

# On regularity properties of a surface growth model

**Jan Burczak**

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8,  
00-656 Warsaw, Poland and Mathematisches Institut, Universität  
Leipzig, Augustusplatz 10, D-04109 Leipzig, Germany  
([burczak@math.uni-leipzig.de](mailto:burczak@math.uni-leipzig.de))

**Wojciech S. Ożański**

Department of Mathematics, University of Southern California, Los  
Angeles, CA 90089, USA  
([ozanski@usc.edu](mailto:ozanski@usc.edu))

**Gregory Seregin**

Oxford University, UK and St Petersburg Department of Steklov  
Mathematical Institute, RAS, Russia  
([seregin@maths.ox.ac.uk](mailto:seregin@maths.ox.ac.uk))

(Received 14 April 2020; revised 9 October 2020; accepted 25 October 2020)

We show local higher integrability of derivative of a suitable weak solution to the surface growth model, provided a scale-invariant quantity is locally bounded. If additionally our scale-invariant quantity is small, we prove local smoothness of solutions.

*Keywords:* Surface growth model, partial regularity, local higher integrability, singular set, box-counting dimension, Hausdorff dimension

2010 *Mathematics subject classification:* 35K25; 35K55; 76D03; 74K35; 35Q35; 35Q30

## 1. Introduction

We consider the one-dimensional scalar surface growth model (SGM)

$$v_t + v_{xxxx} = -\partial_{xx}(v_x)^2, \quad (1.1)$$

© The Author(s), 2021. Published by Cambridge University Press on behalf of The Royal Society of Edinburgh

which is a model of epitaxial growth of monocrystals, with  $v$  being the height of a crystalline layer. For more applicational motivations see [11, 11, 18, 19, 23], and for certain stochastic aspects see [1–4, 7]. The analytical results for the SGM share striking similarities with the 3D incompressible Navier–Stokes equations, which has been explored in the recent years by the authors of [5, 6, 15, 16, 24].

The Cauchy problem associated with (1.1),

$$\begin{cases} v_t + v_{xxxx} = -\partial_{xx}(v_x)^2 & \text{in } \mathbb{R} \times (0, \infty) =: Q_+, \\ v(0) = v_0 & \text{on } \mathbb{R}, \end{cases} \tag{1.2}$$

admits the following notion of an energy weak solution.

DEFINITION 1.1. Let  $v_0 \in L_2(\mathbb{R})$ . We say that a function  $v \in L_\infty(0, \infty; L_2(\mathbb{R}))$  such that  $v_{xx} \in L_2(0, \infty; L_2(\mathbb{R}))$  is an *energy weak solution* to (1.2) provided

(i) (distributional formulation)

$$\int_0^\infty \int_{\mathbb{R}} (v\phi_t - v_{xx}\phi_{xx} - v_x^2\phi_{xx}) = \int_{\mathbb{R}} v_0\phi(0) \tag{1.3}$$

for every  $\phi \in C_0^\infty((-1, \infty) \times \mathbb{R})$ ,

(ii) (energy inequality) for almost every  $t \geq 0$

$$\frac{1}{2} \int_{\mathbb{R}} v(t)^2 + \int_0^t \int_{\mathbb{R}} v_{xx}^2 \leq \frac{1}{2} \int_{\mathbb{R}} v_0^2. \tag{1.4}$$

REMARK 1.2. Any energy weak solution  $v$  of (1.2) can be modified on a set of measure zero so that the energy inequality holds for any  $t \geq 0$  and

(iii)  $v(t)$  is weakly continuous into  $L_2$ , i.e. for any  $w \in L_2$

$$\int_{\mathbb{R}} v(t)w \quad \text{is continuous on } [0, \infty),$$

(iv) (time-truncated distributional formulation) for every  $t > 0$  and  $\phi \in C_0^\infty((-1, \infty) \times \mathbb{R})$

$$\int_{\mathbb{R}} v(t)\phi(t) + \int_0^t \int_{\mathbb{R}} (v\phi_t - v_{xx}\phi_{xx} - v_x^2\phi_{xx}) = \int_{\mathbb{R}} v_0\phi(0), \tag{1.5}$$

(v)  $\|v(t) - v_0\|_{L_2(\mathbb{R})} \rightarrow 0$  as  $t \rightarrow 0^+$ .

Remark 1.2 is proven in appendix 6.1.

In this note we are interested in local regularity properties of weak solutions to (1.1). Throughout the paper we will use the notation  $z = (x, t)$  and we will denote a biparabolic cylinder centred at  $z$  by

$$Q_r(z) := (x - r, x + r) \times (t - r^4, t].$$

We will use the following shorthand notation for

(a) cylinders:

$$Q_r(z) = Q_r \quad \text{as well as} \quad Q_1(0, 0) \equiv Q,$$

where there is no danger of confusing cylinders with different centres, and for

(b) the function spaces:

$$\begin{aligned} L_{p,q}(Q_r) &:= L_q(t - r^4, t; L_p(x - r, x + r)), \\ W_{p,q}^{k,0}(Q_r) &:= L_q(t - r^4, t; W_p^k(x - r, x + r)). \end{aligned}$$

We will also apply the convention that any Sobolev spaces and Lebesgue spaces are considered on  $\mathbb{R}$  unless specified otherwise. Moreover, we write  $\|\cdot\|_q := \|\cdot\|_{L^q}$ ,  $\int := \int_{\mathbb{R}}$ . We are now ready to introduce

DEFINITION 1.3. Function  $v$  is a *suitable weak solution to (1.1) on  $Q$*  provided

- (vi)  $v \in L_{2,\infty}(Q)$  and  $v_{xx} \in L_{2,2}(Q)$ ,
- (vii)  $v$  satisfies (1.1) in the sense of distributions on  $Q$ , i.e.  $\int \int (v(\phi_t - \phi_{xxxx}) - v_x^2 \phi_{xx}) = 0$  for every  $\phi \in C_0^\infty((-1, 1) \times (-1, 0))$ ,
- (viii) (local energy inequality) for any nonnegative  $\phi \in C_0^\infty(Q)$  and almost any  $t \in (-1, 0)$

$$\begin{aligned} &\frac{1}{2} \int_{-1}^1 v^2(t) \phi(t) + \int_{-1}^t \int_{-1}^1 |v_{xx}|^2 \phi \\ &\leq \int_{-1}^t \int_{-1}^1 \left( \frac{1}{2} (\phi_t - \phi_{xxxx}) v^2 + 2|v_x|^2 \phi_{xx} - \frac{5}{3} v_x^3 \phi_x - |v_x|^2 v \phi_{xx} \right). \end{aligned} \tag{1.6}$$

In view of the definition of  $Q$ , the test function  $\phi \in C_0^\infty(Q)$  may not vanish at  $t = 0$ , it vanishes only in the neighbourhood of the parabolic boundary.

REMARK 1.4. As in (iv), we can modify a suitable weak solution  $v$  on a set of measure zero such that the generalized distributional formulation

$$\int_{-1}^1 v(t)\phi(t) + \int_{-1}^t \int_{-1}^1 (v\phi_t - v_{xx}\phi_{xx} - |v_x|^2\phi_{xx}) = 0 \quad (1.7)$$

holds for all  $\phi \in C_0^\infty(Q)$  and  $t \in (-1, 0)$ , and that the local energy inequality (1.6) holds for every  $t \in (-1, 0)$ .

Definition 1.3 admits the obvious generalisation to an arbitrary cylinder  $Q_r(z)$ .

The definitions of weak energy solution and of suitable weak solution are not artificial. Indeed, integration by parts yields the cancellation

$$\int_{\mathbb{R}} v \partial_{xx}(v_x)^2 = - \int_{\mathbb{R}} (v_x)^2 v_{xx} = 0,$$

which enables the following existence result.

THEOREM 1.5. *Given  $v_0 \in L_2(\mathbb{R})$  there exists an energy weak solution to the Cauchy problem (1.2) that is a suitable weak solution on every cylinder  $Q_r(z)$ .*

For the sake of completeness, the proof of theorem 1.5 is given in appendix 6.1.

### 1.1. Supercriticality

Our equation (1.1) enjoys invariance under the scaling

$$v^\lambda(x, t) = v(\lambda x, \lambda^4 t) \quad (1.8)$$

and its total energy

$$E(v) := \sup_{t \geq 0} \int_{\mathbb{R}} |v(t)|^2 + 2 \int_0^\infty \int_{\mathbb{R}} |v_{xx}(s)|^2 ds$$

vanishes on small scales, i.e.  $E(v^\lambda) = \lambda^{-1}E(v) \rightarrow 0$  as  $\lambda \rightarrow \infty$ . In this sense equation (1.1) is supercritical. Hence one expects that standard methods, e.g. a perturbation of linear theory, do not provide a satisfactory answer to well-posedness in the large of (1.2). Observe that the small-scale decay of  $E$  occurs at the same rate  $\lambda^{-1}$  as in the case of 3D Navier–Stokes equations under its scaling. This essentially leads to certain similarities between (1.1) and 3D Navier–Stokes equations, as discussed in the literature.

1.2. Result

We provide regularity statements in relation to the quantity

$$I(z, R) := \frac{1}{R^2} \int_{Q_R(z)} |v_x|^3. \tag{1.9}$$

We note that  $I(z, R)$  is invariant with respect to the scaling (1.8) and it has been studied in the previous work [16], where Hölder continuity of  $v$  in  $Q_{R/2}(z)$  has been deduced from smallness of  $I(z, R)$ . In our main result, i.e. theorem 1.6, we improve integrability of a suitable weak solution for which  $I$  is bounded on small scales, and we show local smoothness if  $I$  is small.

We will denote by  $C_\alpha$  any positive constant that depends on some parameter  $\alpha$ .

**THEOREM 1.6.** *Let  $v$  be a suitable weak solution of the SGM (1.1) on  $Q$ .*

(i) *(higher integrability) If there exists  $M < \infty$  such that*

$$\sup_{Q_R(z) \subset Q} \frac{1}{R^2} \int_{Q_R(z)} |v_x|^3 \leq M, \tag{1.10}$$

*then there exists  $\delta_0 = \delta_0(M) > 0$  such that  $v_x \in L_{\frac{10+\delta_0}{3}, \frac{10+\delta_0}{3}}(Q_{\frac{1}{2}})$  with*

$$\begin{aligned} \left( \int_{Q_{\frac{1}{2}}} |v_x|^{\frac{10+\delta_0}{3}} \right)^{\frac{3}{10+\delta_0}} &\leq C_M \left( \int_Q |v_x|^{\frac{10}{3}} \right)^{\frac{3}{10}} \\ &(\leq CC_M (\|v\|_{2,\infty;Q} + \|v_{xx}\|_{2,2;Q})) \end{aligned} \tag{1.11}$$

(ii) *( $\varepsilon$ -regularity) Given  $\gamma \in (0, 1)$  there exists  $\varepsilon > 0$  such that if*

$$I(0, 1) \equiv \int_{Q_1} |v_x|^3 \leq \varepsilon, \tag{1.12}$$

*then*

$$|v_x(x, t) - v_x(y, s)| \leq C_\gamma \varepsilon^{1/3} \left( |x - y| + |t - s|^{1/4} \right)^\gamma \quad \text{for } (x, t), (y, s) \in Q_{\frac{1}{2}}. \tag{1.13}$$

Let us note that:

- The higher integrability result (1.11) does not follow from regularity of a suitable weak solution by interpolation, etc.
- Our result holds for an arbitrary  $\gamma \in (0, 1)$ , unlike in the case of Navier–Stokes equations (compare [13] and theorem 3.1 in [14]), thanks to lack of pressure-related difficulties.

- Ożański and Robinson [16] showed that smallness of  $I$  implies Hölder continuity of  $v$  (rather than of  $v_x$ ), which is not sufficient to further bootstrap the regularity of  $v$  and exhibits a mismatch between an assumption involving  $v_x$  and a result for  $v$ . Part (ii) fills this gap, as (1.13) guarantees smoothness. Indeed, using for example the regularity condition of [15], one has

COROLLARY 1.7. *Under assumptions of theorem 1.6 (ii),  $v \in C^\infty(Q_{1/8})$ .*

- One can also provide respective partial regularity results based on theorem 1.6. In particular, denoting by

$$S := \{(x, t) \in \mathbb{R} \times (0, \infty) : v \text{ is not infinitely differentiable} \\ \text{on any neighbourhood } U \ni (x, t)\}$$

the singular set of a suitable weak solution  $v$ , one can deduce from part (ii) that  $\mathcal{P}^1(S) = 0$  and that  $d_B(S \cap K) \leq 7/6$  for every compact  $K \subset \mathbb{R} \times (0, \infty)$ , where  $\mathcal{P}^1$  denotes the one-dimensional bipolarabolic Hausdorff measure and  $d_B$  stands for the box-counting dimension. This improves the conclusion of Ożański and Robinson [16], who showed these estimates with  $S$  replaced by the set of points where  $v$  is not Hölder continuous (a subset of  $S$ ).

The remaining part of this note is devoted to proof of theorem 1.6.

## 2. Auxiliary tools

Here we gather two tools needed further: a Campanato-type estimate for a linear equation and a multiplicative inequality.

### 2.1. The linear equation

Consider

$$u_t + u_{xxxx} + \beta u_{xxx} = 0, \tag{2.1}$$

where  $\beta \in \mathbb{R}$  is a parameter. We will use (2.1) as a limiting system in our blowup-type proof of the  $\varepsilon$ -regularity result (i.e. theorem 1.6 (ii)). In fact, one of the main purposes of this work is to demonstrate that (2.1) is a more optimal linearisation of the SGM (1.1) than the biharmonic heat equation  $u_t + u_{xxxx} = 0$  (which was used by Ożański and Robinson [16]). In fact, if one of the  $v_x$ 's in  $(v_x)^2$  in (1.1) is replaced by a constant  $\beta$ , then the nonlinearity becomes  $\beta v_{xxx}$ , which motivates (2.1). In this section, we discuss a Campanato-type estimate for (2.1), which will be later used in closing the blowup-type argument (in §5.1.3).

LEMMA 2.1 Campanato-type estimate for the linear equation. *Suppose that  $u \in L_{2,2}(Q_1)$  with  $u_x \in L_{2,2}(Q_1)$  satisfies (2.1) in the sense of distributions. Then for any  $\theta \in (0, 1/2)$ ,  $p \geq 1$ ,*

$$\left( \int_{Q_\theta} |u_x - (u_x)_\theta|^p \right)^{1/p} \leq c(1 + \beta)\theta \|u_x\|_{L_{2,2}(Q_1)}, \tag{2.2}$$

where  $c > 1$  is a universal constant and  $(u_x)_\theta := \int_{Q_\theta} u_x$ .

*Proof.* See appendix 6.2. □

REMARK 2.2. Observe that there is no lowest-order term  $\|u\|_{L_{2,2}(Q_1)}$  present on the right-hand side of (2.2), which the simplest energy estimate would dictate. Instead (2.2) follows from an introduction of time-dependent oscillations in estimates, inspired by [10], see also [22].

### 2.2. Multiplicative inequality

In the proof of theorem 1.6 (ii) (i.e. our  $\varepsilon$ -regularity result) it is sufficient to use standard interpolation. However for the proof of theorem 1.6 (i) we need the following more precise inequality. Recall  $Q_+ := \mathbb{R} \times (0, \infty)$ .

PROPOSITION 2.3. *Let  $U = Q_r(z)$  or  $U = Q_+$ . There exists a constant  $C > 0$  (independent of  $U$ ) such that*

$$\|f_x\|_{L_{\frac{10}{3}, \frac{10}{3}}(U)} \leq C \|f\|_{L_{2, \infty}(U)}^{\frac{2}{5}} \|f_{xx}\|_{L_{2,2}(U)}^{\frac{3}{5}} \tag{2.3}$$

for every  $f \in L_{2, \infty}(U)$  with  $f_{xx} \in L_{2,2}(U)$  such that, in the case of  $U = Q_r(z)$ ,  $f$  is compactly supported in  $(x - r, x + r)$ .

We note that the compact support requirement may be relaxed to the condition that  $f$  and  $f_x$  vanish at at least one point in space (at almost every  $t$ ) or the condition that they vanish in the sense of spatial average.

*Proof of Proposition 2.3.* From the Gabushin inequality

$$\|u_x\|_{\frac{10}{3}} \leq C \|u\|_2^{\frac{2}{5}} \|u_{xx}\|_2^{\frac{3}{5}},$$

(cf. theorem 1.4 of [12], p.12, or the original [9]) and therefore also on an interval  $I$ , provided  $u$  is compactly supported there (it actually suffices that  $u$  and  $u_x$  vanish somewhere on  $I$ , cf. [12], corollary 1.1, p. 21). Hence by density argument, we have the same result at almost every  $t$  for  $u(t)$ , where  $u \in W_{2,2}^{2,0}(U)$ . Taking both sides to power  $10/3$  and integrating in time yields (2.3). □

**3. Caccioppoli inequality**

Let  $\varphi_0 \in C_0^\infty(-1, 1)$  be a nonnegative and even cut-off function, such that  $\varphi_0 = 1$  on  $(-1/2, 1/2)$ , and, given  $x_0 \in \mathbb{R}$  and  $R > 0$  let

$$\varphi_{x_0,R}(x) := \varphi_0((x - x_0)/R).$$

We introduce the  $\varphi_0$ -related mean of a function  $f$

$$\langle f \rangle_{x_0,R} := \int_{x_0-R}^{x_0+R} f \varphi_{x_0,R} \left( \int_{x_0-R}^{x_0+R} \varphi_{x_0,R} \right)^{-1}. \tag{3.1}$$

REMARK 3.1. If  $v$  is a suitable weak solution on  $Q_R$  then  $\partial_t \langle v \rangle_{x_0,R} \in L_{5/3}(t_0 - R^4, t_0)$  (in particular  $\langle v \rangle_{x_0,R}(t)$  is a continuous function of  $t$ ). Indeed, abbreviating  $\varphi := \varphi_{x_0,R}$ , we have for every  $\psi \in C_0^\infty(t_0 - R^4, t_0)$  via the distributional formulation (vii)

$$\begin{aligned} \left| \int_{t_0-R^4}^{t_0} \langle v \rangle_{x_0,R}(t) \psi'(t) dt \right| &= \frac{1}{\int_{x_0-R}^{x_0+R} \varphi} \left| \int_{Q_R} v \varphi \psi' \right| = C_\varphi \left| \int_{Q_R} v_x^2 \varphi_{xx} \psi - v_x \varphi_{xxx} \psi \right| \\ &\leq C_\varphi \|v_x\|_{L_{10/3,10/3}(Q_R)}^2 \|\psi\|_{L_{5/2}(t_0-R^4, t_0)} \\ &\quad + C_\varphi \|v_x\|_{L_{10/3,10/3}(Q_R)} \|\psi\|_{L_{10/7}(t_0-R^4, t_0)} \\ &\leq C_{\varphi,R,v} \|\psi\|_{L_{5/2}(t_0-R^4, t_0)}, \end{aligned} \tag{3.2}$$

where we used (2.3) and (vi) in the last line.

We will also use a smooth nonnegative time cut-off function  $\chi_0 \in C^\infty(\mathbb{R})$  such that  $\chi_0(t) \equiv 0$  for  $t \leq -1$  and  $\chi_0 \equiv 1$  for  $t \geq -1/16$ . Let

$$\chi_{t_0,R}(t) := \chi_0((t - t_0)/R^4),$$

then  $\chi_{t_0,R}(t) \equiv 0$  for  $t \leq t_0 - R^4$  and  $\chi_{t_0,R}(t) \equiv 1$  for  $t \geq t_0 - (R/2)^4$ . We now set the space-time cutoff by writing

$$\eta_{z_0,R}(x, t) := \chi_{t_0,R}(t) \varphi_{x_0,R}(x). \tag{3.3}$$

Finally, given a function  $f$ ,  $R > 0$  and  $x_0 \in \mathbb{R}$  we will write

$$\hat{f}(x, t) := f(x, t) - \langle f(t) \rangle_{x_0,R}. \tag{3.4}$$



PROPOSITION 3.2 Caccioppoli inequality. *Let  $v$  be a suitable weak solution to (1.1) on  $Q_R(z_0) = (t_0 - R^4, t_0] \times (x_0 - R, x_0 + R)$ , then*

$$\begin{aligned} & \sup_{t \in (t_0 - R^4, t_0)} \int_{x_0 - R}^{x_0 + R} |\hat{v}|^2 \eta_{z_0, R} dx + \int_{Q_R(z_0)} |\hat{v}_{xx}|^2 \eta_{z_0, R} \\ & \leq \frac{C}{R^2} \int_{Q_R(z_0)} |\hat{v}_x|^2 + \frac{C}{R} \int_{Q_R(z_0)} |\hat{v}_x|^3. \end{aligned} \tag{3.5}$$

The main issue in the proof consists of replacing  $v$  with the time-dependent oscillation  $\hat{v}$  in the local energy inequality.

*Proof.* Without loss of generality we can assume  $z_0 = 0$  and  $R = 1$ . The case of general  $z_0$  and  $R > 0$  follows then from dilations and shifts, due to scale invariance of (3.5). We will write  $\eta := \eta_{0,1}$  for brevity. For brevity we will skip the variable under the integrals below; instead every integral ‘ $\int_{-1}^1$ ’ is taken with respect to  $x$  and we will write argument ‘ $(t)$ ’ to point out that the integral is taken at a given time  $t$ . Letting  $c(t) := \langle v(t) \rangle_{0,1}$  we have  $v = \hat{v} + c(t)$ , and so the local energy inequality (1.6) with  $\phi := \eta$  gives

$$\begin{aligned} & \frac{1}{2} \int_{-1}^1 (\hat{v} + c)^2(t) \eta(t) + \int_{-1}^t \int_{-1}^1 |\hat{v}_{xx}|^2 \eta \\ & \leq \int_{-1}^t \int_{-1}^1 \left( \frac{1}{2} (\eta_t - \eta_{xxxx}) (\hat{v} + c)^2 + 2 |\hat{v}_x|^2 \eta_{xx} - \frac{5}{3} \hat{v}_x^3 \eta_x - |\hat{v}_x|^2 (\hat{v} + c) \eta_{xx} \right) \end{aligned}$$

for every  $t \in (-1, 0)$  (recall Remark 1.4). Rearranging this inequality so that all terms involving  $c$  are moved to the right-hand side yields

$$\begin{aligned} & \frac{1}{2} \int_{-1}^1 \hat{v}^2(t) \eta(t) + \int_{-1}^t \int_{-1}^1 |\hat{v}_{xx}|^2 \eta \\ & \quad - \int_{-1}^t \int_{-1}^1 \left( \frac{1}{2} (\eta_t - \eta_{xxxx}) \hat{v}^2 + 2 |\hat{v}_x|^2 \eta_{xx} - \frac{5}{3} \hat{v}_x^3 \eta_x - |\hat{v}_x|^2 \hat{v} \eta_{xx} \right) \\ & \leq -c(t) \int_{-1}^1 \hat{v}(t) \eta(t) - \frac{c^2(t)}{2} \int_{-1}^1 \eta(t) \\ & \quad + \int_{-1}^t \int_{-1}^1 \frac{1}{2} (\eta_t - \eta_{xxxx}) (2c\hat{v} + c^2) - \int_{-1}^t \int_{-1}^1 c |\hat{v}_x|^2 \eta_{xx} \\ & = -c(t) \int_{-1}^1 v(t) \eta(t) + \frac{c^2(t)}{2} \int_{-1}^1 \eta(t) + \int_{-1}^t \int_{-1}^1 \frac{1}{2} \eta_t (2c\hat{v} - c^2) \\ & \quad - \int_{-1}^t \int_{-1}^1 (c\eta)_{xxxx} \hat{v} - \int_{-1}^t \int_{-1}^1 c |v_x|^2 \eta_{xx}, \end{aligned} \tag{3.6}$$

where we substituted  $v = \hat{v} + c(t)$  and used the fact that  $\int_{-1}^1 \eta_{xxxx} = 0$  for the last line.

In order to deal with the second and the third term of the last line of (3.6), let us observe that for every  $t$

$$c(t) \int_{-1}^1 \eta(t) = \frac{\int_{-1}^1 v(t)\varphi_0}{\int_{-1}^1 \varphi_0} \chi_0(t) \int_{-1}^1 \varphi_0 = \int_{-1}^1 \eta(t)v(t), \tag{3.7}$$

by the definition of  $c(t)$  and the fact that  $\eta(x, t) = \chi_0(t)\varphi_0(x)$ . Therefore, first integrating by parts and then using (3.7)

$$\begin{aligned} & \frac{c^2(t)}{2} \int_{-1}^1 \eta(t) + \int_{-1}^t \int_{-1}^1 \frac{1}{2} \eta_t (2cv - c^2) \\ &= \int_{-1}^t \int_{-1}^1 \eta_t cv + \int_{-1}^t c' c \int_{-1}^1 \eta = \int_{-1}^t \int_{-1}^1 (c\eta)_t v. \end{aligned}$$

Using the above identity in the last line of (3.6), we obtain

$$\begin{aligned} & \frac{1}{2} \int_{-1}^1 \hat{v}^2(t)\eta(t) + \int_{-1}^t \int_{-1}^1 |\hat{v}_{xx}|^2 \eta \\ & - \int_{-1}^t \int_{-1}^1 \left( \frac{1}{2} (\eta_t - \eta_{xxxx}) \hat{v}^2 + 2|\hat{v}_x|^2 \eta_{xx} - \frac{5}{3} \hat{v}_x^3 \eta_x - |\hat{v}_x|^2 \hat{v} \eta_{xx} \right) \\ & \leq -c(t) \int_{-1}^1 v(t)\eta(t) + \int_{-1}^t \int_{-1}^1 ((c\eta)_t - (c\eta)_{xxxx})v \\ & - \int_{-1}^t \int_{-1}^1 c|v_x|^2 \eta_{xx} =: I. \end{aligned} \tag{3.8}$$

We will show that  $I$  vanishes. To this end let us observe that we can use  $\phi(x, t) := c(t)\eta(x, t)$  as a test function in (1.7). Indeed, via Remark 3.1  $\partial_t(c\eta) \in L_{5/3,\infty}(Q)$  and  $c(t)$  is continuous. This gives  $I = 0$ , and so (3.8) reduces to

$$\begin{aligned} & \frac{1}{2} \int_{-1}^1 \hat{v}^2(t)\eta(t) + \int_{-1}^t \int_{-1}^1 |\hat{v}_{xx}|^2 \eta \\ & \leq \int_{-1}^t \int_{-1}^1 \left( \frac{1}{2} (\eta_t - \eta_{xxxx}) \hat{v}^2 + 2|\hat{v}_x|^2 \eta_{xx} - \frac{5}{3} \hat{v}_x^3 \eta_x - |\hat{v}_x|^2 \hat{v} \eta_{xx} \right) \end{aligned}$$

for every  $t \in (-1, 0)$ . Thus

$$\begin{aligned} \sup_{t \in (-1, 0)} \int_{-1}^1 \hat{v}(t)^2 \eta(t) + \int_Q |\hat{v}_{xx}|^2 \eta &\leq C \int_Q (\hat{v}^2 + |\hat{v}_x|^2 + |\hat{v}_x|^3 + |\hat{v}_x|^2 |\hat{v}|) \\ &\leq C \int_Q (\hat{v}^2 + |\hat{v}|^3 + |\hat{v}_x|^2 + |\hat{v}_x|^3) \\ &\leq C \int_Q (|\hat{v}_x|^2 + |\hat{v}_x|^3), \end{aligned}$$

as required, where we used the spatial Poincaré-Sobolev inequality (for functions with vanishing  $\varphi$ -related means) in the last line.  $\square$

#### 4. Higher integrability

Here we prove part (i) of theorem 1.6. First we derive a reverse Hölder inequality from the Caccioppoli inequality (3.5). Indeed, (3.5) gives

$$\begin{aligned} \sup_{t \in (t_0 - R^4, t_0)} \int_{x_0 - R}^{x_0 + R} |\hat{v} \eta_{z_0, R}|^2 dx + \int_{Q_R(z_0)} |\partial_{xx}(\hat{v} \eta_{z_0, R})|^2 \\ \leq \frac{C}{R^2} \int_{Q_R(z_0)} |\hat{v}_x|^2 + \frac{C}{R} \int_{Q_R(z_0)} |\hat{v}_x|^3 \\ \leq \frac{C}{R^{\frac{1}{3}}} \left( \int_{Q_R(z_0)} |\hat{v}_x|^3 \right)^{\frac{2}{3}} + \frac{C}{R} \int_{Q_R(z_0)} |\hat{v}_x|^3. \end{aligned} \tag{4.1}$$

Recall that by assumption (1.10)

$$\sup_{Q_R(z_0) \subset Q} \frac{1}{R^2} \int_{Q_R(z_0)} |v_x|^3 \leq M.$$

Thus, (4.1) yields

$$\begin{aligned} \sup_{t \in (t_0 - R^4, t_0)} \int_{x_0 - R}^{x_0 + R} |\hat{v} \eta_{z_0, R}|^2 dx + \int_{Q_R(z_0)} |\partial_{xx}(\hat{v} \eta_{z_0, R})|^2 \\ \leq C \frac{1 + M^{\frac{1}{3}}}{R^{\frac{1}{3}}} \left( \int_{Q_R(z_0)} |\hat{v}_x|^3 \right)^{\frac{2}{3}} \end{aligned} \tag{4.2}$$

for every  $Q_R(z_0) \subset Q$ . Let us use multiplicative inequality (2.3) for  $\hat{v}\eta_{z_0,R}$  to compute

$$\begin{aligned} \left( \int_{Q_{R/2}(z_0)} |\hat{v}_x|^{10/3} \right)^{3/5} &\leq C \|\hat{v}\eta_{z_0,R}\|_{L_{2,\infty}(Q_R(z_0))}^{4/5} \|\partial_{xx}(\hat{v}\eta_{z_0,R})\|_{L_{2,2}(Q_R(z_0))}^{6/5} \\ &\leq \sup_{t \in (t_0 - R^4, t_0)} \int_{x_0 - R}^{x_0 + R} |\hat{v}\eta_{z_0,R}|^2 dx \\ &\quad + \int_{Q_R(z_0)} |\partial_{xx}(\hat{v}\eta_{z_0,R})|^2 \\ &\leq (1 + M^{1/3}) \frac{1}{R^{1/3}} \left( \int_{Q_R(z_0)} |\hat{v}_x|^3 \right)^{\frac{2}{3}}, \end{aligned}$$

where we used (4.2) in the last line. In other words, we obtain the reverse Hölder inequality

$$\left( \int_{Q_{\frac{R}{2}}(z_0)} |\hat{v}_x|^{10/3} \right)^{\frac{3}{10}} \leq C_M \left( \int_{Q_R(z_0)} |\hat{v}_x|^3 \right)^{\frac{1}{3}}$$

for every  $Q_R(z_0) \subset Q$ . Applying the Gehring Lemma (see proposition 1.3 in [10], for example) gives part (i) of theorem 1.6.

### 5. $\varepsilon$ -regularity

In this section, we prove part (ii) of theorem 1.6. It relies on quantifying decay of

$$Y(r, v) := \left( \int_{Q_r} |v_x - (v_x)_r|^3 \right)^{1/3}$$

as  $r \rightarrow 0^+$ , where  $(f)_r := \int_{Q_r} f$ . The needed decay lemma is stated and proved in § 5.1. Then we conclude our proof of the part (ii) of theorem 1.6 in § 5.2, using the Campanato characterisation of Hölder continuity.

#### 5.1. Decay estimate

LEMMA 5.1 Decay Lemma. *For any  $M > 1$  and  $\theta \in (0, 1/4)$  there exists  $\varepsilon_0(\theta, M) > 0$  such that for any suitable weak solution  $v$  to the SGM (1.1) on*

$Q_1$ , if

$$|(v_x)_1| \leq M \quad \text{and} \quad Y(1, v) \leq \varepsilon_0, \tag{5.1}$$

then

$$Y(\theta, v) \leq 6cM\theta Y(1, v),$$

where  $c$  is the universal constant from lemma 2.1.

In § 5.1.1–5.1.4, we prove lemma 5.1: we first compare nonlinear problem (1.1) to a linear one (2.1) using the blow-up technique, and then we use quantitative decay for a linear system, by means of lemma 2.1.

5.1.1. *Proof of Lemma 5.1. The Setup* Suppose that lemma 5.1 is false. Then there exist numbers  $\theta \in (0, 1/4)$ ,  $M > 1$ , a sequence  $\varepsilon_k \rightarrow 0$  and a sequence of suitable weak solutions  $v^{(k)}$  such that

$$\left| (v_x^{(k)})_1 \right| \leq M, \quad Y(1, v^{(k)}) = \varepsilon_k, \quad \text{and} \quad Y(\theta, v^{(k)}) > 6cM\theta\varepsilon_k \tag{5.2}$$

We let

$$u^{(k)} := \varepsilon_k^{-1} \left( v^{(k)} - [v^{(k)}]_1 - (\partial_x v^{(k)})_1 x \right),$$

where we set  $[f]_1 := \int_{-1}^0 \langle f(t) \rangle_{0,1} dt$ , with  $\langle f(t) \rangle_{0,1}$  denoting the  $\varphi$ -related mean (recall (3.1)). Since  $\varphi$  is even we have  $[u^{(k)}]_1 = (u_x^{(k)})_1 = 0$ ;  $u^{(k)}$  also normalises (5.2), i.e.

$$Y(1, u^{(k)}) = 1 \quad \text{and} \quad Y(\theta, u^{(k)}) > 6cM\theta. \tag{5.3}$$

5.1.2. *Proof of Lemma 5.1. Uniform estimate on the rescalings  $u^{(k)}$*  Here we show that

$$\|u^{(k)}\eta_{0,1}\|_{L_{2,\infty}(Q_1)} + \|\partial_{xx}(u^{(k)}\eta_{0,1})\|_{L_{2,2}(Q_1)} \leq C_M. \tag{5.4}$$

uniformly in  $k$ . (Recall (3.3) for the definition of  $\eta_{0,1}$ .)

Letting  $\beta_k := (v_x^{(k)})_1$ , we see that  $u^{(k)}$  satisfies the following *perturbed SGM*:

$$\partial_t u^{(k)} + u_{xxxx}^{(k)} + \varepsilon_k \partial_{xx}(u_x^{(k)})^2 + 2\beta_k u_{xxx}^{(k)} = 0 \tag{5.5}$$

in  $Q_1$  in the sense of distributions. Moreover, the local energy inequality (1.6) for  $v^{(k)}$  gives that

$$\begin{aligned} & \frac{1}{2} \int_{-1}^1 (u^{(k)})^2(t)\phi(t) + \int_{-1}^t \int_{-1}^1 |u_{xx}^{(k)}|^2 \phi \\ & \leq \varepsilon_k \int_{-1}^t \int_{-1}^1 \left( \frac{1}{2}(\phi_t - \phi_{xxxx})(u^{(k)})^2 + 2(u_x^{(k)})^2 \phi_{xx} \right. \\ & \quad \left. - \frac{5}{3}(u_x^{(k)})^3 \phi_x - (u_x^{(k)})^2 u^{(k)} \phi_{xx} \right) \\ & \quad - \beta_k \int_{-1}^t \int_{-1}^1 3(u_x^{(k)})^2 \phi_x - (u^{(k)})^2 \phi_{xxx} \end{aligned} \tag{5.6}$$

for any nonnegative  $\phi \in C_0^\infty(Q_1)$  and every  $t \in (-1, 0)$ . Here, as before, we skipped the variable under the integrals below, and every integral ‘ $\int_{-1}^1$ ’ is taken with respect to  $x$  and we wrote argument ‘ $(t)$ ’ to point out that the integral is taken at a given time  $t$ . Unless specified otherwise, we will apply the same convention in what follows. Letting

$$\hat{u}^{(k)} := u^{(k)} - \langle u^{(k)}(t) \rangle_{0,1}$$

we can use the above inequality to obtain the following Caccioppoli inequality for  $\hat{u}^{(k)}$ , analogously to how (3.5) was proven

$$\begin{aligned} & \sup_{t \in (-1,0)} \int_{-1}^1 \hat{u}^{(k)}(t)^2 \eta_{0,1}(t) + \int_{Q_1} |\hat{u}_{xx}^{(k)}|^2 \eta_{0,1} \\ & \leq \varepsilon_k C \int_{Q_1} |\hat{u}_x^{(k)}|^2 + \varepsilon_k C \int_{Q_1} |\hat{u}_x^{(k)}|^3 + C|\beta_k| \int_{Q_1} |\hat{u}_x^{(k)}|^2 \end{aligned} \tag{5.7}$$

for each  $k$ . Recalling that  $\varepsilon_k \leq 1$ ,  $|\beta_k| = |(v_x^{(k)})_1| \leq M$  (see (5.2)) and observing that (5.3) gives

$$\int_{Q_1} |\hat{u}_x^{(k)}|^3 = \int_{Q_1} |u_x^{(k)}|^3 = 2Y(1, u^{(k)})^3 = 2,$$

we obtain

$$\|\hat{u}^{(k)}\eta_{0,1}\|_{L_{2,\infty}(Q_1)} + \|\partial_{xx}(\hat{u}^{(k)}\eta_{0,1})\|_{L_{2,2}(Q_1)} \leq CM.$$

Comparing this with (5.4), we see that it suffices to show that  $\sup_{t \in (-1,0)} \langle u^{(k)}(t) \rangle_{0,1} \leq CM$ . We will write

$$c(t) := \langle u^{(k)}(t) \rangle_{0,1},$$

for brevity. Similarly as in (3.2) we have for every  $\psi \in C_0^\infty(-1, 0)$

$$\begin{aligned} \left| \int_{-1}^0 c(t)\psi'(t) dt \right| &\leq C\|u_x^{(k)}\|_{L_{3,3}(Q_1)}^2 \|\psi\|_{L_3(-1,0)} + C\|u_x^{(k)}\|_{L_{3,3}(Q_1)} \|\psi\|_{L_{3/2}(-1,0)} \\ &\leq C\|\psi\|_{L_3(-1,0)}, \end{aligned}$$

where we used (5.3). Thus,  $c(t)$  is continuous with  $c'(t) \in L_{3/2}(-1, 0)$ . Since  $\int_{-1}^0 c(t) dt = [u^{(k)}]_1 = 0$  there exists  $t_0 \in (-1, 0)$  such that  $c(t_0) = 0$ , and hence

$$|c(t)| = |c(t) - c(t_0)| = \left| \int_{t_0}^t c'(s) ds \right| \leq \|c'\|_{L_{3/2}(-1,0)} \leq C$$

for every  $t \in (-1, 0)$ , as required. We thus obtained (5.4).

5.1.3. *Proof of Lemma 5.1. Blowup limit equation* Here we extract a sequence of  $u^{(k)}$  converging to a limit  $u$  that satisfies a linear equation and that  $Y(\theta, u) < 6cM\theta$ . Indeed, from (5.3), the interpolation inequality  $\|f\|_{L_{10,10}(U)} \leq C\|f\|_{L_{2,\infty}(U)}^{\frac{4}{5}}\|f\|_{W_{2,2}^{2,0}(U)}^{\frac{1}{5}}$  applied to  $u^{(k)}\eta_{0,1}$  and (5.4) we obtain

$$\int_{Q_1} |u_x^{(k)}|^3 = 2, \quad \int_{Q_{1/2}} (u^{(k)})^{10} \leq C_M. \tag{5.8}$$

This and the fact that  $|\beta_k| \leq M$  allow to extract a subsequence (which we relabel) such that

$$u^{(k)} \rightharpoonup u \quad \text{in } W_{3,3}^{1,0}(Q_{1/2}) \quad \text{and} \quad \beta_k \rightarrow \beta$$

for some  $u \in W_{3,3}^{1,0}(Q_{1/2})$  and  $\beta \in [-M, M]$ . Since also  $\varepsilon_k \rightarrow 0$ , we can take  $k \rightarrow \infty$  in the distributional formulation of perturbed SGM (5.5) to obtain that  $u$  is a distributional solution to the linear equation

$$u_t + u_{xxxx} + 2\beta u_{xxx} = 0 \quad \text{in } Q_{1/2}.$$

Applying lemma 2.1 and using the fact that  $\int_{Q_\theta} |u_x|^3 \leq 2$  for every  $\theta < 1/2$  (a consequence of the weak convergence and (5.8)), we obtain

$$Y(\theta, u) = \left( \int_{Q_\theta} |u_x - (u_x)_\theta|^3 \right)^{1/3} \leq c(1 + 2|\beta|)\theta\|u_x\|_{L_3(Q_1)} < 6cM\theta, \tag{5.9}$$

as required.

5.1.4. *Proof of Lemma 5.1. Compactness and contradiciton* Here we will show that (on a subsequence)

$$u_x^{(k)} \rightarrow u_x \quad \text{in} \quad L_3(Q_{1/2}). \tag{5.10}$$

This will conclude the proof of lemma 5.1 since the strong limit (5.10) implies  $Y(\theta, u^{(k)}) \rightarrow Y(\theta, u)$ , and so the inequality in (5.3) yields

$$6cM\theta \leq Y(\theta, u) \stackrel{(5.9)}{<} 6cM\theta, \tag{5.11}$$

a contradiction. In order to justify (5.10) we will use an Aubin-Lions argument. Applying (5.4) and the interpolation inequality  $\|f\|_{W_{3,3}^{7/6,0}} \leq C\|f\|_{L_{2,\infty}}^{1/3}\|f\|_{W_{2,2}^{2,0}}^{2/3}$  to the function  $u^{(k)}\eta_{0,1}$ , we obtain that

$$\|u^{(k)}\|_{W_{3,3}^{7/6,0}(Q_{1/2})} \leq C_M. \tag{5.12}$$

On the other hand, for every  $\phi \in C_0^\infty(Q_{1/2})$

$$\begin{aligned} \left| \int_{Q_{1/2}} u^{(k)} \partial_t \phi \right| &= \left| - \int_{Q_{1/2}} (u_{xx}^{(k)}) (\phi_{xx} - 2\beta_k \phi_x) - \varepsilon_k \int_{Q_{1/2}} (u_x^{(k)})^2 \phi_{xx} \right| \\ &\leq C_M \|\phi\|_{W_{3,3}^{2,0}(Q_{1/2})} \left( \|u_{xx}^{(k)}\|_{L_{3/2,3/2}(Q_{1/2})} + \|u_x^{(k)}\|_{L_{3,3}(Q_{1/2})}^2 \right) \\ &\leq C_M \|\phi\|_{W_{3,3}^{2,0}(Q_{1/2})}, \end{aligned}$$

where we used (5.4) and (5.8) in the last line. Hence

$$\|u_t^{(k)}\|_{L_{\frac{3}{2}}(-1/16,0;(W_3^2(B_{1/2}))^*)} \leq C_M. \tag{5.13}$$

This, (5.8) and (5.12) allow us to use the Aubin-Lions lemma (see Section 3.2.2 in [25], for example) to extract a subsequence of  $u^{(k)}$  that converges strongly in  $W_{3,3}^{\frac{7}{6}-\delta,0}(Q_{1/2})$  for any fixed  $\delta \in (0, 7/6)$ . (Recall that  $W_{3,3}^{\frac{7}{6}-\delta,0}(Q_r) = L_3(t - r^4, t; W_3^{\frac{7}{6}-\delta}(x - r, x + r))$ .) Taking  $\delta := 1/6$  gives (5.10), as required.

**5.2. Concluding  $\varepsilon$ -regularity proof**

Here we finish the proof of part (ii) of theorem 1.6 by iterating lemma 5.1. Indeed we have the following.



COROLLARY 5.2. *Given  $\gamma \in (0, 1)$  and  $M \geq 1$ , there exist  $\varepsilon_0, \theta \in (0, 1/2)$  with the following property. If  $v$  is a suitable weak solution to the SGM (1.1) on  $Q_1$  such that*

$$|(v_x)_1| \leq M, \quad Y(1, v) \leq \varepsilon_0$$

then

$$\theta^{k-1} |(v_x)_{\theta^{k-1}}| \leq M \quad \text{and} \quad Y(\theta^k, v) \leq \theta^\gamma Y(\theta^{k-1}, v) \tag{5.14}$$

for every  $k \geq 1$ .

*Proof.* We fix  $\theta \in (0, 1/2)$  so small that

$$6cM\theta^{1-\gamma} < 1. \tag{5.15}$$

(Recall that  $c$  is the universal constant from lemma 2.1.) Let  $\varepsilon_0$  be sufficiently small so that lemma 5.1 holds and

$$\varepsilon_0 < \theta^5 M/2. \tag{5.16}$$

The case  $k = 1$  follows from our assumptions and lemma 5.1. For  $k > 1$  we proceed by induction. Suppose that (5.14) holds for  $k' \leq k$ . Then

$$\begin{aligned} \theta^k |(v_x)_{\theta^k}| &\leq \theta^k |(v_x)_{\theta^k} - (v_x)_{\theta^{k-1}}| + \theta^k |(v_x)_{\theta^{k-1}}| \\ &= \theta^k \left| \int_{Q_{\theta^k}} (v_x - (v_x)_{\theta^{k-1}}) \right| + \theta^k |(v_x)_{\theta^{k-1}}| \\ &\leq \theta^{k-5} \int_{Q_{\theta^{k-1}}} |v_x - (v_x)_{\theta^{k-1}}| + \theta M \\ &\leq \theta^{k-5} \left( \int_{Q_{\theta^{k-1}}} |v_x - (v_x)_{\theta^{k-1}}|^3 \right)^{1/3} + M/2 \\ &= \theta^{k-5} Y(\theta^{k-1}, v) + M/2 \leq \theta^{-5} Y(1, v) + M/2 \leq \theta^{-5} \varepsilon_0 + M/2 \leq M, \end{aligned} \tag{5.17}$$

where we used Jensen’s inequality, the fact that  $\theta < \frac{1}{2}$  (so that in particular  $\theta^\gamma < 1$  and  $\theta^k < 1$ ), the inductive assumption (for  $k' = 1, \dots, k - 1$ ), and the choice (5.16).

It remains to show that  $Y(\theta^{k+1}, v) \leq \theta^\gamma Y(\theta^k, v)$ . To this end let us rescale

$$v^k(x, t) := v(\theta^k x, \theta^{4k} t).$$

In particular,  $v^k$  is a suitable weak solution of SGM (1.1) on  $Q_1$ , and

$$|(v_x^k)_1| = \theta^k |(v_x)_{\theta^k}| \leq M \quad \text{and} \quad Y(1, v^k) = \theta^k Y(\theta^k, v) \leq Y(1, v) \leq \varepsilon_0,$$

where we used (5.17), the assumption (5.14) (for  $k' = 1, \dots, k$ ) and the fact that  $\theta < 1$ . Thus, lemma 5.1 gives  $Y(\theta, v^k) \leq 6cM\theta Y(1, v^k)$ , from which we conclude

$$Y(\theta^{k+1}, v) = \theta^{-k} Y(\theta, v^k) \leq 6cM\theta^{1-k} Y(1, v^k) = 6cM\theta Y(\theta^k, v) < \theta^\gamma Y(\theta^k, v),$$

via the choice (5.15). □

We can now conclude the proof of part (ii) of theorem 1.6. Without loss of generality we assume that  $Q = Q_1(0, 0)$ . Recall that we need to show that for any  $\gamma \in (0, 1)$  there exists  $\varepsilon > 0$  such that  $I(Q) \equiv \int_Q |v_x|^3 \leq \varepsilon$  implies that

$$|v_x(x, t) - v_x(y, s)| \leq C_\gamma \varepsilon^{1/3} \left( |x - y| + |t - s|^{1/4} \right)^\gamma \quad \text{for } (x, t), (y, s) \in Q_{\frac{1}{2}}. \tag{5.18}$$

We first deduce from corollary 5.2 that

$$\left( \int_{Q_r(y,s)} |v_x - (v_x)_{Q_r(y,s)}|^3 \right)^{1/3} \leq C_\gamma \varepsilon^{1/3} r^\gamma \tag{5.19}$$

for every  $(y, s) \in Q_{1/2}$ ,  $r \in (0, 1/2)$ , where  $(v_x)_{Q_r(y,s)} := \int_{Q_r(y,s)} v_x$ .

Indeed, let  $\varepsilon_0, \theta \in (0, 1/4)$  be given by corollary 5.2 applied with  $M := 1$ , and let  $\varepsilon := \varepsilon_0^3/16$  and

$$u(x, t) := v(y + x/2, s + t/16).$$

Given  $r \in (0, 1/2)$  let  $K \in \mathbb{N}$  be such that

$$\theta^{K+1} \leq r < \theta^K.$$

By assumption (1.12) and Jensen’s inequality

$$\begin{aligned} |(u_x)_1| &\leq \left( \int_{Q_1} |u_x|^3 \right)^{1/3} = 2^{1/3} \left( \int_{Q_{1/2}(y,s)} |v_x|^3 \right)^{1/3} \leq 2^{1/3} I(Q)^{1/3} \\ &\leq 2^{1/3} \varepsilon^{1/3} \leq 1 = M, \\ Y(1, u) &= \left( \int_{Q_1} |u_x - (u_x)_1|^3 \right)^{1/3} \leq 2 \left( \int_{Q_1} |u_x|^3 \right)^{1/3} \leq 2^{4/3} \varepsilon^{1/3} = \varepsilon_0, \end{aligned}$$

and so corollary 5.2 gives

$$Y(\theta^K, u) \leq \theta^{\gamma K} Y(1, u) \leq \theta^{\gamma K} \varepsilon_0.$$

Noting that  $r/\theta^K \in (\theta, 1)$  and that  $|(u_x)_r - (u_x)_{\theta^K}| \leq \theta^{-5} Y(\theta^K, u)$  (as in (5.17) above), we obtain

$$\begin{aligned} \left( \int_{Q_r} |u_x - (u_x)_r|^3 \right)^{1/3} &\leq \left( \int_{Q_r} |u_x - (u_x)_{\theta^K}|^3 \right)^{1/3} + \theta^{-5} Y(\theta^K, u) \\ &\leq (\theta^{-5/3} + \theta^{-5}) Y(\theta^K, u) \leq (\theta^{-5/3} + \theta^{-5}) \theta^{\gamma K} \varepsilon_0 \\ &\leq (\theta^{-5/3} + \theta^{-5}) \theta^{-\gamma} r^\gamma \varepsilon_0 = C \varepsilon^{1/3} r^\gamma \end{aligned}$$

for every  $r \in (0, 1)$ , where  $C = C(\gamma)$ . (Recall that  $\theta$  is fixed depending on  $\gamma$ , see (5.15).) The claim (5.19) follows by writing the above inequality in terms of  $v$ .

Using (5.19), we obtain (5.18) by applying the following Campanato Lemma with  $p = 3, R = 1$ , cf. the original [8] or lemma A.2 in [16].

LEMMA 5.3 Campanato. *Let  $R \in (0, 1], f \in L_{1,1}(Q_R(0))$  and suppose that there exist positive constants  $\gamma \in (0, 1], N > 0$ , such that*

$$\left( \int_{Q_r(z)} |f(y) - (f)_{Q_r(z)}|^p dy \right)^{1/p} \leq Nr^\gamma$$

for any  $z \in Q_{R/2}(0)$  and any  $r \in (0, R/2)$ , where  $(f)_{Q_r(z)} := \int_{Q_r(z)} f$ . Then  $f$  is Hölder continuous in  $Q_{R/2}(0)$  with

$$|f(x, t) - f(y, s)| \leq cN(|x - y| + |t - s|^{1/\alpha})^\gamma$$

for all  $(x, t), (y, s) \in Q_{R/2}(0)$ .

## 6. Appendices

### 6.1. Appendix on weak solutions

*Proof of Remark 1.2.* The distributional formulation (1.3) yields

$$\left| \int_0^\infty \int_{\mathbb{R}} v \phi_t \right| \leq \int_0^\infty \left( \|v_{xx}(t)\|_2 \|\phi_{xx}(t)\|_2 + \|v_x(t)\|_{10/3}^2 \|\phi_{xx}(t)\|_{5/2} \right) dt.$$

In particular, for  $\phi(x, t) = \varphi(x)\chi(t)$ , where  $\chi \in C_0^\infty((0, \infty))$  and  $\phi \in C_0^\infty(\mathbb{R})$ , using (2.3), and the energy inequality (1.4)

$$\left| \int_0^\infty \int_{\mathbb{R}} v \varphi \chi_t \right| \leq C_{u_0} \|\varphi\|_{W_2^{5/2}} \|\chi\|_{L_{5/2}},$$

i.e.  $v_t \in L_{\frac{5}{3}}((0, \infty); (W_2^{5/2}(\mathbb{R}))^*)$ . Hence redefining  $v$  on a set of measure zero, we have  $v \in C((0, T); (W_2^{5/2}(\mathbb{R}))^*)$ . Since also  $v \in L_{2, \infty}$ , we have weak  $L_2$  continuity of  $v$ , via e.g. lemma 2.2.5. of [17], which shows (iii).

We now verify that the energy inequality (1.4) holds for every  $t > 0$ . Choose any  $t \geq 0$  and a sequence  $t_n \rightarrow t^+$  such that the energy inequality holds at each  $t_n$ . Since  $v(t_n) \rightarrow v(t)$  (by (iii)) we can take  $\liminf_{t_n \rightarrow t}$  of the energy inequalities and use lower weak semicontinuity of the norm to write  $\|v(t)\|^2 \leq \liminf_{t_n \rightarrow t^+} \|v(t_n)\|^2 \leq \|v_0\|^2 - 2 \int_0^t \int_{\mathbb{R}} |v_{xx}|^2$ .

As for (iv), given  $\phi \in C_0^\infty((-1, \infty) \times \mathbb{R})$  we can multiply  $\phi$  by a cutoff in time (as in lemma 3.14 in [20], for example) to obtain (1.5) for almost every  $t > 0$ . Weak continuity (iii) guarantees that (1.5) holds for every  $t > 0$ .

As for (v), note that (iv) implies weak convergence  $v(t) \rightarrow v_0$  as  $t \rightarrow 0^+$ . Moreover  $\|v(t)\| \rightarrow \|v_0\|$  since  $\liminf_{t \rightarrow 0^+} \|v(t)\| \geq \|v_0\|$  (from the weak convergence) and  $\limsup_{t \rightarrow 0^+} \|v(t)\| \leq \|v_0\|$  (from the energy inequality (1.4)), and so strong convergence follows, cf. p. 106 in [21]. □

*Proof of Theorem 1.5.* Given  $l > 0$  let  $v_{0,l} \in C_0^\infty(-l, l)$  be such that  $\|v_{0,l}\| \leq \|v_0\|$  and  $v_{0,l} \rightarrow v_0$  strongly in  $L^2$  as  $l \rightarrow \infty$ , and let  $T_l > 0$  be such that  $T_l \rightarrow \infty$  as  $l \rightarrow \infty$ . Let us denote by  $\dot{W}_2^2(-l, l)$  the completion of  $C_0^\infty(-l, l)$  in the  $W_2^2$  norm. By a straightforward modification of the arguments in theorem 4.3 in [24] and theorem 2.4 in [16], for each  $l$  there exists a suitable weak solution of the initial-boundary value problem

$$\begin{cases} \partial_t v_l + \partial_{xxxx} v_l + \partial_{xx}(v_l)^2 = 0 & \text{on } (-l, l) \times (0, T) \\ v_l(0) = v_{0,l} & \\ v_l = \partial_x v_l = 0 & \text{on } \{-l, l\} \times (0, T), \end{cases}$$

namely there exists  $v_l \in L_\infty((0, T_l); L_2(-l, l)) \cap L_2((0, T_l); \dot{W}_2^2(-l, l))$  such that, after null-extending  $v_l$  from  $(-l, l)$  to  $\mathbb{R}$ , we have

$$\int_0^\infty \int (v_l \phi_t - (\partial_{xx} v_l) \phi_{xx} - (\partial_x v_l)^2 \phi_{xx}) = \int v_{0,l} \phi(0) \tag{6.1}$$

for every  $\phi \in C_0^\infty((-1, T_l) \times (-l, l))$ , and

$$\begin{aligned} \int_0^\infty \int (\partial_{xx} v_l)^2 \phi &\leq \int_0^\infty \int \left( \frac{1}{2} (\phi_t - \phi_{xxxx}) v_l^2 + 2 (\partial_x v_l)^2 \phi_{xx} \right. \\ &\quad \left. - \frac{5}{3} (\partial_x v_l)^3 \phi_x - (\partial_x v_l)^2 v_l \phi_{xx} \right) \end{aligned} \tag{6.2}$$

for every  $\phi \in C_0^\infty((-1, T_l) \times (-l, l))$  with  $\phi \geq 0$ , and

$$\|v_l(t)\|^2 + 2 \int_0^t \|\partial_{xx} v_l(\tau)\|^2 d\tau \leq \|v_{0,t}\|^2 \leq \|v_0\|^2 \tag{6.3}$$

for almost every  $t > 0$ . Here we wrote  $\|\cdot\| \equiv \|\cdot\|_{L_2(\mathbb{R})}$  for brevity. Consequently, there exists  $v$  such that

$$\begin{aligned} v_l &\overset{*}{\rightharpoonup} v && \text{in } L_\infty((0, \infty); L_2) \\ \partial_{xx} v_l &\rightharpoonup v_{xx} && \text{in } L_2((0, \infty); L_2) \end{aligned} \tag{6.4}$$

as  $l \rightarrow \infty$  and, via lower weak semi-continuity

$$\|v(t)\|^2 + 2 \int_0^t \|v_{xx}(\tau)\|^2 d\tau \leq \|v_0\|^2 \tag{6.5}$$

for almost every  $t > 0$ , in particular  $v \in L_\infty(0, \infty; L_2(\mathbb{R})) \cap L_2(0, \infty; W_2^2(\mathbb{R}))$ . The multiplicative inequality (2.3) controls via (6.5) the nonlinear term, and thus for any  $\phi \in C_0^\infty((-1, \infty) \times \mathbb{R})$  we can pass to the limit in (6.1). We obtained  $v$  satisfying the definition of energy weak solution to the Cauchy problem (1.2).

Let us show that  $v$  is a suitable weak solution. Fix any bounded cylinder  $Q_z(r) \subset (-L, L) \times (0, T)$ . Identity (6.1), the energy bound (6.3) and the reasoning as in proof of Remark 1.2 imply

$$\|v_t\|_{L_{\frac{5}{3}}(0,T;(W_{5/2}^2(-L,L))^*)} d\tau \leq C_{T,L,\|v_0\|}, \tag{6.6}$$

with the only difference now being  $C(L)$ , due to the Hölder inequality  $\|v_{xx}(t)\|_{L_{\frac{5}{3}}(-L,L)} \leq (2L)^{\frac{1}{5}} \|v_{xx}(t)\|_{L_2(-L,L)}$ . Applying the Aubin-Lions Lemma (see for example theorem 2.1 on p. 184 in [25]) with  $X_0 := H^2(-L, L)$ ,  $X := W_\infty^1(-L, L)$ ,  $X_1 := (W_{5/2}^2(-L, L))^*$ ,  $\alpha_0 := 2$ ,  $\alpha_1 := 5/3$ , we see that (along a subsequence)

$$v_l \rightarrow v \quad \text{in } L_2((0, T); W_\infty^1(-L, L)).$$

Thus we can take  $\lim_{l \rightarrow \infty}$  on the r.h.s. of (6.2) and  $\liminf_{l \rightarrow \infty}$  on its l.h.s. to obtain via l.w.s.c.

$$\int_0^L \int_{-L}^L v_{xx}^2 \phi \leq \int_0^L \int_{-L}^L \left( \frac{1}{2} (\phi_t - \phi_{xxxx}) v^2 + 2v_x^2 \phi_{xx} - \frac{5}{3} v_x^3 \phi_x - v_x^2 v_l \phi_{xx} \right) \tag{6.7}$$

for any  $\phi \in C_0^\infty(Q)$ . Applying a cut-off procedure in time, i.e. rewriting (6.7) for  $\phi(x,t)\chi(t)$ , with  $\chi$  being a bump function around  $t$ , we obtain that

$$\begin{aligned} & \frac{1}{2} \int v(t)^2 \phi(t) + \int_0^t \int v_{xx}^2 \phi \\ & \leq \int_0^t \int \left( \frac{1}{2} (\phi_t - \phi_{xxxx}) v^2 + 2v_x^2 \phi_{xx} - \frac{5}{3} v_x^3 \phi_x - v_x^2 v_t \phi_{xx} \right) \end{aligned} \tag{6.8}$$

for almost every  $t > 0$ . More precisely, for every Lebesgue point of  $f(t) := \int v(t)^2 \phi(t)$ , so the times  $t > 0$  where (6.8) holds depends on the choice of  $\phi$ . However, since  $v(t)$  is weakly continuous in  $L_2$  (by (iii), cf. Remark 1.2), the same is true of  $v(t)\sqrt{\phi(t)}$  and so we can extend (6.8) for all  $t > 0$ .  $\square$

**6.2. Proof of Lemma 2.1**

We show now a Campanato-type estimate for the linear equation (2.1). We may assume that  $u \in C^\infty(Q_{3/4})$ . Otherwise first we use the standard mollification  $u^{(\varepsilon)}$  and then our claim follows from taking the limit  $\varepsilon \rightarrow 0$  in (2.2).

Let  $\phi \in C_0^\infty(Q_{3/4})$  be such that  $\phi = 1$  on  $Q_{5/8}$ , multiplying (2.1) by  $u\phi$  and integrating by parts, we obtain

$$\int_{Q_{3/4}} u_{xx}^2 \phi = \int_{Q_{3/4}} \left( \frac{1}{2} u^2 (\phi_t - \phi_{xxxx}) + u_x^2 \left( 2\phi_{xx} + \frac{3\beta}{2} \phi_x \right) - \beta u_x u \phi_{xx} \right) \tag{6.9}$$

Thus

$$\|u_{xx}\|_{L_{2,2}(Q_{5/8})} \leq C_\beta \left( \|u\|_{L_{2,2}(Q_{3/4})} + \|u_x\|_{L_{2,2}(Q_{3/4})} \right). \tag{6.10}$$

Since any space derivative  $\partial_x^m u$  ( $m \geq 0$ ) satisfies (2.1) on  $Q_{3/4}$  and any mixed derivatives  $\partial_t^k \partial_x^m u$  ( $k, m \geq 0$ ) can be expressed in terms of pure space derivatives via (2.1) itself, we obtain

$$\|\partial_t^k \partial_x^m u\|_{L_{2,2}(Q_{1/2})} \leq C_{k,m,\beta} \left( \|u\|_{L_{2,2}(Q_{3/4})} + \|u_x\|_{L_{2,2}(Q_{3/4})} \right) \tag{6.11}$$

for any  $k \geq 0, m \geq 2$ , by bootstrapping the inequalities of type (6.10) on a sequence of decreasing cylinders. An embedding and (6.11) imply

$$\|u_x\|_{W_{\infty,\infty}^{1,0}(Q_{1/2})} \leq C_\beta \left( \|u\|_{L_{2,2}(Q_{3/4})} + \|u_x\|_{L_{2,2}(Q_{3/4})} \right).$$

Hence

$$\begin{aligned} \left( \int_{Q_\theta} |u_x - (u_x)_\theta|^p \right)^{1/p} &\leq \max_{Q_\theta} |u_x - (u_x)_\theta| = \max_{Q_\theta} |u_x - u_x(z_0)| \\ &\leq C\theta \|u_x\|_{W_{\infty,\infty}^{1,0}(Q_{1/2})} \\ &\leq C_\beta \theta \left( \|u\|_{L_{2,2}(Q_{3/4})} + \|u_x\|_{L_{2,2}(Q_{3/4})} \right) \end{aligned}$$

since by Darboux property there exists  $z_0 \in Q_\theta$  such that  $u_x(z_0) = (u_x)_\theta$ . It remains to estimate  $\|u\|_{L_{2,2}(Q_{3/4})}$  in terms of  $\|u_x\|_{L_{2,2}(Q_1)}$  above. To this end, we introduce  $\hat{u}$  in place of  $u$  in (6.9) and along the lines of our proofs of Caccioppoli inequalities (3.5) or (5.7) (this case is easier, since problem is linear and solutions are smooth). Next we repeat the above proof with  $\hat{u}$ .

## 7. Financial Support

J. Burczak was supported by MNiSW ‘Mobilność Plus’ grant 1289/MOB/IV/2015/0. W. S. Ożański was supported by postdoctoral funding from ERC 616797, the AMS Simons Travel Grant and by funding from Charles Simonyi Endowment at the Institute for Advanced Study. G. Seregin was supported by the grant RFBR 20-01-00397.

## References

- 1 D. Blömker, F. Flandoli and M. Romito. Markovianity and ergodicity for a surface growth PDE. *Ann. Probab.* **37** (2009), 275–313.
- 2 D. Blömker and C. Gugg. Thin-film-growth-models: on local solutions, in *Recent developments in stochastic analysis and related topics* Sergio Albeverio, Zhi-Ming Ma and Michael Roeckner (eds), pp. 66–77 (Hackensack, NJ: World Sci. Publ., 2004).
- 3 D. Blömker, C. Gugg and M. Raible. Thin-film-growth models: roughness and correlation functions. *European J. Appl. Math.*, **13** (2002), 385–402.
- 4 D. Blömker and M. Hairer. Stationary solutions for a model of amorphous thin-film growth’. *Stochastic Anal. Appl.* **22** (2004), 903–922.
- 5 D. Blömker and M. Romito. Regularity and blow up in a surface growth model. *Dyn. Partial Differ. Equ.* **6** (2009), 227–252.
- 6 D. Blömker and M. Romito. Local existence and uniqueness in the largest critical space for a surface growth model. *Nonlinear Differ. Equ. Appl.* **19** (2012), 365–381.
- 7 D. Blömker and M. Romito. Stochastic PDEs and lack of regularity: a surface growth equation with noise: existence, uniqueness, and blow-up’. *Jahresber. Dtsch. Math.-Ver.* **117** (2015), 233–286.
- 8 S. Campanato. Proprietà di hölderianità di alcune classi di funzioni. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **17** (1963), 175–188.
- 9 V. N. Gabushin. Inequalities for norms of a function and its derivatives in lp metrics. *Mat. Zam.* **1** (1967), 291–298.
- 10 M. Giaquinta and M. Struwe. On the partial regularity of weak solutions of nonlinear parabolic systems. *Math. Z.* **179** (1982), 437–451.

- 11 R. H. W. Hoppe and E. M. Nash. A combined spectral element/finite element approach to the numerical solution of a nonlinear evolution equation describing amorphous surface growth of thin films. *J. Numer. Math.* **10** (2002), 127–136.
- 12 M. K. Kwong and A. Zettl. *Norm Inequalities for Derivatives and Differences* (Berlin Heidelberg: Springer, 1993).
- 13 O. A. Ladyzhenskaya and G. A. Seregin. On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations. *J. Math. Fluid Mech.* **1** (1999), 356–387.
- 14 F. Lin. A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Comm. Pure Appl. Math.* **51** (1998), 241–257.
- 15 W. S. Ożański. A sufficient integral condition for local regularity of solutions to the surface growth model. *J. Funct. Anal.* **276** (2019), 2990–3013.
- 16 W. S. Ożański and J. C. Robinson. Partial regularity for a surface growth model. *SIAM J. Math. Anal.* **51** (2019), 228–255.
- 17 M. Pokorný. Navier–Stokes Equations’, available online at [https://www.karlin.mff.cuni.cz/~pokorny/NavierandStokes\\_eng.pdf](https://www.karlin.mff.cuni.cz/~pokorny/NavierandStokes_eng.pdf), 2020.
- 18 M. Raible, S. J. Linz and P. Hänggi. Amorphous thin film growth: Minimal deposition equation. *Phys. Rev. E* **62** (2000), 1691–1694.
- 19 M. Raible, S. G. Mayr, S. J. Linz, M. Moske, P. Hänggi and K. Samwer. Amorphous thin film: Theory compared with experiment. *Europhys. Lett.* **50** (2000), 61–67.
- 20 J. C. Robinson, J. L. Rodrigo and W. Sadowski. *The three-dimensional Navier-Stokes equations*, Vol. 157 of *Cambridge Studies in Advanced Mathematics* (Cambridge: Cambridge University Press, 2016).
- 21 G. Seregin. *Lecture Notes on Regularity Theory for the Navier-Stokes Equations*, WSPC, 2014.
- 22 G. A. Seregin, L. Silvestre, V. Šverák and A. Zlatoš. On divergence-free drifts. *J. Differential Equations* **252** (2012), 505–540.
- 23 M. Siegert and M. Plischke. Solid-on-solid models of molecular-beam epitaxy. *Physical Review E* **50** (1994), 917–931.
- 24 O. Stein and M. Winkler. Amorphous molecular beam epitaxy: global solutions and absorbing sets. *European J. Appl. Math.* **16** (2005), 767–798.
- 25 R. Temam. *Navier-Stokes equations, theory and numerical analysis* (Providence, RI: AMS Chelsea Publishing, 2001). Reprint of the 1984 edition.