}\hat{\Omega}^1_{\mathcal{O}_{K_0}}(\mathcal{O}_{M_0})\longrightarrow\hat{\Omega}^1_{\mathcal{O}_{K_0}}(\mathcal{O}_M)\longrightarrow\Omega^1_{\mathcal{O}_M/\mathcal{O}_{M_0}}\longrightarrow 0.$$

Proof We notice that $\mathcal{O}_{M_0}$ is a henselian discrete valuation ring with perfect residue field. Let M_{ur} be the maximal unramified subextension of M/M_0, and let $f\in\mathcal{O}_{M_{\mathrm{ur}}}[X]$ be the monic minimal polynomial of a uniformizer ϖ of $\mathcal{O}_M$. Then we have $\mathcal{O}_M=\mathcal{O}_{M_{\mathrm{ur}}}[X]/(f(X))$. For a sufficiently large finite subextension L_1 of M_{ur}/K_0 such that $f\in\mathcal{O}_{L_1}[X]$, $L_2=L_1(\varpi)$ is totally ramified over L_1. The same argument as in

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Proposition 3.3 gives us a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_{L_2} \otimes_{\mathcal{O}_{L_1}} \hat{\Omega}^1_{\mathcal{O}_{L_1}} \longrightarrow \hat{\Omega}^1_{\mathcal{O}_{L_2}} \longrightarrow \Omega^1_{\mathcal{O}_{L_2}/\mathcal{O}_{L_1}} \longrightarrow 0.$$

By taking colimit over L_1, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_M \otimes_{\mathcal{O}_{M_{\mathrm{ur}}}} \hat{\Omega}^1_{\mathcal{O}_{K_0}}(\mathcal{O}_{M_{\mathrm{ur}}}) \longrightarrow \hat{\Omega}^1_{\mathcal{O}_{K_0}}(\mathcal{O}_M) \longrightarrow \Omega^1_{\mathcal{O}_M/\mathcal{O}_{M_{\mathrm{ur}}}} \longrightarrow 0. \tag{3.5.1}$$

A similar colimit argument shows that $\hat{\Omega}^1_{\mathcal{O}_{K_0}}(\mathcal{O}_{M_{\mathrm{ur}}}) = \mathcal{O}_{M_{\mathrm{ur}}} \otimes_{\mathcal{O}_{M_0}} \hat{\Omega}^1_{\mathcal{O}_{K_0}}(\mathcal{O}_{M_0})$. The conclusion follows from (3.5.1). ∎

Proposition 3.6 ([Hyo86, 4-2-2]) *There is an exact sequence of $\mathcal{O}_{\overline{K}}$- G_K-modules that splits as a sequence of $\mathcal{O}_{\overline{K}}$-modules,*

$$0 \longrightarrow \overline{K}/\mathfrak{a}(1) \xrightarrow{\vartheta} \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}}) \longrightarrow \overline{K} \otimes_{\mathcal{O}_K} (\oplus_{i\in I}\mathcal{O}_K)^\wedge \longrightarrow 0, \tag{3.6.1}$$

where $\mathfrak{a} = \{x \in \overline{K} \mid v_p(x) \geq -1/(p-1)\}$, and $\vartheta(p^{-k} \otimes (\zeta_n)_n) = \mathrm{d}\log \zeta_k$ for any $k \in \mathbb{N}$ and any $(\zeta_n)_n \in \mathbb{Z}_p(1)$. The map $\overline{K} \otimes_{\mathcal{O}_K} (\oplus_{i\in I}\mathcal{O}_K)^\wedge \to \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})$, sending $p^{-m} \otimes e_i$ to $\mathrm{d}\log w_{im}$ for any $i \in I$ and $m \in \mathbb{N}$, gives a splitting of the sequence.

Proof With the same notation as in Lemma 3.4, let M run through all finite subextensions of $\overline{K}/M_0$. We get from Lemma 3.5 an exact sequence of $\mathcal{O}_{\overline{K}}$-modules

$$0 \longrightarrow \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_{M_0}} \hat{\Omega}^1_{\mathcal{O}_{K_0}}(\mathcal{O}_{M_0}) \longrightarrow \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}}) \longrightarrow \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_{M_0}} \longrightarrow 0. \tag{3.6.2}$$

We identify its first term with $\overline{K} \otimes_{\mathcal{O}_K} (\oplus_{i\in I}\mathcal{O}_K)^\wedge$ by Lemma 3.4. Let $\overline{\mathbb{Q}_p}$ be the algebraic closure of $\mathbb{Q}_p$ in $\overline{K}$, $\overline{\mathbb{Z}_p}$ the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$. By Fontaine's computation [Fon82, Théorème 1′], we have an isomorphism of $\overline{\mathbb{Z}_p}$-modules

$$\overline{\mathbb{Q}_p}/\mathfrak{a}_0(1) \xrightarrow{\sim} \Omega^1_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p}, \quad p^{-k} \otimes (\zeta_n)_n \longmapsto \mathrm{d}\log \zeta_k, \quad \forall k \in \mathbb{N}, \quad \forall (\zeta_n)_n \in \mathbb{Z}_p(1),$$

where $\mathfrak{a}_0 = \{x \in \overline{\mathbb{Q}_p} \mid v_p(x) \geq -1/(p-1)\}$, and we have an isomorphism of $\mathcal{O}_{\overline{K}}$-modules

$$\overline{K}/\mathfrak{a}(1) \xrightarrow{\sim} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_{M_0}}, \quad p^{-k} \otimes (\zeta_n)_n \longmapsto \mathrm{d}\log \zeta_k, \quad \forall k \in \mathbb{N}, \quad \forall (\zeta_n)_n \in \mathbb{Z}_p(1),$$

where $\mathfrak{a} = \{x \in \overline{K} \mid v_p(x) \geq -1/(p-1)\}$. Hence, the composition of

$$\overline{K}/\mathfrak{a}(1) \xrightarrow{\sim} \mathcal{O}_{\overline{K}} \otimes_{\overline{\mathbb{Z}_p}} \Omega^1_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p} \longrightarrow \Omega^1_{\mathcal{O}_{\overline{K}}/\mathbb{Z}_p} \longrightarrow \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})$$

gives a splitting of (3.6.2). Thus, we obtain the splitting sequence (3.6.1) of $\mathcal{O}_{\overline{K}}$-modules. We notice that the Galois conjugates of ζ_n, w_{im} are of the form $\zeta_n^a, \zeta_m^b w_{im}$, respectively, which implies that (3.6.1) is G_K-equivariant. ∎

3.7 As $\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})$ is p-divisible, we have an exact sequence $0 \to T_p(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})) \to V_p(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})) \to \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}}) \to 0$. After inverting p, we get an exact sequence

$$\begin{aligned} 0 \longrightarrow C(1) \longrightarrow \overline{K} \otimes_{\mathcal{O}_{\overline{K}}} V_p(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})) \\ \longrightarrow \overline{K} \otimes_{\mathcal{O}_{\overline{K}}} \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}}) \longrightarrow 0, \end{aligned} \tag{3.7.1}$$

where we identified $\overline{K} \otimes_{\mathcal{O}_{\overline{K}}} T_p(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}}))$ with $C(1)$ by (3.6.1).

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Theorem 3.8 ([Hyo86, Theorem 1 and Remark 3])

(i) *The composition of*

$$K \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \xrightarrow{\varepsilon} (\overline{K} \otimes_{\mathcal{O}_{\overline{K}}} \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}}))^{G_K} \xrightarrow{\delta} H^1(G_K, C(1)), \tag{3.8.1}$$

where ε is the canonical map and δ is the connecting map associated with (3.7.1), is an isomorphism. Moreover, for any integer q, the cup product induces an isomorphism

$$(\wedge^q H^1(G_K, C(1)))^\wedge \xrightarrow{\sim} H^q(G_K, C(q)).$$

(ii) *The K-module $H^1(G_K, C)$ is free of rank 1. Moreover, for any integer q, the cup product induces an isomorphism*

$$H^1(G_K, C) \otimes_K (\wedge^{q-1} H^1(G_K, C(1)))^\wedge \xrightarrow{\sim} H^q(G_K, C(q-1)).$$

(iii) *For any integers r and q such that $r \neq q$ or $q-1$, we have $H^q(G_K, C(r)) = 0$.*

Remark 3.9 By Proposition 3.3, we have an isomorphism

$$K \otimes_{\mathcal{O}_K} (\oplus_{i\in I} \mathcal{O}_K)^\wedge \xrightarrow{\sim} K \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K}, \quad 1 \otimes e_i \longmapsto 1 \otimes \mathrm{d}\log u_i, \quad \forall i \in I. \tag{3.9.1}$$

By composing it with (3.8.1), we get an isomorphism

$$K \otimes_{\mathcal{O}_K} (\oplus_{i\in I} \mathcal{O}_K)^\wedge \xrightarrow{\sim} H^1(G_K, C(1)), \quad 1 \otimes e_i \longmapsto [f_i],$$

where f_i is a 1-cocycle sending each $\sigma \in G_K$ to $\sigma(1 \otimes (\mathrm{d}\log w_{im})_m) - 1 \otimes (\mathrm{d}\log w_{im})_m$ in view of (3.7.1).

4 Faltings Extension

Lemma 4.1 *Let $M = \bigcup_{i\in I, m\geq 0} K(w_{im}) \subseteq \overline{K}$. Then there is an isomorphism of $\mathcal{O}_M$-modules*

$$\oplus_{i\in I} M/\mathcal{O}_M \xrightarrow{\sim} \Omega^1_{\mathcal{O}_M/\mathcal{O}_K}, \quad p^{-m} e_i \longmapsto \mathrm{d}\log w_{im}, \quad \forall i \in I, m \in \mathbb{N}.$$

Proof For any $N \geq 0$, we set $M_N = \bigcup_{i\in I} K(w_{iN})$. Since $(\overline{u_i})$ form a p-base of the residue field k, the elements of the form $\prod_{i\in I} \overline{w_{iN}}^{k_i}$ where $0 \leq k_i < p^N$ with finitely many nonvanishing, are linearly independent over k. Therefore, $\mathcal{O}_{M_N} = \mathcal{O}_K[T_i]_{i\in I}/(T_i^{p^N} - u_i)$, where T_i maps to w_{iN}. Hence,

$$\Omega^1_{\mathcal{O}_{M_N}/\mathcal{O}_K} = \oplus_{i\in I} \mathcal{O}_{M_N}/p^N \mathcal{O}_{M_N} = \oplus_{i\in I} p^{-N} \mathcal{O}_{M_N}/\mathcal{O}_{M_N},$$

where $p^{-N} e_i$ corresponds to $\mathrm{d}\log w_{iN}$. The conclusion follows by taking colimit over N. ■

Proposition 4.2 *With the same notation as in Lemma 4.1, there is an exact sequence of $\mathcal{O}_{\overline{K}}$-modules*

$$0 \longrightarrow \oplus_{i\in I} \overline{K}/\mathcal{O}_{\overline{K}} \xrightarrow{\theta} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \longrightarrow \overline{K}/\mathfrak{b}(1) \longrightarrow 0,$$

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where $\theta(p^{-m}e_i) = \mathrm{d}\log w_{im}$ *for any* $i \in I$ *and* $m \in \mathbb{N}$, *and* $\mathfrak{b} = \{x \in \overline{K} \mid v_p(x) \geq -v_p(\mathfrak{D}_{M/M_1}) - 1/(p-1)\}$, *where* M_1 *is the fraction field of the Witt ring with coefficients in the residue field of* M, *and* $\mathfrak{D}_{M/M_1}$ *is the different ideal of* M/M_1.

Proof We notice that $\mathcal{O}_M$ is a henselian discrete valuation ring with perfect residue field. Thus, the sequence of modules of differentials of $\mathcal{O}_{\overline{K}}/\mathcal{O}_M/\mathcal{O}_K$,

$$0 \longrightarrow \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_M} \Omega^1_{\mathcal{O}_M/\mathcal{O}_K} \longrightarrow \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \longrightarrow \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_M} \longrightarrow 0, \tag{4.2.1}$$

is exact by Lemma 3.1. We identify its first term with $\oplus_{i\in I}\overline{K}/\mathcal{O}_{\overline{K}}$ by Lemma 4.1. By Fontaine's computation ([Fon82, Théorème 1′]), we have an isomorphism of $\mathcal{O}_{\overline{K}}$-modules

$$\overline{K}/\mathfrak{b}(1) \xrightarrow{\sim} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_M}, \quad p^{-k} \otimes (\zeta_n)_n \longmapsto \mathrm{d}\log \zeta_k, \quad \forall k \in \mathbb{N}, \quad \forall (\zeta_n)_n \in \mathbb{Z}_p(1).$$

The conclusion follows from (4.2.1). ∎

Lemma 4.3 *The canonical map*

$$K \otimes_{\mathcal{O}_K} (\oplus_{i\in I}\mathcal{O}_K)^\wedge \longrightarrow (C \otimes_{\mathcal{O}_C} (\oplus_{i\in I}\mathcal{O}_C)^\wedge)^{G_K}$$

is an isomorphism.

Proof It follows from the following descriptions

$$\begin{aligned} &C \otimes_{\mathcal{O}_C} (\oplus_{i\in I}\mathcal{O}_C)^\wedge \\ &\quad = \Big\{(x_i) \in \prod_{i\in I} C \mid \forall N > 0,\ \exists \text{ finite } J \subseteq I, |x_i|_p < 1/N,\ \forall i \notin J\Big\}, \end{aligned} \tag{4.3.1}$$

$$\begin{aligned} &K \otimes_{\mathcal{O}_K} (\oplus_{i\in I}\mathcal{O}_K)^\wedge \\ &\quad = \Big\{(x_i) \in \prod_{i\in I} K \mid \forall N > 0,\ \exists \text{ finite } J \subseteq I, |x_i|_p < 1/N,\ \forall i \notin J\Big\}. \end{aligned} \tag{4.3.2}$$

∎

Theorem 4.4 *There is a canonical exact sequence of* C*-* G_K*-modules that splits as a sequence of* C*-modules,*

$$0 \longrightarrow C(1) \xrightarrow{\iota} V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \xrightarrow{\nu} C \otimes_{\mathcal{O}_C} (\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^\wedge \longrightarrow 0, \tag{4.4.1}$$

where $\iota(1 \otimes (\zeta_n)_n) = (\mathrm{d}\log \zeta_n)_n$ *for any* $(\zeta_n)_n \in \mathbb{Z}_p(1)$. *There is an isomorphism of* C*-*G_K*-modules*

$$\begin{aligned} C \otimes_{\mathcal{O}_C} (\oplus_{i\in I}\mathcal{O}_C)^\wedge &\xrightarrow{\sim} C \otimes_{\mathcal{O}_C} (\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^\wedge, \\ 1 \otimes e_i &\longmapsto 1 \otimes 1 \otimes \mathrm{d}\log u_i, \quad \forall i \in I, \end{aligned} \tag{4.4.2}$$

and the map $C \otimes_{\mathcal{O}_C} (\oplus_{i\in I}\mathcal{O}_C)^\wedge \to V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$, *sending* $1 \otimes e_i$ *to* $(\mathrm{d}\log w_{im})_m$ *for any* $i \in I$, *gives a* C*-linear section of* ν.

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Proof We consider the sequence of modules of differentials of $\mathcal{O}_L/\mathcal{O}_K/\mathbb{Z}_p$, where L/K is a finite subextension of $\overline{K}/K$, and pass to p-adic completions. Since $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ is killed by a power of p, we still get an exact sequence [Sta20, 0315, 0BNG]

$$\mathcal{O}_L \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \longrightarrow \hat{\Omega}^1_{\mathcal{O}_L} \longrightarrow \Omega^1_{\mathcal{O}_L/\mathcal{O}_K} \longrightarrow 0.$$

By taking colimit over all such L, we get an exact sequence

$$\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \xrightarrow{\alpha} \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}}) \xrightarrow{\beta} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \longrightarrow 0. \tag{4.4.3}$$

Combining with Propositions 3.3, 3.6, and 4.2, we get a commutative diagram:

(4.4.4)

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} (\oplus_{i\in I}\mathcal{O}_K)^\wedge & \longrightarrow & \overline{K} \otimes_{\mathcal{O}_K} (\oplus_{i\in I}\mathcal{O}_K)^\wedge & \longrightarrow & \oplus_{i\in I}\overline{K}/\mathcal{O}_{\overline{K}} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} & \xrightarrow{\alpha} & \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}}) & \xrightarrow{\beta} & \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\mathcal{O}_{K_0}} & & \overline{K}/\mathfrak{a}(1) & \longrightarrow & \overline{K}/\mathfrak{b}(1) & \longrightarrow 0,
\end{array}$$

where the rows and columns are exact, and the middle column splits. We set $D = \mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$. We see that $\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} (\oplus_{i\in I}\mathcal{O}_K)^\wedge \to D$ is injective, whose cokernel is killed by a power of p. Now for any $n > 0$, by applying $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p/p^n\mathbb{Z}_p, -)$ to (4.4.3), we get an exact sequence of $\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$-modules

$$\begin{aligned} 0 \longrightarrow D[p^n] \longrightarrow \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})[p^n] \longrightarrow \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}[p^n] \longrightarrow D/p^nD \\ \longrightarrow \hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})/p^n\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}}) = 0. \end{aligned} \tag{4.4.5}$$

We notice that the inverse system $(D[p^n])_n$ is Artin–Rees null, and that $(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})[p^n])_n$ satisfies Mittag–Leffler condition. Therefore, by taking the inverse limit of (4.4.5), we get an exact sequence of $\mathcal{O}_C$-modules

$$0 \longrightarrow T_p(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})) \longrightarrow T_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \longrightarrow D^\wedge \longrightarrow 0. \tag{4.4.6}$$

By applying $T_p(-)$ to the middle column of (4.4.4), we get $T_p(\hat{\Omega}^1_{\mathcal{O}_K}(\mathcal{O}_{\overline{K}})) = \widehat{\mathfrak{a}}(1)$. On the other hand, we notice that $\oplus_{i\in I}\overline{K}/\mathcal{O}_{\overline{K}}$ is p-divisible, and that $((\oplus_{i\in I}\overline{K}/\mathcal{O}_{\overline{K}})[p^n])_n$ satisfies the Mittag–Leffler condition. Therefore, by applying $T_p(-)$ to the right column of (4.4.4), we get an exact sequence of $\mathcal{O}_C$-modules

$$0 \longrightarrow (\oplus_{i\in I}\mathcal{O}_C)^\wedge \longrightarrow T_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \longrightarrow \widehat{\mathfrak{b}}(1) \longrightarrow 0. \tag{4.4.7}$$

As $\Omega^1_{\mathcal{O}_K/\mathcal{O}_{K_0}}$ is killed by a power of p, the map $(\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K})^\wedge \to D^\wedge$ becomes an isomorphism after inverting p. Afterwards, we get from (4.4.6) a canonical exact sequence of C-modules

$$0 \longrightarrow C(1) \to V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \longrightarrow C \otimes_{\mathcal{O}_C} (\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K})^\wedge \longrightarrow 0, \tag{4.4.8}$$

and from (4.4.7) an exact sequence of C-modules

$$0 \longrightarrow C \otimes_{\mathcal{O}_C} (\oplus_{i\in I}\mathcal{O}_C)^{\wedge} \longrightarrow V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \longrightarrow C(1) \longrightarrow 0. \tag{4.4.9}$$

The latter gives a splitting of (4.4.8) and an isomorphism $C \otimes_{\mathcal{O}_C} (\oplus_{i\in I}\mathcal{O}_C)^{\wedge} \xrightarrow{\sim} C \otimes_{\mathcal{O}_C} (\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K})^{\wedge}$ by sending $1 \otimes e_i$ to $1 \otimes 1 \otimes \mathrm{d}\log u_i$ by diagram chasing. We notice that the Galois conjugates of ζ_n, w_{im} are of the form $\zeta_n^a, \zeta_m^b w_{im}$ respectively, which implies that (4.4.8) is G_K-equivariant. Hence, (4.4.8) gives us the exact sequence (4.4.1) of C- G_K-modules that splits as a sequence of C-modules. ∎

Corollary 4.5 *The canonical map* $K \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \to (C \otimes_{\mathcal{O}_C} (\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^{\wedge})^{G_K}$ *is an isomorphism, and the connecting map of the sequence* (4.4.1)

$$\delta : K \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \longrightarrow H^1(G_K, C(1)) \tag{4.5.1}$$

is an isomorphism that coincides with (3.8.1). *In particular,*

$$V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})^{G_K} = 0. \tag{4.5.2}$$

Proof By (3.9.1), (4.4.2), and 4.3, we see that the canonical map $K \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \to (C \otimes_{\mathcal{O}_C} (\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K})^{\wedge})^{G_K}$ is an isomorphism. Now (4.5.1) follows from Theorem 3.8(i) and Remark 3.9. And (4.5.2) follows from the fact that $C(1)^{G_K} = 0$. ∎

Definition 4.6. We call sequence (4.4.1) the *Faltings extension of* $\mathcal{O}_K$ *over* $\mathbb{Z}_p$.

5 Fontaine's Injection

5.1 For any proper model $\mathfrak{X}$ of the abelian variety X over $\mathcal{O}_K$ (*i.e.*, a proper $\mathcal{O}_K$-scheme whose generic fiber is X), we identify $\mathfrak{X}(\mathcal{O}_{\overline{K}})$ with $X(\overline{K})$ by valuative criterion. Pullback of Kähler differentials defines a map

$$H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \longrightarrow \mathrm{Map}_{G_K}(X(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}), \quad \omega \longmapsto (u \longmapsto u^*\omega). \tag{5.1.1}$$

We notice that $H^0(X, \Omega^1_{X/K}) = K \otimes_{\mathcal{O}_K} H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$, and that any differential form over X is invariant under translations. Hence, we can take an integer $r > 0$ big enough, such that for any $\omega \in p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$ and $u_1, u_2 \in \mathfrak{X}(\mathcal{O}_{\overline{K}})$, $(u_1 + u_2)^*\omega = u_1^*\omega + u_2^*\omega$ (*cf.* [Fon82, Proposition 3]). Therefore, (5.1.1) induces a homomorphism of $\mathcal{O}_K$-modules

$$\rho_1 : p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}), \quad \omega \longmapsto (u \longmapsto u^*\omega).$$

We can also assume that $p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$ has no p-torsion for further use.

5.2 The functor $V_p(-)$ gives us an injective homomorphism

$$\rho_2 : \mathrm{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G_K]}\big(V_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})\big), \tag{5.2.1}$$

since $X(\overline{K})$ is p-divisible (*cf.* [Fon82, 3.5, Lemme 1]).

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5.3 The composition $\rho_2 \circ \rho_1$ induces a homomorphism of K-modules

$$(5.3.1)\qquad H^0(X, \Omega^1_{X/K}) = K \otimes_{\mathcal{O}_K} p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})).$$

As the category of $\mathcal{O}_K$-proper models of X is connected, this composition does not depend on the choice of the model and number r (*cf.* [Fon82, Proposition 4]). We conclude by the following lemma that (5.3.1) is injective.

Lemma 5.4 ([Fon82, 3.5, Lemme 1]) *There is a proper model $\mathfrak{X}$ of X such that ρ_1 is injective.*

Proof We follow closely the proof of [Fon82, 3.5, Lemme 1], which does not essentially use the assumption that the residue field k is perfect. We briefly sketch how to adapt Fontaine's proof.

(a) Let u be the origin of X and let d be the dimension of X. We first take a closed immersion $X \to \mathbb{P}^n_K$, and then we take an open immersion $\mathbb{P}^n_K \to \mathbb{P}^n_{\mathcal{O}_K}$ described later (all the morphisms are over $\mathcal{O}_K$). Let $\mathfrak{X}$ be the scheme theoretic image of the composition $X \to \mathbb{P}^n_{\mathcal{O}_K}$, which is thus a proper model of X. Let $\overline{u}$ be the special point of the scheme theoretic image of u. It is a k-point. After a linear transformation of coordinates, we can at first choose an open immersion $\mathbb{P}^n_K \to \mathbb{P}^n_{\mathcal{O}_K}$ such that $\mathcal{O}_{\mathfrak{X},\overline{u}}$ is a $(d+1)$-dimensional regular local ring (*cf.* [Fon82, 3.6, Lemme 3]).

(b) The $\mathfrak{m}_{\mathfrak{X},\overline{u}}$-adic completion of the local ring $\mathcal{O}_{\mathfrak{X},\overline{u}}$ is isomorphic to $\mathcal{O}_K[\![T_1, \dots, T_d]\!]$, denoted by $\widehat{\mathcal{O}}_{\mathfrak{X},\overline{u}}$. The $\mathfrak{m}_{\mathfrak{X},\overline{u}}$-adic completion of $\Omega^1_{\mathcal{O}_{\mathfrak{X},\overline{u}}/\mathcal{O}_K}$ is a free $\widehat{\mathcal{O}}_{\mathfrak{X},\overline{u}}$-module of rank d, denoted by $\hat{\Omega}^1_{\mathcal{O}_{\mathfrak{X},\overline{u}}/\mathcal{O}_K}$. The invariance of differential forms over X and the fact that $p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \subseteq H^0(X, \Omega^1_{X/K})$ imply that the canonical map $p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \Omega^1_{\mathcal{O}_{\mathfrak{X},\overline{u}}/\mathcal{O}_K}$ is injective (*cf.* [Fon82, 3.7]). We remark that the canonical map $\Omega^1_{\mathcal{O}_{\mathfrak{X},\overline{u}}/\mathcal{O}_K} \to \hat{\Omega}^1_{\mathcal{O}_{\mathfrak{X},\overline{u}}/\mathcal{O}_K}$ is injective, as $\Omega^1_{\mathcal{O}_{\mathfrak{X},\overline{u}}/\mathcal{O}_K}$ is of finite type over the Noetherian local ring $\mathcal{O}_{\mathfrak{X},\overline{u}}$.

(c) We have the following commutative diagram

$$\begin{array}{ccc} p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) & \xrightarrow{\rho_1} & \mathrm{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \\ \big\downarrow & & \big\downarrow \\ \widehat{\Omega}^1_{\mathcal{O}_{\mathfrak{X},\overline{u}}/\mathcal{O}_K} & \xrightarrow{\rho_1'} & \mathrm{Map}(\mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\widehat{\mathcal{O}}_{\mathfrak{X},\overline{u}}, \mathcal{O}_{\overline{K}}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}), \end{array}$$

where we identify the set of continuous $\mathcal{O}_K$-algebra homomorphisms from $\widehat{\mathcal{O}}_{\mathfrak{X},\overline{u}}$ to $\mathcal{O}_{\overline{K}}$ with a subset of $\mathfrak{X}(\mathcal{O}_{\overline{K}}) = X(\overline{K})$. To show the injectivity of ρ_1, it suffices to show that of ρ_1'. More precisely, we need to show that for any nonzero formal differential form $\sum_{i=1}^d \alpha_i(T_1, \dots, T_d)\mathrm{d}T_i$ where $\alpha_i \in \mathcal{O}_K[\![T_1, \dots, T_d]\!]$, there are $x_1, \dots, x_d \in \mathfrak{m}_{\overline{K}}$ such that $\sum_{i=1}^d \alpha_i(x_1, \dots, x_d)\mathrm{d}x_i$ is not zero in $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$.

(d) For $d = 1$, suppose $\alpha(T) = \sum_{k\geq 0} a_k T^k$ where $a_k \in \mathcal{O}_K$ not all zero. Let k_0 be the minimal number such that $v_p(a_{k_0})$ is minimal. For a sufficiently large integer N,

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we take $x = \varpi^{1/p^N} \in \mathfrak{m}_{\overline{K}}$, where ϖ is a uniformizer of $\mathcal{O}_K$, such that $v_p(a_{k_0}x^{k_0}) < v_p(a_k x^k)$ for any $k \neq k_0$. Let $M = \bigcup_{i\in I, m\geq 0} K(w_{im}) \subseteq \overline{K}$. The annihilator of $\mathrm{d}x$ in $\Omega^1_{\mathcal{O}_{M(x)}/\mathcal{O}_M}$ is generated by $p^N x^{p^N-1}$. As $\mathcal{O}_M$ is a henselian discrete valuation ring with perfect residue field, Lemma 3.1 implies that the annihilator of $\mathrm{d}x$ in $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_M}$ is also generated by $p^N x^{p^N-1}$. When N is big enough, $\alpha(x)\mathrm{d}x$ is not zero in $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ (*cf.* [Fon82, 3.7, Lemme 4]).

(e) As $\mathcal{O}_K$ is an infinite domain, there are formal series $\beta_1, \ldots, \beta_d \in \mathcal{O}_K[\![T]\!]$ without constant term, such that $\sum_{i=1}^d \alpha_i(\beta_1, \ldots, \beta_d)\cdot \beta_i' \in \mathcal{O}_K[\![T]\!]$ is still nonzero. Hence, the general case reduces to the case $d = 1$ (*cf.* [Fon82, 3.7, Lemme 5]). ∎

5.5 As $X(\overline{K})$ is p-divisible, we have a canonical exact sequence

$$0 \longrightarrow T_p(X) \longrightarrow V_p(X) \longrightarrow X(\overline{K}) \longrightarrow 0.$$

After applying the functor $\mathrm{Hom}_{\mathbb{Z}[G_K]}(-, V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}))$, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G_K]}\left(X(\overline{K}), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})\right) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G_K]}\left(V_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})\right) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G_K]}\left(T_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})\right).$$

Let $f : X(\overline{K}) \to V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$ be a G_K-equivariant homomorphism. For any finite extension L/K, we denote by $G_L = \mathrm{Gal}(\overline{K}/L)$ the absolute Galois group of L. Then f maps $X(L)$ to $V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})^{G_L}$. We notice that the kernel of the surjection $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_L}$ is killed by a power of p, which indicates that the map $V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \to V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_L})$ is an isomorphism. Now, by applying (4.5.2) to L, we get

$$V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})^{G_L} = V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_L})^{G_L} = 0.$$

Hence, $f(X(\overline{K})) = \bigcup_{L/K} f(X(L)) = 0$, which indicates that we have an injective map (*cf.* [Fon82, 3.5, Lemme 2])

$$\rho_3 : \mathrm{Hom}_{\mathbb{Z}[G_K]}\left(V_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})\right) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G_K]}\left(T_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})\right).$$

Remark that any element in the image of $\rho_3 \circ \rho_2 \circ \rho_1$ is $\mathbb{Z}_p$-linear. All in all, we have generalized Fontaine's injection ([Fon82, Théorème 2′]) to the imperfect residue field case.

Theorem 5.6 *There is a canonical K-linear injective homomorphism*

(5.6.1) $$\rho : H^0(X, \Omega^1_{X/K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p[G_K]}\left(T_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})\right).$$

6 Weak Hodge–Tate Representations

Definition 6.1 For any C- G_K-module V of finite dimension, let

(6.1.1) $$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$$

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be a composition series of V, *i.e.*, V_{i+1}/V_i is an irreducible C- G_K-module for any i. The set of factors $\{V_{i+1}/V_i\}_{0\le i<n}$ does not depend on the choice of the composition series by Schreier refinement theorem. We call the multiset

$$\mathrm{wt}(V) = \{r_i \mid V_{i+1}/V_i \cong C(r_i),\ 0 \le i < n\} \tag{6.1.2}$$

the multiset of *weak Hodge–Tate weights* of V. If all the factors are Tate twists of C, *i.e.*, $\dim_C V$ equals the cardinality of $\mathrm{wt}(V)$, then we call V a *weak Hodge–Tate C-representation of* G_K. We denote by $\mathscr{C}$ the full subcategory of finite-dimensional C-G_K-modules formed by weak Hodge–Tate representations.

Proposition 6.2 *Let V be a finite-dimensional C- G_K-module.*

(i) *For any short exact sequence of finite-dimensional C- G_K-modules $0 \to V' \to V \to V'' \to 0$, we have $\mathrm{wt}(V) = \mathrm{wt}(V') \sqcup \mathrm{wt}(V'')$. In particular, $\mathscr{C}$ is a closed under taking subrepresentation, quotient and extension.*
(ii) *For the dual representation $V^* = \mathrm{Hom}_C(V, C)$, we have $\mathrm{wt}(V^*) = -\mathrm{wt}(V)$.*

Proof The first assertion follows from the basic properties of composition series. The second assertion follows from the basic fact $C(r)^* = C(-r)$. ■

Proposition 6.3 *For $s \in \mathbb{N}$ and $r \in \mathbb{Z}$, the subrepresentations and quotients of $C(r)^{\oplus s}$ in $\mathscr{C}$ are direct summands of $C(r)^{\oplus s}$ of the form $C(r)^{\oplus t}$ for some $t \in \mathbb{N}$.*

Proof After twisting by $-r$, we can assume that $r = 0$. For any subrepresentation V of $C^{\oplus s}$, we set $W = C^{\oplus s}/V$. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V^{G_K} \otimes_K C & \longrightarrow & C^{\oplus s} & \longrightarrow & W^{G_K} \otimes_K C & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & C^{\oplus s} & \longrightarrow & W & \longrightarrow & 0.
\end{array}$$

We see that the first and third vertical maps are injective, because K-linearly independent G_K-invariant elements are also C-linearly independent. But the middle map is identity, which shows that $V = V^{G_K} \otimes_K C$, $W = W^{G_K} \otimes_K C$. Then any splitting of $0 \to V^{G_K} \to K^{\oplus s} \to W^{G_K} \to 0$ induces a splitting of $0 \to V \to C^{\oplus s} \to W \to 0$, which completes our proof. ■

Proposition 6.4 *For $s, t \in \mathbb{N}$ and integers r_1, r_2 such that $r_1 - r_2 \neq 1$ or 0, any extension of $C(r_2)^{\oplus s}$ by $C(r_1)^{\oplus t}$ in $\mathscr{C}$ is trivial.*

Proof After twisting by $-r_2$, we can assume that $r_2 = 0$ and $r_1 = r \neq 1$ or 0. Given an exact sequence $0 \to C(r)^{\oplus t} \to V \to C^{\oplus s} \to 0$, take G_K-invariants; then we obtain an exact sequence

$$0 = \left(C(r)^{\oplus t}\right)^{G_K} \longrightarrow V^{G_K} \longrightarrow K^{\oplus s} \longrightarrow H^1\left(G_K, C(r)^{\oplus t}\right) = 0,$$

from which we get an isomorphism $V^{G_K} \xrightarrow{\sim} K^{\oplus s}$. Hence, $V = C(r)^{\oplus t} \oplus C^{\oplus s}$. ■

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7 Hodge–Tate Filtration for Abelian Varieties

7.1 We keep the following simplified notation in this section:

$$G = G_K,\ \Omega = \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K};$$
$$K_I = K \otimes_{\mathcal{O}_K} \hat{\Omega}^1_{\mathcal{O}_K} \xleftarrow{\sim} K \otimes_{\mathcal{O}_K} (\oplus_{i\in I}\mathcal{O}_K)^\wedge \text{ (by (3.9.1))};$$
$$C_I = C \otimes_{\mathcal{O}_C} (\mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K/\mathbb{Z}_p})^\wedge \xleftarrow{\sim} C \otimes_{\mathcal{O}_C} (\oplus_{i\in I}\mathcal{O}_C)^\wedge \text{ (by (4.4.2))};$$
$$E = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C),\ E^G(1) = \mathrm{Hom}_{\mathbb{Z}_p[G]}(T_p(X), C) \otimes_K C(1) \subseteq E(1).$$

We remark that the Tate module $T_p(X)$ of the abelian variety X is a finite free $\mathbb{Z}_p$-module. By applying the functor $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), -) = E \otimes_C -$ to the Faltings extension (4.4.1), we get an exact sequence of C- G_K-modules

(7.1.1)
$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), V_p(\Omega)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C_I) \longrightarrow 0.$$

We also write it as $0 \to E(1) \to E \otimes_C V_p(\Omega) \to E \otimes_C C_I \to 0$.

We choose a C-linear retraction of ι in (4.4.1) and denote by

(7.1.2)
$$\pi : \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), V_p(\Omega)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}\big(T_p(X), C(1)\big)$$

the induced C-linear homomorphism.

We denote by $\tilde{\rho}$ the composition of

$$H^0(X, \Omega^1_{X/K}) \xrightarrow{\rho} \mathrm{Hom}_{\mathbb{Z}_p[G]}(T_p(X), V_p(\Omega)) \xrightarrow{\pi} E(1) \longrightarrow E(1)/E^G(1),$$

where ρ is the Fontaine's injection (5.6.1).

Lemma 7.2 *The canonical map*

$$E^G \otimes_K K_I \longrightarrow (E \otimes_C C_I)^G$$

is an isomorphism.

Proof Since E is a finite-dimensional C-vector space, the complete absolute value on C extends to a complete absolute value on E uniquely up to equivalence. We fix such an absolute value and still denote it by $|\ |_p$. Following (4.3.2) and (4.3.3), the conclusion follows from the following descriptions

$$E \otimes_C C_I = \Big\{(x_i) \in \prod_{i\in I} E \mid \forall N > 0,\ \exists \text{ finite } J \subseteq I, |x_i|_p < 1/N,\ \forall i \notin J\Big\},$$
$$E^G \otimes_K K_I = \Big\{(x_i) \in \prod_{i\in I} E^G \mid \forall N > 0,\ \exists \text{ finite } J \subseteq I, |x_i|_p < 1/N,\ \forall i \notin J\Big\}. \qquad \blacksquare$$

Lemma 7.3 *The map $\tilde{\rho}$ is injective, and its image lies in the G-invariants of $E(1)/E^G(1)$. Moreover, $\tilde{\rho}$ does not depend on the choice of π. Hence, we have a canonical K-linear injective homomorphism*

$$\tilde{\rho} : H^0(X, \Omega^1_{X/K}) \longrightarrow \big(E(1)/E^G(1)\big)^G.$$

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Proof We take a K-basis $\{h_l\}$ of E^G. For any $\omega \in H^0(X, \Omega^1_{X/K})$, thanks to Lemma 7.2, we denote by $\sum h_l \otimes \alpha_l \in E^G \otimes_K K_I$ the image of ω in $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C_I)$ via Fontaine's injection ρ (5.6.1) and (7.1.1). Take any lifting $\beta_l \in V_p(\Omega)$ of α_l in the Faltings extension (4.4.1). Consider the element

$$\rho(\omega) - \sum h_l \otimes \beta_l \in \mathrm{Hom}_{\mathbb{Z}_p}\big(T_p(X), V_p(\Omega)\big) = E \otimes_C V_p(\Omega).$$

In fact, it lies in $E(1)$. For any $\sigma \in G$,

$$\sigma(\rho(\omega) - \sum h_l \otimes \beta_l) - (\rho(\omega) - \sum h_l \otimes \beta_l) = \sum h_l \otimes (\beta_l - \sigma(\beta_l)) \in E^G(1).$$

Therefore, $\rho(\omega) - \sum h_l \otimes \beta_l$ is G-invariant modulo $E^G(1)$; *i.e.*, it defines an element in $(E(1)/E^G(1))^G$. Moreover, this element does not depend on the choice of the lifting β_l. Indeed, suppose β_l and β_l' two liftings of α_l, then $\beta_l' - \beta_l \in C(1)$, which shows that $(\rho(\omega) - \sum h_l \otimes \beta_l) - (\rho(\omega) - \sum h_l \otimes \beta_l') \in E^G(1)$. In particular, $\tilde{\rho}$ does not depend on the choice of π.

Now we show the injectivity of $\tilde{\rho}$. Suppose that $\rho(\omega) - \sum h_l \otimes \beta_l = \sum h_l \otimes \gamma_l \in E^G(1)$. Then for any $\sigma \in G$,

$$\sum h_l \otimes (\sigma(\beta_l + \gamma_l) - (\beta_l + \gamma_l)) = 0,$$

which implies that $\beta_l + \gamma_l \in V_p(\Omega)^G$=0 by (4.5.2). Hence, $\rho(\omega) = 0$, which forces ω to be zero, since ρ is injective. ■

Theorem 7.4 *There is a canonical exact sequence of C- G_K-modules*

(7.4.1)

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \otimes_K C(1) \xrightarrow{\psi} \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) \xrightarrow{\phi} H^0(X, \Omega^1_{X/K}) \otimes_K C \longrightarrow 0.$$

Proof We set $d = \dim X = \dim_K H^0(X, \Omega^1_{X/K})$. Then $T_p(X)$ is a free $\mathbb{Z}_p$-module of rank $2d$. Lemma 7.3 implies that the weak Hodge–Tate weight 0 of $E(1)$ has multiplicity $\geq d$. Let X' be the dual abelian variety of X, and we set $E' = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X'), C)$. Due to the fact that $E' = E(1)^*$ (by Weil pairing) and Proposition 6.2, the weak Hodge–Tate weight 1 of $E(1)$ has multiplicity $\geq d$. But $\dim_C E(1) = 2d$, which forces these inequalities to be equalities. In particular, $\tilde{\rho} : H^0(X, \Omega^1_{X/K}) \to (E(1)/E^G(1))^G$ is an isomorphism. Since $C(1)$ has only trivial extension by $C^{\oplus d}$ (Proposition 6.4), we see that $C^{\oplus d}$ is a quotient representation of $E(1)$. By duality again, we see that $C(1)^{\oplus d}$ is a subrepresentation of $E(1)$, and thus the canonical injection $(E(1)/E^G(1))^G \otimes_K C \to E(1)/E^G(1)$ is an isomorphism. Therefore, we have a canonical surjection

$$E(1) \longrightarrow H^0(X, \Omega^1_{X/K}) \otimes_K C.$$

By duality, $H^1(X, \mathcal{O}_X) \otimes_K C(1) = H^0(X', \Omega^1_{X'/K})^* \otimes_K C(1)$ canonically identifies with a subrepresentation of $E(1)$. Now (7.4.1) follows from the avoidance of $C(1)^{\oplus d}$ and $C^{\oplus d}$. ■

7.5 Let us complete the proof of the Main Theorem 1.3. We choose a retraction of ι in the Faltings extension (4.4.1). By our construction, we have the following

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commutative diagram

(7.5.1)
$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) & \xrightarrow{\ \phi\ } & H^0(X, \Omega^1_{X/K}) \otimes_K C \\ & \nwarrow^{\pi} & \big\downarrow \rho \\ & & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), V_p(\Omega)), \end{array}$$

where ϕ is the surjection in the Hodge–Tate filtration (7.4.1), π is induced by the chosen retraction, and ρ is the Fontaine's injection (5.6.1). Consider the following diagram:

(7.5.2)
$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{Z}_p[G]}(T_p(X), C(1)) & \xrightarrow{\ \phi\ } & H^0(X, \Omega^1_{X/K}) & \xrightarrow{\ \delta'\ } & H^1(G, H^1(X, \mathcal{O}_X) \otimes_K C(1)) \\ & & \big\downarrow \rho & & \big\downarrow \\ \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C(1)) & \xleftarrow{\ \pi\ } & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), V_p(\Omega)) & \xrightarrow{\ \pi'\ } & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C_I), \end{array}$$

where δ' is the connecting map associated to (7.4.1), where $-\pi'$ is the surjection in (7.1.1), and where we identify $H^1(X, \mathcal{O}_X)$ with $\mathrm{Hom}_{\mathbb{Z}_p[G]}(T_p(X), C)$ by (7.4.1) and identify $H^1(G, C(1))$ with K_I by (4.5.1), which gives the right vertical arrow. Let $\{h_l\}$ be a K-basis of $H^1(X, \mathcal{O}_X)$. For any $\omega \in H^0(X, \Omega^1_{X/K})$, we write $-\pi'(\rho(\omega)) = \sum h_l \otimes \alpha_l$ by 7.2, where $\alpha_l \in K_I$. Let $\beta_l \in V_p(\Omega)$ be the lifting of α_l via the chosen splitting of the Faltings extension. We see by the diagram (7.5.1) that $\rho(\omega) - \sum h_l \otimes \beta_l$ is a lifting of ω via ϕ. Thus, $\delta'(\omega)$ is represented by the following 1-cocycle:

$$\sigma \longmapsto \sum h_l \otimes \big(\beta_l - \sigma(\beta_l)\big), \quad \forall \sigma \in G.$$

We notice that $\alpha_l \in K_I$ corresponds to a class in $H^1(G, C(1))$ represented by the following 1-cocycle:

$$\sigma \longmapsto \sigma(\beta_l) - \beta_l, \quad \forall \sigma \in G.$$

In conclusion, diagram (7.5.2) is commutative.

7.6 Now we can prove Corollary 1.4 to the main theorem. If the sequence (7.4.1) splits, then the ϕ in (7.5.2) is surjective. Hence, δ' is zero map, and so is $\pi' \circ \rho$. Thus, the image of the Fontaine's injection ρ lies in $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C(1))$. We easily see that, conversely, if the image of the Fontaine's injection ρ lies in $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), C(1))$, then sequence (7.4.1) splits. Moreover, the splitting is unique by the avoidance of $C(1)^{\oplus d}$ and $C^{\oplus d}$.

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