Journal of the Inst. of Math. Jussieu (2002) **1**(2), 175–277 © Cambridge University Press 175
DOI:10.1017/S1474748002000051 Printed in the United Kingdom

A STABLE TRACE FORMULA. I. GENERAL EXPANSIONS

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(Received 20 June 2000; accepted 8 November 2000)

Abstract This is the first of three articles designed to stabilize the global trace formula. The results apply to any group for which the fundamental lemma (and its variants for weighted orbital integrals) is valid. The main purpose of this paper is to establish a series of expansions that are parallel to the expansions in the trace formula. We shall also formulate the local and global theorems required to interpret the terms in these expansions. The proofs of the theorems will be given in the subsequent two articles. The expansions of this paper will then yield both a stable trace formula, and a decomposition of the ordinary trace formula into a linear combination of stable trace formulae.

Keywords: automorphic representations; trace formula; (weighted) orbital integrals; (weighted) characters; stable conjugacy; transfer

AMS 2000 *Mathematics subject classification:* Primary 22E55
Secondary 11R39

Introduction

This paper is the first of three articles designed to stabilize the trace formula. The goal is to stabilize the global trace formula for a general connected group, subject to a condition on the fundamental lemma that has been established in some special cases. The role of this paper will be to investigate the underlying structure of the process. We shall establish a series of expansions that are parallel to the expansions on each side of the trace formula. In the subsequent articles, we shall show that these expansions provide both a stable trace formula and a decomposition of the ordinary trace formula into a linear combination of stable trace formulae.

We ought to stand back for a moment in order to recall some of the reasons for studying the trace formula and its stabilization. In fact, it might be a good idea to begin with a brief historical introduction to the problem of stabilization. We will then be in good position to outline the contents of the paper. More detailed descriptions of the results will be given later, in the remarks that introduce individual sections.

Suppose that G is a connected reductive group over a number field F. One can form the group of points in G with values in the adele ring $\mathbb{A}$ of F. This gives a locally compact group $G(\mathbb{A})$, in which $G(F)$ embeds diagonally as a discrete subgroup. Automorphic representation theory is the study of the regular representation of $G(\mathbb{A})$ on the Hilbert space

$L^2(G(F)\backslash G(\mathbb{A}))$. Automorphic representations of $G(\mathbb{A})$ are the irreducible constituents of this representation, and are thought to carry fundamental arithmetic information. One can investigate their properties by applying methods of harmonic analysis to the decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$.

The trace formula is an analogue for the quotient $G(F)\backslash G(\mathbb{A})$ of the familiar Poisson summation formula. In general, the decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$ into irreducible representations has both continuous and discrete spectra. The general trace formula is therefore quite complicated. It is the identity given by two different expansions of a certain linear form $I(f)$, where f is an appropriate test function. The geometric expansion is a linear combination of distributions

$$I(f) = \sum_M |W_0^M||W_0^G|^{-1} \sum_\gamma a^M(\gamma) I_M(\gamma, f), \tag{0.1}$$

parametrized by conjugacy classes γ in Levi subgroups M of G. The spectral expansion is a linear combination of distributions

$$I(f) = \sum_M |W_0^M||W_0^G|^{-1} \int a^M(\pi) I_M(\pi, f)\, \mathrm{d}\pi, \tag{0.2}$$

parametrized by representations π of Levi subgroups M. One can try to gain information about the terms in (0.2) by studying the terms in (0.1).

Some of the terms in the two expansions are easy to describe. For example, suppose that $M = G$, and that γ is a semisimple elliptic conjugacy class in $G(F)$. The corresponding term on the geometric side is then equal to the product of the volume

$$a^G(\gamma) = \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A})^1),$$

with Harish-Chandra's invariant orbital integral

$$f_G(\gamma) = I_G(\gamma, f) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, \mathrm{d}x, \quad f \in C_c^\infty(G(\mathbb{A})).$$

As is customary, we have written G_γ for the connected centralizer in G of a representative of γ. Similarly, suppose that $M = G$ again, and that π is an irreducible representation of $G(\mathbb{A})$ that does not occur in any of the continuous spectra. The corresponding term on the spectral side is then equal to the product of the multiplicity

$$a^G(\pi) = m(\pi)$$

of π in the discrete spectrum, with the character

$$f_G(\pi) = I_G(\pi, f) = \mathrm{tr}(\pi(f)), \quad f \in C_c^\infty(G(\mathbb{A})),$$

of π. If G is anisotropic over F, or equivalently if $G(F)\backslash G(\mathbb{A})$ is compact, these are the only terms. For in this case, $M = G$ is the only Levi subgroup. Moreover, the elements in

$G(F)$ are all semisimple elliptic, and the entire spectrum is discrete. The identity between (0.1) and (0.2) reduces to Selberg's original trace formula

$$\sum_{\gamma} \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, \mathrm{d}x = \sum_{\pi} m(\pi)\, \mathrm{tr}(\pi(f)). \qquad (0.3)$$

The analogy with the Poisson summation formula is clear. Unfortunately, it is not possible to remain within the category of anisotropic groups, even if one were interested only in these groups. However, the identity (0.3) is very useful for suggesting how one might go about applying the general trace formula.

If G is allowed to vary, the arithmetic data wrapped up in the associated families of automorphic representations are not independent. In fact, it is believed that there are fundamental relationships among automorphic representations of different groups. These are summarized by Langlands's principle of functoriality, a far reaching conjecture that includes a non-abelian generalization of class field theory. The trace formula seems to be the most powerful tool for attacking those aspects of functoriality that are at all accessible. The general strategy is to use the structure theory of algebraic groups to transfer conjugacy classes γ between different groups. One would then hope to define corresponding relationships among terms in the geometric expansions. Since the geometric expansion (0.1) equals the spectral expansion (0.2), for any given G, this ought to imply relationships among the terms in the spectral expansions.

The strategy was carried out by Jacquet and Langlands [**19**, § 16] for the groups D^* and GL(2), where D is a quaternion algebra over F. Langlands then considered the problem for general G. By studying how to transfer the geometric terms in (0.3), he was lead to some remarkable new ideas. Langlands published his results in the monograph [**28**], where he outlined a general program for transfer that has subsequently become known as the theory of endoscopy. He also gave a solution of the problem, subject to some conjectural local conditions, for the semisimple elliptic terms in the trace formula that are strongly regular. These are the geometric terms in (0.3) for which the centralizer of γ in G is a torus. Kottwitz [**22**] later extended Langlands's results to elliptic singular terms. At the same time, Shelstad had been working on the local foundations of the theory of endoscopy. The culmination of her work appeared in the paper [**35**], where she solved the local problems for archimedean fields.

The theory of endoscopy is based on the notion of a stable distribution. Two strongly regular elements in a local factor $G(F_v)$ of $G(\mathbb{A})$ are said to be stably conjugate if they are conjugate over an algebraic closure $G(\bar{F}_v)$. Stable conjugacy reduces to conjugacy in the groups D^* and GL(2) above, but is usually a weaker equivalence relation. In general, if δ_v is a stable conjugacy class of strongly regular elements in $G(F_v)$, one defines the stable orbital integral

$$f_v^G(\delta_v) = \sum_{\gamma_v} f_{v,G}(\gamma_v), \quad f_v \in C_c^\infty(G(F_v)),$$

where γ_v is summed over the finite set of $G(F_v)$-conjugacy classes in the stable class δ_v, and $f_{v,G}(\gamma_v)$ is the orbital integral of f_v at γ_v. A distribution on $G(F_v)$ is said to be stable if it lies in the closed linear span of the stable orbital integrals $f_v^G(\delta_v)$. It is easy to explain the rationale for such a definition. The natural way to transfer elements between a pair of groups G and G', related for example by inner twisting, is through invariant theory. But invariant theory works in this context only over an algebraically closed field. One can therefore transfer only stable conjugacy classes. We should then expect only to be able to transfer stable orbital integrals and stable distributions.

On the other hand, the geometric expansion (0.1) is not generally a stable distribution on $G(\mathbb{A})$. That is, $I(f)$ is not generally a tensor product of stable distributions on the local components $G(F_v)$. For example, suppose that G is anisotropic, and that γ is a strongly regular elliptic element in $G(F)$. If $\gamma' = \prod_v \gamma'_v$ is an element in $G(\mathbb{A})$ such that each component γ'_v is stably conjugate to γ in $G(F_v)$, then γ' need not be $G(\mathbb{A})$-conjugate to an element in $G(F)$. In other words, the orbital integral at the stable conjugate γ' of γ need not occur in (0.3), even though the orbital integral at γ does. It follows easily that the left-hand side $I(f)$ of (0.3) is not generally a stable distribution.

Given this background, we can pose the general problem informally as a question. Can one write $I(f)$ as the sum of a stable distribution together with an explicit error term? At this point of the discussion, we assume for simplicity that the derived group of G is simply connected. Langlands's study of the regular elliptic terms lead him to attach a family of quasisplit groups $\{G'\}$ to G. These objects are known as endoscopic groups for G, or more properly endoscopic data for G, since they come with extra structure. For any G', Langlands also defined a conjectural correspondence

$$f = \prod_v f_v \to f' = \prod_v f'_v,$$

from functions $f \in C_c^\infty(G(\mathbb{A}))$ to functions $f' \in C_c^\infty(G'(\mathbb{A}))$, where f' is determined only up to the values taken by its stable orbital integrals. For example, the quasisplit inner form G^* of G is the largest of its endoscopic groups. In this case, the transfer f^* of f is defined by the stable orbital integrals of the local components f_v of f. In particular, if S^* is a stable distribution on $G^*(\mathbb{A})$, and $\hat{S}^*$ is the corresponding linear form on the space of stable orbital integrals on $G^*(\mathbb{A})$, the distribution $f \to \hat{S}^*(f^*)$ on $G(\mathbb{A})$ is stable.

The general problem of stabilization can now be stated more precisely as follows. Given G, find a decomposition

$$I(f) = \sum_{G'} \iota(G, G')\hat{S}'(f'), \tag{0.4}$$

for stable distributions $S' = S^{G'}$ on the endoscopic groups G' for G. The coefficient $\iota(G, G')$ comes with a simple formula [**21**, Theorem 8.3.1], and equals 1 if $G' = G^*$. The decomposition one seeks can therefore be written as

$$I(f) = \hat{S}^*(f^*) + \sum_{G' \neq G^*} \iota(G, G')\hat{S}'(f').$$

The distribution $f \to \hat{S}^*(f^*)$ is to be regarded as the stable part of $I(f)$, while the summands with $G' \neq G^*$ can be considered the error terms. The essential point is that S' is assumed to depend only on the group G', rather than the group G from which

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G' arises as an endoscopic datum. Given the conjectural local correspondence $f \to f'$, the construction of the decomposition (0.4) is then a well-defined problem. One would assume inductively the existence of the distribution S' for each elliptic endoscopic datum $G' \neq G$. If G is quasisplit, in which case we take $G^* = G$, we simply define

$$S^G(f) = I(f) - \sum_{G' \neq G^*} \iota(G, G')\hat{S}'(f').$$

The problem in this case is to show that the right-hand side is stable. If G is not quasisplit, all of the terms on the right-hand side of (0.4) are given by the inductive definition. The decomposition (0.4) then takes the role of an identity to be proved.

There seems to be no direct way to establish (0.4). One has first to establish corresponding decompositions for each of the terms in the geometric expansion (0.1). Langlands and Kottwitz treated the terms with $M = G$ and γ semisimple, as we have already mentioned, and in the process laid down foundations in Galois cohomology. In §§ 6 and 7, we shall describe the decompositions that would have to be established for the remaining terms.

The reason for trying to establish a decomposition (0.4) is to gain information about the coefficients $a^G(\pi)$ in the spectral expansion. Such information would take the form of identities that relate the spectral coefficients with corresponding coefficients for endoscopic groups. In fact, we should expect decompositions for each of the terms in (0.2) that are completely parallel to the decompositions of the terms in (0.1). These will also be described in §§ 6 and 7. We shall formulate the various problems as a series of theorems, corresponding to each of the terms in the trace formula. The theorems that apply to the distributions $I_M(\gamma, f)$ and $I_M(\pi, f)$ are essentially local in nature, and will be stated in § 6. The theorems that apply to the coefficients $a^M(\gamma)$ and $a^M(\pi)$ are global, and will be stated in § 7. Taken together, the theorems represent a stabilization of all the terms in the trace formula. They are our main results, and will not be proved until the last of the three articles.

The test function f will actually be taken from a Hecke algebra on the subgroup

$$G_V = G(F_V) = \prod_{v \in V} G(F_v)$$

of $G(\mathbb{A})$, where V is a finite set of valuations on F that contains the ramified places. This represents a slight departure from earlier papers [**2**, **8**]. It is also best to build the invariant linear form $I(f)$ out of the canonical weighted characters of [**11**], rather than the weighted characters in [**8**] that depend on normalizing factors for intertwining operators. This simplifies the stabilization problem. In particular, it frees us from having to compare normalizing factors attached to different endoscopic groups. In § 2, we shall derive the geometric expansion (0.1) from the earlier expansion of [**8**, § 3]. In § 3, we shall derive the spectral expansion (0.2) from the expansion of [**8**, § 4]. Sections 2 and 3 both depend on some simple notions introduced in § 1. These include an abstract basis $\Gamma(M_V, \zeta_V)$, which will be used in place of conjugacy classes on M_V, and which is better suited to endoscopic transfer.

In § 4, we shall recall the Langlands–Shelstad transfer factors $\Delta(\delta', \gamma)$. These objects are defined for strongly regular conjugacy classes γ in G_V and strongly G-regular stable conjugacy classes δ' in an endoscopic group G'_V. They play the role of a transformation matrix in the definition

$$f'(\delta') = \sum_{\gamma} \Delta(\delta', \gamma) f_G(\gamma)$$

of the conjectural transfer mapping. Since V contains the ramified places, the function $\Delta(\delta', \gamma)$ is really a global object. It is independent of the choice of base point that is part of the definition of local transfer factors. From § 4 on, it will be convenient to let G stand for a slightly more general object, which we call a global K-group. The transfer factors then satisfy adjoint relations (4.7) and (4.8) that allow us to invert the transfer mappings $f \to f'$.

The Langlands–Shelstad transfer conjecture asserts that for any f, $f'(\delta')$ represents the stable orbital integral of a suitable function on G'_V. Waldspurger [**38**] has reduced this conjecture to the fundamental lemma. The fundamental lemma may in turn be regarded as a variant of the transfer conjecture for unramified places. We shall impose it, in a generalized form that applies to weighted orbital integrals, as a hypothesis on G (Assumption 5.2). The hypothesis is known to hold in a limited number of cases, which include the groups GSp(4), SO(5) and SO(4) of rank 2. In particular, it is valid for the classical groups whose representations one would hope to classify in terms of those of GL(4). After introducing the hypothesis in the first part of § 5, we shall then describe some consequences of the transfer conjecture.

The last three sections of the paper represent the first stage of the proof of the theorems. In § 8, we shall deal with the unramified terms in the trace formula. These terms do not appear explicitly in the expansions (0.1) and (0.2). They are actually buried in the definitions of the coefficients $a^M(\gamma)$ and $a^M(\pi)$ in §§ 2 and 3. They have nonetheless to be stabilized. The unramified geometric terms are taken care of by the generalized fundamental lemma. The fundamental lemma is thus required here in its own right, as well as for the Langlands–Shelstad transfer conjecture. The stabilization of the unramified spectral terms is not so deep. It is provided by the combinatorial identity of the paper [**13**].

The main results of the paper are contained in the final two sections. In § 10, we shall establish the expansions mentioned at the beginning of the introduction. The argument in this last section relies at a key point on the cancellation of certain terms obtained by transfer from the original trace formula. We shall establish the required cancellation in § 9. The result, Theorem 9.1, is the global analogue of the local vanishing theorem [**12**, Theorem 8.3]. It bears the same relation to the global trace formula as the latter does to the local trace formula. Like its local counterpart, Theorem 9.1 depends ultimately on some internal signs in the Langlands–Shelstad transfer factors. However, it is somewhat more delicate, for reasons having to do with the rational global base point.

The expansions of § 10 are either geometric or spectral in nature. They can be divided along another line as well. In common with the objects in §§ 6 and 7, the expansions of § 10 separate into two categories that we call 'endoscopic' and 'stable'.

The endoscopic expansions are stated in parts (a) of Theorems 10.1 and 10.6. They come from the right-hand side of (0.4), and represent a decomposition of the trace formula into a linear combination of stable trace formulae. Parts (a) of the theorems in §§ 6 and 7 apply to the constituents of these expansions. They assert term by term identities between the endoscopic expansions and the corresponding expansions (0.1) and (0.2). In particular, the theorems imply that the 'endoscopic' trace formula, obtained by identifying the two endoscopic expansions, reduces to the ordinary trace formula. Of course this is after the fact. The endoscopic trace formula will have a central role to play in the proof of the theorems.

The stable expansions are restricted to the case that G is quasisplit. They are stated in parts (b) of Theorems 10.1 and 10.6, and represent two different expansions of the leading term $S^G(f) = \hat{S}^{G^*}(f^*)$ on the right-hand side of (0.4). Parts (b) of the theorems in §§ 6 and 7 apply to the terms in these expansions. They imply a reduction of the stable expansions to expressions

$$S(f) = \sum_M |W_0^M||W_0^G|^{-1} \sum_\delta b^M(\delta) S_M(\delta, f) \tag{0.5}$$

and

$$S(f) = \sum_M |W_0^M||W_0^G|^{-1} \int b^M(\phi) S_M(\phi, f)\, \mathrm{d}\phi \tag{0.6}$$

that are completely parallel to (0.1) and (0.2). The theorems also assert that the distributions $S_M(\delta, f)$ and $S_M(\phi, f)$ in these expansions are stable. The identity obtained from the right-hand sides of (0.5) and (0.6) will thus be an explicit formula whose terms are stable distributions. It is the stable trace formula we are looking for.

It would have been better to stabilize the more general twisted trace formula. This ought to be within reach, given the results of Kottwitz and Shelstad [**24**] and Labesse [**25**]. However, there are still a number of properties for twisted groups that remain to be established. Rather than write a series of papers that depend on more than the fundamental lemma, it seemed advisable at this time just to deal with the standard trace formula.

1. Functions and distributions

Throughout the paper, F will be a field of characteristic 0, and we shall often write $\Gamma = \mathrm{Gal}(\bar{F}/F)$ for the Galois group of an algebraic closure $\bar{F}$ over F. For the time being, we take G to be a *connected*, reductive algebraic group of F. We write A_G for the F-split component of the centre of G, and we set

$$\mathfrak{a}_G = \mathrm{Hom}(X(G)_F, \mathbb{R}).$$

Then $\mathfrak{a}_G$ is a real vector space, of dimension equal to that of the torus A_G. If c belongs to G, we shall write $G_{c,+}$ for the centralizer of c in G. This leaves the symbol G_c free to denote the connected component of 1 in $G_{c,+}$. A semisimple element c in $G(F)$ is said to be elliptic (over F) if $A_{G_c} = A_G$.

By a Levi subgroup M of G, we mean an F-rational Levi component of a parabolic subgroup of G over F. For any such M,

$$W(M) = W^G(M) = \mathrm{Norm}_G(M)/M$$

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denotes the Weyl group of (G, A_M). We also follow the standard notation of writing $\mathcal{L}(M) = \mathcal{L}^G(M)$ for the finite set of Levi subgroups of G that contain M, and $\mathcal{L}^0(M)$ for the complement of G in $\mathcal{L}(M)$. Similarly, $\mathcal{F}(M) = \mathcal{F}^G(M)$ stands for the finite set of parabolic subgroups

$$P = M_P N_P, \quad M_P \in \mathcal{L}(M),$$

over F that contain M, and

$$\mathcal{P}(M) = \mathcal{P}^G(M) = \{P \in \mathcal{P}(M) : M_P = M\}$$

is the subset of parabolic subgroups in $\mathcal{F}(M)$ with Levi component M. We shall frequently assume that we have fixed a minimal Levi subgroup M_0 of G, in which case we write $W_0 = W_0^G = W^G(M_0)$, $\mathcal{L} = \mathcal{L}^G = \mathcal{L}(M_0)$ and $\mathcal{L}^0 = \mathcal{L}^0(M_0)$.

For the rest of the paper, F will actually be a local or a global field. For purposes of induction, it will be convenient to fix a pair (Z, ζ) as in [**12**]. Then Z is a central induced torus in G over F, whose quotient G/Z we shall denote by $\bar{G}$. The second component ζ is a character on $Z(F)$ if F is local, and an automorphic character of $Z(\mathbb{A})$ if F is global. In the case of F global, we shall write $V_{\mathrm{ram}}(G, \zeta)$ for the finite set of valuations of F at which G, Z or ζ ramify. This set contains V_∞, the subset of archimedean valuations.

Suppose now that F is global. We shall be concerned with the trace formula on $G(\mathbb{A})$. However, the introduction of the pair (Z, ζ) forces us to work in a slightly different setting from [**8**]. For any connected reductive subgroup H of G over F, there is a canonical map from $\mathfrak{a}_H$ to $\mathfrak{a}_G$. There is also the usual canonical map H_G from $G(\mathbb{A})$ to $\mathfrak{a}_G$. If Δ is a subset of $G(\mathbb{A})$, we shall write

$$\Delta^H = \{x \in \Delta : H_G(x) \in \mathrm{image}(\mathfrak{a}_H \to \mathfrak{a}_G)\}.$$

For example, $G(\mathbb{A})^G = G(\mathbb{A})^{M_0} = G(\mathbb{A})$. In the opposite extreme of $H = 1$, we have

$$G(\mathbb{A})^1 = \{x \in G(\mathbb{A}) : H_G(x) = 0\}.$$

This matches the notation of [**8**]. Observe that if H contains Z, $G(\mathbb{A})^H$ contains $Z(\mathbb{A})$. For most of the paper, we shall in fact take $H = Z$.

Suppose that V is a finite set of valuations of F. For simplicity, we generally write

$$G_V = G(F_V) = \prod_{v \in V} G(F_v)$$

for the group of points in G with values in the ring

$$F_V = \prod_{v \in V} F_v.$$

This notation can of course be applied to the quotient $\bar{G} = G/Z$, and since Z is an induced torus, the group $\bar{G}_V = (G/Z)(F_V)$ equals G_V/Z_V [**20**, Lemma 1.1(3)]. We also write ζ_V for the restriction of ζ to the subgroup Z_V of $Z(\mathbb{A})$. If $H \subset G$ is as above, we write $\mathcal{C}(G_V^H, \zeta_V)$ for the space of ζ_V^{-1}-equivariant Schwartz functions on G_V^H. We shall usually confine ourselves to functions in the Hecke algebra $\mathcal{H}(G_V^H, \zeta_V)$. This is a subalgebra of $\mathcal{C}(G_V^H, \zeta_V)$, which depends on a choice of maximal compact subgroup $K_\infty = K_{V_\infty}$ of G_{V_∞}. We shall also generally assume that V contains $V_{\mathrm{ram}}(G,\zeta)$, and $H = Z$. To emphasize this special case, we write

$$\mathcal{H}(G,V,\zeta) = \mathcal{H}(G_V^Z, \zeta_V).$$

In §§ 2 and 3, we are going to reformulate the global trace formula as an identity of linear forms on $\mathcal{H}(G,V,\zeta)$. As such, it will depend on a choice of maximal compact subgroup of $G(\mathbb{A}^V)$, where $\mathbb{A}^V$ denotes the subring of elements in $\mathbb{A}$ that are equal to zero at each $v \in V$.

Let

$$K^{\mathrm{ram}} = \prod_{v \in V_{\mathrm{ram}}(G,\zeta)} K_v$$

be a fixed, open maximal compact subgroup of $G(\mathbb{A}^{V_{\mathrm{ram}}(G,\zeta)})$. We assume that each K_v is hyperspecial, and in good position relative to some underlying minimal Levi subgroup M_0. If V is a finite set of valuations that contains $V_{\mathrm{ram}}(G,\zeta)$, $K^V = \prod_{v \notin V} K_v$ then is a maximal compact subgroup of $G^V = G(\mathbb{A}^V)$. We shall generally reserve the symbol f for a function on G_V. This leaves us in need of other notation for functions on $G(\mathbb{A})$. Let $u^V = u^{V,\zeta}$ be the function on G^V, with support equal to $K^V Z^V$, such that

$$u^V(kz) = \zeta(z)^{-1}, \quad k \in K^V, \quad z \in Z^V.$$

The map

$$f \to \dot{f} = f \times u^V, \quad f \in \mathcal{H}(G,V,\zeta),$$

then sends functions in $\mathcal{H}(G,V,\zeta)$ to functions in the space $\mathcal{H}(G,\zeta) = \mathcal{H}(G(\mathbb{A})^Z, \zeta)$. In particular, if $Z = 1$, and f belongs to the space $\mathcal{H}(G,V) = \mathcal{H}(G_V^Z)$, $\dot{f} = f \times u^V$ is a function in the space $\mathcal{H}(G) = \mathcal{H}(G(\mathbb{A})^1)$. We shall use similar notation if S is some finite set of valuations that contains V. Then $K_S^V = \prod_{v \in S-V} K_v$ is a maximal compact subgroup of $G_S^V = \prod_{v \in S-V} G(F_v)$, and $u_S^V = u_S^{V,\zeta}$ denotes the function on G_S^V, with support $K_S^V Z_S^V$, such that

$$u_S^V(kz) = \zeta(z)^{-1}, \quad k \in K_S^V, \quad z \in Z_S^V.$$

In this case,

$$f \to \dot{f}_S = f \times u_S^V, \quad f \in \mathcal{H}(G,V,\zeta),$$

is a map from $\mathcal{H}(G,V,\zeta)$ to $\mathcal{H}(G,S,\zeta)$. In the study of the geometric side of the trace formula, S will be a large finite set of valuations that depends on the support of a given f.

We need to define a certain subspace of $\mathcal{H}(G,S,\zeta)$. The polynomial

$$\det(1 + t - \mathrm{Ad}(x)) = \sum_{k=0}^{d} D_k(x) t^k, \quad x \in G,$$

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provides a morphism

$$\mathcal{D} = (D_0, \dots, D_d) : G \to (\mathbb{G}_a)^{d+1}, \quad d = \dim G,$$

over F from G to affine $(d+1)$-space. If $X = (X_0, \dots, X_d)$ is a non-zero point in $(\mathbb{G}_a)^{d+1}$, set $X_{\min} = X_k$, where k is the smallest integer with $X_k \neq 0$. Then

$$D(x) = \mathcal{D}(x)_{\min}, \quad x \in G,$$

is the generalized Weyl discriminant of G. The minimal k in this case is of course bounded below by the rank of G. If S is a finite set of valuations that contains V_∞, set

$$\mathfrak{o}^S = \bigoplus_{v \notin S} \mathfrak{o}_v,$$

where $\mathfrak{o}_v$ is the ring of integers in F_v. We shall say that a subset C_S of $F_S^{d+1} - \{0\}$ is *admissible* if any point X in the intersection

$$F^{d+1} \cap (C_S \times (\mathfrak{o}^S)^{d+1}),$$

has the property that $|X_{\min}|_v = 1$ for each $v \notin S$. Assume now that S contains $V_{\mathrm{ram}}(G, \zeta)$, and that

$$Z(\mathbb{A}) = Z(F) Z_S Z(\mathfrak{o}^S).$$

We shall say that a subset Δ_S of G_S is *admissible* if $\mathcal{D}(\Delta_S)$ is admissible in F_S^{d+1}. This condition implies that

$$|D(\dot\gamma)|_v = 1, \quad \dot\gamma \in G(F) \cap (\Delta_S \times K^S), \quad v \notin S.$$

It is clear that if Δ_S is admissible, so is the larger set obtained by taking G_S-stable conjugates of elements in Δ_S. The same is true of the set $\Delta_S Z_S$. In particular, Δ_S is admissible if and only if its projection $\bar\Delta_S$ onto $\bar G_S = G_S/Z_S$ is admissible. We shall write $\mathcal{H}_{\mathrm{adm}}(G, S, \zeta)$ for the subspace of functions in $\mathcal{H}(G, S, \zeta)$ whose support is an admissible subset of G_S.

There is an adelic variant of the notion of admissibility, which we can apply to subsets Δ of $G(\mathbb{A})$. We shall say that Δ is *S-admissible,* for a finite set S as above, if there is an admissible subset C_S of F_S^{d+1} such that $\mathcal{D}(\Delta)$ is contained in $C_S \times (\mathfrak{o}^S)^{d+1}$. We claim that any compact subset Δ of $G(\mathbb{A})$ is S-admissible, for some S. To see this, we first embed Δ in a compact set of the form $\Delta_V \times K^V$, for a finite set of valuations $V \supset V_{\mathrm{reg}}(G, \zeta)$. We then choose $S \supset V$ such that $|X_{\min}|_v$ equals 1, for each $v \notin S$, and for every X in the finite set

$$F^{d+1} \cap (\mathcal{D}(\Delta_V) \times (\mathfrak{o}^V)^{d+1}).$$

We now recall some simple objects from the paper [**14**] that will eventually be required for the study of the geometric terms in the trace formula. For a finite set V, and $H \subset G$ as above, let $\mathcal{D}(G_V^H, \zeta_V)$ be the vector space of distributions D on G_V^H that satisfy the following three conditions.

(i) D is invariant under conjugation by G^H_V.

(ii) D is ζ_V-equivariant under translation by Z_V.

(iii) D is supported on the preimage in G^H_V of a finite union of conjugacy classes in $\bar{G}^H_V = G^H_V/Z_V$.

Suppose that c belongs to the set $\Gamma_{\mathrm{ss}}(\bar{G}^H_V)$ of semisimple conjugacy classes in $\bar{G}^H_V$. We write $\mathcal{D}_c(G^H_V, \zeta_V)$ for the subspace of distributions D in $\mathcal{D}(G^H_V, \zeta_V)$ for which the conjugacy classes in (iii) all have semisimple parts equal to c. This space could be zero. However, it is easy to characterize the subset $\Gamma_{\mathrm{ss}}(\bar{G}^H_V, \zeta_V)$ of classes c in $\Gamma_c(\bar{G}^H_V)$ such that $\mathcal{D}_c(G^H_V, \zeta_V)$ is non-zero. It consists of images of semisimple conjugacy classes in G^H_V whose stabilizer in Z_V lies in the kernel of ζ_V. The original space obviously has a direct sum decomposition

$$\mathcal{D}(G^H_V, \zeta_V) = \bigoplus_c \mathcal{D}_c(G^H_V, \zeta_V)$$

over the classes c in $\Gamma_{\mathrm{ss}}(\bar{G}^H_V, \zeta_V)$. We shall say that a distribution in $\mathcal{D}(G^H_V, \zeta_V)$ is *unipotent* if it lies in the subspace $\mathcal{D}_{\mathrm{unip}}(G^H_V, \zeta_V) = \mathcal{D}_1(G^H_V, \zeta_V)$ of $\mathcal{D}(G^H_V, \zeta_V)$. In general, one can define the *semisimple* part of any $D \in \mathcal{D}(G^H_V, \zeta_V)$ to be the union of those classes $c \in \Gamma_{\mathrm{ss}}(\bar{G}^H_V)$ for which the image of D in $\mathcal{D}_c(G^H_V, \zeta_V)$ is non-zero. If V contains V_∞, we define D to be *admissible* if its semisimple part is admissible, in the sense defined above.

The space $\mathcal{D}(G^H_V, \zeta_V)$ contains the familiar invariant orbital integrals. Suppose that $\gamma_V = \prod_{v \in V} \gamma_v$ belongs to G^H_V, and that f is a smooth function of compact support on G^H_V. The orbital integral of f at γ_V is defined as

$$f_G(\gamma_V) = |D(\gamma_V)|^{1/2} \int_{G_{\gamma_V} \cap G^H_V \backslash G^H_V} f(x^{-1}\gamma_V x)\, \mathrm{d}x,$$

for $G_{\gamma_V} = \prod_v G_{\gamma_v}$ and $|D(\gamma_V)|_V = \prod_v |D(\gamma_v)|_v$, and for some choice of invariant measure on the quotient $G_{\gamma_V} \cap G^H_V \backslash G^H_V$. We can also define the ζ_V-equivariant orbital integral at γ_V. It is the distribution

$$f \to \int_{Z_V} \zeta_V(z) f_G(z\gamma_V)\, \mathrm{d}z \tag{1.1}$$

in $\mathcal{D}(G^H_V, \zeta_V)$. We shall write $\mathcal{D}_{\mathrm{orb}}(G^H_V, \zeta_V)$ for the subspace of $\mathcal{D}(G^H_V, \zeta_V)$ spanned by distributions of this form. If V consists entirely of p-adic valuations, the theory of Shalika germs implies that $\mathcal{D}_{\mathrm{orb}}(G^H_V, \zeta_V)$ equals $\mathcal{D}(G^H_V, \zeta_V)$. If V contains archimedean places, however, one can also take radial derivatives of orbital integrals. In this case, $\mathcal{D}_{\mathrm{orb}}(G^H_V, \zeta_V)$ is a proper subspace of $\mathcal{D}(G^H_V, \zeta_V)$. The larger space is necessary for questions of endoscopic transfer.

To specify more general elements in $\mathcal{D}(G^H_V, \zeta_V)$ explicitly, one would have to introduce more elaborate notation. Rather than do so, we prefer simply to fix some basis of $\mathcal{D}(G^H_V, \zeta_V)$. This is the point of view of the paper [**14**]. It is not really much of a departure from the usual practice. For example, even orbital integrals depend on implicit choices of invariant measures on conjugacy classes. For the duration of this paper, we shall write $\Gamma(G_V^H, \zeta_V)$ for a fixed basis of the space $\mathcal{D}(G_V^H, \zeta_V)$.

We assume implicitly that the elements in $\Gamma(G_V^H, \zeta_V)$ have been chosen to satisfy various natural compatibility conditions. (See [**14**].) For example, $\Gamma(G_V^H, \zeta_V)$ is supposed to be a subset of a basis $\Gamma(G_V, \zeta_V)$ of $\mathcal{D}(G_V, \zeta_V)$. Moreover, any element γ in $\Gamma(G_V, \zeta_V)$ is assumed to have a decomposition

$$\gamma = \prod_{v \in V} \gamma_v, \quad \gamma_v \in \Gamma(G_v, \zeta_v),$$

relative to fixed bases $\Gamma(G_v, \zeta_v)$ of the spaces $\mathcal{D}(G_v, \zeta_v)$. It is also required that each subset

$$\Gamma_c(G_V^H, \zeta_V) = \Gamma(G_V^H, \zeta_V) \cap \mathcal{D}_c(G_V^H, \zeta_V), \quad c \in \Gamma_{\mathrm{ss}}(\bar{G}_V^H, \zeta_V),$$

be a basis of $\mathcal{D}_c(G_V^H, \zeta_V)$. In other words, the semisimple part of any element in $\Gamma(G_V^H, \zeta_V)$ is a single class c. In addition, the elements in the set

$$\Gamma_{\mathrm{orb}}(G_V^H, \zeta_V) = \Gamma(G_V^H, \zeta_V) \cap \mathcal{D}_{\mathrm{orb}}(G_V^H, \zeta_V)$$

are required to be orbital integrals, and to be a basis of $\mathcal{D}_{\mathrm{orb}}(G_V^H, \zeta_V)$. It follows from this that there is a bijection between $\Gamma_{\mathrm{orb}}(G_V^H, \zeta_V)$ and the set $\Gamma_{\mathrm{orb}}(\bar{G}_V^H, \zeta_V)$ of conjugacy classes in $\bar{G}_V^H$ whose semisimple part lies in $\Gamma_{\mathrm{ss}}(\bar{G}_V, \zeta_V)$. We can therefore define a chain of subsets

$$\Gamma_{\mathrm{orb}}(G_V^H, \zeta_V) \supset \Gamma_{\mathrm{ss}}(G_V^H, \zeta_V) \supset \Gamma_{\mathrm{reg}}(G_V^H, \zeta_V) \supset \Gamma_{\mathrm{reg,ell}}(G_V^H, \zeta_V) \tag{1.2}$$

of $\Gamma_{\mathrm{orb}}(G_V^H, \zeta_V)$, corresponding to subsets of classes in $\Gamma_{\mathrm{orb}}(\bar{G}_V^H, \zeta_V)$ that are, respectively, semisimple, strongly regular, and strongly regular elliptic. In particular, the semisimple part c of a general element γ in $\Gamma(G_V^H, \zeta_V)$ can be identified with a distribution in the subset $\Gamma_{\mathrm{ss}}(G_V^H, \zeta_V)$ of $\Gamma(G_V^H, \zeta_V)$. We assume, in fact, that γ has been constructed in the natural way from c, and from an element α in a fixed basis $\Gamma_{\mathrm{unip}}(G_{c,V}^H, \zeta_V)$ of the space $\mathcal{D}_{\mathrm{unip}}(G_{c,V}^H, \zeta_V)$ of unipotent distributions for $G_{c,V}^H$. A general element $\gamma \in \Gamma(G_V^H, \zeta_V)$ therefore has a *Jordan decomposition*, which we write formally as

$$\gamma = c\alpha, \qquad c \in \Gamma_{\mathrm{ss}}(\bar{G}_V^H, \zeta_V), \qquad \alpha \in \Gamma_{\mathrm{unip}}(G_{c,V}^H, \zeta_V). \tag{1.3}$$

The distributions in $\mathcal{D}(G_V^H, \zeta_V)$ are tempered as well as ζ_V-equivariant. Therefore, any element γ in $\mathcal{D}(G_V^H, \zeta_V)$ transfers to a continuous linear form

$$f \to f_G(\gamma), \quad f \in \mathcal{C}(G_V^H, \zeta_V),$$

on $\mathcal{C}(G_V^H, \zeta_V)$. The transfer actually depends on an implicit choice of Haar measure on Z_V, but this is obviously harmless. Suppose that γ is an element in $\Gamma_{\mathrm{orb}}(\bar{G}_V^H, \zeta_V)$, and that γ_V is a conjugacy class in G_V^H that maps to γ. The ζ_V-equivariant orbital integral at γ_V provides a distribution in $\mathcal{D}(G_V^H, \zeta_V)$, and hence a continuous linear form

$$f \to f_G(\gamma_V), \quad f \in \mathcal{C}(G_V^H, \zeta_V).$$

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This of course depends on the choice of representative γ_V of γ, as well as a choice of invariant measure on the G_V^H-conjugacy class γ_V. On the other hand, we may as well identify γ with the distribution in $\Gamma_{\text{orb}}(G_V^H, \zeta_V)$ to which it corresponds. The linear forms $f_G(\gamma_V)$ and $f_G(\gamma)$ then differ by a scalar multiple. We obtain

$$f_G(\gamma_V) = (\gamma_V/\gamma) f_G(\gamma), \quad f \in \mathcal{C}(G_V^H, \zeta_V), \tag{1.4}$$

where (γ_V/γ) is the ratio of the invariant measure on γ_V and the signed measure on γ_V that comes with γ. As we mentioned earlier, we shall usually restrict our attention to functions f in the Hecke algebra $\mathcal{H}(G_V^H, \zeta_V)$.

What is the spectral analogue of the space $\mathcal{D}(G_V^H, \zeta_V)$? There is some ambiguity in the question, but one could argue plausibly that the answer is the space $\mathcal{F}(G_V^H, \zeta_V)$ of generalized characters on G_V^H, with Z_V-central character equal to ζ_V. By a generalized character, here, we mean a finite, complex linear combination of irreducible characters. As with $\mathcal{D}(G_V^H, \zeta_V)$, we identify an element π in $\mathcal{F}(G_V^H, \zeta_V)$ with a linear form

$$f \to f_G(\pi), \quad f \in \mathcal{H}(G_V^H, \zeta_V),$$

on $\mathcal{H}(G_V^H, \zeta_V)$, which in this case depends on an implicit choice of Haar measure on G_V^H/Z_V. The space $\mathcal{F}(G_V^H, \zeta_V)$ already has a canonical basis. It is the set $\Pi(G_V^H, \zeta_V)$ of irreducible characters with Z_V-central character equal to ζ_V. We write $\Pi_{\text{unit}}(G_V^H, \zeta_V)$ for the subset of characters in $\Pi(G_V^H, \zeta_V)$ that are unitary. Then, in partial analogy with (1.2), we define a chain of subsets

$$\Pi_{\text{unit}}(G_V^H, \zeta_V) \supset \Pi_{\text{temp}}(G_V^H, \zeta_V) \supset \Pi_{\text{temp,ell}}(G_V^H, \zeta_V) \tag{1.5}$$

of characters in $\Pi_{\text{unit}}(G_V^H, \zeta_V)$ that are, respectively, tempered, and tempered elliptic.

For each f in $\mathcal{H}(G_V^H, \zeta_V)$, we have been regarding f_G as both a linear function on $\mathcal{D}(G_V^H, \zeta_V)$, and a linear function on $\mathcal{F}(G_V^H, \zeta_V)$. The former is determined by its restriction to the subset $\Gamma_{\text{reg}}(G_V^H, \zeta_V)$ of $\mathcal{D}(G_V^H, \zeta_V)$, while the latter is determined by its restriction to the subset $\Pi_{\text{temp}}(G_V^H, \zeta_V)$ of $\mathcal{F}(G_V^H, \zeta_V)$. It is known that either of the functions is determined by the other, so the notation is consistent. In the usual fashion, we can form the invariant Hecke space

$$\mathcal{I}(G_V^H, \zeta_V) = I\mathcal{H}(G_V^H, \zeta_V) = \{f_G : f \in \mathcal{H}(G_V^H, \zeta_V)\}$$

of functions obtained from $\mathcal{H}(G_V^H, \zeta_V)$. This space comes with the topology that makes the surjective map $f \to f_G$ from $\mathcal{H}(G_V^H, \zeta_V)$ to $\mathcal{I}(G_V^H, \zeta_V)$ open and continuous. There is also the invariant Schwartz space

$$\mathcal{I}(G_V^H, \zeta_V) = I\mathcal{C}(G_V^H, \zeta_V) = \{f_G : f \in \mathcal{C}(G_V^H, \zeta_V)\},$$

which comes with a similar topology. However, we shall use the overlapping notation $\mathcal{I}(\cdot)$ only if the context is clear. In either case, we can use the familiar notation

$$I(f) = \hat{I}(f_G),$$

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for any invariant linear form I that lies in the image of the transpose of the map $f \to f_G$.

For later use, we also introduce the stably invariant Hecke space. Recall that a distribution on G_V^H is *stable* if it lies in the closed linear span of the strongly regular, stable orbital integrals

$$f^G(\delta_V) = \sum_{\gamma_V \to \delta_V} f_G(\gamma_V).$$

Here, δ_V is any strongly regular, stable conjugacy class in G_V^H, and γ_V is summed over the finite set of conjugacy classes in δ_V. Let $S\mathcal{D}(G_V^H, \zeta_V)$ and $S\mathcal{F}(G_V^H, \zeta_V)$ be the subspaces of stable distributions in $\mathcal{D}(G_V^H, \zeta_V)$ and $\mathcal{F}(G_V^H, \zeta_V)$, respectively. Suppose that δ belongs to the set $\Delta_{\mathrm{reg}}(\bar{G}_V^H)$ of strongly regular, stable conjugacy classes in $\bar{G}_V^H$. Then there is a corresponding tempered distribution

$$f \to f^G(\delta) = \sum_{\gamma} f_G(\gamma), \quad f \in \mathcal{C}(G_V^H, \zeta_V),$$

where γ is summed over those classes in the set $\Gamma_{\mathrm{reg}}(\bar{G}_V^H) = \Gamma_{\mathrm{reg}}(\bar{G}_V^H, \zeta_V)$ that map to δ, and $f_G(\gamma)$ is the corresponding linear form in $\Gamma_{\mathrm{reg}}(G_V^H, \zeta_V)$. One of the requirements in [**14**] on the choice of basis $\Gamma(G_V^H, \zeta_V)$ is a simple compatibility condition on the summands that insures an identity

$$f^G(\delta_V) = (\delta_V/\delta) f^G(\delta), \tag{1.6}$$

where δ_V is any stable class in G_V^H that maps to δ. The ratio (δ_V/δ) equals (γ_V/γ), where γ_V is the conjugacy class in δ_V that maps to the given class γ in the sum. In particular, the distribution δ is stable. We have thus identified $\Delta_{\mathrm{reg}}(\bar{G}_V^H)$ with a subset $\Delta_{\mathrm{reg}}(G_V^H, \zeta_V)$ of $S\mathcal{D}(G_V^H, \zeta_V)$. The stably invariant Hecke space is the space of functions

$$S\mathcal{I}(G_V^H, \zeta_V) = SI\mathcal{H}(G_V^H, \zeta_V) = \{f^G : f \in \mathcal{H}(G_V^H, \zeta_V)\}$$

on $\Delta_{\mathrm{reg}}(G_V^H, \zeta_V)$. The stably invariant Schwartz space is defined in the same way.

Suppose that M is a Levi subgroup of G. Assuming that M contains H, we of course take $\Pi(M_V^H, \zeta_V)$ to be the basis of $\mathcal{F}(M_V^H, \zeta_V)$. We assume that we have also chosen a basis $\Gamma(M_V^H, \zeta_V)$ of $\mathcal{D}(M_V^H, \zeta_V)$, as well as corresponding subsets (1.2), as above. Consider the case that $H = M$, so that $M_V^H = M_V$ and $G_V^H = G_V$. We recall that there is a canonical map $f \to f_M$ from $\mathcal{H}(G_V, \zeta_V)$ to $\mathcal{I}(M_V, \zeta_V)$, which factors through the map $f \to f_G$. This map allows us to define induction operations. We obtain canonical linear maps

$$\mu \in \mathcal{D}(M_V, \zeta_V) \to \mu^G \in \mathcal{D}(G_V, \zeta_V)$$

and

$$\rho \in \mathcal{F}(M_V, \zeta_V) \to \rho^G \in \mathcal{F}(G_V, \zeta_V),$$

https://doi.org/10.1017/S1474748002000051 Published online by Cambridge University Press

such that

$$f_G(\mu^G) = f_M(\mu) \tag{1.7}$$

and

$$f_G(\rho^G) = f_M(\rho), \tag{1.8}$$

for any $f \in \mathcal{H}(G_V, \zeta_V)$. The choice of bases also determine adjoint restriction maps. These are the unique linear maps

$$\gamma \in \mathcal{D}(G_V, \zeta_V) \to \gamma_M \in \mathcal{D}(M_V, \zeta_V)$$

and

$$\pi \in \mathcal{F}(G_V, \zeta_V) \to \pi_M \in \mathcal{F}(M_V, \zeta_V),$$

such that

$$\sum_{\gamma \in \Gamma(G_V, \zeta_V)} a_M(\gamma_M) b_G(\gamma) = \sum_{\mu \in \Gamma(M_V, \zeta_V)} a_M(\mu) b_G(\mu^G) \tag{1.9}$$

and

$$\sum_{\pi \in \Pi(G_V, \zeta_V)} c_M(\pi_M) d_G(\pi) = \sum_{\rho \in \Pi(M_V, \zeta_V)} c_M(\rho) d_G(\rho^G), \tag{1.10}$$

for any linear functions $a_M \in \mathcal{D}(M_V, \zeta_V)^*$, $b_G \in \mathcal{D}(G_V, \zeta_V)^*$, $c_M \in \mathcal{F}(M_V, \zeta_V)^*$ and $d_G \in \mathcal{F}(G_V, \zeta_V)^*$ such that the right-hand inner products converge.

The notions we have been reviewing come from [**14**] and earlier papers. We shall make free use of any obvious variants of the notation. For example, the notation $\Gamma(G_V, \zeta_V)$, $\mathcal{D}(G_V, \zeta_V)$, $\mathcal{F}(G_V, \zeta_V)$, etc., in the case above that $H = M$, is in obvious recognition of the fact that H plays no role. As we noted earlier, we shall generally assume that V contains $V_{\mathrm{ram}}(G, \zeta)$, and that H equals Z. In this case, let $\mathfrak{a}^*_{G,Z}$ denote the subspace of linear forms on $\mathfrak{a}_G$ that are trivial on the image of $\mathfrak{a}_Z$ in $\mathfrak{a}_G$. Then there is an action

$$\lambda : \pi \to \pi_\lambda, \quad \pi \in \Pi_{\mathrm{unit}}(G_V, \zeta_V), \quad \lambda \in i\mathfrak{a}^*_{G,Z},$$

of $i\mathfrak{a}^*_{G,Z}$ on $\Pi_{\mathrm{unit}}(G_V, \zeta_V)$, whose orbits can be identified with the set $\Pi_{\mathrm{unit}}(G^Z_V, \zeta_V)$. We have agreed to write $\mathcal{H}(G, V, \zeta) = \mathcal{H}(G^Z_V, \zeta_V)$. This is the space of test functions we will be using for our global study. We shall also write

$$\mathcal{I}(G, V, \zeta) = \mathcal{I}(G^Z_V, \zeta_V)$$

for the corresponding invariant space. In the next two sections, we shall single out subsets $\Gamma(G, V, \zeta)$ and $\Pi(G, V, \zeta)$ of $\Gamma_{\mathrm{orb}}(G^Z_V, \zeta_V)$ and $\Pi_{\mathrm{unit}}(G^Z_V, \zeta_V)$, respectively, that also have special global significance.

2. Global trace formula: geometric side

The first main task of the paper will be to recast the global trace formula of [**8**] in somewhat different terms. There are three reasons for doing so. The first concerns how we make the trace formula invariant. We want to use the canonically normalized weighted characters of [**11**], rather than the weighted characters of [**8**] that depend on a choice of normalizing factors for intertwining operators. Secondly, it will be convenient to work with test functions on a finite product $G_V = G(F_V)$ of local groups, rather than on the adele group $G(\mathbb{A})$. Finally, we have to set things up for ζ^{-1}-equivariant test functions, in order to allow for future induction arguments. The result will be a formulation of the trace formula that is quite natural, and that clearly displays the remarkable duality between terms on the geometric and spectral sides.

Until further notice, F will be a global field, (G,ζ) will be a fixed pair over F as in § 1, and V will be a finite set of valuations of F that contains the ramified set $V_{\mathrm{ram}}(G,\zeta)$. The formula of [**8**] is the identity provided by two different expansions of a certain continuous linear form on $\mathcal{H}(G) = \mathcal{H}(G(\mathbb{A})^1)$. The formula we want will be an identity given by two expansions of a continuous linear form on $\mathcal{H}(G,V,\zeta) = \mathcal{H}(G_V^Z,\zeta_V)$. We shall first describe a formal process for passing from the former to the latter. The bulk of this section will then be devoted to the explicit construction of the new geometric expansion. The next section will be reserved for the construction of the new spectral expansion.

There is a natural projection

$$\dot{f}^1 \to \dot{f}^\zeta, \quad \dot{f}^1 \in \mathcal{H}(G),$$

from $\mathcal{H}(G)$ onto the space $\mathcal{H}(G,\zeta) = \mathcal{H}(G(\mathbb{A})^Z,\zeta)$. The image of $\dot{f}^1$ is defined to be the function

$$\dot{f}^\zeta(x) = \int_{Z(\mathbb{A})^x} \dot{f}^1(zx)\zeta(z)\,\mathrm{d}z, \quad x \in G(\mathbb{A})^Z,$$

where we have written

$$Z(\mathbb{A})^x = \{z \in Z(\mathbb{A}) : H_G(zx) = 0\}.$$

Suppose that J is a continuous, $Z(F)$-invariant linear form on $\mathcal{H}(G)$. If $\dot{f}^1_z$ denotes the translate of a function $\dot{f}^1 \in \mathcal{H}(G)$ by a point $z \in Z(\mathbb{A})^1$, the integral

$$J^\zeta(\dot{f}^1) = \int_{Z(F)\backslash Z(\mathbb{A})^1} J(\dot{f}^1_z)\zeta(z)\,\mathrm{d}z$$

converges, and depends only on the image $\dot{f}^\zeta$ of $\dot{f}^1$ in $\mathcal{H}(G,\zeta)$. We write

$$J(\dot{f}^\zeta) = J^\zeta(\dot{f}^1).$$

We then define a linear form on $\mathcal{H}(G,V,\zeta)$ by setting

$$J(f) = J(\dot{f}), \quad f \in \mathcal{H}(G,V,\zeta),$$

where $\dot{f} = f \times u^V$ is the function in $\mathcal{H}(G,\zeta)$ defined in § 1. We are using the symbol J here to denote three different objects: the original $Z(F)$-invariant linear form on $\mathcal{H}(G)$,

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the projection of this form onto the space of linear forms on $\mathcal{H}(G,\zeta)$, and the projection of the second object onto the space of linear forms on $\mathcal{H}(G,V,\zeta)$. Since the three linear forms act on three different spaces, there is no ambiguity in denoting them by the same symbol.

We shall apply this general procedure to the basic linear form that is the foundation of the trace formula of [**8**]. We need refer to [**8**] only for the non-invariant trace formula, since we will be using a different process to make it invariant. The relevant formula from [**8**] consists then of two different expansions of the continuous (non-invariant) linear form

$$J(\dot{f}^1) = \sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}(\dot{f}^1) = \sum_{\chi\in\mathcal{X}} J_{\chi}(\dot{f}^1), \qquad \dot{f}^1 \in \mathcal{H}(G), \tag{2.1}$$

on $\mathcal{H}(G)$ that appears in [**8**, (2.1)]. The two expansions are derived in §§ 3 and 4 of [**8**] by refining the terms in the respective sums over $\mathfrak{o}$ and χ. Our aim here is to convert these expansions into two expansions of an invariant linear form on $\mathcal{H}(G,V,\zeta)$.

It is an immediate consequence of the general constructions in [**1**] that the linear form (2.1) is $Z(F)$-invariant. The process above therefore provides a (non-invariant) linear form

$$J(f) = J(\dot{f}) = J^{\zeta}(\dot{f}^1), \quad f \in \mathcal{H}(G,V,\zeta),$$

on $\mathcal{H}(G,V,\zeta)$, where $\dot{f}^1$ is any function in $\mathcal{H}(G)$ whose projection $\dot{f}^{\zeta}$ onto $\mathcal{H}(G,\zeta)$ equals $\dot{f} = f \times u^V$. We then define an invariant linear form $I = I^G$ on $\mathcal{H}(G,V,\zeta)$ inductively by setting

$$I(f) = J(f) - \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1}\hat{I}^M(\phi_M(f)), \tag{2.2}$$

for certain maps

$$\phi_M : \mathcal{H}(G,V,\zeta) \to \mathcal{I}(M,V,\zeta) \tag{2.3}$$

constructed from the normalized weighted characters of [**11**]. To describe the maps precisely, suppose first that $\tilde{f}$ belongs to the Schwartz space $\mathcal{C}(G_V,\zeta_V)$. Then $\phi_M(\tilde{f})$ is defined to be the function on $\Pi_{\mathrm{temp}}(M_V,\zeta_V)$ whose value at a representation

$$\tilde{\pi} = \bigotimes_v \pi_v, \quad \pi_v \in \Pi_{\mathrm{temp}}(M_v,\zeta_v),$$

equals

$$\mathrm{tr}(\mathcal{M}_M(\tilde{\pi},P)\mathcal{I}_P(\tilde{\pi},\tilde{f})), \quad P \in \mathcal{P}(M).$$

The operator

$$\mathcal{M}_M(\tilde{\pi},P) = \lim_{\Lambda\to 0} \sum_{Q\in\mathcal{P}(M)} \Big(\bigotimes_{v\in V} \mathcal{M}_Q(\Lambda,\pi_v,P)\Big)\theta_Q(\Lambda)^{-1},$$

with

$$\theta_Q(\Lambda) = \mathrm{vol}(\mathfrak{a}_M^G/Z(\Delta_Q^\vee))^{-1} \prod_{\alpha\in\Delta_Q} \Lambda(\alpha^\vee),$$

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is defined as part of the theory of (G, M)-families [**2**, § 6]. The relevant (G, M)-family here is a tensor product of the (G, M)-families

$$\mathcal{M}_Q(\Lambda, \pi_v, P) = \mu_Q(\Lambda, \pi_v, P)\mathcal{J}_Q(\Lambda, \pi_v, P), \quad Q \in \mathcal{P}(M), \quad \Lambda \in i\mathfrak{a}_M^*,$$

defined for π_v in general position on p. 37 of [**11**]. It follows from [**11**, Lemma 3.1] that ϕ_M maps $\mathcal{C}(G_V, \zeta_V)$ continuously to $I\mathcal{C}(M_V, \zeta_V)$. If f and π are the restrictions of $\tilde{f}$ and $\tilde{\pi}$ to G_V^Z and M_V^Z, respectively, we set

$$\phi_M(f, \pi) = \int_{i\mathfrak{a}_{M,Z}^*} \phi_M(\tilde{f}, \tilde{\pi}_\lambda)\, \mathrm{d}\lambda.$$

Our concern here is in the case that f belongs to the subspace $\mathcal{H}(G, V, \zeta)$ of $\mathcal{C}(G_V^Z, \zeta_V)$. The argument of [**9**, § 12] is easily modified to show that ϕ_M maps $\mathcal{H}(G, V, \zeta)$ continuously to the subspace $\mathcal{I}(M, V, \zeta) = I\mathcal{H}(M_V^Z, \zeta_V)$ of $I\mathcal{C}(M_V^Z, \zeta_V)$.

We shall now derive a geometric expansion of $I(f)$ from the geometric expansion in [**8**, § 3] of (2.1). We have first to describe the local and global ingredients of the new expansion. We will then be able to apply the methods of [**8**, § 3].

The local terms in the geometric expansion of $I(f)$ are essentially the invariant distributions of [**11**, § 3]. They are invariant linear forms $I_M(\gamma) = I_M^G(\gamma)$ on $\mathcal{H}(G, V, \zeta)$, which are parametrized by Levi subgroups $M \in \mathcal{L}$ and elements $\gamma \in \Gamma(M_V^Z, \zeta_V)$. For any $f \in \mathcal{H}(G, V, \zeta)$, $I_M(\gamma, f)$ is defined inductively by the usual formula

$$I_M(\gamma, f) = J_M(\gamma, f) - \sum_{L \in \mathcal{L}^0(M)} \hat{I}_M^L(\gamma, \phi_L(f)),$$

where $\phi_L(f)$ is the map (2.3), and $J_M(\gamma, f)$ is the weighted orbital integral, defined for $\gamma \in \Gamma_{\mathrm{orb}}(M_V^Z, \zeta_V)$ as in [**6**], and for general γ in [**14**].

The global terms in the geometric expansion appear as coefficients. They require rather more discussion. We begin by recalling the global coefficients

$$a^G(S, \dot{\gamma}), \quad \dot{\gamma} \in (G(F))_{G,S},$$

of [**8**], which were defined in [**5**, (8.1)]. Here $S \supset V_{\mathrm{ram}}(G)$ is a large finite set of valuations, and $(G(F))_{G,S}$ is the set of what we called the (G, S)-equivalence classes in $G(F)$. (We are using the dot notation $\dot{\gamma}$ for elements in $G(F)$, since γ will generally be reserved for elements in G_V.) We recall that two elements $\dot{\gamma}$ and $\dot{\gamma}_1$ in $G(F)$, with standard Jordan decompositions $\dot{\gamma} = c\dot{\alpha}$ and $\dot{\gamma}_1 = c_1\dot{\alpha}_1$, were defined to be (G, S)-equivalent if there is an element $\dot{\delta} \in G(F)$ such that $\dot{\delta}^{-1}c_1\dot{\delta} = c$, and such that $\dot{\delta}^{-1}\dot{\alpha}_1\dot{\delta}$ is conjugate to $\dot{\alpha}$ in $G_c(F_S)$. For a general element $\dot{\gamma} = c\dot{\alpha}$, the coefficient was defined in [**5**, (8.1)] by a descent formula

$$a^G(S, \dot{\gamma}) = i^G(S, c)|\operatorname{Stab}(c, \dot{\alpha})|^{-1} a^{G_c}(S, \dot{\alpha}). \tag{2.4}$$

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We write $\mathrm{Stab}(c, \dot{\alpha})$ here for the stabilizer of $\dot{\alpha}$ in the finite group $(G_{c,+}(F)/G_c(F))$, which acts on the set of unipotent conjugacy classes in $G_c(F_S)$. The symbol $i^G(S, c)$ is as in [**8**, (3.2)]. It equals 1 if c is F-elliptic in G and the $G(\mathbb{A}^S)$-conjugacy class of c meets K^S, and is otherwise equal to 0. (We neglected to mention the second condition on c explicitly in [**5**, (8.1)], although we included it in the proof.) The point of the descent formula is to reduce the study of $a^G(S, \dot{\gamma})$ to the case of unipotent elements treated in [**4**].

As explained in [**5**] and [**8**], the finite set S has to be large in a sense that depends on the semisimple part of $\dot{\gamma}$. In this paper, we require a quantitative criterion for the choice of S. It is provided by the next lemma, and the definitions of § 1.

Lemma 2.1. *If $\dot{\gamma}$ belongs to $G(F)$, the coefficient $a^G(S, \dot{\gamma})$ is defined for any finite set S such that $\dot{\gamma}$ is S-admissible.*

Proof. Suppose that $\dot{\gamma}$ is S-admissible. We must show that the definition of $a^G(S, \dot{\gamma})$ in [**5**] and [**8**] is valid for the given S. For the definition in [**5**], the requirements to verify are the conditions (i)–(iv) on p. 203. The condition (i) follows from the definition of S-admissible, while conditions (iii) and (iv) follow from [**5**, Lemma 7.1]. The condition (ii), in the untwisted case we are considering in this paper, asserts that $\dot{\gamma}_v$ belongs to K_v for each $v \notin S$. This condition was removed in [**8**, § 3] by simply setting $a^G(S, \dot{\gamma}) = 0$ if $\dot{\gamma}$ is not $G(\mathbb{A}^S)$-conjugate to an element in K^S. Therefore, $a^G(S, \dot{\gamma})$ is defined whenever $\dot{\gamma}$ is S-admissible. □

In this paper, we would like to index the coefficients by admissible elements in $\Gamma(G_S^Z, \zeta_S)$, rather than by classes in $(G(F))_{G,S}$. To help us make the transition, we set

$$I_{\mathrm{ell}}(\dot{f}_S^1) = \sum_{\dot{\gamma} \in (G(F))_{G,S}} a^G(S, \dot{\gamma}) \dot{f}_{S,G}^1(\dot{\gamma}), \quad \dot{f}_S^1 \in \mathcal{H}_{\mathrm{adm}}(G, S). \tag{2.5}$$

This is the term with $M = G$ in the geometric expansion [**8**, (3.3)], and can be regarded as the G-elliptic part of the expansion. (We use 'elliptic' to refer to the semisimple components of the classes in $G(F)$ that index the summands.) One consequence of the descent formula (2.4) is that

$$a^G(S, z\dot{\gamma}) = a^G(S, \dot{\gamma}), \quad z \in Z_{S,\mathfrak{o}},$$

where

$$Z_{S,\mathfrak{o}} = Z(F) \cap Z_S Z(\mathfrak{o}^S).$$

We can obviously embed $Z_{S,\mathfrak{o}}$ as a discrete subgroup of Z_S. The linear form $I_{\mathrm{ell}}(\dot{f}_S^1)$ on $\mathcal{H}_{\mathrm{adm}}(G, S)$ is then $Z_{S,\mathfrak{o}}$-invariant. Applying a variant of the process at the beginning of this section, we define a linear form

$$I_{\mathrm{ell}}(\dot{f}_S) = \int_{Z_{S,\mathfrak{o}} \backslash Z_S^1} I_{\mathrm{ell}}(\dot{f}_{S,z}^1) \zeta(z) \, \mathrm{d}z, \quad \dot{f}_S \in \mathcal{H}_{\mathrm{adm}}(G, S, \zeta),$$

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on $\mathcal{H}_{\mathrm{adm}}(G,S,\zeta)$, in which $\dot{f}_S^1$ is any function in $\mathcal{H}_{\mathrm{adm}}(G,S)$ whose projection $\dot{f}_S^{\zeta_S}$ onto $\mathcal{H}(G,S,\zeta)$ equals $\dot{f}_S$. It follows from the $Z_{S,\mathfrak{o}}$-invariance of $a^G(S,\dot{\gamma})$ that

$$\begin{aligned} I_{\mathrm{ell}}(\dot{f}_S) &= \sum_{\dot{\gamma}\in(G(F))_{G,S}} a^G(S,\dot{\gamma}) \int_{Z_{S,\mathfrak{o}}\backslash Z_S^1} \dot{f}_{S,G}^1(z\dot{\gamma})\zeta(z)\,\mathrm{d}z \\ &= \sum_{\{\dot{\gamma}\}} |Z(F,\dot{\gamma})|^{-1} a^G(S,\dot{\gamma}) \int_{Z_S^1} \dot{f}_{S,G}^1(z\dot{\gamma})\zeta(z)\,\mathrm{d}z, \end{aligned}$$

where $\{\dot{\gamma}\}$ stands for a set of representatives of $Z_{S,\mathfrak{o}}$-orbits in $(G(F))_{G,S}$, and

$$Z(F,\dot{\gamma}) = \{z\in Z(F) : z\dot{\gamma} = \dot{\gamma}\} = \{z \in Z_{S,\mathfrak{o}} : z\dot{\gamma} = \dot{\gamma}\}.$$

The integral

$$\int_{Z_S^1} \dot{f}_{S,G}^1(z\dot{\gamma})\zeta(z)\,\mathrm{d}z$$

is easily evaluated in terms of $\dot{f}_S$. It vanishes unless $\dot{\gamma}$ maps to an element $\dot{\gamma}_S$ in $\Gamma_{\mathrm{orb}}(G_S^Z,\zeta_S)$, which is to say that the conjugacy class of $\dot{\gamma}$ in $\bar{G}_S^Z = G_S^Z/Z_S$ lies in $\Gamma_{\mathrm{orb}}(\bar{G}_S^Z,\zeta_S)$, and in addition, the $G(F_v)$-orbit of $\dot{\gamma}$ meets K_v for each $v\notin S$. If the two conditions hold, the integral simply equals

$$(\dot{\gamma}/\dot{\gamma}_S)\dot{f}_{S,G}(\dot{\gamma}_S),$$

where $(\dot{\gamma}/\dot{\gamma}_S)$ is the ratio in (1.4).

We have converted the expansion (2.5) for $I_{\mathrm{ell}}(\dot{f}_S^1)$ into an expansion for $I_{\mathrm{ell}}(\dot{f}_S)$. To describe the latter, we first define a coefficient $a_{\mathrm{ell}}^G(\dot{\gamma}_S)$. If $\dot{\gamma}_S$ is any admissible element in $\Gamma(G_S,\zeta_S)$, we set

$$a_{\mathrm{ell}}^G(\dot{\gamma}_S) = \sum_{\{\dot{\gamma}\}} |Z(F,\dot{\gamma})|^{-1} a^G(S,\dot{\gamma})(\dot{\gamma}/\dot{\gamma}_S), \tag{2.6}$$

where $\{\dot{\gamma}\}$ is summed over those $Z_{S,\mathfrak{o}}$-orbits in $(G(F))_{G,S}$ that map to $\dot{\gamma}_S$, and such that the $G(\mathbb{A}^S)$-conjugacy class of $\dot{\gamma}$ in $G(\mathbb{A}^S)$ meets K^S. This coefficient obviously vanishes on the complement of $\Gamma_{\mathrm{orb}}(G_S^Z,\zeta_S)$ in $\Gamma(G_S,\zeta_S)$. It is in fact instructive to introduce a subset of $\Gamma_{\mathrm{orb}}(G_S^Z,\zeta_S)$ that can serve as a more manageable domain. If V is any finite set containing $V_{\mathrm{ram}}(G,\zeta)$, we write $\Gamma_{\mathrm{ell}}(G,V,\zeta)$ for the collection of elements $\gamma\in\Gamma_{\mathrm{orb}}(G_V^Z,\zeta_V)$ such that there is a $\dot{\gamma}\in G(F)$ that satisfies the following three conditions.

(i) The semisimple part of $\dot{\gamma}$ is F-elliptic in G.

(ii) The conjugacy class of $\dot{\gamma}$ in G_V maps to γ.

(iii) The element $\dot{\gamma}$ is bounded at each $v\notin V$. In other words, for each $v\notin V$, the image of $\dot{\gamma}$ in G_v lies in a compact subgroup.

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The subset $\Gamma_{\mathrm{ell}}(G,V,\zeta)$ is discrete in the natural topology on $\Gamma(G_V^Z,\zeta_V)$. Taking $V=S$, we see that $a_{\mathrm{ell}}^G(\dot\gamma_S)$ is supported on the set of admissible elements in $\Gamma_{\mathrm{ell}}(G,S,\zeta)$. The expansion for $I_{\mathrm{ell}}(\dot f_S)$ is just

$$I_{\mathrm{ell}}(\dot f_S)=\sum_{\dot\gamma_S\in\Gamma_{\mathrm{ell}}(G,S,\zeta)} a_{\mathrm{ell}}^G(\dot\gamma_S)\dot f_{S,G}(\dot\gamma_S),\quad \dot f_S\in\mathcal{H}_{\mathrm{adm}}(G,S,\zeta). \tag{2.7}$$

The definition (2.6) is only a part of our reformulation of the global coefficients. We are going to define coefficients that depend on elements $\gamma\in\Gamma(G_V^Z,\zeta_V)$, where V is an arbitrary finite set of valuations that contains $V_{\mathrm{ram}}(G,\zeta)$. The role of S will be simply that of some finite set containing V that is large relative to γ. First, let us define a discrete subset of $\Gamma(G_V^Z,\zeta_V)$ that will serve as a suitable domain for the new coefficients. We have already defined the 'elliptic' set $\Gamma_{\mathrm{ell}}(G,V,\zeta)$. If M is a Levi subgroup of G, and μ belongs to $\Gamma(M_V^Z,\zeta_V)$, the induced distribution μ^G is a finite linear combination of elements in $\Gamma(G_V^Z,\zeta_V)$. We write $\Gamma(G,V,\zeta)$ for the set of elements so obtained, as M ranges over $\mathcal{L}$ and μ runs over the elements in $\Gamma_{\mathrm{ell}}(M,V,\zeta)$. This will be the domain.

The new coefficients will combine the elliptic coefficients (2.6) with unramified weighted orbital integrals at places v in $S-V$. Let us write $\mathcal{K}(\bar G_S^V)$ for the set of conjugacy classes in $\bar G_S^V=G_S^V/Z_S^V$ that are bounded. Since ζ_S^V is trivial on $Z(\mathfrak{o}_S^V)$, $\mathcal{K}(\bar G_S^V)$ is contained in $\Gamma_{\mathrm{orb}}(\bar G_S^V,\zeta_S^V)$, so by the conventions of § 1, any element $k\in\mathcal{K}(\bar G_S^V)$ provides a distribution $\gamma_S^V(k)$ in the subset $\Gamma_{\mathrm{orb}}(G_S^V,\zeta_S^V)$ of $\Gamma(G_S^V,\zeta_S^V)$. If k belongs to $\mathcal{K}(\bar G_S^V)$ and γ is an element in $\Gamma(G_V^Z,\zeta_V)$, we shall write

$$\gamma\times k=\gamma\times\gamma_S^V(k)$$

for the associated element in $\Gamma(G_S^Z,\zeta_S)$. It is then clear that for any k, the preimage of $\Gamma_{\mathrm{ell}}(G,S,\zeta)$ under the map $\gamma\to\gamma\times k$ is contained in $\Gamma_{\mathrm{ell}}(G,V,\zeta)$. To emphasize the duality with the spectral expansion in next section, we shall also write $\mathcal{K}_{\mathrm{ell}}^V(\bar G,S)$ for the set of k in $\mathcal{K}(\bar G_S^V)$ such that $\gamma\times k$ belongs to $\Gamma_{\mathrm{ell}}(G,S,\zeta)$ for some γ. These sets are of course defined if G is replaced by a Levi subgroup $M\in\mathcal{L}$. The unramified weighted orbital integrals $J_M(\cdot,u_S^V)$ will appear in the form of a function

$$r_M^G(k)=J_M(\gamma_S^V(k),u_S^V),\quad k\in\mathcal{K}(\bar M_S^V),$$

on $\mathcal{K}(\bar M_S^V)$.

We can now define the coefficients that will occur in our geometric expansion. If γ belongs to $\Gamma(G_V^Z,\zeta_V)$, we set

$$a^G(\gamma)=\sum_{M\in\mathcal{L}}|W_0^M||W_0^G|^{-1}\sum_{k\in\mathcal{K}_{\mathrm{ell}}^V(\bar M,S)} a_{\mathrm{ell}}^M(\gamma_M\times k)r_M^G(k), \tag{2.8}$$

where $S\supset V$ is any finite set of valuations such that the set $\gamma\times K^V$ is S-admissible. The summands on the right-hand side are defined, by Lemma 2.1, and since $\mathcal{K}_{\mathrm{ell}}^V(\bar M,S)$ is discrete in the relevant topology in $\mathcal{K}(\bar M_S^V)$, the sum over k can be taken over a finite

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set. Of course $\gamma_M \times k$ is understood to be a finite linear combination of elements $\dot{\gamma}_S$ in $\Gamma(M_S^Z, \zeta_S)$, and $a_{\mathrm{ell}}^M(\gamma_M \times k)$ is the corresponding finite linear combination of values $a_{\mathrm{ell}}^M(\dot{\gamma}_S)$. It follows from the definitions that $a^G(\gamma)$ is supported on the discrete subset $\Gamma(G, V, \zeta)$ of $\Gamma(G_V^Z, \zeta_V)$.

Proposition 2.2. *Suppose that $f \in \mathcal{H}(G, V, \zeta)$. Then the linear form $I(f)$ in (2.2) has a geometric expansion*

$$I(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma) I_M(\gamma, f), \tag{2.9}$$

in which the inner sum can be taken over a finite set that depends only on the support of f.

Proof. As a function of γ in $\Gamma(M_V^Z, \zeta_V)$, $I_M(\gamma, f)$ has compact support. This assertion is valid only because we are considering elements γ attached to the closed subgroup M_V^Z of M_V. It follows from the usual splitting and descent formulae satisfied by $I_M(\gamma, f)$, as for example in the proof of [**8**, Lemma 3.2]. Since $\Gamma(M, V, \zeta)$ is a discrete subset of $\Gamma_{\mathrm{ell}}(M_V^Z, \zeta_V)$, the inner summand on the right-hand side of (2.9) therefore has finite support.

The proof of (2.9) is similar to that of [**8**, Theorem 3.3]. The main step will be to establish a parallel expansion for the linear form

$$J(f) = J^\zeta(\dot{f}^1) = \int_{Z(F)\backslash Z(\mathbb{A})^1} J(\dot{f}_z^1)\zeta(z)\,\mathrm{d}z,$$

in which $\dot{f}^1$ is any function in $\mathcal{H}(G)$ whose projection $\dot{f}^\zeta$ onto $\mathcal{H}(G, \zeta)$ equals the function $\dot{f} = f \times u^V$.

We recall the geometric expansion for $J(\dot{f}^1)$ that is provided by [**5**, Theorem 9.2]. Let $S \supset V$ be a finite set of valuations such that support of $\dot{f}^1$ is S-admissible, and such that $\dot{f}^1$ is of the form

$$\dot{f}_S^1 \times u^{S,1}, \quad \dot{f}_S^1 \in \mathcal{H}(G, S).$$

Then

$$J(\dot{f}^1) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\gamma} \in (M(F))_{M,S}} a^M(S, \dot{\gamma}) J_M(\dot{\gamma}, \dot{f}_S^1),$$

where $J_M(\dot{\gamma}, \dot{f}_S^1)$ is the weighted orbital integral of $\dot{f}_S^1$ over the conjugacy class of $\dot{\gamma}$ in G_S. The term corresponding to M depends on S, but the sum over M does not. For a given choice of S, the linear form $J(\dot{f}^1)$ is obviously K^S-invariant. We obtain

$$\begin{aligned} J(f) &= \int_{Z(F)Z(\mathfrak{o}^S)\backslash Z(\mathbb{A})^1} J(\dot{f}_z^1)\zeta(z)\,\mathrm{d}z \\ &= \int_{Z_{S,\mathfrak{o}}\backslash Z_S^1} J(\dot{f}_z^1)\zeta(z)\,\mathrm{d}z \\ &= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot{\gamma} \in (M(F))_{M,S}} a^M(S, \dot{\gamma}) \int_{Z_{S,\mathfrak{o}}\backslash Z_S^1} J_M(z\dot{\gamma}, \dot{f}_S^1)\zeta(z)\,\mathrm{d}z, \end{aligned}$$

since

$$Z(\mathbb{A}) = Z(F)Z_S Z(\mathfrak{o}^S),$$

and

$$J_M(\dot\gamma, \dot f^1_{S,z}) = J_M(z\dot\gamma, \dot f^1_S), \quad z \in Z_S.$$

Following the derivation of (2.7) from (2.5) above, we conclude that

$$J(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot\gamma_S \in \Gamma_{\mathrm{ell}}(M,S,\zeta)} a^M_{\mathrm{ell}}(\dot\gamma_S) J_M(\dot\gamma_S, \dot f_S),$$

where $\dot f_S$ equals the projection $\dot f_S^{\zeta_S}$ of $\dot f^1_S$ onto $\mathcal{H}(G,S,\zeta)$.

It is a consequence of the definitions that $\dot f_S$ is equal to the product $f \times u_S^V$, where $f \in \mathcal{H}(G,V,\zeta)$ is the function we started with. Taking a corresponding decomposition $\dot\gamma_V \times \dot\gamma_S^V$ of $\dot\gamma_S$, we see from the usual splitting and descent properties that

$$J_M(\dot\gamma_S, \dot f_S) = \sum_{L\in\mathcal{L}(M)} J_M^L(\dot\gamma_S^V, (u_S^V)_L) J_L(\dot\gamma_V^L, f).$$

(See [**7**, Proposition 9.1 and Corollary 8.2]. We are dealing with weighted orbital integrals here, rather than the corresponding invariant distributions, so it does not matter that the set $S - V$ fails to have the closure property of [**8**]. We have also used the fact that $(u_S^V)_L = (u_S^V)_Q$ is the unit in the invariant unramified Hecke algebra on L_V^S, and is therefore independent of $Q \in \mathcal{P}(L)$.) Now $J_M^L(\dot\gamma_S^V, (u_S^V)_L)$ vanishes unless $\dot\gamma_S^V = \gamma_S^V(k)$ for some k in $\mathcal{K}(\bar M_S^V)$, in which case it equals $r_M^L(k)$ by definition. With this condition on $\dot\gamma_S^V$, $a^M_{\mathrm{ell}}(\dot\gamma_S)$ vanishes unless $\mu = \dot\gamma_V$ lies $\Gamma_{\mathrm{ell}}(M,V,\zeta)$, and k lies in $\mathcal{K}^V_{\mathrm{ell}}(\bar M, S)$. We find that $J(f)$ equals

$$\sum_{L\in\mathcal{L}} \sum_{M\in\mathcal{L}^L} |W_0^M||W_0^G|^{-1} \sum_\mu \sum_k a^M_{\mathrm{ell}}(\mu\times k) r_M^L(k) J_L(\mu^L, f),$$

where μ and k are summed over $\Gamma_{\mathrm{ell}}(M,V,\zeta)$ and $\mathcal{K}^V_{\mathrm{ell}}(\bar M, S)$, respectively. But we can write

$$\begin{aligned}\sum_{M\in\mathcal{L}^L} &|W_0^M||W_0^G|^{-1} \sum_\mu \sum_k a^M_{\mathrm{ell}}(\mu\times k) r_M^L(k) J_L(\mu^L, f)\\ &= \sum_M |W_0^M||W_0^G|^{-1} \sum_{\gamma\in\Gamma(L,V,\zeta)} \sum_k a^M_{\mathrm{ell}}(\gamma_M\times k) r_M^L(k) J_L(\gamma, f)\\ &= |W_0^L||W_0^G|^{-1} \sum_{\gamma\in\Gamma(L,V,\zeta)} a^L(\gamma) J_L(\gamma, f),\end{aligned}$$

by (1.9) and (2.8). We obtain an expansion

$$J(f) = \sum_{L\in\mathcal{L}} |W_0^L||W_0^G|^{-1} \sum_{\gamma\in\Gamma(L,V,\zeta)} a^L(\gamma) J_L(\gamma, f) \tag{2.10}$$

of the form we want.

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It is now a simple matter to convert the expansion (2.10) for $J(f)$ into an expansion for $I(f)$. By definition,

$$I(f) = J(f) - \sum_{L \in \mathcal{L}^0} |W_0^L||W_0^G|^{-1} \hat{I}^L(\phi_L(f)).$$

Assume inductively that the required expansion (2.9) holds if G is replaced by any group $L \in \mathcal{L}^0$. Combining this with the formula (2.10) for $J(f)$, we see that $I(f)$ equals

$$\sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma)\Big(J_M(\gamma, f) - \sum_{L \in \mathcal{L}^0(M)} \hat{I}_M^L(\gamma, \phi_L(f))\Big).$$

By the definition of $I_M(\gamma, f)$, this in turn equals

$$\sum_M |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma) I_M(\gamma, f),$$

the required expansion for $I(f)$. □

We shall later have reason to consider the simpler linear form

$$I_{\mathrm{orb}}(f) = \sum_{\gamma \in \Gamma(G,V,\zeta)} a^G(\gamma) f_G(\gamma), \quad f \in \mathcal{H}(G, V), \tag{2.11}$$

defined by the term with $M = G$ in the expansion (2.9). It is the purely 'orbital' part of $I(f)$, and consists of a linear combination of invariant orbital integrals. However, the coefficients of this expansion are defined by a formula (2.8) that seems to depend on the set $S \supset V$.

Corollary 2.3. *The coefficients $\{a^G(\gamma)\}$ are in fact independent of S, and so therefore is $I_{\mathrm{orb}}(f)$.*

Proof. Recall that we constructed $I(f)$ from the linear form (2.1) that does not depend on S. We can assume inductively that if M is a proper Levi subgroup of G, the coefficients $\{a^M(\gamma)\}$ are independent of S. The corresponding term on the right-hand side of (2.9) is therefore also independent of S. This leaves only the term with $M = G$ on the right-hand side of (2.9), which is just $I_{\mathrm{orb}}(f)$. The corollary follows. □

Remark. The corollary could easily be proved directly from the admissibility of $\gamma \times K^V$, and properties of the weighted orbital integrals $r_M^G(k)$.

3. Global trace formula: spectral side

We turn now to the spectral side. As before, V will be any finite set of valuations that contains $V_{\mathrm{ram}}(G, \zeta)$. We shall convert the spectral expansion [**8**, § 4] for the right-hand side of (2.1) into an expansion for the linear form (2.2) on $\mathcal{H}(G, V, \zeta)$.

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We should recall that the spectral expansion for (2.1) in [**8**] is only conditionally convergent. There is first an absolutely convergent sum

$$J(\dot{f}^1) = \sum_{t \geqslant 0} J_t(\dot{f}^1), \quad \dot{f}^1 \in \mathcal{H}(G), \tag{3.1}$$

for the linear form (2.1). The terms

$$J_t(\dot{f}^1) = \sum_{\{\chi \in \mathcal{X} : \| \operatorname{Im}(\nu_\chi) \| = t\}} J_\chi(\dot{f}^1), \quad t \geqslant 0,$$

in this sum are obtained from those summands on the right-hand side of (2.1) whose archimedean infinitesimal characters ν_χ are of height t [**8**, § 4]. Each $J_t(\dot{f}^1)$ in turn has an absolutely convergent spectral expansion. A strengthening of results of Müller [**32**] would establish that the resulting expansion of $J(\dot{f}^1)$ is actually absolutely convergent (as a double integral). However, for the comparison problems of this and subsequent papers, it will be no trouble for us to treat the spectral expansion as a conditionally convergent double integral.

We first have to apply the formal process at the beginning of § 2 to each of the terms $J_t(\dot{f}^1)$. It follows without difficulty from the original definition [**2**] of $J_\chi(\dot{f}^1)$ that each linear form $J_t(\dot{f}^1)$ on $\mathcal{H}(G, V, \zeta)$ is $Z(F)$-invariant. We obtain a (non-invariant) linear form

$$J_t(f) = J_t(\dot{f}) = J_t^\zeta(\dot{f}^1), \quad f \in \mathcal{H}(G, V, \zeta),$$

on $\mathcal{H}(G, V, \zeta)$, in which $\dot{f}^1$ is any function in $\mathcal{H}(G)$ whose projection $\dot{f}^\zeta$ onto $\mathcal{H}(G, V)$ equals $\dot{f} = f \times u^V$. We then define a corresponding invariant linear form $I_t = I_t^G$ on $\mathcal{H}(G, V, \zeta)$ inductively by setting

$$I_t(f) = J_t(f) - \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \hat{I}_t^M(\phi_M(f)), \tag{3.2}$$

with ϕ_M being the map of (2.3).

To treat the conditional convergence of the spectral expansion, we shall exploit the multiplier convergence estimate of [**16**, (2.15.2)] and [**10**, Lemma 7.1]. Recall that a multiplier for G is a function α in $C_c^\infty(\mathfrak{h}^Z)^{W_\infty}$, where

$$\mathfrak{h} = i\mathfrak{h}_K \oplus \mathfrak{h}_0$$

is a Cartan subalgebra of a split form of the real group $G_\infty = G_{V_\infty}$, and W_∞ is the corresponding Weyl group. (See [**16**, § 2.15] or [**10**, § 7].) We have written $\mathfrak{h}^Z$ for the subspace of points in $\mathfrak{h}$ whose projection onto $\mathfrak{a}_G$ lies in the image of $\mathfrak{a}_Z$. The Fourier transform $\hat{\alpha}$ is then a W_∞-invariant function in the Paley–Wiener space on

$$\mathfrak{h}_{\mathbb{C}}^*/\mathfrak{a}_{G,Z,\mathbb{C}}^* = \mathfrak{h} \otimes \mathbb{C}/\mathfrak{a}_{G,Z}^* \otimes \mathbb{C}.$$

If f belongs to $\mathcal{H}(G, V, \zeta)$, one can transform the archimedean components of f by α. This provides a second function f_α in $\mathcal{H}(G, V, \zeta)$, which is characterized by the property that

$$f_{\alpha,G}(\pi) = \hat{\alpha}(\nu) f_G(\pi),$$

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for any representation $\pi \in \Pi(G_V^Z, \zeta_V)$ whose archimedean infinitesimal character corresponds to the point $\nu \in \mathfrak{h}_{\mathbb{C}}^*/\mathfrak{a}_{G,Z,\mathbb{C}}^*$.

The convergence estimate is given by the values of $\hat{\alpha}$ on a subset

$$\mathfrak{h}_u^*(r,T) = \{\nu \in \mathfrak{h}_u^* : \|\operatorname{Re}(\nu)\| \leqslant r,\ \|\operatorname{Im}(\nu)\| \geqslant T\}$$

of $\mathfrak{h}_{\mathbb{C}}^*/i\mathfrak{a}_{G,Z,\mathbb{C}}^*$, where $\mathfrak{h}_u^*$ is a subset of $\mathfrak{h}_{\mathbb{C}}^*/i\mathfrak{a}_{G,Z}^*$ that embeds into $\mathfrak{h}_{\mathbb{C}}^*/\mathfrak{a}_{G,Z,\mathbb{C}}^*$, and contains the infinitesimal characters of all unitary representations. (The definitions are essentially those of [**8**, p. 536] and [**10**, p. 558]. In particular, the norm $\|\cdot\|$ is assumed to come from a fixed, W_∞-invariant, Euclidean inner product on $\mathfrak{h}^Z$.) Suppose that

$$A_t(f), \quad f \in \mathcal{H}(G,V,\zeta), \quad t \geqslant 0,$$

is a family of linear forms such that for any f and α, the function

$$t \to A_t(f_\alpha), \quad t \geqslant 0,$$

is supported on a discrete set that is independent of α. We shall say that the family satisfies the *multiplier convergence estimate* if for each $f \in \mathcal{H}(G,V,\zeta)$, we can choose constants C, k and r with the following property. For any positive numbers T and N, and any α in

$$C_N^\infty(\mathfrak{h}^Z)^{W_\infty} = \{\alpha \in C_c^\infty(\mathfrak{h}^Z)^{W_\infty} : \|\operatorname{supp}(\alpha)\| \leqslant N\},$$

the estimate

$$\sum_{t>T} |A_t(f_\alpha)| \leqslant C\mathrm{e}^{kN} \sup_{\nu \in \mathfrak{h}_u^*(r,T)} (|\hat{\alpha}(\nu)|) \tag{3.3}$$

holds.

Proposition 3.1. *The linear forms*

$$I_t(f), \quad f \in \mathcal{H}(G,V,\zeta), \quad t \geqslant 0,$$

satisfy the multiplier convergence estimate (3.3), and the formula

$$I(f) = \sum_t I_t(f). \tag{3.4}$$

Proof. The non-invariant linear forms $J(f)$ and $J_t(f)$ are continuous images of the linear forms in (3.1). It follows from (3.1) that

$$J(f) = \sum_{t \geqslant 0} J_t(f).$$

The formula (3.4) then follows inductively from the definitions (2.2) and (3.2). The multiplier convergence estimate follows in the same way from the parallel estimate for the linear forms

$$J_t(\dot{f}^1), \quad \dot{f}^1 \in \mathcal{H}(G), \quad t \geqslant 0,$$

that was the main step in the proof of [**8**, Lemma 6.3]. □

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Suppose now that $t \geqslant 0$ is fixed. We shall derive a spectral expansion for $I_t(f)$ from the expansion in [**8**, § 4] for $J_t(\dot{f}^1)$. As in § 2, we shall first describe the local and global ingredients of the new expansion.

The local terms are similar to those in [**8**, § 4], except that they are defined by means of the canonically normalized weighted characters of [**11**]. They are invariant linear forms $I_M(\pi) = I_M^G(\pi)$ on $\mathcal{H}(G, V, \zeta)$, parametrized by Levi subgroups $M \in \mathcal{L}$ and representations $\pi \in \Pi_{\mathrm{unit}}(M_V^Z, \zeta_V)$. For any $f \in \mathcal{H}(G, V, \zeta)$, $I_M(\pi, f)$ is defined inductively by the formula

$$I_M(\pi, f) = J_M(\pi, f) - \sum_{L \in \mathcal{L}^0(M)} \hat{I}_M^L(\pi, \phi_L(f)),$$

where ϕ_L is again the map (2.3), and $J_M(\pi, f)$ is the weighted character defined in [**11**] and [**15**]. We recall that if $\tilde{\pi} \in \Pi_{\mathrm{unit}}(M_V, \zeta_V)$ is a unitary representation of M_V whose restriction to M_V^Z is π, $J_M(\pi, f)$ is defined by an integral

$$\int_{i\mathfrak{a}_{M,Z}^*} \mathrm{tr}(\mathcal{M}_M(\tilde{\pi}_\lambda, P)\mathcal{I}_P(\tilde{\pi}_\lambda, \tilde{f}))\, \mathrm{d}\lambda,$$

where $\tilde{f}$ is a function in $\mathcal{H}(G_V, \zeta_V)$ whose restriction to G_V^Z equals f. This matches the definition of $\phi_M(f, \pi)$ if π is tempered, in which case $I_M(\pi, f)$ vanishes if $M \neq G$. In the general case, however, $I_M(\pi, f)$ is given by a complicated combination of residues of weighted characters in the complex domain.

The global terms in our spectral expansion again appear as coefficients. They are to be constructed from the basic coefficients $a_{\mathrm{disc}}^G(\dot{\pi})$ defined in [**8**, § 4]. Recall that for each $t \geqslant 0$, there is a discrete subset $\Pi_{t,\mathrm{disc}}(G) = \Pi_{\mathrm{disc}}(G, t)$ of $\Pi_{\mathrm{unit}}(G(\mathbb{A})^1)$ that supports a linear combination

$$I_{t,\mathrm{disc}}(\dot{f}^1) = \sum_{\dot{\pi} \in \Pi_{t,\mathrm{disc}}(G)} a_{\mathrm{disc}}^G(\dot{\pi}) \dot{f}_G^1(\dot{\pi}), \quad \dot{f}^1 \in \mathcal{H}(G), \tag{3.5}$$

of characters on $\mathcal{H}(G)$. The linear form $I_{t,\mathrm{disc}}(\dot{f}^1)$ is the 'discrete part' of $I_t(\dot{f}^1)$, relative to the spectral variable $\dot{\pi}$, and is defined by an explicit expression [**8**, (4.3)]. We recall that t equals the norm of the imaginary part of the archimedean infinitesimal character of any representation in $\Pi_{t,\mathrm{disc}}(G)$. Given $\Pi_{t,\mathrm{disc}}(G)$ from the construction of [**8**, § 4], we define $\Pi_{t,\mathrm{disc}}(G, \zeta)$ to be the set of representations in $\Pi_{\mathrm{unit}}(G(\mathbb{A})^Z, \zeta)$ whose restrictions to $G(\mathbb{A})^1$ lie in $\Pi_{t,\mathrm{disc}}(G)$. The restriction map identifies $\Pi_{t,\mathrm{disc}}(G, \zeta)$ with the subset of representations in $\Pi_{t,\mathrm{disc}}(G)$ whose central character on $Z(\mathbb{A})^1$ coincides with ζ. We can also define a linear form

$$I_{t,\mathrm{disc}}(\dot{f}) = I_{t,\mathrm{disc}}^\zeta(\dot{f}^1) = \int_{Z(F)\backslash Z(\mathbb{A})^1} I_{t,\mathrm{disc}}(\dot{f}_z^1)\zeta(z)\, \mathrm{d}z$$

by the general procedure of § 2, with $\dot{f}^1$ being any function in $\mathcal{H}(G)$ whose projection $\dot{f}^\zeta$ onto $\mathcal{H}(G, \zeta)$ equals $\dot{f}$. It comes with an expansion

$$I_{t,\mathrm{disc}}(\dot{f}) = \sum_{\dot{\pi} \in \Pi_{t,\mathrm{disc}}(G,\zeta)} a_{\mathrm{disc}}^G(\dot{\pi}) \dot{f}_G(\dot{\pi}), \quad \dot{f} \in \mathcal{H}(G, \zeta). \tag{3.6}$$

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For later use, we agree to extend the domain of $a^G_{\mathrm{disc}}(\dot{\pi})$ to $\Pi_t(G(\mathbb{A})^Z,\zeta)$, the subset of $\Pi(G(\mathbb{A})^Z,\zeta)$ associated to t, by setting it equal to zero on the complement of $\Pi_{t,\mathrm{disc}}(G,\zeta)$.

As in the geometric case, we are going to index spectral coefficients by objects associated with G_V, in this case representations $\pi \in \Pi_{\mathrm{unit}}(M_V,\zeta_V)$. Our general spectral coefficients will combine the discrete coefficients above with terms that come from unramified automorphic L-functions for G^V. The corresponding terms in the spectral expansion of [**8**, § 4] come from complete automorphic L-functions for $G(\mathbb{A})$, or rather, global normalizing factors that are conjectured to be quotients of L-functions. The source of the discrepancy will be our use here of the canonically normalized weighted characters of [**11**].

To describe the terms that come from unramified L-functions, we review some simple definitions from [**13**]. Let $\mathcal{C}(G^V,\zeta^V)$ be the set of families

$$c = \{c_v : v \notin V\},$$

with c_v being a semisimple conjugacy class in the L-group ${}^LG_v = \hat{G} \rtimes W_{F_v}$ whose image in W_{F_v} is a Frobenius element, that satisfy the following two conditions. First of all, each c_v must be compatible with the unramified character ζ_v on Z_v. In other words, the image of c_v under the projection ${}^LG_v \to {}^LZ_v$ is the conjugacy class in ${}^LZ_v$ associated to ζ_v. Secondly, we require that c satisfy an estimate

$$|A(c_v)| \leqslant q_v^{r_A}, \quad v \notin V,$$

for every $\hat{G}$-invariant polynomial A on ${}^LG$. As usual, q_v is the order of the residue field of F_v, while r_A is some constant that depends only on A. Suppose that c belongs to $\mathcal{C}(G^V,\zeta^V)$, and that ρ is a finite dimensional representation of ${}^LG$. We can then form the Euler product

$$L(s,c,\rho) = \prod_{v\notin V} \det(1-\rho(c_v)q_v^{-s})^{-1}, \quad s \in \mathbb{C},$$

which converges, and defines an analytic function of s in some right half plane. We note that there is a natural action

$$c \to c_\lambda = \{c_{v,\lambda} : v \notin V\}, \quad \lambda \in \mathfrak{a}^*_{G,Z,\mathbb{C}},$$

of $\mathfrak{a}^*_{G,Z,\mathbb{C}}$ on $\mathcal{C}(G^V,\zeta^V)$. As a function of (s,λ), $L(s,c_\lambda,\rho)$ is analytic for the real part of s large relative to the real part of λ.

By the theory of the Satake transform, any element $c \in \mathcal{C}(G^V,\zeta^V)$ can be identified with a K^V-unramified representation $\pi^V(c)$ in $\Pi(G^V,\zeta^V)$. If c belongs to $\mathcal{C}(G^V,\zeta^V)$, and π is a representation in $\Pi(G_V,\zeta_V)$, we shall write

$$\pi \times c = \pi \otimes \pi^V(c)$$

for the associated representation in $\Pi(G(\mathbb{A}),\zeta)$. We shall use the same notation if π belongs to the quotient $\Pi(G^Z_V,\zeta_V)$ of $\Pi(G_V,\zeta_V)$, with the understanding that π is identified with a representative in $\Pi(G_V,\zeta_V)$, and $\pi\times c$ is identified with a corresponding representative in $\Pi(G(\mathbb{A})^1,\zeta)$. Any use we make of this convention will depend ultimately only on π as an element in $\Pi(G_V^Z, \zeta_V)$. For example, it makes sense to define $\Pi_{t,\mathrm{disc}}(G, V, \zeta)$ as the set of representations $\pi \in \Pi_{\mathrm{unit}}(G_V^Z, \zeta_V)$ such that $\pi \times c$ belongs to $\Pi_{t,\mathrm{disc}}(G, \zeta)$, for some element $c \in \mathcal{C}(G^V, \zeta^V)$. We also define $\mathcal{C}^V_{\mathrm{disc}}(G, \zeta)$ to be the set of $c \in \mathcal{C}(G^V, \zeta^V)$ such that $\pi \times c$ belongs to $\Pi_{t,\mathrm{disc}}(G, \zeta)$, for some t and some $\pi \in \Pi_{t,\mathrm{disc}}(G, V, \zeta)$. The set $\mathcal{C}^V_{\mathrm{disc}}(G, \zeta)$ is invariant under the action of $i\mathfrak{a}^*_{G,Z}$.

Suppose that $M \in \mathcal{L}$ is a fixed Levi subgroup of G, and that $\hat{M} \subset \hat{G}$ is a dual Levi subgroup [**12**, § 1]. Then there is a bijection $P \to \hat{P}$ from the set $\mathcal{P}(M)$ of parabolic subgroups of G with Levi component M to the set $\mathcal{P}(\hat{M})$ of Γ-stable parabolic subgroups of $\hat{G}$ with Levi component $\hat{M}$. If $P, Q \in \mathcal{P}(M)$, let $\rho_{Q|P}$ denote the adjoint representation of ${}^L M$ on the Lie algebra of the intersection of the unipotent radicals of $\hat{P}$ and $\hat{\bar{Q}}$. Suppose that c belongs to $\mathcal{C}^V_{\mathrm{disc}}(M, \zeta)$. It follows from a theorem of Shahidi [**34**] that $L(s, c, \rho_{Q|P})$ has analytic continuation as a meromorphic function of $s \in \mathbb{C}$, and that for any fixed s, $L(s, c_\lambda, \rho_{Q|P})$ is a meromorphic function of λ in $\mathfrak{a}^*_{M,Z,\mathbb{C}}$. Following the usual prescription, we define unramified normalizing factors

$$r_{Q|P}(c_\lambda) = L(0, c_\lambda, \rho_{Q|P}) L(1, c_\lambda, \rho_{Q|P})^{-1}, \quad P, Q \in \mathcal{P}(M).$$

We then form the (G, M)-family of functions

$$r_Q(\Lambda, c_\lambda) = r_{Q|\bar{Q}}(c_\lambda)^{-1} r_{Q|\bar{Q}}(c_{\lambda+\Lambda/2}), \quad Q \in \mathcal{P}(M), \tag{3.7}$$

of $\Lambda \in i\mathfrak{a}^*_M$, as in [**13**, § 4]. The limit

$$r^G_M(c_\lambda) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} r_Q(\Lambda, c_\lambda) \theta_Q(\Lambda)^{-1} \tag{3.8}$$

is then defined as a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,Z,\mathbb{C}}$.

Lemma 3.2. *Assume that $c \in \mathcal{C}^V_{\mathrm{disc}}(M, \zeta)$. Then $r^G_M(c_\lambda)$ is an analytic function of $\lambda \in i\mathfrak{a}^*_{M,Z}$, and satisfies an inequality*

$$\int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} r^G_M(c_\lambda)(1 + \|\lambda\|)^{-N} \,\mathrm{d}\lambda < \infty, \tag{3.9}$$

for some N.

Proof. Since c belongs to $\mathcal{C}^V_{\mathrm{disc}}(M, \zeta)$, there is a representation

$$\dot{\pi} = \pi \times c, \quad \pi \in \Pi_{t,\mathrm{disc}}(M, V, \zeta),$$

that lies in $\Pi_{t,\mathrm{disc}}(M, \zeta)$ for some $t \geqslant 0$. The automorphic representation $\dot{\pi}$ and the representation π of G_V can both be assigned their own sets of (non-canonical) normalizing factors $\{r_{Q|P}(\dot{\pi}_\lambda)\}$ and $\{r_{Q|P}(\pi_\lambda)\}$. (See [**8**, § 4], for example.) Let $r_Q(\Lambda, \dot{\pi}_\lambda)$ and $r_Q(\Lambda, \pi_\lambda)$ be corresponding (G, M)-families, defined by analogues of (3.7). Then

$$r_Q(\Lambda, \pi_\lambda) r_Q(\Lambda, c_\lambda) = r_Q(\Lambda, \dot{\pi}_\lambda).$$

It was actually the (G, M)-family

$$r_Q(\Lambda, \dot{\pi}_\lambda, P) = r_{Q|P}(\dot{\pi}_\lambda)^{-1} r_{Q|P}(\dot{\pi}_{\lambda+\Lambda}), \quad \Lambda \in i\mathfrak{a}_M^*, \quad Q \in \mathcal{P}(M),$$

defined for a fixed $P \in \mathcal{P}(M)$, that was used in [**8**] and earlier papers. With this in mind, we rewrite the function $r_Q(\Lambda, \dot{\pi}_\lambda)$ in the form

$$r_Q(\Lambda, \pi_\lambda) r_Q(\Lambda, c_\lambda) = \nu_Q(\Lambda, \dot{\pi}_\lambda, P) r_Q(\Lambda, \dot{\pi}_\lambda, P), \tag{3.10}$$

where

$$\nu_Q(\Lambda, \dot{\pi}_\lambda, P) = r_Q(\Lambda, \dot{\pi}_\lambda) r_Q(\Lambda, \dot{\pi}_\lambda, P)^{-1}.$$

We claim that the limit

$$\nu_M^L(\dot{\pi}_\lambda, P) = \lim_{\Lambda \to 0} \sum_{\{Q \in \mathcal{P}(M): Q \subset Q_L\}} \nu_Q(\Lambda, \dot{\pi}_\lambda, P) \theta_{Q \cap L}(\Lambda)^{-1}, \quad Q_L \in \mathcal{P}(L),$$

vanishes for any $L \in \mathcal{L}(M)$ with $L \neq M$. Indeed, a global version of the argument at the end of the proof of [**11**, Lemma 2.1] tells us that

$$\nu_M^L(\dot{\pi}_\lambda, P) = \mu_M^L(\dot{\pi}_\lambda, P),$$

where $\mu_M^L(\dot{\pi}_\lambda, P)$ is obtained from the (G, M)-family

$$\mu_Q(\Lambda, \dot{\pi}_\lambda, P) = \mu_{Q|P}(\dot{\pi}_\lambda)^{-1} \mu_{Q|P}(\dot{\pi}_{\lambda+\Lambda/2})$$

that is constructed from global Plancherel densities

$$\mu_{Q|P}(\dot{\pi}_\lambda) = (r_{Q|P}(\dot{\pi}_\lambda) r_{P|Q}(\dot{\pi}_\lambda))^{-1}.$$

By the functional equation of the global normalizing factors $r_{Q|P}(\dot{\pi}_\lambda)$ [**8**, p. 519], $\mu_{Q|P}(\dot{\pi}_\lambda)$ equals 1 for every P and Q. The claim follows from the fact that

$$\lim_{\Lambda \to 0} \sum_{\{Q \in \mathcal{P}(M): Q \subset Q_L\}} \theta_{Q \cap L}(\Lambda)^{-1} = 0, \quad L \neq M.$$

If we apply the splitting formula [**7**, Corollary 7.4] to each side of (3.10), we obtain an identity

$$\sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) r_M^{L_1}(\pi_\lambda) r_M^{L_2}(c_\lambda) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) \nu_M^{L_1}(\dot{\pi}_\lambda, P) r_M^{L_2}(\dot{\pi}_\lambda, P),$$

which then reduces to

$$r_M^G(c_\lambda) = r_M^G(\dot{\pi}_\lambda, P) - \sum_{\{L_1, L_2 \in \mathcal{L}(M): L_2 \neq G\}} d_M^G(L_1, L_2) r_M^{L_1}(\pi_\lambda) r_M^{L_2}(c_\lambda).$$

The assertions of the lemma are known to hold if $r_M^G(c_\lambda)$ is replaced by the function $r_M^G(\dot{\pi}_\lambda, P)$. (See the discussion on [**8**, p. 519], which is based on [**3**, Proposition 7.5 and

Lemma 8.4].) Since the representation $\pi = \bigotimes_{v\in V} \pi_v$ is unitary, the assertions also hold if $r_M^G(c_\lambda)$ is replaced by any of the functions $r_M^{L_1}(\pi_\lambda)$. This follows from [**11**, Corollary 2.4], and the growth properties of local normalizing factors. Finally, we can assume inductively that the assertions of the lemma hold if $r_M^G(c_\lambda)$ is replaced by any of the functions $r_M^{L_2}(c_\lambda)$, with $L_2 \neq G$. In particular, the contribution to (3.9) of a pair (L_1, L_2), with $L_2 \neq G$ and $d_M^G(L_1, L_2) \neq 0$, will be given by a convergent double integral over the space

$$(i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{L_1,Z}^*) \oplus (i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{L_2,Z}^*) \cong i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{G,Z}^*.$$

The lemma follows. □

Before defining the general spectral coefficients, we first construct a subset $\Pi_t(G,V,\zeta)$ of $\Pi_{\mathrm{unit}}(G_V^Z, \zeta_V)$ from the sets $\Pi_{t,\mathrm{disc}}(M,V,\zeta)$. Let $\tilde{\Pi}_{t,\mathrm{disc}}(G,V,\zeta)$ be the preimage of $\Pi_{t,\mathrm{disc}}(G,V,\zeta)$ in $\Pi_{\mathrm{unit}}(G_V,\zeta_V)$. Then $\Pi_{t,\mathrm{disc}}(G,V,\zeta)$ can be identified with the set of $i\mathfrak{a}_{G,Z}^*$-orbits in $\tilde{\Pi}_{t,\mathrm{disc}}(G,V,\zeta)$. There is of course also a similar set if G is replaced by M. We write $\Pi_{t,\mathrm{disc}}^G(M,V,\zeta)$ for the set of $i\mathfrak{a}_{G,Z}^*$-orbits in $\tilde{\Pi}_{t,\mathrm{disc}}(M,V,\zeta)$. Then $i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{G,Z}^*$ has a free action $\rho \to \rho_\lambda$ on $\Pi_{t,\mathrm{disc}}^G(M,V,\zeta)$, whose orbits can be identified with $\Pi_{t,\mathrm{disc}}(M,V,\zeta)$. Any element ρ in $\Pi_{t,\mathrm{disc}}^G(M,V,\zeta)$ is an irreducible representation of $M_V \cap G_V^Z$, from which one can form the parabolically induced representation ρ^G of G_V^Z. We define $\Pi_t(G,V,\zeta)$ to be the union, over $M \in \mathcal{L}$ and $\rho \in \Pi_{t,\mathrm{disc}}^G(M,V,\zeta)$, of the irreducible constituents of ρ^G. This space comes with a Borel measure $\mathrm{d}\pi$, defined by setting

$$\int_{\Pi_t(G,V,\zeta)} h(\pi)\,\mathrm{d}\pi = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\rho\in\Pi_{t,\mathrm{disc}}(M,V,\zeta)} \int_{i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{G,Z}^*} h(\rho_\lambda^G)\,\mathrm{d}\lambda, \quad (3.11)$$

for any $h \in C_c(\Pi_t(G,V,\zeta))$.

We now define the general spectral coefficients. If π belongs to $\Pi_t(G_V^Z, \zeta_V)$, we set

$$a^G(\pi) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{c\in\mathcal{C}_{\mathrm{disc}}^V(M,\zeta)} a_{\mathrm{disc}}^M(\pi_M \times c) r_M^G(c). \quad (3.12)$$

Of course $\pi_M \times c$ is a finite sum of representations $\dot{\pi}$ in $\Pi_{\mathrm{unit}}(M(\mathbb{A}),\zeta)$, and $a_{\mathrm{disc}}^M(\pi_M \times c)$ is the corresponding sum of values $a_{\mathrm{disc}}^M(\dot{\pi})$. A similar convention applies to the integrand $h(\rho_\lambda^G)$ in (3.11). It follows from the definitions that $a^G(\pi)$ is supported on the subset $\Pi_t(G,V,\zeta)$ of $\Pi_t(G_V^Z,\zeta_V)$.

We have formulated the definition (3.12) in obvious analogy with that of the geometric coefficients (2.8). We could have made it slightly simpler. The role of $\Pi_t(G,V,\zeta)$ in the trace formula will be strictly that of a measure space, which means that we can ignore sets of measure 0. If ρ belongs to $\Pi_{t,\mathrm{disc}}^G(M,V,\zeta)$, and λ lies in the complement of a set of measure 0 in $i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{G,Z}^*$, the representation ρ_λ^G is irreducible. Moreover, if M_1 belongs to $\mathcal{L}$, and λ again lies in the complement of a set of measure 0, the irreducible

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components of $(\rho^G_\lambda)_{M_1}$ are disjoint from $\Pi^G_{t,\mathrm{disc}}(M_1, V, \zeta)$ unless M_1 lies in the W_0-orbit of M, in which case

$$(\rho^G_\lambda)_{M_1} = \bigoplus_w w\rho_\lambda, \quad w \in W^G_0/W^M_0, \quad wM = M_1.$$

It is easy to check from the original definition in [**8**] that the coefficients $a^M_{\mathrm{disc}}(\dot\pi)$ are invariant under isomorphisms of M. In particular,

$$a^{M_1}_{\mathrm{disc}}(w\rho_\lambda \times wc) = a^M_{\mathrm{disc}}(\rho_\lambda \times c), \quad wM = M_1, c \in \mathcal{C}^V_{\mathrm{disc}}(M, \zeta).$$

We could therefore have defined $\Pi_t(G, V, \zeta)$ to be the disjoint union of induced representations ρ^G, as ρ ranges over the set of W_0-orbits in

$$\coprod_{M \in \mathcal{L}} (\Pi^G_{t,\mathrm{disc}}(M, V, \zeta)).$$

The coefficient $a^G_{\mathrm{disc}}(\rho^G)$ would then be defined as

$$\sum_{c \in \mathcal{C}^V_{\mathrm{disc}}(M,\zeta)} a^M_{\mathrm{disc}}(\rho \times c) r^G_M(c).$$

(See [**8**, p. 519].) The two formulations are the same under an isomorphism of measure spaces.

Proposition 3.3. *Suppose that $f \in \mathcal{H}(G, V, \zeta)$ and $t \geqslant 0$. Then the linear form $I_t(f)$ in (3.2) has a spectral expansion*

$$I_t(f) = \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \int_{\Pi_t(M,V,\zeta)} a^M(\pi) I_M(\pi, f)\, \mathrm{d}\pi, \tag{3.13}$$

in which the integral converges absolutely.

Proof. As a function on $\Pi_t(M, V, \zeta)$, $I_M(\pi, f)$ is rapidly decreasing. The absolute convergence of the integral then follows from Lemma 3.2.

The proof of (3.13) follows the same steps as that of Proposition 2.2. Again the main point is to construct a parallel expansion for the non-invariant linear form

$$J_t(f) = J^\zeta_t(\dot f^1) = \int_{Z(F)\backslash Z(\mathbb{A})^1} J_t(\dot f^1_z)\zeta(z)\, \mathrm{d}z,$$

where $\dot f^1$ is any function in $\mathcal{H}(G)$ whose projection onto $\mathcal{H}(G, \zeta)$ equals $\dot f = f \times u^V$.

It follows from [**3**, Theorem 8.2] and the definitions of [**8**, § 4] that $J_t(f)$ has an expansion

$$\int_{Z(F)\backslash Z(\mathbb{A})^1} \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\dot\pi \in \Pi_{t,\mathrm{disc}}(M)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} a^M_{\mathrm{disc}}(\dot\pi_\lambda) J_M(\dot\pi_\lambda, \dot f^1_z)\zeta(z)\, \mathrm{d}\lambda\, \mathrm{d}z,$$

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where

$$J_M(\dot\pi_\lambda, \dot f_z^1) = \operatorname{tr}(\mathcal{J}_M(\dot\pi_\lambda, P)\mathcal{I}_P(\dot\pi_\lambda, \dot f_z^1))$$

is the global *unnormalized* weighted character. The operator

$$\mathcal{J}_M(\dot\pi_\lambda, P) = \lim_{\Lambda\to 0} \sum_{Q\in\mathcal{P}(M)} \mathcal{J}_Q(\Lambda, \dot\pi_\lambda, P)\theta_Q(\Lambda)^{-1}$$

is obtained from the (G, M)-family

$$\mathcal{J}_Q(\Lambda, \dot\pi_\lambda, P) = J_{Q|P}(\dot\pi_\lambda)^{-1} J_{Q|P}(\dot\pi_{\lambda+\Lambda}),$$

in which

$$J_{Q|P}(\dot\pi_\lambda) : \mathcal{H}_P(\dot\pi) \to \mathcal{H}_Q(\dot\pi)$$

is the global (unnormalized) intertwining operator that comes from the theory of Eisenstein series. The integral over $Z(F)\backslash Z(\mathbb{A})^1$ simply annihilates the contributions from those $\dot\pi$ in the complement of $\Pi_{t,\mathrm{disc}}(M,\zeta)$ in $\Pi_{t,\mathrm{disc}}(M)$. Consequently, $J_t(f)$ equals

$$\sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\dot\pi\in\Pi_{t,\mathrm{disc}}(M,\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} a^M_{\mathrm{disc}}(\dot\pi_\lambda) J_M(\dot\pi_\lambda, \dot f)\,\mathrm{d}\lambda. \tag{3.14}$$

Consider a representation $\dot\pi$ in $\Pi_{t,\mathrm{disc}}(M,\zeta)$ that is unramified outside of V. Then we can write

$$\dot\pi = \dot\pi_V \otimes \pi^V(c) = \pi\times c, \quad \pi\in\Pi_{t,\mathrm{disc}}(M,V,\zeta), \quad c\in\mathcal{C}^V_{\mathrm{disc}}(M,\zeta).$$

We would like to express $J_M(\dot\pi_\lambda, \dot f)$ in terms of the local *normalized* weighted characters

$$J_L(\pi^L_\lambda, f) = \operatorname{tr}(\mathcal{M}_L(\pi^L_\lambda, P_L)\mathcal{I}_{P_L}(\pi^L_\lambda, f)), \quad L\in\mathcal{L}(M), \quad P_L\in\mathcal{P}(L).$$

(We have allowed the same symbol J to denote two different linear forms on the two different spaces $\mathcal{H}(G)$ and $\mathcal{H}(G,V,\zeta)$.) The problem is obviously one of comparison between two operator valued (G,M)-families $\{\mathcal{J}_Q(\Lambda,\dot\pi_\lambda,P)\}$ and $\{\mathcal{M}_Q(\Lambda,\pi_\lambda,P)\}$. Since $\dot\pi$ is unramified outside of V, $\mathcal{J}_Q(\Lambda,\dot\pi_\lambda,P)$ is a scalar multiple of $\mathcal{M}_Q(\Lambda,\pi_\lambda,P)$. More precisely,

$$\begin{aligned}\mathcal{J}_Q(\Lambda,\dot\pi_\lambda,P) &= r_Q(\Lambda,c_\lambda,P)\mathcal{J}_Q(\lambda,\pi_\lambda,P)\\ &= r_Q(\Lambda,c_\lambda,P)\mu_Q(\Lambda,\pi_\lambda,P)^{-1}\mathcal{M}_Q(\Lambda,\pi_\lambda,P),\end{aligned}$$

in the notation of [**11**, § 2] and the proof of Lemma 3.2. Since

$$\mu_Q(\Lambda,\pi_\lambda,P)\mu_Q(\Lambda,c_\lambda,P) = \mu_Q(\Lambda,\dot\pi_\lambda,P) = 1,$$

by the triviality of global Plancherel densities noted in the proof of Lemma 3.2, we obtain

$$\mathcal{J}_Q(\Lambda,\dot\pi_\lambda,P) = (r_Q(\Lambda,c_\lambda,P)\mu_Q(\Lambda,c_\lambda,P))\mathcal{M}_Q(\Lambda,\pi_\lambda,P).$$

We apply the splitting formula [**2**, Lemma 6.5] to this product of (G,M)-families. A variant of [**11**, Lemma 2.1], which applies to the function u^V on $G(\mathbb{A}^V)$, asserts that the limit

$$\lim_{\Lambda\to 0}\sum_{Q\in\mathcal{P}(M)}(r_Q(\Lambda,c_\lambda,P)\mu_Q(\Lambda,c_\lambda,P))\theta_Q(\Lambda)^{-1}$$

equals $r_M^G(c_\lambda)$. Indeed, the left-hand side of the analogue of [**11**, (2.6)] for u^V equals the given limit, while the summand corresponding to $L\in\mathcal{L}(M)$ on the right-hand side vanishes unless $L=G$, in which case it equals $r_M^G(c_\lambda)$. A similar assertion holds if G is replaced by any $L\in\mathcal{L}(M)$. The splitting formula then yields the identity

$$\mathcal{J}_M(\dot{\pi}_\lambda,P)=\sum_{L\in\mathcal{L}(M)}r_M^L(c_\lambda)\mathcal{M}_L(\pi_\lambda^L,P).$$

It follows that

$$J_M(\dot{\pi}_\lambda,\dot{f})=\sum_{L\in\mathcal{L}(M)}r_M^L(c_\lambda)J_L(\pi_\lambda^L,f).$$

Since $\dot{f}$ equals $f\times u^V$, the term $J_M(\dot{\pi}_\lambda,\dot{f})$ in (3.14) vanishes unless $\dot{\pi}$ is unramified outside of V. We can therefore replace the sum over $\dot{\pi}\in\Pi_{t,\mathrm{disc}}(M,\zeta)$ with a double sum over $\pi\in\Pi_{t,\mathrm{disc}}(M,V,\zeta)$ and $c\in\mathcal{C}^V_{\mathrm{disc}}(M,\zeta)$. At the same time, we can substitute the formula we have just obtained for $J_M(\dot{\pi}_\lambda,\dot{f})$. Ignoring sets of measure 0, as in the remark following the definition (3.12), we write

$$a^L(\pi_\lambda^L)=\sum_{c\in\mathcal{C}^V_{\mathrm{disc}}(M,\zeta)}a^M_{\mathrm{disc}}(\pi_\lambda\times c)r_M^L(c)=\sum_c a^M_{\mathrm{disc}}(\pi_\lambda\times c_\lambda)r_M^L(c_\lambda),$$

for any point $\lambda\in i\mathfrak{a}^*_{M,Z}$ in general position. The expansion (3.14) for $J_t(f)$ becomes

$$\sum_{L\in\mathcal{L}}|W_0^L||W_0^G|^{-1}\sum_{M\in\mathcal{L}^L}|W_0^M||W_0^L|^{-1}\sum_{\pi\in\Pi_{t,\mathrm{disc}}(M,V,\zeta)}\int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}}a^L(\pi_\lambda^L)J_L(\pi_\lambda^L,f)\,\mathrm{d}\lambda.$$

Now the coefficient $a^L(\pi_\lambda^L)$, and the integral

$$J_L(\pi^L_{\lambda_1})=\int_{i\mathfrak{a}^*_{L,Z}/i\mathfrak{a}^*_{G,Z}}J_L(\pi^L_{\lambda+\Lambda},f)\,\mathrm{d}\Lambda,$$

both depend only on the image λ_1 of λ in $i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{L,Z}$. In other words, they depend only on the restriction $\pi^L_{\lambda_1}$ of the representation π^L_λ to L^Z_V. Changing notation, we write π instead of $\pi^L_{\lambda_1}$. Then π runs over representations in $\Pi_t(L,V,\zeta)$. Recalling the definition (3.11) of the measure $\mathrm{d}\pi$ on $\Pi_t(L,V,\zeta)$, we obtain at last an expansion

$$J_t(f)=\sum_{L\in\mathcal{L}}|W_0^L||W_0^G|^{-1}\int_{\Pi_t(L,V,\zeta)}a^L(\pi)J_L(\pi,f)\,\mathrm{d}\pi \tag{3.15}$$

of the form we want.

We can now argue exactly as at the end of the proof of Proposition 2.2. Assuming inductively that (3.13) holds if G is replaced by a group $L \in \mathcal{L}^0$, we obtain

$$\begin{aligned} I_t(f) &= J_t(f) - \sum_{L \in \mathcal{L}^0} |W_0^L||W_0^G|^{-1} \hat{I}_t^L(\phi_L(f)) \\ &= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi_t(M,V,\zeta)} a^M(\pi)\Big(J_M(\pi, f) - \sum_{L \in \mathcal{L}^0(M)} \hat{I}_M^L(\pi, \phi_L(f))\Big)\, \mathrm{d}\pi \\ &= \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \int_{\Pi_t(M,V,\zeta)} a^M(\pi) I_M(\pi, f)\, \mathrm{d}\pi. \end{aligned}$$

This is the required expansion. □

If we recall this stage of the discussion of the geometric side, we can imagine what to do next. We introduce the linear form

$$I_{t,\mathrm{unit}}(f) = \int_{\Pi_t(G,V,\zeta)} a^G(\pi) f_G(\pi)\, \mathrm{d}\pi, \quad f \in \mathcal{H}(G, V, \zeta), \tag{3.16}$$

defined by the term with $M = G$ in the expansion (3.13). This is the purely 'unitary' part of $I_t(f)$, which consists of a (continuous) linear combination of irreducible unitary characters. Not surprisingly, $I_{t,\mathrm{unit}}(f)$ will play a role that is parallel to that of $I_{\mathrm{orb}}(f)$.

4. *K*-groups and transfer factors

To study the transfer properties of the various objects in the trace formula, it is best to work with several groups simultaneously. From now on, G will be a *multiple group* over the field F, in the sense of [**12**, § 1]. Thus

$$G = \coprod_{\alpha} G_\alpha, \quad \alpha \in \pi_0(G),$$

is a variety whose connected components G_α are reductive groups over F, equipped with an equivalence class of frames

$$(\psi, u) = \{(\psi_{\alpha\beta}, u_{\alpha\beta}) : \alpha, \beta \in \pi_0(G)\}.$$

Recall that $\psi_{\alpha\beta}$: $G_\beta \to G_\alpha$ is an isomorphism over $\bar{F}$, and that $u_{\alpha\beta}$: $\Gamma \to G_{\alpha,\mathrm{sc}}$ is a locally constant function from the Galois group $\Gamma = \mathrm{Gal}(\bar{F}/F)$ to $G_{\alpha,\mathrm{sc}}$. (As usual, $G_{\alpha,\mathrm{sc}}$ stands for the simply connected cover of the derived group $G_{\alpha,\mathrm{der}}$ of G_α.) A given pair $(\psi_{\alpha\beta}, u_{\alpha\beta})$ is required to satisfy some compatibility conditions, while equivalence of frames (ψ, u) is defined in a natural way in terms of conjugacy.

We shall make free use of the notation and terminology of [**12**]. For example, a homomorphism between multiple groups G and $\bar{G}$ over F is a morphism

$$\theta = \coprod_{\alpha} (\theta_\alpha : G_\alpha \to \bar{G}_{\bar{\alpha}})$$

https://doi.org/10.1017/S1474748002000051 Published online by Cambridge University Press

from G to $\bar{G}$ (as varieties over F) that preserves all the structure. In other words, there are frames (ψ, u) and $(\bar{\psi}, \bar{u})$ for G and $\bar{G}$ such that $\theta_\alpha \circ \psi_{\alpha\beta} = \bar{\psi}_{\bar{\alpha}\bar{\beta}} \circ \theta_\beta$ and $\bar{u}_{\bar{\alpha}\bar{\beta}} = \theta_{\alpha,\mathrm{sc}}(u_{\alpha\beta})$, for each $\alpha, \beta \in \pi_0(G)$. An *isomorphism* of multiple groups is of course an invertible homomorphism. In this paper, we shall also make use of a weaker notion of isomorphism. We shall say that a map $\theta\colon G \to \bar{G}$ is a *weak isomorphism* if it satisfies all the requirements of an isomorphism *except* for the condition relating $\bar{u}_{\bar{\alpha}\bar{\beta}}$ with $u_{\alpha\beta}$. We introduce this notion in order to be able to identify multiple groups that differ only in the choices of functions $\{u_{\alpha\beta}\}$.

Another notion from [**12**] is that of a *Levi subgroup* M of the multiple group G. For any such M, we construct the associated objects $W(M)$, $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\mathcal{F}(M)$ as in [**12**]. We can also form a dual group $\hat{G}$ for G, and a dual Levi subgroup $\hat{M} \subset \hat{G}$ for M. Any such $\hat{M}$ comes with a bijection $L \to \hat{L}$ and $P \to \hat{P}$ from $\mathcal{L}(M)$ to $\mathcal{L}(\hat{M})$ and from $\mathcal{P}(M)$ to $\mathcal{P}(\hat{M})$. (Recall that $\mathcal{P}(\hat{M})$, $\mathcal{L}(\hat{M})$ and $\mathcal{F}(\hat{M})$ consist of groups that are Γ-stable.) Finally, we have the notion of a *quasisplit inner twist* G^* for G, and of a Levi subgroup M^* of G^* *corresponding* to M. Then G^* can be regarded as a component of a multiple group $G \amalg G^*$, and M^* is a component of a Levi subgroup $M \amalg M^*$. Any such M^* comes with bijections $L \to L^*$ and $P \to P^*$ from $\mathcal{L}(M)$ to $\mathcal{L}(M^*)$ and from $\mathcal{P}(M)$ to $\mathcal{P}(M^*)$.

Suppose for a moment that F is a local field. Following suggestions of Kottwitz (which were in turn motivated by ideas of Vogan), we introduced a notion in [**12**, § 2] that we called a K-group. By definition, a K-group over F is a multiple group G such that the functions $u_{\alpha\beta}\colon \Gamma \to G_{\alpha,\mathrm{sc}}$ are all 1-cocycles, and such that for each α, the map

$$\{u_{\alpha_\beta} : \beta \in \pi_0(G)\} \to H^1(F, G_\alpha)$$

is a bijection onto the image of $H^1(F, G_{\alpha,\mathrm{sc}})$ in $H^1(F, G_\alpha)$. A K-group over a p-adic field F is just an ordinary connected group. However, a K-group over $F = \mathbb{R}$ can have several connected components.

We assume for the rest of this section that F is a global field. Suppose that G is a multiple group over F that satisfies the global analogue of the property above. That is, for every frame (ψ, u), the functions $u_{\alpha\beta}\colon \Gamma \to G_{\alpha,\mathrm{sc}}$ are 1-cocycles, and for any α, the map $\{u_{\alpha_\beta}\} \to H^1(F, G_\alpha)$ is a bijection onto the image of $H^1(F, G_{\alpha,\mathrm{sc}})$ in $H^1(F, G_\alpha)$. We shall be interested in representing G as a product of local K-groups. We define a *local product structure* on G to be a family of local K-groups (G_v, F_v), indexed by the valuations of F, and a family of (multiple group) homomorphisms $G \to G_v$ over F_v whose restricted direct product

$$G \to \prod_v G_v$$

is an isomorphism of schemes over $\mathbb{A}$. Such a structure determines a surjective map

$$\alpha \to \alpha_V = \prod_{v \in V} \alpha_v, \quad \alpha \in \pi_0(G), \quad \alpha_v \in \pi_0(G_v),$$

of components, for any finite set V of valuations, which is bijective if V contains V_∞. We also obtain a group theoretic injection of $G_\alpha(F)$ into $G_{\alpha_V}(F_V)$, for each $\alpha \in \pi_0(G)$. We shall often write

$$G_V = \coprod_{\alpha_V} G_{V,\alpha_V} = \coprod_{\alpha_V} G_{V,\alpha_V}(F_V),$$

a set we can also represent as a product

$$G_V = \prod_{v \in V} G_v = \prod_{v \in V} G_v(F_v).$$

It is easy to see from the Hasse principle for the groups $G_{\alpha,\mathrm{sc}}$, together with Lemmas 4.3.1(b) and 4.3.2(b) of [**21**], that G does have a local product structure.

We define a *K-group* over the global field F to be a multiple group over F, as above, together with a local product structure. Suppose that G is a K-group over F, and that G^* is a quasisplit inner twist of G. By definition, G^* is a connected quasisplit group over F, together with a G^*-inner class of inner twists ψ_α: $G_\alpha \to G^*$ and a corresponding family of functions u_α: $\Gamma \to G^*_{\mathrm{sc}}$, for $\alpha \in \pi_0(G)$. Then G^* determines a quasisplit inner twist G^*_v of each of the local K-groups G^*_v. Following [**12**], we shall sometimes refer to G as an inner *K-form* of G^*. We shall say that G is quasisplit if one of its components is quasisplit (over F).

We emphasize that K-groups have been introduced only to streamline some aspects of the study of connected groups. If we are given a connected reductive group G_1 over F, we can find a K-group G over F such that $G_{\alpha_1} = G_1$ for some $\alpha_1 \in \pi_0(G)$. There could of course be several such G, but the weak isomorphism class of G is uniquely determined by G_1. In particular, any connected quasisplit group G^* has a quasisplit inner K-form G, which is unique up to weak isomorphism.

Suppose that G is a K-group over F, with quasisplit inner twist G^*. As in the local case, any Levi subgroup M of G inherits the structure of a K-group. We shall investigate the case that M is minimal.

For each v, we write $\hat{\zeta}_{G,v}$ for the character attached to the local K-group G_v that was denoted by ζ_{G_v} in [**12**, (2.2)]. Then $\hat{\zeta}_{G,v}$ is a character on the group $\hat{Z}_{\mathrm{sc}}^{\Gamma_v}$ of invariants of the local Galois group $\Gamma_v = \mathrm{Gal}(\bar{F}_v/F_v)$ in the centre $\hat{Z}_{\mathrm{sc}} = Z(\hat{G}_{\mathrm{sc}})$. It is trivial unless v belongs to $V_{\mathrm{ram}}(G) = V_{\mathrm{ram}}(G_\alpha)$, $\alpha \in \pi_0(G)$. The tensor product

$$\hat{\zeta}_G = \bigotimes_v \hat{\zeta}_{G,v} : \prod_v (\hat{Z}_{\mathrm{sc}}^{\Gamma_v}) \to \mathbb{C}^*$$

over v of these characters is invariant on the diagonal image of $\hat{Z}_{\mathrm{sc}}^{\Gamma}$ in $\prod_v \hat{Z}_{\mathrm{sc}}^{\Gamma_v}$. (See [**22**, § 2].) Now the canonical based root datum

$$(X_G, \Delta_G, X_G^\vee, \Delta_G^\vee)$$

for G is canonically isomorphic to its counterpart for G^*. The canonical isomorphism

$$\hat{Z}_{\mathrm{sc}} \cong X_{G^*_{\mathrm{sc}}}/X_{G^*_{\mathrm{ad}}}$$

leads to a map

$$\alpha \to z_\alpha, \quad \alpha \in \Delta_G/\Gamma,$$

from the set of Γ-orbits of simple roots into Z_{sc}^{Γ}. The definition is identical to the case of a local K-group treated in [**12**, § 2]. This in turn determines a Γ-stable subset

$$\Delta_0 = \{\alpha \in \Delta_G : \hat{\zeta}_{G,v}(z_\alpha) = 1,\ v \in V_{\mathrm{ram}}(G)\}$$

of Δ_G. On the other hand, the set of simple roots of any parabolic subgroup P of G over F also determines a Γ-stable subset Δ_P of Δ_G.

Lemma 4.1. *Suppose that Δ is a Γ-stable subset of Δ_G. Then there is a parabolic subgroup P of G over F with $\Delta_P = \Delta$ if and only if Δ_P is contained in Δ_0.*

Proof. The proof of the lemma is essentially the same as that of its local analogue [**12**, Lemma 2.1]. The only difference is that in place of the local map

$$K_v : H^1(F_v, G^*_{v,\mathrm{ad}}) \to \hat{Z}_{\mathrm{sc}}^{\Gamma_v},$$

whose kernel is the image of $H^1(F_v, G^*_{v,\mathrm{sc}})$ in $H^1(F_v, G^*_{v,\mathrm{ad}})$, we use the composition of maps

$$K : H^1(F, G^*_{\mathrm{ad}}) \to \prod_v H^1(F_v, G^*_{v,\mathrm{ad}}) \to \prod_v \hat{Z}_{\mathrm{sc}}^{\Gamma_v},$$

of which the kernel is the image of $H^1(F, G^*_{\mathrm{sc}})$ in $H^1(F, G^*_{\mathrm{ad}})$. In particular, the role of the local character $\hat{\zeta}_{G,v}$ in the earlier proof is taken here by the global product $\hat{\zeta}_G$. The definition of a global K-group is such that the proof of [**12**, Lemma 2.17] applies directly to the global situation here. □

As in the local case [**12**, Corollary 2.2], the proof of the lemma also provides a corollary.

Corollary 4.2. *Suppose that R is a Levi subgroup of G^*, with a dual Levi subgroup $\hat{R} \subset \hat{G}$. Then R corresponds to a Levi subgroup M of G (with dual Levi subgroup $\hat{M} = \hat{R}$) if and only if for each v, $\hat{\zeta}_{G,v}$ is trivial on the subgroup*

$$\hat{Z}_{\mathrm{sc}}^{\Gamma_v} \cap (Z(\hat{R}_{\mathrm{sc}})^{\Gamma})^0 = \hat{Z}_{\mathrm{sc}}^{\Gamma_v} \cap (Z(\hat{M}_{\mathrm{sc}})^{\Gamma})^0$$

*of $\hat{Z}_{\mathrm{sc}}^{\Gamma_v}$. (As in [**12**, § 1], $\hat{M}_{\mathrm{sc}}$ stands for the preimage of $\hat{M}$ in $\hat{G}_{\mathrm{sc}}$.)*

If M is a Levi subgroup of G, with dual Levi subgroup $\hat{M} \subset \hat{G}$, $\hat{\zeta}_G$ is the pullback of a character $\hat{\zeta}_G^M = \prod_v \hat{\zeta}_{G,v}^M$ on $\prod_v \pi_0(Z(\hat{M}_{\mathrm{sc}})^{\Gamma_v})$. The Levi subgroup will be *minimal* if and only if for any $P \in \mathcal{P}(M)$, Δ_P equals Δ_0. In this case, we write $M_0 = M$, and we denote the character $\hat{\zeta}_G^M$ by

$$\hat{\zeta}_G^0 = \prod_v \hat{\zeta}_{G,v}^0.$$

For the rest of this section, G will be a fixed K-group over the global field F, together with a quasisplit inner twist G^*. Following § 1, we fix a central induced torus Z for the K-group G, and a character ζ on $Z(\mathbb{A})/Z(F)$. The notion is the same as that for a local K-group [**12**, § 3]. Thus, Z is an induced torus over F, together with central embeddings

$$Z \xrightarrow{\sim} Z_\alpha \subset G_\alpha, \quad \alpha \in \pi_0(G),$$

over F that are compatible with the isomorphisms $\psi_{\alpha_\beta}\colon G_\beta \to G_\alpha$. For each α, ζ determines a character ζ_α on $Z_\alpha(\mathbb{A})/Z_\alpha(F)$. As in [**12**], we shall make free use of obvious extensions to G of notation for connected groups. In particular, we have the quotient

$$\bar{G} = G/Z = \coprod_{\alpha \in \pi_0(G)} (G_\alpha/Z_\alpha),$$

which is easily seen to be a K-group. We also have the notion of a central extension

$$\tilde{G} = \coprod_{\alpha \in \pi_0(G)} \tilde{G}_\alpha$$

of G by an induced torus $\tilde{Z}$, which is a K-group such that $\tilde{G}/\tilde{Z} = G$. Other examples are the vector spaces

$$\mathcal{H}(G,\zeta) = \bigoplus_{\alpha \in \pi_0(G)} \mathcal{H}(G_\alpha,\zeta_\alpha) = \bigoplus_{\alpha \in \pi_0(G)} \mathcal{H}(G_\alpha(\mathbb{A}),\zeta_\alpha)$$

and

$$\mathcal{I}(G,\zeta) = \bigoplus_{\alpha \in \pi_0(G)} \mathcal{I}(G_\alpha,\zeta_\alpha) = \bigoplus_{\alpha \in \pi_0(G)} \mathcal{I}(G_\alpha(\mathbb{A}),\zeta_\alpha).$$

Similarly, if V is a finite set of valuations of F,

$$\Gamma(G_V,\zeta_V) = \coprod_{\alpha_V} \Gamma(G_{V,\alpha_V},\zeta_{V,\alpha_V})$$

stands for a basis of the vector space

$$\mathcal{D}(G_V,\zeta_V) = \bigoplus_{\alpha_V} \mathcal{D}(G_{V,\alpha_V},\zeta_{V,\alpha_V})$$

of ζ_V-equivariant distributions on G_V, while

$$\Gamma_{\mathrm{reg}}(G_V,\zeta_V) = \coprod_{\alpha_V} \Gamma_{\mathrm{reg}}(G_{V,\alpha_V},\zeta_{V,\alpha_V})$$

is the subset of orbital distributions in $\Gamma(G_V,\zeta_V)$ that have strongly regular support. Moreover,

$$\Pi(G_V,\zeta_V) = \coprod_{\alpha_V} \Pi(G_{V,\alpha_V},\zeta_{V,\alpha_V})$$

stands for a basis of the vector space

$$\mathcal{F}(G_V,\zeta_V) = \bigoplus_{\alpha_V} \mathcal{F}(G_{V,\alpha_V},\zeta_{V,\alpha_V})$$

of ζ_V-equivariant distributions on G_V, while

$$\Pi_{\mathrm{temp}}(G_V,\zeta_V) = \coprod_{\alpha_V} \Pi_{\mathrm{temp}}(G_{V,\alpha_V},\zeta_{V,\alpha_V})$$

is the subset of tempered distributions in $\Pi(G_V, \zeta_V)$.

We turn now to transfer factors. We would eventually like to be able to transfer general elements in $\Gamma(G_V, \zeta_V)$. Before we can consider this, however, we must first describe the canonical adelic transfer factors for strongly G-regular conjugacy classes. If $\alpha \in \pi_0(G)$ is a component of G, the strongly regular set $G_{\alpha,\mathrm{reg}}$ in G_α is Zariski open. The set $G_{\alpha,\mathrm{reg}}(\mathbb{A})$ of adelic points is therefore defined, as is the corresponding set $\Gamma_{\mathrm{reg}}(G_\alpha(\mathbb{A}))$ of strongly regular conjugacy classes. We shall describe transfer factors attached to elements in

$$\Gamma_{\mathrm{reg}}(G(\mathbb{A})) = \coprod_\alpha \Gamma_{\mathrm{reg}}(G_\alpha(\mathbb{A})).$$

This is largely a review. It combines a mild generalization [**12**] of the local Langlands–Shelstad transfer factors with the global definitions of [**31**, § 6].

As in the local case, an endoscopic datum for G is defined entirely in terms of the dual group $\hat{G}$, and is therefore the same as an endoscopic datum for G^*. It consists of a connected quasisplit group G' over F, embedded in a larger datum $(G', \mathcal{G}', s', \xi')$ [**31**, (1.2)]. We shall write $\mathcal{E}(G)$ for the set of isomorphism classes of endoscopic data for G over F that are locally relevant to G. In other words, for every v, $G'(F_v)$ has a strongly G-regular element that is an image (in the sense of [**31**, (1.3)]) of some class in

$$\Gamma_{\mathrm{reg}}(G_v) = \coprod_{\alpha_v \in \pi_0(G_v)} \Gamma_{\mathrm{reg}}(G_{v,\alpha_v}(F_v)).$$

As usual, we generally denote an element in $\mathcal{E}(G)$ by G', even though G' is really only the first component of a representative of an equivalence class. If V is a finite set of valuations of F that contains $V_{\mathrm{ram}}(G)$, we write $\mathcal{E}(G, V)$ for the subset of elements $G' \in \mathcal{E}(G)$ that are unramified outside of V. We also write $\mathcal{E}_{\mathrm{ell}}(G)$ and $\mathcal{E}_{\mathrm{ell}}(G, V)$ for the subset of elements in $\mathcal{E}(G)$ and $\mathcal{E}(G, V)$, respectively, that are elliptic over F.

If G' belongs to $\mathcal{E}_{\mathrm{ell}}(G)$, we can fix a central extension $\tilde{G}' \to G'$ and an L-embedding $\tilde{\xi}' \colon \mathcal{G}' \to {}^L\tilde{G}'$ that satisfy the conditions of [**10**, Lemma 2.1]. In this paper, it will be convenient to write $\tilde{C}'$ for the induced torus that is the kernel of the projection $\tilde{G}' \to G'$, and $\tilde{\eta}'$ for the character on $\tilde{C}'(\mathbb{A})/\tilde{C}'(F)$ that is dual to the global Langlands parameter obtained from the composition

$$W_F \to \mathcal{G}' \xrightarrow{\tilde{\xi}'} {}^L\tilde{G}' \to {}^L\tilde{C}',$$

where W_F is the Weil group of F, and $W_F \to \mathcal{G}'$ is any section. (The local forms of these objects were denoted by $\tilde{Z}'$ and $\tilde{\zeta}'$ in [**10**] and [**12**].) We reserve the symbol $\tilde{Z}'$ for the preimage of Z in $\tilde{G}'$. Global analogues of the local constructions in [**31**, (4.4)] lead to a canonical extension of $\tilde{\eta}'$ to a character on $\tilde{Z}'(\mathbb{A})/\tilde{Z}'(F)$. We shall reserve the symbol $\tilde{\zeta}'$ for the character on $\tilde{Z}'(\mathbb{A})/\tilde{Z}'(F)$ obtained from the product of $\tilde{\eta}'$ with the pullback of ζ. (The local forms of these last objects were denoted by $\tilde{Z}'Z$ and $\tilde{\zeta}'_Z\zeta$ in [**10**] and [**12**].)

The global transfer factors are products of the local transfer factors of [**31**]. However, we have to agree how to normalize them.

Suppose that $G' \in \mathcal{E}(G)$. Since G' is locally relevant to G, we can find a maximal torus $\bar{T}'_v$ in G' over F_v, for any v, that transfers over F_v to G_v. Choose a finite set of valuations

https://doi.org/10.1017/S1474748002000051 Published online by Cambridge University Press

$V \supset V_{\mathrm{ram}}(G)$, and let U_V be the set of elements in $G'_V = \prod_{v \in V} G'_v(F_v)$ that are G'_V-conjugate to G-strongly regular elements in $\bar{T}'_V = \prod_v \bar{T}'_v(F_v)$. Then U_V is an open subset of G'_V whose closure contains 1. Now the closure of the diagonal image of $G'(F)$ in G'_V is known to contain an open neighbourhood of 1. (See [**23**, Theorem 1].) It follows that there is a strongly G-regular element in $G'(F)$ that for each $v \in V$ is a local image of some point in $G_v(F_v)$. This element is automatically also a local image from any of the quasisplit groups G_v, $v \notin V$. Let $\bar{\delta}'$ be a point in its preimage in $\tilde{G}'(F)$. The projection of $\bar{\delta}'$ onto $\bar{G}'(F)$ is then an adelic image of a strongly regular element $\bar{\gamma} = \prod_v \bar{\gamma}_v$ in $G_{\bar{\alpha}}(\mathbb{A})$, for some $\bar{\alpha} \in \pi_0(G)$. We fix the two elements $\bar{\delta}'$ and $\bar{\gamma}$. The pair $(\bar{\delta}', \bar{\gamma})$ will serve as a base point for the global transfer factor.

Let $\bar{T}'$ be the centralizer in G' of the projection of $\bar{\delta}'$ onto G'. Then $\bar{T}'$ is a maximal torus in G' over F. Choose an admissible embedding $\bar{T}' \to \bar{T}$ of $\bar{T}'$ into a maximal torus in G^*, and let $\bar{\delta}^* \in \bar{T}(F)$ be the corresponding image of $\bar{\delta}'$. For the element $\bar{\gamma} \in G_{\bar{\alpha}}(\mathbb{A})$, we choose a point $\bar{h} \in G^*_{\mathrm{sc}}(\overline{\mathbb{A}})$ such that $\bar{h}\psi_{\bar{\alpha}}(\bar{\gamma})\bar{h}^{-1} = \bar{\delta}^*$. The function

$$\bar{v}(\tau) = \bar{h} u_{\bar{\alpha}}(\tau)\tau(\bar{h})^{-1}, \quad \tau \in \Gamma,$$

takes values in $\bar{T}_{\mathrm{sc}}(\mathbb{A})$, where $\bar{T}_{\mathrm{sc}}$ is the preimage of $\bar{T}$ in G^*_{sc}. This function need not be a cocycle, but its boundary $\partial\bar{v}$ equals $\partial u_{\bar{\alpha}}$, and takes values in $\bar{T}_{\mathrm{sc}}(\bar{F})$. One can therefore project $\bar{v}$ onto an element $\mu_{\bar{T}} = \mu_{\bar{T}}(\bar{\delta}^*, \bar{\gamma})$ in $H^1(F, \bar{T}_{\mathrm{sc}}(\overline{\mathbb{A}})/\bar{T}_{\mathrm{sc}}(\bar{F}))$. On the other hand, the admissible embedding $\bar{T}' \to \bar{T}$, and the semisimple element $s' \in \hat{G}$ attached to G', determine a point $s'_{\bar{T}}$ in $\hat{\bar{T}}$ [**31**, (3.1)]. This point projects to an element $\bar{s}'_{\bar{T}}$ in $\pi_0(\hat{\bar{T}}^{\Gamma}_{\mathrm{ad}})$, where $\hat{\bar{T}}_{\mathrm{ad}}$ is a dual torus for $\bar{T}_{\mathrm{sc}}$. As in [**31**, p. 268], we set

$$d(\bar{\delta}', \bar{\gamma}) = \langle \mu_{\bar{T}}, \bar{s}'_{\bar{T}} \rangle,$$

where the right-hand side is given by the global Tate–Nakayama pairing for the torus $\bar{T}_{\mathrm{sc}}$.

Now suppose that $\delta' \in \tilde{G}'(\mathbb{A})$ is strongly G-regular and that $\gamma \in G(\mathbb{A})$ is strongly regular. We define the relative global transfer factor as a product

$$\Delta(\delta', \gamma; \bar{\delta}', \bar{\gamma}) = \prod_v \Delta(\delta'_v, \gamma_v; \bar{\delta}'_v, \bar{\gamma}_v)$$

of relative transfer factors for the local K-groups G_v. The local factors were defined in [**12**, § 2] by a natural variant of the basic construction in [**31**, (3.7)], and are easily seen to equal 1 for almost all v. Following [**24**, (7.3)], we define the absolute global transfer factor

$$\Delta(\delta', \gamma) = \Delta(\delta', \gamma; \bar{\delta}', \bar{\gamma}) d(\bar{\delta}', \bar{\gamma})^{-1}. \tag{4.1}$$

It depends only on the image of γ in $\Gamma_{\mathrm{reg}}(G(\mathbb{A}))$ and on the image of δ' in the set $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'(\mathbb{A}))$ of strongly G-regular stable conjugacy classes in $\tilde{G}'(\mathbb{A})$.

Lemma 4.3. *The absolute global transfer factor $\Delta(\delta', \gamma)$ is independent of the base point $(\bar{\delta}', \bar{\gamma})$.*

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Proof. Suppose that $(\bar{\delta}', \bar{\gamma})$ is replaced by another base point $(\bar{\bar{\delta}}', \bar{\bar{\gamma}})$, with $\bar{\bar{\delta}}' \in \tilde{G}'(F)$ strongly G-regular. Then

$$\Delta(\delta', \gamma; \bar{\bar{\delta}}', \bar{\bar{\gamma}}) = \Delta(\delta', \gamma; \bar{\delta}', \bar{\gamma})\Delta(\bar{\delta}', \bar{\gamma}; \bar{\bar{\delta}}', \bar{\bar{\gamma}}),$$

by the analogue of [**31**, Lemma 4.1.A] for local K-groups. Furthermore,

$$\Delta(\bar{\delta}', \bar{\gamma}; \bar{\bar{\delta}}', \bar{\bar{\gamma}}) = d(\bar{\bar{\delta}}', \bar{\bar{\gamma}})d(\bar{\delta}', \bar{\gamma})^{-1},$$

by [**31**, Lemma 6.3.B]. The lemma follows. □

Corollary 4.4. *The absolute global transfer factor depends only on the weak isomorphism class of G.*

Proof. It follows from the definitions that the transfer factors are invariant under isomorphisms of K-groups. To deal with the more general case of weak isomorphisms, we need only show that the absolute global transfer factor for G remains unchanged if we modify the functions $\{u_{\alpha_\beta}\}$. The relative local transfer factors, from which the global factor is formed, are actually sensitive to the choice of $\{u_{\alpha_\beta}\}$. The dependence is through the term [**12**, (3.4)], defined on p. 222 of [**12**]. However, if the two points γ and $\bar{\gamma}$ lie in the same component G_α, this term is the same as the factor in [**31**, (3.4)]. The latter is defined in terms of the function u_α, (which was denoted by u in [**31**]), although it is actually independent of the choice of this function. The functions $\{u_{\alpha\beta}\}$ have no role at all in this special case. The corollary therefore follows from the lemma. □

Our concern in this paper will be primarily with products of local transfer factors over finite sets $V \supset V_{\mathrm{ram}}(G)$. Suppose that $G' \in \mathcal{E}_{\mathrm{ell}}(G)$ is fixed. We recall that the local transfer factor at any v is defined by

$$\Delta(\delta'_v, \gamma_v) = \Delta(\delta'_v, \gamma_v; \bar{\delta}'_v, \bar{\gamma}_v)\Delta(\bar{\delta}'_v, \bar{\gamma}_v),$$

where $\Delta(\bar{\delta}'_v, \bar{\gamma}_v)$ is an arbitrary preassigned value at the local base point $(\bar{\delta}'_v, \bar{\gamma}_v)$. Consider the case that v does not belong to $V_{\mathrm{ram}}(G)$. In particular, G_v is a quasisplit group. As such, it has canonical transfer factors that depend only on a choice of splitting for G_v [**31**, (3.7)]. As observed by Hales [**17**, § 7], our choice of hyperspecial maximal compact subgroup K_v determines a family of splittings of G_v for which the associated transfer factors are the same. We obtain a transfer factor $\Delta_{K_v}(\delta'_v, \gamma_v)$ for (G_v, G'_v) that depends only on K_v. For our preassigned value at v, we set $\Delta(\bar{\delta}'_v, \bar{\gamma}_v)$ equal to $\Delta_{K_v}(\bar{\delta}'_v, \bar{\gamma}_v)$. Since

$$\Delta_{K_v}(\delta'_v, \gamma_v) = \Delta(\delta'_v, \gamma_v; \bar{\delta}'_v, \bar{\gamma}_v)\Delta_{K_v}(\bar{\delta}'_v, \bar{\gamma}_v)$$

by definition, we obtain

$$\Delta(\delta'_v, \gamma_v) = \Delta_{K_v}(\delta'_v, \gamma_v). \tag{4.2}$$

In particular, $\Delta(\delta'_v, \gamma_v)$ is independent of $(\bar{\delta}'_v, \bar{\gamma}_v)$. For the places $v \in V_{\mathrm{ram}}(G)$, we choose any preassigned values $\Delta(\bar{\delta}'_v, \bar{\gamma}_v)$, subject only to the condition

$$\prod_{v \in V_{\mathrm{ram}}(G)} \Delta(\bar{\delta}'_v, \bar{\gamma}_v) = d(\bar{\delta}', \bar{\gamma})^{-1} \prod_{v \notin V_{\mathrm{ram}}(G)} \Delta_{K_v}(\bar{\delta}'_v, \bar{\gamma}_v)^{-1}. \tag{4.3}$$

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The absolute global transfer factor will then be given by a product

$$\Delta(\delta', \gamma) = \prod_v \Delta(\delta'_v, \gamma_v),$$

almost all of whose factors are easily seen to be equal to 1.

Consider a finite set of valuations V that contains $V_{\mathrm{ram}}(G)$. Suppose that $\gamma_V \in \Gamma_{\mathrm{reg}}(G_V)$ is a strongly regular conjugacy class in $G_V(F_V)$, and that $\delta'_V \in \Delta_{G\text{-reg}}(\tilde{G}'_V)$ is a strongly G-regular stable conjugacy class in $\tilde{G}'_V$. The transfer factor for γ_V and δ'_V is defined by a product

$$\Delta(\delta'_V, \gamma_V) = \prod_{v \in V} \Delta(\delta'_v, \gamma_v) \tag{4.4}$$

of local transfer factors, chosen as above. We can certainly assume that δ'_V and γ_V are projections of adelic elements δ' and γ. We obtain a representation

$$\Delta(\delta'_V, \gamma_V) = \Delta(\delta', \gamma) \prod_{v \notin V} (\Delta_{K_v}(\delta'_v, \gamma_v))^{-1}$$

that is independent of the base point $(\bar{\delta}', \bar{\gamma})$. The transfer factor $\Delta(\delta'_V, \gamma_V)$ is thus a canonical object, that depends only on the hyperspecial maximal compact subgroup K^V. This will be the general setting for our study of global transfer.

What makes the transfer factor (4.4) remain a global object is the fact that the endoscopic datum G' is over F. We can of course also consider local endoscopic data. Suppose that V is *any* finite set of valuations. We write $\mathcal{E}(G_V)$ for the set of products

$$G'_V = \prod_{v \in V} G'_v, \quad G'_v \in \mathcal{E}(G_v),$$

of local endoscopic data. If $\delta'_V = \prod_{v \in V} \delta'_v$ belongs to the set

$$\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V) = \prod_{v \in V} \Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_v),$$

and γ_v lies in $\Gamma_{\mathrm{reg}}(G_V)$, the transfer factor $\Delta(\delta'_V, \gamma_V)$ can still be defined by a product (4.4). As a local object, however, it does depend on a preassigned value at a local base point $(\bar{\delta}'_V, \bar{\gamma}_V)$. It of course also depends on the products $\tilde{G}'_V = \prod \tilde{G}'_v$ and $\tilde{\xi}'_V = \prod_v \tilde{\xi}'_v$ of auxiliary data.

For each valuation v, we write $\tilde{\Delta}^{\mathcal{E}}_{\mathrm{reg,ell}}(G_v)$, $\Delta^{\mathcal{E}}_{\mathrm{reg,ell}}(G_v)$, $\tilde{\Delta}^{\mathcal{E}}_{\mathrm{reg}}(G_v)$ and $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G_v)$ for the endoscopic sets of [**14**]. We can then form the sets $\tilde{\Delta}^{\mathcal{E}}_{\mathrm{reg,ell}}(G_V)$, $\Delta^{\mathcal{E}}_{\mathrm{reg,ell}}(G_V)$, $\tilde{\Delta}^{\mathcal{E}}_{\mathrm{reg}}(G_V)$ and $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G_V)$ as products over the places v in a given finite set V. Thus, $\tilde{\Delta}^{\mathcal{E}}_{\mathrm{reg}}(G_V)$ is a quotient of the family of elements in

$$\{(G'_V, \tilde{\xi}'_V, \delta'_V) : G'_V \in \mathcal{E}(G_V),\ \tilde{\xi}'_V : \mathcal{G}'_V \to {}^L\tilde{G}'_V,\ \delta'_V \in \Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V)\}$$

that are G_V-images, taken with respect to a certain natural equivalence relation. This is the set denoted by $\tilde{\Gamma}^{\mathcal{E}}(G_V)$ in [**12**, § 2] and [**10**, § 2] (in the special case that V contains one element), apart from the fact that the latter did not have variable embeddings $\tilde{\xi}'_V$ built into the definition. The set $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G_V)$ is the corresponding quotient of the family of elements in

$$\{(G'_V, \delta'_V) : G'_V \in \mathcal{E}(G_V),\ \delta'_V \in \Delta_{G\text{-}\mathrm{reg}}(G'_V)\}$$

that are G_V-images. It is equal to the set denoted by $\Gamma^{\mathcal{E}}(G_V)$ in [**12**, § 2] and [**10**, § 2].

The point of introducing these endoscopic sets is that the transfer factor attached to (δ'_V, γ_V) depends only on the image δ_V of δ'_V in $\tilde{\Delta}^{\mathcal{E}}_{\mathrm{reg}}(G_V)$, or rather, of the triplet $(G'_V, \tilde{\xi}'_V, \delta'_V)$ represented by δ'_V. We can therefore regard the transfer factor as a function

$$\Delta(\delta_V, \gamma_V) = \Delta(\delta'_V, \gamma_V)$$

on $\tilde{\Delta}^{\mathcal{E}}_{\mathrm{reg}}(G_V) \times \Gamma_{\mathrm{reg}}(G_V)$. Now the group $\tilde{C}'_V = \prod_v \tilde{C}'_v$ acts simply transitively on the fibres of the map $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V) \to \Delta_{G\text{-}\mathrm{reg}}(G'_V)$. Moreover, the group

$$H^1(W_{F_V}, Z(\hat{\tilde{G}}'_V)) = \bigoplus_{v \in V} H^1(W_{F_v}, Z(\hat{\tilde{G}}'_v))$$

acts simply transitively on the set of $Z(\hat{\tilde{G}}'_V)$-orbits of admissible embeddings $\tilde{\xi}'_V : \mathcal{G}'_V \to {}^L\tilde{G}'_V$. If $a_V z_V \delta_V$ is the image in $\tilde{\Delta}^{\mathcal{E}}_{\mathrm{reg}}(G_V)$ of a point

$$(G'_V, a_V \tilde{\xi}'_V, z_V \delta'_V), \quad a_V \in H^1(W_{F_V}, Z(\hat{\tilde{G}}'_V)), \quad z_V \in \tilde{C}'_V,$$

the transfer factor satisfies

$$\Delta(a_V z_V \delta_V, \gamma_V) = \chi_{a_V}(\delta'_V)\tilde{\eta}'_V(z_V)\Delta(\delta_V, \gamma_V), \tag{4.5}$$

where $\tilde{\eta}'_V = \prod_v \tilde{\eta}'_v$ is the canonical character on $\tilde{C}'_V$, and $\chi_{a_V} = \prod_v \chi_{a_v}$ is a character on $\tilde{G}'_V$ that can be defined in terms of the Langlands correspondence for tori [**30**] from the parameter a_V. Since the embeddings $\tilde{\xi}'_V$ are assumed implicitly to be of unitary type, χ_{a_V} is indeed a unitary character. It follows that the product of $\Delta(\delta_V, \gamma_V)$ with the adjoint transfer factor

$$\Delta(\gamma_V, \delta_V) = |\mathcal{K}_{\gamma_V}|^{-1}\overline{\Delta(\delta_V, \gamma_V)}, \quad |\mathcal{K}_{\gamma_V}| = \prod_{v \in V} |\mathcal{K}_{\gamma_v}|, \tag{4.6}$$

of [**12**, § 2] depends only on the image of δ_V in $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G_V)$. As in [**12**, Lemma 2.3], we obtain adjoint relations

$$\sum_{\delta_V \in \Delta^{\mathcal{E}}_{\mathrm{reg}}(G_V)} \Delta(\gamma_V, \delta_V)\Delta(\delta_V, \gamma_{V,1}) = \delta(\gamma_V, \gamma_{V,1}), \quad \gamma_V, \gamma_{V,1} \in \Gamma_{\mathrm{reg}}(G_V), \tag{4.7}$$

and

$$\sum_{\gamma_V \in \Gamma_{\mathrm{reg}}(G_V)} \Delta(\delta_V, \gamma_V)\Delta(\gamma_V, \delta_{V,1}) = \tilde{\delta}(\delta_V, \delta_{V,1}), \quad \delta_V, \delta_{V,1} \in \tilde{\Delta}^{\mathcal{E}}_{\mathrm{reg}}(G_V). \tag{4.8}$$

Our concern in this paper is really with the general basis $\Gamma(G_V,\zeta_V)$ of $\mathcal{D}(G_V,\zeta_V)$. We shall set up transfer factors for elements in this basis in § 5, under the assumption of the fundamental lemma. In the meantime, we consider distributions in the subset $\Gamma_{\mathrm{reg}}(G_V,\zeta_V)$ of $\Gamma(G_V,\zeta_V)$. For each $G'_V \in \mathcal{E}(G_V)$, we fix a subset $\Delta_{\mathrm{reg}}(\tilde{G}'_V,\tilde{\zeta}'_V)$ of $S\mathcal{D}(\tilde{G}'_V,\tilde{\zeta}'_V)$, as at the end of § 1. This set in turn has a subset $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V,\tilde{\zeta}'_V)$ of G-regular elements. We shall convert the basic transfer factor above to a function on $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V,\tilde{\zeta}'_V)\times\Gamma_{\mathrm{reg}}(G_V,\zeta_V)$.

If G'_V belongs to $\mathcal{E}(G_V)$, the transfer factor determines a map

$$f \to f'(\delta'_V) = \sum_{\gamma_V\in\Gamma_{\mathrm{reg}}(G_V)} \Delta(\delta'_V,\gamma_V) f_G(\gamma_V), \quad \delta'_V \in \Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V),$$

from functions $f\in\mathcal{H}(G_V)$ to functions $f' = f^{G'}$ on $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V)$. The image f' is $(\tilde{\eta}'_V)^{-1}$-equivariant under translation by $\tilde{C}'_V$. If we apply a variant of the projection (1.1) to f', we obtain a function that is $(\tilde{\zeta}')^{-1}$-equivariant under translation by $\tilde{Z}'_V$. This determines a function on $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V,\tilde{\zeta}'_V)$, which we denote again by f'. The new function depends only on the image of f_G in $\mathcal{I}(G_V,\zeta_V)$, which we denote again by f_G. How are these two new functions related? The answer is clearly given by (1.4) and (1.6). If $\delta'\in\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V,\tilde{\zeta}'_V)$ and $\gamma\in\Gamma_{G\text{-}\mathrm{reg}}(G_V,\zeta_V)$, we define a modified transfer factor by

$$\begin{aligned}\Delta(\delta',\gamma) &= \sum_{\gamma_V}(\delta'_V/\delta')^{-1}\Delta(\delta'_V,\gamma_V)(\gamma_V/\gamma)^{-1}\\ &= \sum_{\delta'_V}(\delta'_V/\delta')^{-1}\Delta(\delta'_V,\gamma_V)(\gamma_V/\gamma)^{-1},\end{aligned}$$

where in the first formula, for example, δ'_V is any representative of δ' in $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V)$, and γ_V is summed over all representatives of γ in $\Gamma_{\mathrm{reg}}(G_V)$. The new functions f' and f_G are then related by

$$f'(\delta') = \sum_{\gamma\in\Gamma_{\mathrm{reg}}(G_V,\zeta_V)} \Delta(\delta',\gamma) f_G(\gamma), \quad \delta'\in\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V,\tilde{\zeta}'_V). \tag{4.9}$$

The Langlands–Shelstad conjecture amounts to the assertion that for any $f\in\mathcal{H}(G_V,\zeta_V)$, the corresponding function f' belongs to $S\mathcal{I}(\tilde{G}'_V,\tilde{\zeta}'_V)$.

If we choose the bases $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V,\tilde{\zeta}'_V)$ appropriately, we can also construct an analogue for ζ_V-equivariant distributions of the set $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G_V)$. One defines $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G_V,\zeta_V)$ as a quotient of the subset of elements in

$$\{(G'_V,\delta') : G'\in\mathcal{E}(G_V),\ \delta'\in\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V,\zeta'_V)\}$$

that are relevant to G_V. Any element δ in this quotient can be represented by a number of triplets

$$(G'_V,\tilde{\xi}'_V,\delta'), \quad G'_V\in\mathcal{E}(G_V), \quad \tilde{\xi}'_V : \mathcal{G}'_V \to {}^L\tilde{G}'_V, \quad \delta'\in\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}'_V,\tilde{\zeta}'_V),$$

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but it is not hard to show from the definitions that the modified transfer factor

$$\Delta(\delta, \gamma) = \Delta(\delta', \gamma)$$

depends only on δ. In particular, the elements in $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G_V, \zeta_V)$ provide linear forms

$$f \to f^{\mathcal{E}}_G(\delta) = f'(\delta'), \quad f \in \mathcal{H}(G_V, \zeta_V).$$

We obtain a space

$$\mathcal{I}^{\mathcal{E}}(G_V, \zeta_V) = \{f^{\mathcal{E}}_G : f \in \mathcal{H}(G_V, \zeta_V)\},$$

of functions on $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G_V, \zeta_V)$. Obviously,

$$f^{\mathcal{E}}_G(\delta) = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(G_V, \zeta_V)} \Delta(\delta, \gamma) f_G(\gamma), \quad f \in \mathcal{H}(G_V, \zeta_V),$$

and it follows from the adjoint relations (4.7) and (4.8) that the map $f_G \to f^{\mathcal{E}}_G$ is an isomorphism from $\mathcal{I}(G_V, \zeta_V)$ onto $\mathcal{I}^{\mathcal{E}}(G_V, \zeta_V)$. We note that the sets $\Gamma_{\mathrm{reg}}(G_V, \zeta_V)$ and $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G_V, \zeta_V)$ represent a pair of bases of the subspace of distributions in $\mathcal{D}(G_V, \zeta_V)$ that are supported on the strongly regular set in G_V.

This completes our summary of the basic transfer factors

$$\Delta_G(\cdot, \cdot) = \Delta(\cdot, \cdot)$$

attached to strongly regular classes in G_V. The discussion has been largely formal. We have tried to set things up in a way that will ease the transition to more general transfer factors, which we shall introduce in the next section under the hypothesis of the fundamental lemma.

5. An assumption: the fundamental lemma

The fundamental lemma is a misnomer. It is not a lemma at all, but a general conjecture of Langlands. In this paper, it will be the basic assumption on which the main theorems depend. We shall actually require a generalization of the usual fundamental lemma, which applies weighted orbital integrals of the unit unramified spherical function.

To describe the fundamental lemma, we take F to be a local field. Suppose for a moment that F is arbitrary, and that G is a K-group over F. Suppose also that Z is a central induced torus in G over F, and that ζ is a character on $Z(F)$. We then write

$$\mathcal{H}(G, \zeta) = \mathcal{H}(G(F), \zeta), \qquad \mathcal{I}(G(F), \zeta) = \mathcal{I}(G, \zeta),$$
$$\Gamma_{\mathrm{reg}}(G, \zeta) = \Gamma_{\mathrm{reg}}(G(F), \zeta), \quad \Delta_{\mathrm{reg}}(G, \zeta) = \Delta_{\mathrm{reg}}(G(F), \zeta),$$

etc., for the various objects attached to $G(F)$. The basic local transfer factor

$$\Delta(\delta', \gamma) = \Delta_G(\delta', \gamma), \quad \gamma \in \Gamma_{\mathrm{reg}}(G, \zeta), \quad \delta' \in \Delta_{G\text{-}\mathrm{reg}}(\tilde{G}', \tilde{\zeta}'),$$

is defined as a function on $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}', \tilde{\zeta}') \times \Gamma_{\mathrm{reg}}(G, \zeta)$.

Assume now that G, Z and ζ are unramified over F. In particular, F is non-archimedean, and G is a connected reductive group. Following § 2, we write $\mathcal{K}_{\mathrm{reg}}(\bar{G})$ for the set of strongly regular conjugacy classes in $\bar{G}(F) = G(F)/Z(F)$ that are bounded, and $k \to \gamma(k)$ for the canonical injection from $\mathcal{K}_{\mathrm{reg}}(\bar{G})$ to $\Gamma_{\mathrm{reg}}(G,\zeta)$. We also write $\mathcal{L}_{\mathrm{reg}}(\bar{G})$ for the set of strongly regular stable conjugacy classes in $\bar{G}(F)$ that are bounded, and $\ell \to \delta(\ell)$ for the corresponding injection from $\mathcal{L}_{\mathrm{reg}}(\bar{G})$ to $\Delta_{\mathrm{reg}}(G,\zeta)$. Suppose that K is a hyperspecial maximal compact subgroup of $G(F)$. If G' is any endoscopic datum for G over F, the normalized transfer factor $\Delta_K(\delta',\gamma)$ attached to K is a canonical function on $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}',\tilde{\zeta}') \times \Gamma_{\mathrm{reg}}(G,\zeta)$. It does depend on the auxiliary data $(\tilde{G}',\tilde{\xi}')$ attached to G'. However, if G' is unramified, there is a canonical class of admissible embeddings of ${}^L G'$ into ${}^L G$. (See [**17**, § 6].) This means that we can set $\tilde{G}' = G'$. The embedding $\tilde{\xi}'$ must still be chosen. It takes the form of an L-isomorphism of $\mathcal{G}'$ with ${}^L G'$ that is uniquely determined up to the action of the group $H^1(\Gamma_{\mathrm{un}}, Z(\hat{G}'))$, where Γ_{un} is the Galois group of the maximal unramified extension of F. Having made these choices, we set

$$\Delta_K(\ell',k) = \Delta_K(\delta'(\ell'),\gamma(k)), \quad \ell' \in \mathcal{L}_{G\text{-}\mathrm{reg}}(\bar{G}'), \quad k \in \mathcal{K}_{\mathrm{reg}}(\bar{G}),$$

for the unramified endoscopic datum G'. As a function on $\mathcal{L}_{G\text{-}\mathrm{reg}}(\bar{G}') \times \mathcal{K}_{\mathrm{reg}}(\bar{G})$, Δ_K is independent of both the character ζ and the choice of $\tilde{\xi}'$.

Suppose that M is a Levi subgroup of G that is in good position relative to K. As in § 2, we set

$$r^G_M(k) = J_M(k,u), \quad k \in \mathcal{K}_{G\text{-}\mathrm{reg}}(\bar{M}),$$

where $u = u^\zeta$ is the unit in the Hecke algebra attached to K and ζ. The function r^G_M depends of course on K, but it is independent of Z and ζ. If M' is an unramified endoscopic datum for M, the transfer factor

$$\Delta_{K\cap M}(\ell',k), \quad \ell' \in \mathcal{L}_{G\text{-}\mathrm{reg}}(\bar{M}'), \quad k \in \mathcal{K}_{G\text{-}\mathrm{reg}}(\bar{M}),$$

is defined. If we sum its product with $r^G_M(k)$ over k, we obtain a function of ℓ' that is easily seen to be independent of K, as well as Z and ζ.

We now state the generalized fundamental lemma as a conjecture on the unramified groups G and M. We may as well take Z and ζ to be trivial, since the functions we have defined are independent of these objects. The conjecture takes the form of a family of identities, indexed by unramified, elliptic endoscopic data M' for M. The identity corresponding to M' is given by a sum over the set $\mathcal{E}_{M'}(G)$ of endoscopic data for G introduced in [**11**, § 4] and [**12**, § 3], with coefficients

$$\iota_{M'}(G,G') = |Z(\hat{M}')^\Gamma/Z(\hat{M})^\Gamma||Z(\hat{G}')^\Gamma/Z(\hat{G})^\Gamma|^{-1}, \quad G' \in \mathcal{E}_{M'}(G).$$

Conjecture 5.1. *For each G and M, there is a function*

$$s^G_M(\ell), \quad \ell \in \mathcal{L}_{G\text{-}\mathrm{reg}}(M),$$

with the property that for any G, M and K, any unramified elliptic endoscopic datum M' for M, and any element $\ell' \in \mathcal{L}_{G\text{-}\mathrm{reg}}(M')$, the transfer

$$\sum_{k\in\mathcal{K}_{G\text{-}\mathrm{reg}}(M)} \Delta_{K\cap M}(\ell',k) r^G_M(k) \tag{5.1}$$

equals

$$\sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') s_{M'}^{G'}(\ell'). \tag{5.2}$$

Observe that the function $s_M^G(\ell)$ is uniquely determined by the required identity. For if $M' = M$, the quasisplit group G belongs to $\mathcal{E}_{M'}(G)$. The required identity can be written

$$s_M^G(\ell) = \sum_k \Delta_{K \cap M}(\ell, k) r_M^G(k) - \sum_{G' \neq G} \iota_M(G, G') s_M^{G'}(\ell)$$

in this case, and serves as an inductive definition of $s_M^G(\ell)$. To establish the conjecture, one would need to prove the additional identities that come from elements $M' \neq M$.

Consider the case that $M = G$. Then $G' = M'$ is an elliptic endoscopic datum for G, and $\mathcal{E}_{M'}(G)$ consists of G' alone. The expression (5.2) equals $s_{G'}(\ell') = s_{G'}^{G'}(\ell')$. If $G' \neq G$, this has to satisfy two formulae. On the one hand, $s_{G'}(\ell')$ is supposed to equal the stable orbital integral on $G'(F)$, and on the other hand, it is required to equal the unstable orbital on $G(F)$. This is the standard fundamental lemma that was conjectured by Langlands, and that assumed a precise form with the definition of the transfer factors in [**31**] (normalized as in [**17**]). It has been established in a limited number of cases. If $G = \mathrm{GL}(n)$ or $\mathrm{PGL}(n)$, $G' = G$ is the only elliptic endoscopic group, and there is nothing to prove. For $\mathrm{SL}(2)$ and $U(3)$, the fundamental lemma was established in [**26**] and [**33**], respectively, in the course of stabilizing the trace formulae for these groups. For $G = \mathrm{SL}(n)$, the standard fundamental lemma was established by Waldspurger [**36**], and for $G = \mathrm{Sp}(4)$, $\mathrm{GSp}(4)$ and $\mathrm{SO}(5)$, it was established by Hales [**18**] and Waldspurger [**37**].

At the other extreme, we could take M to be a minimal Levi subgroup. The assertion of the conjecture is then trivial, since $M' = M$ is the only endoscopic datum. If M is neither minimal nor equal to G, there can be non-trivial elliptic endoscopic data M' for M. The general identities have not been investigated in this case, and seem to be as difficult as in the standard case that $M = G$. However, there are a few examples in which there is nothing more to prove. If $G = \mathrm{GSp}(4)$, $\mathrm{SO}(5)$ or $\mathrm{SL}(p)$, with p prime, no Levi subgroup $M \neq G$ has a proper elliptic endoscopic datum. The conjecture then holds in these examples, since it has been established for $M = G$.

We have really to be more precise in discussing whether the conjecture applies to a given case, since the definition is inductive. We shall introduce some sets that include the unramified endoscopic groups for a given G. To allow room for future constructions, we ask that the sets include groups taken over unramified extensions, and groups that are centralizers of semisimple elements. Consider a collection $\mathcal{U}$ of pairs (G, F) such that if $\mathcal{U}$ contains (G, F), then it contains any pair (G_1, F_1) obtained from (G, F) in one of the following three ways.

(i) F_1 equals F, and G_1 is an unramified endoscopic group for G.

(ii) F_1 is an unramified extension of F, and $G_1 = G \times_F F_1$.

(iii) F_1 equals F, and $G_1 = G_c$ is the connected centralizer of a semisimple element $c \in G(F)$ such that $\mathcal{D}(c) \in \mathfrak{o}_F^d$, and $|D(c)| = 1$. (It follows from [**22**, Proposition 7.1] that G_1 is quasisplit.)

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If G_* is a fixed unramified group over a local field F_*, let $\mathcal{U}(G_*, F_*)$ denote the smallest family $\mathcal{U}$ that contains (G_*, F_*), and satisfies the hereditary properties (i), (ii) and (iii). We shall say that Conjecture 5.1 holds for $\mathcal{U}(G_*, F_*)$ if the given identities hold for any (G, M), where (G, F) belongs to $\mathcal{U}(G_*, F_*)$, and M is a Levi subgroup of G.

Conjecture 5.1 has an analogue for the Lie algebra $\mathfrak{g}$ of G. The assertion is essentially the same, except that K is replaced by a hyperspecial lattice in $\mathfrak{g}(F)$. We shall be concerned only with the standard case that $M = G$, which we require in order to apply the results of Waldspurger [**38**]. The case of the Lie algebra is closely related to that of the group, but we shall not consider the question of how to pass from one to the other. We shall simply impose the Lie algebra variant as an extra condition on the family $\mathcal{U}(G_*, F_*)$.

Having completed our discussion of the fundamental lemma, we return to the case that F is an arbitrary local or global field. We shall introduce a formal assumption on which future results will rest. We state it first as a hypothesis on a pair (G, F), where G is a *connected* reductive group over F.

Assumption 5.2.

(i) If F is global, both Conjecture 5.1 and the standard form of its Lie algebra analogue hold for any of the families

$$\mathcal{U}(G_v, F_v), \quad v \notin V_{\mathrm{fund}}(G),$$

where $V_{\mathrm{fund}}(G)$ is some finite set of valuations of F that contains $V_{\mathrm{ram}}(G)$.

(ii) If F is local, (G, F) is isomorphic to a localization $(\dot{G}_u, \dot{F}_u)$ of some global pair $(\dot{G}, \dot{F})$ that satisfies (i).

We have set things up so that the conditions remain in force for groups derived from G by natural operations. For example, if (G, F) satisfies Assumption 5.2, so do any pairs $(\tilde{G}, F)$ and $(\bar{G}, F)$ obtained from extensions and quotients of G by induced central tori. This is clear if F is global, and can be established for local F by a global approximation argument. Other examples are given in the following lemma, which is modelled on the properties of the sets $\mathcal{U}$ above.

Lemma 5.3. *Suppose that Assumption 5.2 holds for (G, F). Then it also holds for any pair (G_1, F_1) obtained from (G, F) in one of the following three ways.*

(i) *F_1 equals F, and G_1 is an inner twist of an endoscopic group for G.*

(ii) *F_1 is a finite extension of F, and $G_1 = G \times_F F_1$.*

(iii) *F_1 equals F, and $G_1 = G_c$ is the connected centralizer of a semisimple element $c \in G(F)$.*

Proof. Suppose that F is global. For (i), we take $V_{\mathrm{fund}}(G_1)$ to be the union of $V_{\mathrm{fund}}(G)$ with $V_{\mathrm{ram}}(G_1)$. For (ii), we take $V_{\mathrm{fund}}(G_1)$ to be the set of places of F_1 that either ramify in F_1, or lie above a place in $V_{\mathrm{fund}}(G)$. For (iii), we take $V_{\mathrm{fund}}(G_1)$ to be any finite set

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$S \supset V_{\mathrm{fund}}(G)$ such that c is S-admissible. The global form of the lemma then follows from the properties of the sets $\mathcal{U}(G_v, F_v)$.

Suppose that F is local. The lemma in this case follows from a standard local–global argument. We shall just give a brief sketch. Let $E \supset F$ be a finite Galois extension over which G and G_1 both split, and which contains F_1 in case (ii). Let $(\dot{G}, \dot{F})$ be the global pair provided by part (ii) of Assumption 5.2, and let $\dot{E}$ be a finite Galois extension over which $\dot{G}$ splits. Enlarging E and $\dot{E}$, if necessary, we can assume that $E = \dot{E}_w$, for some place w over u. Then

$$\mathrm{Gal}(E/F) = \mathrm{Gal}(\dot{E}_w/\dot{F}_u) \subset \mathrm{Gal}(\dot{E}/\dot{F}).$$

Replacing $\dot{F}$ by the fixed field in $\dot{E}$ of $\mathrm{Gal}(\dot{E}_w/\dot{F}_u)$, we can assume that $\mathrm{Gal}(\dot{E}/\dot{F})$ actually equals $\mathrm{Gal}(E/F)$. The local forms of (i) and (ii) then follow from the global versions we have established. To deal with (iii), we replace $(\dot{E}, \dot{F})$ by $(\dot{E}\dot{F}', \dot{F}')$, where $\dot{F}'$ is a suitable extension of $\dot{F}$ in which u splits completely. This allows us to assume that $\mathrm{Gal}(\dot{E}/\dot{F}) = \mathrm{Gal}(\dot{E}_{u_i}/\dot{F}_{u_i})$ for several places u_i of $\dot{F}$. It then follows from [**23**, Lemma 1(b)] that $\dot{G}(\dot{F})$ is dense in $\dot{G}(\dot{F}_u)$. A simple argument, whose details we shall omit, then establishes the local form of (iii) from its global version. □

Suppose that G is a K-group over the local or global field F. We shall say that G satisfies Assumption 5.2 if the assumption holds for each of the components

$$(G_\alpha, F), \quad \alpha \in \pi_0(G).$$

The main results of this and subsequent papers will apply to any such G. For example, G could be an inner K-form of one of the split groups $\mathrm{SO}(5)$, $\mathrm{GSp}(4)$ or $\mathrm{SL}(p)$, with p prime. Then Assumption 5.2, or at least the group theoretic part of it, holds for G. The Lie algebraic part of the assumption can undoubtedly be established from this, so we can be confident that the results of the paper apply at least to these groups.

From now on, unless we state otherwise, G will stand for a K-group over F that satisfies Assumption 5.2. We return to the general setting of § 4, in which Z is a central induced torus in G over F, and ζ is a character on $Z(F)$ or $Z(\mathbb{A})/Z(F)$ (according to whether F is local or global). We recall also that if $G' \in \mathcal{E}(G)$ is an endoscopic datum, $\tilde{G}'$ stands for a central extension of G' by a central induced torus $\tilde{C}'$, and $\tilde{\zeta}'$ is a character on either $\tilde{Z}'(F)$ or $\tilde{Z}'(\mathbb{A})/\tilde{Z}'(F)$.

Suppose that F is local. Then we can assume that the Langlands–Shelstad conjecture holds for $G(F)$. In the formulation at the end of § 4, this means that for any $G' \in \mathcal{E}(G)$, the map that takes $f \in \mathcal{H}(G, \zeta)$ to the function

$$f'(\delta') = f^{G'}(\delta') = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(G,\zeta)} \Delta(\delta', \gamma) f_G(\gamma), \quad \delta' \in \Delta_{G\text{-}\mathrm{reg}}(\tilde{G}', \tilde{\zeta}'),$$

on $\Delta_{G\text{-}\mathrm{reg}}(\tilde{G}', \tilde{\zeta}')$ sends $\mathcal{H}(G, \zeta)$ continuously to the space $S\mathcal{I}(\tilde{G}', \tilde{\zeta}')$. If F is non-archimedean, this is the main result of Waldspurger's paper [**38**]. It depends on the Lie algebraic part of Assumption 5.2. If F is archimedean, the result was proved unconditionally in the paper [**35**] of Shelstad.

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The existence of the local transfer mapping will be used repeatedly throughout the paper. In particular, we shall make use of results and constructions of [**10**] and [**14**] that are conditional of the Langlands–Shelstad conjecture.

We first recall a convention from [**12**]. Two elements $x_1 \in G_{\alpha_1,\mathrm{reg}}(F)$ and $x_2 \in G_{\alpha_2,\mathrm{reg}}(F)$ in $G_{\mathrm{reg}}(F)$ are said to be *stably conjugate* if for any frame (ψ, u), $\psi_{\alpha_1\alpha_2}(x_2)$ is conjugate in $G_{\alpha_1}(\bar{F})$ to x_1. A *stable* distribution on $G(F)$ can then be defined as a distribution that lies in the closed linear span of the set of stable orbital integrals, taken over strongly regular stable conjugacy classes in $G(F)$. We write $S\mathcal{D}(G,\zeta)$ and $S\mathcal{F}(G,\zeta)$ for the subspaces of stable distributions in $\mathcal{D}(G,\zeta)$ and $\mathcal{F}(G,\zeta)$, respectively. If the basis $\Gamma_{\mathrm{reg}}(G,\zeta)$ is suitably chosen, it has a partition whose equivalence classes $\Delta_{\mathrm{reg}}(G,\zeta)$ are bijective with the strongly regular stable conjugacy classes in $\bar{G}(F)$, and parametrize distributions

$$f \to f^G(\delta) = \sum_{\gamma\in\delta} f_G(\gamma), \quad \delta \in \Delta_{\mathrm{reg}}(G,\zeta), \quad f \in \mathcal{C}(G,\zeta),$$

that are stable. There is a canonical injection $\delta \to \delta^*$ from $\Delta_{\mathrm{reg}}(G,\zeta)$ to $\Delta_{\mathrm{reg}}(G^*,\zeta^*)$, which is a bijection if G is quasisplit, such that

$$f^G(\delta) = f^*(\delta^*), \quad f \in \mathcal{C}(G,\zeta).$$

In particular, we can identify $\Delta_{\mathrm{reg}}(G,\zeta)$ with the subset

$$\Delta^{\mathcal{E}}_{\mathrm{reg}}(G,\zeta) \cap S\mathcal{D}(G,\zeta)$$

of the basis $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G,\zeta)$ defined at the end of § 4. In general, we shall say that a linear form S on $\mathcal{H}(G,\zeta)$ is *stable* if its value at any f depends only on f^*. This matches the definition above. If G is quasisplit, there is a unique linear form $\hat{S}$ on $S\mathcal{I}(G^*,\zeta^*)$ attached to any such S, with the property that

$$\hat{S}(f^*) = S(f), \quad f \in \mathcal{H}(G,\zeta).$$

This convention is familiar from the earlier case of invariant distributions. We shall use it both in the form described here, and also as it applies to global K-groups.

The existence of transfer mappings allows one to construct extended transfer factors. These are functions

$$\Delta(\delta',\gamma), \quad G' \in \mathcal{E}(G), \quad \delta' \in \Delta(\tilde{G}',\tilde{\zeta}'), \quad \gamma \in \Gamma(G,\zeta),$$

defined for fixed bases $\Delta(\tilde{G}',\tilde{\zeta}')$ of the spaces $S\mathcal{D}(\tilde{G}',\tilde{\zeta}')$, such that

$$f'(\delta') = \sum_{\gamma\in\Gamma(G,\zeta)} \Delta(\delta',\gamma) f_G(\gamma), \quad \delta' \in \Delta(\tilde{G}',\tilde{\zeta}'), \quad f \in \mathcal{H}(G,\zeta).$$

The rest of the discussion at the end of § 4 extends to this setting. One defines the set $\Delta^{\mathcal{E}}(G,\zeta)$ as a quotient of the subset of G-relevant pairs in

$$\{(G',\delta') : G' \in \mathcal{E}(G),\ \delta' \in \Delta(G',\zeta')\}.$$

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If δ belongs to $\Delta^{\mathcal{E}}(G,\zeta)$, the extended transfer factor

$$\Delta(\delta,\gamma) = \Delta(\delta',\gamma)$$

and the linear form

$$f \to f_G^{\mathcal{E}}(\delta) = f'(\delta'), \quad f \in \mathcal{H}(G,\zeta),$$

are both independent of whatever triplet $(G',\tilde{\xi}',\delta')$ is used to represent δ. In particular, $\Delta^{\mathcal{E}}(G,\zeta)$ can be identified with a family of ζ-equivariant distributions on $G(F)$, which one proves is a basis of $\mathcal{D}(G,\zeta)$. The subset

$$\Delta(G,\zeta) = \Delta^{\mathcal{E}}(G,\zeta) \cap S\mathcal{D}(G,\zeta)$$

of $\Delta^{\mathcal{E}}(G,\zeta)$ forms a basis of the subspace $S\mathcal{D}(G,\zeta)$ of $\mathcal{D}(G,\zeta)$. It corresponds to pairs of the form (G^*,δ^*), and comes with an injection $\delta \to \delta^*$ into $\Delta(G^*,\zeta^*)$ that is a bijection if G is quasisplit. Obviously,

$$f_G^{\mathcal{E}}(\delta) = \sum_{\gamma \in \Gamma(G,\zeta)} \Delta(\delta,\gamma) f_G(\gamma), \quad f \in \mathcal{H}(G,\zeta), \tag{5.3}$$

and since $f_G \to f_G^{\mathcal{E}}$ is an isomorphism from $\mathcal{I}(G,\zeta)$ to $S\mathcal{I}(G,\zeta)$, we can also write

$$f_G(\gamma) = \sum_{\delta \in \Delta^{\mathcal{E}}(G,\zeta)} \Delta(\gamma,\delta) f_G^{\mathcal{E}}(\delta), \quad f \in \mathcal{H}(G,\zeta),$$

for complex numbers $\Delta(\gamma,\delta)$. The extended transfer factor $\Delta(\delta,\gamma)$ and its adjoint companion $\Delta(\gamma,\delta)$ satisfy adjoint relations

$$\sum_{\delta \in \Delta^{\mathcal{E}}(G,\zeta)} \Delta(\gamma,\delta)\Delta(\delta,\gamma_1) = \delta(\gamma,\gamma_1), \quad \gamma,\gamma_1 \in \Gamma(G,\zeta), \tag{5.4}$$

and

$$\sum_{\gamma \in \Gamma(G,\zeta)} \Delta(\delta,\gamma)\Delta(\gamma,\delta_1) = \delta(\delta,\delta_1), \quad \delta,\delta_1 \in \Delta^{\mathcal{E}}(G,\zeta). \tag{5.5}$$

We refer the reader to the forthcoming paper [**14**] for details of the various constructions.

Suppose that M is a Levi subgroup of G. The canonical induction map $\gamma \to \gamma^G$ from $\mathcal{D}(M,\zeta)$ to $\mathcal{D}(G,\zeta)$ sends the subspace $S\mathcal{D}(M,\zeta)$ of $\mathcal{D}(M,\zeta)$ to the subspace $S\mathcal{D}(G,\zeta)$ of $\mathcal{D}(G,\zeta)$. The endoscopic basis $\Delta^{\mathcal{E}}(G,\zeta)$ of $\mathcal{D}(G,\zeta)$ provides an adjoint restriction map $\delta \to \delta_M$ from $\mathcal{D}(G,\zeta)$ to $\mathcal{D}(M,\zeta)$, which sends the subspace $S\mathcal{D}(G,\zeta)$ to $S\mathcal{D}(M,\zeta)$, and satisfies the analogue of (1.9). Unlike induction, however, the restriction operator depends on the basis in question. The map $\delta \to \delta_M$ here is different from the map $\gamma \to \gamma_M$ of § 1, even though we have not made a distinction in the notation. The generalized transfer factors for G and M do satisfy natural reciprocity formulae

$$\Delta_G(\nu^G,\gamma) = \Delta_M(\nu,\gamma_M), \quad \nu \in \Delta^{\mathcal{E}}(M,\zeta), \quad \gamma \in \Gamma(G,\zeta),$$

and

$$\Delta_G(\delta, \mu^G) = \Delta_M(\delta_M, \mu), \quad \delta \in \Delta^{\mathcal{E}}(G,\zeta), \quad \mu \in \Gamma(M,\zeta),$$

These are a reflection of the fact that the canonical maps $f_G \to f_M$ and $f_G^{\mathcal{E}} \to f_M^{\mathcal{E}}$ commute with the corresponding two transfer maps.

The situation on the spectral side is entirely parallel. Here we have an endoscopic basis $\Phi^{\mathcal{E}}(G,\zeta)$ of $\mathcal{F}(G,\zeta)$, and a subset

$$\Phi(G,\zeta) = \Phi^{\mathcal{E}}(G,\zeta) \cap S\mathcal{F}(G,\zeta)$$

that forms a basis of $S\mathcal{F}(G,\zeta)$. These objects are studied in [**10**] and [**15**]. We recall that they are defined in terms of abstract bases $\Phi_{\mathrm{ell}}(M,\zeta)$ of the cuspidal spaces $S\mathcal{I}_{\mathrm{cusp}}(M,\zeta)$, and similar objects for endoscopic groups M' of M. In the special case that the Levi subgroup M is abelian [**30**], or the field F is archimedean [**29**], we can take the particular bases afforded by Langlands parameters. In these cases, $\Phi_{\mathrm{ell}}(M,\zeta)$ is to be identified with the set of equivalence classes of cuspidal Langlands parameters

$$\phi : W_F \to {}^L M$$

that are compatible with ζ, in the sense that the composition of ϕ with the projection ${}^L M \to {}^L Z$ is the Langlands parameter defined by ζ. The stable distribution associated with such a parameter is then the sum of the standard characters in the corresponding L-packet. In general, if ϕ' is an element in $\Phi(\tilde{G}', \tilde{\zeta}')$ with image ϕ in $\Phi^{\mathcal{E}}(G,\zeta)$, we have spectral transfer factors

$$\Delta(\phi,\pi) = \Delta(\phi',\pi), \quad \pi \in \Pi(G,\zeta),$$

in terms of which the linear form

$$f \to f_G^{\mathcal{E}}(\phi) = f'(\phi'), \quad f \in \mathcal{H}(G,\zeta),$$

has an expansion

$$f_G^{\mathcal{E}}(\phi) = \sum_{\pi \in \Pi(G,\zeta)} \Delta(\phi,\pi) f_G(\pi). \tag{5.6}$$

There is also an inverse expansion

$$f_G(\pi) = \sum_{\phi \in \Phi^{\mathcal{E}}(G,\zeta)} \Delta(\pi,\phi) f_G^{\mathcal{E}}(\phi), \quad f \in \mathcal{H}(G,\zeta),$$

for complex numbers $\Delta(\pi,\phi)$ that satisfy adjoint relations

$$\sum_{\phi \in \Phi^{\mathcal{E}}(G,\zeta)} \Delta(\pi,\phi)\Delta(\phi,\pi_1) = \delta(\pi,\pi_1), \quad \pi,\pi_1 \in \Pi(G,\zeta), \tag{5.7}$$

and

$$\sum_{\pi \in \Pi(G,\zeta)} \Delta(\phi,\pi)\Delta(\pi,\phi_1) = \delta(\phi,\phi_1), \quad \phi,\phi_1 \in \Phi^{\mathcal{E}}(G,\zeta). \tag{5.8}$$

For archimedean F, these results are implicit in [**35**], while for p-adic F, they follow from [**10**, Theorems 6.1 and 6.2]. We refer the reader to the forthcoming paper [**15**] for more details.

Suppose that F is global. Let V be a finite set of valuations that contains the set $V_{\mathrm{ram}}(G,\zeta)$ of ramified places. We define bases $\Delta^{\mathcal{E}}(G_V,\zeta_V)$, $\Delta(G_V,\zeta_V)$, $\Phi^{\mathcal{E}}(G_V,\zeta_V)$ and $\Phi(G_V,\zeta_V)$ of the respective spaces $\mathcal{D}(G_V,\zeta_V)$, $S\mathcal{D}(G_V,\zeta_V)$, $\mathcal{F}(G_V,\zeta_V)$ and $S\mathcal{F}(G_V,\zeta_V)$, as tensor products over $v\in V$ of the local bases chosen above. We can then define extended transfer factors as corresponding products

$$\Delta(\delta,\gamma)=\prod_{v\in V}\Delta(\delta_v,\gamma_v),\quad \delta\in\Delta^{\mathcal{E}}(G_V,\zeta_V),\quad \gamma\in\Gamma(G_V,\zeta_V),$$

and

$$\Delta(\phi,\pi)=\prod_{v\in V}\Delta(\phi_v,\pi_v),\quad \phi\in\Phi^{\mathcal{E}}(G_V,\zeta_V),\quad \pi\in\Pi(G_V,\zeta_V),$$

of their local analogues. We can define adjoint transfer factors $\Delta(\gamma,\delta)$ and $\Delta(\pi,\phi)$ the same way, and it is clear that the obvious variants of the relations (5.3)–(5.8) all hold. The operations of induction and restriction exist in this context, and are compatible with the extended transfer factors. Suppose that $\delta\in\Delta^{\mathcal{E}}(G_V,\zeta_V)$ is the image of an element $\delta'\in\Delta(\tilde{G}'_V,\tilde{\zeta}'_V)$, where $G'\in\mathcal{E}(G)$ is a global endoscopic datum (as opposed to a general element in $\mathcal{E}(G_V)$). Then $\Delta(\delta,\gamma)$ is independent of any choice of base point, although it does of course depend on a choice of bases $\Delta^{\mathcal{E}}(G_V,\zeta_V)$ and $\Gamma(G_V,\zeta_V)$. A similar remark applies to the spectral transfer factors $\Delta(\phi,\pi)$.

Suppose that $H\supset Z$ is as in § 1. Then we can arrange that subsets $\Delta^{\mathcal{E}}(G_V^H,\zeta_V)$ and $\Delta(G_V^H,\zeta_V)$ of $\Delta^{\mathcal{E}}(G_V,\zeta_V)$ and $\Delta(G_V,\zeta_V)$ give bases of $\mathcal{D}(G_V^H,\zeta_V)$ and $S\mathcal{D}(G_V^H,\zeta_V)$, respectively, and that quotients $\Phi^{\mathcal{E}}(G_V^H,\zeta_V)$ and $\Phi(G_V^H,\zeta_V)$ of $\Phi^{\mathcal{E}}(G_V,\zeta_V)$ and $\Phi(G_V,\zeta_V)$ form bases of $\mathcal{F}(G_V^H,\zeta_V)$ and $S\mathcal{F}(G_V^H,\zeta_V)$. We can also arrange that each of these four bases has a chain of subsets that is parallel to (1.2) or (1.5). For elements $\delta\in\Delta^{\mathcal{E}}(G_V^H,\zeta_V)$ and $\gamma\in\Gamma(G_V^H,\zeta_V)$, the extended transfer factors $\Delta(\delta,\gamma)$ and $\Delta(\gamma,\delta)$ are defined simply as the restrictions of the ones above. For spectral objects $\phi\in\Phi^{\mathcal{E}}(G_V^H,\zeta_V)$ and $\pi\in\Pi(G_V^H,\zeta_V)$, however, we have to define $\Delta(\phi,\pi)$ and $\Delta(\pi,\phi)$ as an average of transfer factors above. We may as well assume that $H=Z$ at this point, and that ϕ and π have unitary central characters. This allows us to identify ϕ and π with orbits ϕ_λ and π_λ of $i\mathfrak{a}^*_{G,Z}$ in $\Phi^{\mathcal{E}}(G_V,\zeta_V)$ and $\Pi(G_V,\zeta_V)$, respectively. We then define the transfer factors by formulae

$$\Delta(\phi,\pi)=\sum_\lambda\Delta(\phi,\pi_\lambda)=\sum_\lambda\Delta(\phi_\lambda,\pi)$$

and

$$\Delta(\pi,\phi)=\sum_\lambda\Delta(\pi_\lambda,\phi)=\sum_\lambda\Delta(\pi,\phi_\lambda),$$

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in which λ is summed over $i\mathfrak{a}^*_{G,Z}$. Each of the sums has at most one non-zero term, and the resulting functions depend only on ϕ and π. The extended transfer factors we have just introduced describe the correspondence $f_G \leftrightarrow f_G^{\mathcal{E}}$ between $\mathcal{I}(G_V^Z,\zeta_V)$ and $\mathcal{I}^{\mathcal{E}}(G_V^Z,\zeta_V)$, through the appropriate variant of (5.3) or (5.6). We shall use them in § 10 to stabilize the geometric and spectral sides of the trace formula.

On the spectral side, we shall also require adelic transfer factors. It is clear how to define adelic families $\Phi(G(\mathbb{A})^Z,\zeta)$ and $\Phi^{\mathcal{E}}(G(\mathbb{A})^Z,\zeta)$. One simply augments elements in local families $\Phi(G_V^Z,\zeta_V)$ and $\Phi^{\mathcal{E}}(G_V^Z,\zeta_V)$ by products of unramified Langlands parameters $\{\phi_v : v \notin V\}$. Suppose that $\dot\pi \in \Pi(G(\mathbb{A})^Z,\zeta)$, and that for some $G' \in \mathcal{E}(G)$, $\dot\phi'$ is an element in $\Phi(\tilde G'(\mathbb{A})^{\tilde Z'},\tilde\zeta')$ with image $\dot\phi$ in $\Phi^{\mathcal{E}}(G(\mathbb{A})^Z,\zeta)$. We then define

$$\Delta(\dot\phi',\dot\pi) = \Delta(\dot\phi,\dot\pi) = \varinjlim_S \Delta(\dot\phi_S,\dot\pi_S)$$

and

$$\Delta(\dot\pi,\dot\phi) = \Delta(\dot\pi,\dot\phi) = \varinjlim_S \Delta(\dot\pi_S,\dot\phi_S),$$

where the limits are over large finite sets of valuations S, and $\dot\phi_S$ and $\dot\pi_S$ denote the images of $\dot\phi$ and $\dot\pi$ in $\Phi^{\mathcal{E}}(G_S^Z,\zeta_S)$ and $\Pi(G_S^Z,\zeta_S)$, respectively. The limits actually stabilize, since if ϕ_v' is an unramified Langlands parameter for an unramified endoscopic datum G_v', the function $\Delta(\phi_v,\pi_v)$ equals either 0 or 1. This property does not hold in general for geometric elements δ_v and γ_v, which is the reason we cannot work with adelic transfer factors on the geometric side.

6. Statement of Local Theorems 1 and 2

As we mentioned in the introduction, this paper is the first of a series of three articles designed to stabilize the terms in the trace formula. We are in a position now to state the main theorems. They apply to objects F, G and ζ, chosen as in the last section. Thus, F is a local or global field (of characteristic 0), G is a K-group over F that satisfies Assumption 5.2, and ζ is a character attached to a central induced torus Z for G over F.

There are four basic theorems, of which two are local and two are global. Within each of the two categories, there is in turn of a geometric result and a spectral result. The four theorems are in fact designed for the four different kinds of terms in the trace formula. It will be convenient to supplement each of the four theorems with a companion result that more directly describes the relevant terms. We will be able to describe how to resolve the supplementary theorems in terms of the original ones before the end of this paper. The proofs of the primary theorems, however, will require a long and detailed comparison of trace formulae that will have to be carried out over the full course of the three articles.

We shall state the local theorems in this section, and the global theorems in the next. The local results will be stated as separate theorems, following the scheme above, but they will actually all be consequences of the main transfer identity that was stated as a conjecture in [**12**, § 3]. Local Theorem 1 is in fact the assertion that this conjecture

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holds (for any local K-group that satisfies Assumption 5.2). This is to be regarded as the fundamental local result.

We are going to describe endoscopic and stable analogues of the local terms in geometric expansion (2.9). The full construction, which is given in [**12**] and [**14**], is a more elaborate version of the definition of the functions used to state Conjecture 5.1. In particular, it is based on a global form of the set $\mathcal{E}_{M'}(G)$ that indexes the sum in (5.2). We recall some properties of this set.

At this point, G stands for a K-group over a global field F, with a fixed Levi subgroup M, while M' represents an elliptic endoscopic datum $(M', \mathcal{M}', s'_M, \xi'_M)$ for M. It is assumed that $\mathcal{M}'$ is an L-subgroup of ${}^LM$ and that ξ'_M is the identity embedding of $\mathcal{M}'$ into ${}^LM$. Then $\mathcal{E}_{M'}(G)$ is the set of endoscopic data $(G', \mathcal{G}', s', \xi')$ for G over F, taken up to translation of s' by $Z(\hat{G})^\Gamma$, in which s' lies in $s'_M Z(\hat{M})^\Gamma$, $\hat{G}'$ is the connected centralizer of s' in $\hat{G}$, $\mathcal{G}'$ equals $\mathcal{M}'\hat{G}'$, and ξ' is the identity embedding of $\mathcal{G}'$ into ${}^LG$. The definition is taken from [**13**, § 3], and is identical to its local analogue in [**11**, § 4] and [**12**, § 3]. The one possible complication for the global case is easily resolved [**13**, Lemma 2], and one sees that $\mathcal{E}_{M'}(G)$ is in bijection with the set $Z(\hat{M})^\Gamma/Z(\hat{G})^\Gamma$ [**13**, Corollary 3]. It follows that for any valuation v of F, there is a map $G' \to G'_v$ from $\mathcal{E}_{M'}(G)$ to $\mathcal{E}_{M'_v}(G_v)$. The endoscopic datum M'_v for M_v here need not actually be elliptic over F_v, but this property is not required for the construction of $\mathcal{E}_{M'_v}(G_v)$.

Following the notation in [**12**] for local fields, we introduce the subset

$$\mathcal{E}^0_{M'}(G) = \begin{cases} \mathcal{E}_{M'}(G) - \{G^*\}, & \text{if } G \text{ is quasisplit,} \\ \mathcal{E}_{M'}(G), & \text{otherwise,} \end{cases}$$

of $\mathcal{E}_{M'}(G)$, and the factor

$$\mathcal{E}(G) = \begin{cases} 1, & \text{if } G \text{ is quasisplit,} \\ 0, & \text{otherwise.} \end{cases}$$

We also form coefficients

$$\iota_{M'}(G, G') = |Z(\hat{M}')^\Gamma/Z(\hat{M})^\Gamma||Z(\hat{G}')^\Gamma/Z(\hat{G})^\Gamma|^{-1}, \quad G' \in \mathcal{E}_{M'}(G).$$

These are not to be confused with the Langlands global coefficients

$$\iota(G, G') = \iota(G_\alpha, G'), \quad \alpha \in \pi_0(G), \quad G' \in \mathcal{E}_{\text{ell}}(G),$$

which have a more interesting formula [**21**, Theorem 8.3.1]. We shall use the latter in the next section, along with obvious variants of the notation such as

$$\mathcal{E}^0_{\text{ell}}(G, V) = \begin{cases} \mathcal{E}_{\text{ell}}(G, V) - \{G^*\}, & \text{if } G \text{ is quasisplit,} \\ \mathcal{E}_{\text{ell}}(G, V), & \text{otherwise,} \end{cases}$$

to state the global theorems.

The elements in $\mathcal{E}_{M'}(G)$ have to be fitted with extra structure before they can be used to construct new linear forms. For each G' in $\mathcal{E}_{M'}(G)$, we fix an embedding $M' \subset G'$

as a Levi subgroup for which $\hat{M}' \subset \hat{G}'$ is a dual Levi subgroup. We also fix the usual auxiliary data $\tilde{G}' \to G'$ and $\tilde{\xi}'$: $\mathcal{G}' \to {}^L\tilde{G}'$ for G'. Given these data, we set $\tilde{M}'$ equal to the preimage of M' in $\tilde{G}'$. Then $\tilde{M}'$ is a Levi subgroup of $\tilde{G}'$ for which $\hat{\tilde{M}}' = \hat{M}'Z(\hat{\tilde{G}}')$ is a dual Levi subgroup. The restriction of $\tilde{\xi}'$ provides an L-embedding of $\mathcal{M}'$ into an L-group ${}^L\tilde{M}' = \hat{\tilde{M}}'W_F$ of $\tilde{M}'$. The pair $\tilde{M}' \to M'$ and $\tilde{\xi}'$: $\mathcal{M}' \to {}^L\tilde{M}'$ are auxiliary data for M', for which the objects $\tilde{C}'$, $\tilde{Z}'$, $\tilde{\eta}'$ and $\tilde{\zeta}'$ described in § 4 match the corresponding objects for G'.

The data $(\tilde{M}', \tilde{\xi}')$ for M' vary with G'. However, for our purposes, they are really equivalent. The point is that for any V, the vector spaces $S\mathcal{D}(\tilde{M}'_V, \tilde{\zeta}'_V)$ are all canonically isomorphic. To see this in concrete terms, we can choose auxiliary data $\tilde{\tilde{M}}' \to M'$ and $\tilde{\tilde{\xi}}'_M$: $\mathcal{M}' \to {}^L\tilde{\tilde{M}}'$ for M that are independent of G', but for which $\tilde{\tilde{M}}'$ is equipped with a factorization $\tilde{\tilde{M}}' \to \tilde{M}' \to M'$, for each $G' \in \mathcal{E}_{M'}(G)$. For example, we could take $\tilde{\tilde{M}}'$ to be the fibre product of the extensions $\tilde{M}' \to M'$, as G' ranges over the finite set of elliptic elements in $\mathcal{E}_{M'}(G)$. Given the choice of $(\tilde{\tilde{M}}', \tilde{\tilde{\xi}}'_M)$, we obtain an admissible L-embedding

$$\tilde{\varepsilon}'_M : {}^L\tilde{M}' \to {}^L\tilde{\tilde{M}}',$$

for each G', with the property that

$$\tilde{\tilde{\xi}}'_M = \tilde{\varepsilon}'_M \tilde{\xi}'.$$

Now $\tilde{\varepsilon}'_M$ differs from the standard embedding ${}^L\tilde{M}' \to {}^L\tilde{\tilde{M}}'$ by a 1-cocycle from W_F to $Z(\tilde{\tilde{M}}')$. This in turn determines an automorphic character $\tilde{\chi}'_M$ in $\tilde{\tilde{M}}'(\mathbb{A})$. Identifying any function on $\tilde{M}'_V$ with its pullback to $\tilde{\tilde{M}}'_V$, we obtain a topological isomorphism

$$f \to \tilde{\chi}'_M f, \quad f \in \mathcal{C}(\tilde{M}'_V, \tilde{\zeta}'_V),$$

from $\mathcal{C}(\tilde{M}'_V, \tilde{\zeta}'_V)$ onto $\mathcal{C}(\tilde{\tilde{M}}'_V, \tilde{\tilde{\zeta}}'_V)$. This determines an isomorphism from $\mathcal{D}(\tilde{\tilde{M}}'_V, \tilde{\tilde{\zeta}}'_V)$ onto $\mathcal{D}(\tilde{M}'_V, \tilde{\zeta}'_V)$ that maps $S\mathcal{D}(\tilde{\tilde{M}}'_V, \tilde{\tilde{\zeta}}'_V)$ onto $S\mathcal{D}(\tilde{M}'_V, \tilde{\zeta}'_V)$. We can obviously compose any one such isomorphism with the inverse of another. In this way, we obtain an isomorphism between any two of the spaces $S\mathcal{D}(\tilde{M}'_V, \tilde{\zeta}'_V)$ that is easily seen to be independent of the choice of $\tilde{\tilde{M}}'$ and $\tilde{\tilde{\xi}}'_M$. We can assume that the bases $\Delta(\tilde{M}'_V, \tilde{\zeta}'_V)$ of the spaces $S\mathcal{D}(\tilde{M}'_V, \tilde{\zeta}'_V)$ are compatible with the isomorphisms.

We are interested in linear forms in $f \in \mathcal{H}(G, V, \zeta)$, where V is a finite set of valuations that contains $V_{\mathrm{ram}}(G, \zeta)$. We may as well assume at this point that the data M' and $\{(\tilde{G}', \tilde{\xi}')\}$ are also unramified away from V.

If δ belongs to the endoscopic basis $\Delta^{\mathcal{E}}(M^Z_V, \zeta_V)$, we first construct a linear form

$$I_M(\delta, f) = \sum_{\gamma} \Delta_M(\delta, \gamma) I_M(\gamma, f) \tag{6.1}$$

from $I_M(\gamma, f)$, simply by summing γ over $\Gamma(M^Z_V, \zeta_V)$. However, the true endoscopic analogue of $I_M(\gamma, f)$ is a more interesting object, which is constructed from the elements

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in $\mathcal{E}_{M'}(G)$. It is based on an inductive definition of linear forms

$$\hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f'), \quad G' \in \mathcal{E}_{M'}(G), \quad \delta' \in \Delta(\tilde{M}'_V, \tilde{\zeta}'_V),$$

on the spaces $S\mathcal{I}(\tilde{G}'_V, \tilde{\zeta}'_V)$. Suppose that

$$\{(G', \delta') : G' \in \mathcal{E}_{M'}(G),\ \delta' \in \Delta((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)\}$$

is a family of matching elements, relative to the bijections among the bases $\Delta(\tilde{M}'_V, \tilde{\zeta}'_V)$. We can write δ' for both the family, and the corresponding element in the set $\Delta((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$ attached to a given G'. We define linear forms $I^{\mathcal{E}}_M(\delta', f)$ and $S^G_M(M', \delta', f)$ inductively by the basic formula

$$I^{\mathcal{E}}_M(\delta', f) = \sum_{G' \in \mathcal{E}^0_{M'}(G)} \iota_{M'}(G, G') \hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f') + \varepsilon(G) S^G_M(M', \delta', f), \tag{6.2}$$

together with the supplementary requirement that

$$I^{\mathcal{E}}_M(\delta', f) = I_M(\delta, f), \tag{6.3}$$

in the case that G is quasisplit and δ' maps to the element δ in $\Delta^{\mathcal{E}}(M^Z_V, \zeta_V)$. The coefficient $\iota_{M'}(G, G')$ vanishes unless G' belongs to the finite subset of elliptic elements in $\mathcal{E}^0_{M'}(G)$, so the sum in (6.2) can be taken over a finite set. Since $S^G_M(M', \delta', f)$ is defined only if G is quasisplit, the two identities determine the two linear forms uniquely.

To complete the inductive definition, one has still to prove something in the special case that G is quasisplit and M' equals M^*. Then $\delta' = \delta^*$ belongs to $\Delta((M^*_V)^{Z^*}, \zeta^*_V)$, and the image δ of δ' in $\Delta^{\mathcal{E}}(M^Z_V, \zeta_V)$ lies in the subset $\Delta(M^Z_V, \zeta_V)$. The problem in this case is to show that the linear form

$$S^G_M(\delta, f) = S^G_M(M^*, \delta^*, f)$$

is stable. Only then would one have a linear form

$$\hat{S}^{G^*}_{M^*}(\delta^*, f^*) = S^G_M(\delta, f)$$

on $S\mathcal{I}((G^*_V)^{Z^*}, \zeta^*_V)$ that is the analogue for (G^*, M^*) of the terms $\hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f')$ in (6.2). This stability condition will be part of the local theorems we are preparing to state. It will not be established until a future article, in which the theorems will be proved. In the meantime, we carry it as an induction hypothesis on the groups $\tilde{G}'$ that occur in the sum (6.2).

Remark. The definitions (6.1)–(6.3) are taken from [**14**]. They extend the earlier treatment in [**11**, § 4] and [**12**, § 3], which applies to G-regular conjugacy classes, rather than distributions in the general bases $\Delta(\tilde{M}'_V, \tilde{\zeta}'_V)$. We were a bit careless in setting up the definitions in [**12**] (and [**11**]). The remark at the top of p. 242 of [**12**] notwithstanding, it is not generally possible to choose the auxiliary data $(\tilde{G}', \tilde{\xi}')$ so that $(\tilde{M}', \tilde{\xi}')$ is independent of G'. The definitions do make sense in the context of [**12**],but one has to take

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δ' to be an element in $\Delta_{G\text{-reg}}(\tilde{\tilde{M}}'_V)$. The results of [**12**] remain valid as stated, with the understanding that the linear forms $\hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta')$ in the analogue [**12**, (3.5)] of (6.2) depend on the embeddings $\tilde{\varepsilon}'_M$: ${}^L\tilde{M}' \to {}^L\tilde{\tilde{M}}'$, as well as δ'. It is easy to see from the definition in [**12**] that

$$\hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta', f') = \tilde{\chi}'_M(\delta')\hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta'_G, f'),$$

where δ'_G is the image in $\tilde{M}'_V$ of the element $\delta' \in \tilde{\tilde{M}}'_V$, and $\hat{S}^{\tilde{G}'}_{\tilde{M}'}(\delta'_G)$ is taken with respect to the standard embedding ${}^L\tilde{M}' \to {}^L\tilde{M}'$.

The distribution $S^G_M(\delta, f)$ is meant to be a stable form of the local term $I_M(\gamma, f)$ on the geometric side of the trace formula. The endoscopic form is a distribution

$$I^{\mathcal{E}}_M(\gamma, f), \quad \gamma \in \Gamma(M^Z_V, \zeta_V),$$

that like the original term, depends on an element in $\Gamma(M^Z_V, \zeta_V)$. It has the property that if δ' is relevant to M_V, then

$$I^{\mathcal{E}}_M(\delta', f) = \sum_{\gamma} \Delta_M(\delta', \gamma) I^{\mathcal{E}}_M(\gamma, f), \tag{6.4}$$

with γ summed over $\Gamma(M^Z_V, \zeta_V)$. In particular, the distribution

$$I^{\mathcal{E}}_M(\delta, f) = I^{\mathcal{E}}_M(\delta', f) \tag{6.5}$$

depends only on the image δ of δ' in $\Delta^{\mathcal{E}}(M^Z_V, \zeta_V)$. Observe that $I^{\mathcal{E}}_M(\gamma, f)$ is not uniquely determined by (6.4), since the endoscopic datum $M' \in \mathcal{E}_{\text{ell}}(M)$ was assumed to be over F. However, the construction makes sense if M' is replaced by an endoscopic datum $M'_V \in \mathcal{E}(M_V)$ over F_V. The distribution $I^{\mathcal{E}}_M(\gamma, f)$ can then be defined by inversion from (6.4). (See [**12**, § 5], [**14**].)

Local Theorem 1 actually applies to a simpler version of the linear forms above. To state it, we take F to be a local field, and M' to be a local endoscopic datum for M. We also take δ, δ' and γ to be elements in the subsets $\Delta^{\mathcal{E}}_{G\text{-reg,ell}}(M, \zeta)$, $\Delta_{G\text{-reg,ell}}(\tilde{M}', \tilde{\zeta}')$ and $\Gamma_{G\text{-reg,ell}}(M, \zeta)$ of G-regular, F-elliptic elements in the general bases $\Delta^{\mathcal{E}}(M, \zeta)$, $\Delta(\tilde{M}', \tilde{\zeta}')$ and $\Gamma(M, \zeta)$ attached to F. The associated distributions $I_M(\delta, f)$, $I^{\mathcal{E}}_M(\delta', f)$, $S^G_M(M', \delta', f)$, $I^{\mathcal{E}}_M(\gamma, f)$, and $I^{\mathcal{E}}_M(\delta, f)$ are essentially those of [**12**, § 3]. They are defined by the equations (6.1)–(6.5), stated exactly as above, but with the sums taken over γ and G' in the local sets $\Gamma_{G\text{-reg,ell}}(M, \zeta)$ and $\mathcal{E}_{M'}(G)$.

Local Theorem 1. *Assume that F is local.*

(a) *If G is arbitrary,*

$$I^{\mathcal{E}}_M(\gamma, f) = I_M(\gamma, f), \quad \gamma \in \Gamma_{G\text{-reg,ell}}(M, \zeta), \quad f \in \mathcal{H}(G, \zeta).$$

(b) *Suppose that G is quasisplit, and that δ' belongs to $\Delta_{G\text{-reg}}(\tilde{M}', \tilde{\zeta}')$, for some $M' \in \mathcal{E}_{\text{ell}}(M)$. Then the linear form*

$$f \to S^G_M(M', \delta', f), \quad f \in \mathcal{H}(G, \zeta),$$

vanishes unless $M' = M^*$, in which case it is stable.

As we mentioned above, this theorem is essentially Conjecture 3.3 of [**12**] (which was a slight generalization of Conjecture 4.1 of [**11**]). Our supplementary theorem will have a similar statement, except that it applies to the compound linear forms.

Local Theorem 1′. *Suppose that F is global, and that V is a finite set of valuations containing $V_{\mathrm{ram}}(G,\zeta)$.*

(a) *If G is arbitrary,*

$$I_M^{\mathcal{E}}(\gamma, f) = I_M(\gamma, f), \quad \gamma \in \Gamma(M_V^Z, \zeta_V), \quad f \in \mathcal{H}(G, V, \zeta).$$

(b) *Suppose that G is quasisplit, and that δ' belongs to $\Delta((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$, for some $M' \in \mathcal{E}_{\mathrm{ell}}(M, V)$. Then the linear form*

$$f \to S_M^G(M', \delta', f), \quad f \in \mathcal{H}(G, V, \zeta),$$

vanishes unless $M' = M^*$, in which case it is stable.

The next proposition is a consequence of the general results of [**14**], which extend the splitting and descent formulae in [**12**, § 6-7], and reduce the compound linear forms of Local Theorem 1′ to the simple linear forms of Local Theorem 1. For the special case of strongly G-regular conjugacy classes, the reader can refer to [**12**, Proposition 7.4].

Proposition 6.1. *Local Theorem 1 implies Local Theorem 1′.*

The proof of Local Theorem 1 will have to wait. We shall establish it in a subsequent article by global methods, as in the special case treated in [**16**].

There is an important point to mention before we turn to the local spectral terms. The element δ' in (6.2) does not have to be relevant to M. More generally, the definition (6.2) makes sense if we assume that the endoscopic datum M' belongs to $\mathcal{E}_{\mathrm{ell}}(M^*, V)$, rather than the subset $\mathcal{E}_{\mathrm{ell}}(G, V)$ of $\mathcal{E}_{\mathrm{ell}}(M^*, V)$. However, the more general linear forms are subject to the following vanishing property, which we shall require in § 10.

Proposition 6.2. *Suppose that $V \supset V_{\mathrm{ram}}(G,\zeta)$, that M' represents a class in $\mathcal{E}_{\mathrm{ell}}(M^*, V)$, and that δ' belongs to $\Delta((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$. Then $I_M^{\mathcal{E}}(\delta', f)$ vanishes unless M' and δ' are both locally relevant to M.*

Proof. Using the appropriate splitting and descent formulae, together with the relevant germ expansions [**14**], it is easy to translate the proposition to a corresponding assertion in which F is a local field, M' is an elliptic endoscopic for M^* over F, and $\delta' \in \Delta(\tilde{M}', \tilde{\zeta}')$ is strongly G-regular. In the local case, an elliptic endoscopic datum for M^* is automatically relevant to M. If δ' is not relevant to M, the local vanishing theorem [**12**, Theorem 8.6] asserts that $I_M^{\mathcal{E}}(\delta', f) = 0$. The proposition follows. □

The proposition asserts that $I_M^{\mathcal{E}}(\delta', f)$ equals zero unless δ' maps to an element δ in $\Delta^{\mathcal{E}}(M_V^Z, \zeta_V)$. In this case, as we have already noted (6.5), the distribution depends only on δ. Recall that as a function of δ', the transfer factor $\Delta_M(\delta', \gamma)$ also has these

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properties. It follows that (6.4) remains valid for M' and δ' chosen according to the general criteria of Proposition 6.2.

The other two local theorems have the same form, except that they apply to the residual linear forms on the spectral side. If F is global, there are stable and endoscopic analogues of the linear forms $I_M(\pi, f)$ in the spectral expansion (3.13). They are defined [**15**] by a construction that is dual to the one above. In fact, one defines linear forms $I_M(\phi, f)$, $I_M^{\mathcal{E}}(\phi', f)$, $S_M^G(M', \phi', f)$, $I_M^{\mathcal{E}}(\pi, f)$ and $I_M^{\mathcal{E}}(\phi, f)$ simply by replacing the elements δ, δ' and γ in (6.1)–(6.5) by elements $\phi \in \Phi^{\mathcal{E}}(M_V^Z, \zeta_V)$, $\phi' \in \Phi((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$ and $\pi \in \Pi(M_V^Z, \zeta_V)$. These objects are actually specializations at $X = 0$ of more general linear forms that depend on a point X in $\mathfrak{a}_M^Z$. If F is local, one constructs simpler but more fundamental objects, in which it is convenient to consider the dependence on this extra parameter. For elements $\phi \in \Phi^{\mathcal{E}}(M, \zeta)$, $\phi' \in \Phi(\tilde{M}', \tilde{\zeta}')$ and $\pi \in \Pi(M, \zeta)$, and a point X in the subgroup

$$\mathfrak{a}_{M,F} = \{H_G(x) : x \in G(F)\}$$

of $\mathfrak{a}_M$, one defines linear forms $I_M(\phi, X, f)$, $I_M^{\mathcal{E}}(\phi', X, f)$, $S_M^G(M', \phi', X, f)$, $I_M^{\mathcal{E}}(\pi, X, f)$ and $I_M^{\mathcal{E}}(\phi, X, f)$ on $\mathcal{H}(G, \zeta)$ once again by the obvious variants of the formulae (6.1)–(6.5). (See [**15**].)

Local Theorem 2. *Assume that F is local, and that X lies in $\mathfrak{a}_{M,F}$.*

(a) *If G is arbitrary,*

$$I_M^{\mathcal{E}}(\pi, X, f) = I_M(\pi, X, f), \quad \pi \in \Pi(M, \zeta), \quad f \in \mathcal{H}(G, \zeta).$$

(b) *Suppose that G is quasisplit, and that ϕ' belongs to $\Phi(\tilde{M}', \tilde{\zeta}')$, for some $M' \in \mathcal{E}_{\mathrm{ell}}(M)$. Then the linear form*

$$f \to S_M^G(M', \phi', X, f), \quad f \in \mathcal{H}(G, \zeta),$$

vanishes unless $M' = M^*$, in which case it is stable.

Again we have a supplementary theorem, which applies to the compound linear forms that arise from the local terms in the spectral expansion.

Local Theorem 2′. *Assume that F is global, and that V is a finite set of valuations containing $V_{\mathrm{ram}}(G, \zeta)$.*

(a) *If G is arbitrary,*

$$I_M^{\mathcal{E}}(\pi, f) = I_M(\pi, f), \quad \pi \in \Pi(M_V^Z, \zeta_V), \quad f \in \mathcal{H}(G, V, \zeta).$$

(b) *Suppose that G is quasisplit, and that ϕ' belongs to $\Phi((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}')$, for some $M' \in \mathcal{E}_{\mathrm{ell}}(M, V)$. Then the linear form*

$$f \to S_M^G(M', \phi', f), \quad f \in \mathcal{H}(G, V, \zeta),$$

vanishes unless $M' = M^*$, in which case it is stable.

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In the forthcoming paper [**15**], we shall establish spectral splitting and descent formulae that are parallel to the geometric formulae in [**14**]. They yield the following proposition.

Proposition 6.3. *Local Theorem 2 implies Local Theorem 2′.*

Similarly, the results of [**15**] yield a spectral vanishing property that is parallel to Proposition 6.2.

Proposition 6.4. *Suppose that V and M' are as in Proposition 6.2, and that ϕ' belongs to $\Phi((\tilde{M}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$. Then $I^{\mathcal{E}}_M(\phi', f)$ vanishes unless M' and ϕ' are both locally relevant to M.*

It follows that the spectral analogue

$$I^{\mathcal{E}}_M(\phi', f) = \sum_{\pi} \Delta_M(\phi', \pi) I^{\mathcal{E}}_M(\pi, f) \tag{6.6}$$

of (6.4), in which π is summed over $\Pi(M^Z_V, \zeta_V)$, is valid for any M' and ϕ' chosen as in Proposition 6.4.

7. Statement of Global Theorems 1 and 2

The main global theorems concern the coefficients on each side of the trace formula. We shall define endoscopic and stable forms of the various coefficients. The theorems will then assert identities among these new objects.

We are assuming that G is a K-group over the global field F. Before we define the new coefficients, we should first take care of a point having to do with the objects $\tilde{G}' \to G'$ and $\tilde{\xi}'\colon \mathcal{G}' \to {}^L\tilde{G}'$ attached to any given G'. We assume henceforth these auxiliary data have been chosen according to the following lemma.

Lemma 7.1. *Suppose that $G' \in \mathcal{E}_{\mathrm{ell}}(G, V)$, for a finite set $V \supset V_{\mathrm{ram}}(G)$. Then the global auxiliary data $\tilde{G}'$ and $\tilde{\xi}'$ for G' can be chosen so that they are unramified at any $v \notin V$.*

Proof. We have to be able to choose $(\tilde{G}', \tilde{\xi}')$ in such a way that for any $v \notin V$, the local embedding $\tilde{\xi}'_v\colon \mathcal{G}'_v \to {}^L\tilde{G}'_v$ is unramified. This means that $\tilde{G}'_v$ is an unramified group over F_v, and that $\tilde{\xi}'_v$ maps any Frobenius element in $\mathcal{G}'_v$ to a Frobenius element in ${}^L\tilde{G}'_v$. A Frobenius element is of course one that projects into the Frobenius coset in W_{F_v}. The proof of the property is implicit in the discussion on pp. 718–720 of the Langlands's paper [**27**]. In particular, we shall take $\tilde{G}'$ to be a z-extension of G'. The centre $Z(\hat{\tilde{G}}')$ of $\hat{\tilde{G}}'$ is then connected. By [**27**, Lemma 4], the inflation to the Weil group W_F of any 2-cocycle from $\Gamma = \Gamma_F$ to $Z(\hat{\tilde{G}}')$ splits.

Let K be a finite Galois extension of F over which G' splits, and which is unramified outside of V. Let T'_{sc} be a maximally split, maximal torus in G'_{sc} over F. Since G' is quasisplit, T'_{sc} is an induced torus, which splits over K. The construction in [**27**, pp. 721, 722] then yields a z-extension $\tilde{G}'$ of G' by the torus $\tilde{C}' = T'_{\mathrm{sc}}$, which is quasisplit over F and splits over K. As in [**27**] and [**24**, § 2.2], the groups $\mathcal{G}'$ and ${}^L\tilde{G}'$ determine a 2-cocycle

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a from $\Gamma_{K/F} = \mathrm{Gal}(K/F)$ to $Z(\hat{\tilde{G}}')$. One obtains an embedding $\tilde{\xi}'\colon \mathcal{G}' \to {}^L\tilde{G}'$ from any splitting of this cocycle over the Weil group $W_{K/F}$. The embedding will be unramified outside of V if and only if the splitting is defined over the largest quotient $W_{K/F,V}$ of $W_{K/F}$ that is unramified outside of V. Our task is therefore to show that the inflation of a to the extension $W_{K/F,V}$ of $\Gamma_{K/F}$ splits.

Following the proof of [**27**, Lemma 4], we set $X = X^*(Z(\hat{\tilde{G}}'))$, and we form a short exact sequence

$$1 \to X_2 \to X_1 \to X \to 1$$

of $\Gamma_{K/F}$-modules that are free over $\mathbb{Z}$, such that X_1 is also free over $\Gamma_{K/F}$. At this stage of the argument in [**27**], Langlands introduces the idele class group C_K of K. We shall instead use the quotient

$$C_{K,V} = \mathbb{A}_K^*/K^*(\mathcal{O}_K^V)^*,$$

where $\mathcal{O}_K^V$ is the maximal compact subring of $\mathbb{A}_K^V$. In particular, we apply the functor

$$\mathrm{Hom}_{\Gamma_{K/F}}(—, C_{K,V})$$

to the dual short exact sequence

$$1 \longleftarrow X_2^\vee \longleftarrow X_1^\vee \longleftarrow X^\vee \longleftarrow 1.$$

This yields a short exact sequence

$$1 \to T_2(C_{K,V})^{\Gamma_{K/F}} \to T_1(C_{K,V})^{\Gamma_{K/F}} \to T(C_{K,V})^{\Gamma_{K/F}},$$

where T_* denotes the torus $\mathrm{Hom}(X_*^\vee, -)$ (regarded as a $\mathbb{Z}$-scheme with action of $\Gamma_{K/F}$). It follows from the injectivity of the middle arrow, and standard facts about extensions of characters, that the dual map

$$(T_1(C_{K,V})^{\Gamma_{K/F}})^* \to (T_2(C_{K,V})^{\Gamma_{K/F}})^*$$

of character groups is surjective. The main global theorem of [**30**] gives a functorial isomorphism from $(T_*(C_K)^{\Gamma_{K/F}})^*$ onto the continuous cohomology group $H_c^1(W_{K/F}, \hat{T}_*)$. The long exact sequence of cohomology implies that

$$T_*(C_{K,V})^{\Gamma_{K/F}} = T_*(C_K)^{\Gamma_{K/F}}/T_*(\mathcal{O}_F^V),$$

since $H^1(\Gamma_{K/F}, T_*(\mathcal{O}_K^V))$ is trivial, so that the group $(T_*(C_{K,V})^{\Gamma_{K/F}})^*$ is a subgroup of $(T_*(C_K)^{\Gamma_{K/F}})^*$. The main point for us is that the global isomorphism takes this subgroup onto the subgroup $H_c^1(W_{K/F,V}, \hat{T}_*)$ of $H_c^1(W_{K/F}, \hat{T}_*)$. This follows easily from the local correspondence in [**30**] for unramified characters. We therefore conclude that the map

$$H_c^1(W_{K/F,V}, \hat{T}_1) \to H_c^1(W_{K/F,V}, \hat{T}_2)$$

is surjective.

The rest of the proof is identical to the last paragraph in the proof of Lemma 4 of [**27**]. We obtain a splitting over $W_{K/F,V}$ for the original 2-cocycle a. This yields an L-embedding $\tilde{\xi}'$ that is unramified outside of V. □

We remark that if G_{der} is simply connected, we can choose the auxiliary data $\tilde{G}'$ and $\tilde{\xi}'$ of the lemma in such a way that $\tilde{G}' = G'$. For if a is the 2-cocycle from $\Gamma_{K/F}$ to $Z(\hat{G}')$ described in the proof above, the argument in [**27**, pp. 705–715] establishes that the projection of a onto $Z(\hat{G}')/Z(\hat{G})$ splits. Since $Z(\hat{G}) = Z(\hat{G}')^0$ is connected, one can construct required embedding $\tilde{\xi}'\colon \mathcal{G}' \to {}^L G'$ as in the proof of the lemma. It follows that for arbitrary G, we can choose the auxiliary data $\tilde{G}'$ and $\tilde{\xi}'$ of the lemma in such a way that $\tilde{G}'$ is the endoscopic datum for a fixed z-extension $\tilde{G} \to G$ attached to G'.

We consider now the global coefficients. We shall begin with the general family of geometric coefficients $a^G(\gamma)$. Recall that $a^G(\gamma)$ is defined on $\Gamma(G^Z_V, \zeta_V)$, and is in fact supported on the discrete subset $\Gamma(G, V, \zeta)$ of this domain. We shall construct parallel families of coefficients $a^{G,\mathcal{E}}(\gamma)$ and $b^G(\delta)$, the latter only in the case that G is quasisplit, on the respective domains $\Gamma(G^Z_V, \zeta_V)$ and $\Delta^{\mathcal{E}}(\Delta^Z_V, \zeta_V)$. If γ lies in $\Gamma(G^Z_V, \zeta_V)$, we set

$$a^{G,\mathcal{E}}(\gamma) = \sum_{G'} \sum_{\delta'} \iota(G, G') b^{\tilde{G}'}(\delta') \Delta_G(\delta', \gamma) + \varepsilon(G) \sum_{\delta} b^G(\delta) \Delta_G(\delta, \gamma), \tag{7.1}$$

with G', δ' and δ summed over $\mathcal{E}^0_{\mathrm{ell}}(G, V)$, $\Delta((\tilde{G}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$ and $\Delta^{\mathcal{E}}(G^Z_V, \zeta_V)$, respectively, and with coefficients $b^{\tilde{G}'}(\delta')$ defined inductively by the requirement that

$$a^{G,\mathcal{E}}(\gamma) = a^G(\gamma), \tag{7.2}$$

in the case that G is quasisplit. The process is similar to those of § 6. If G is quasisplit, the relations (7.1) and (7.2), together with the local inversion formulae (5.5), provide a formula for $b^G(\delta)$ as a function on $\Delta^{\mathcal{E}}(G^Z_V, \zeta_V)$. To complete the inductive definition, we define the function

$$b^{G^*}(\delta^*) = b^G(\delta), \quad \delta \in \Delta(G^Z_V, \zeta_V),$$

from the restriction of b^G to the subset $\Delta(G^Z_V, \zeta_V)$ of $\Delta^{\mathcal{E}}(G^Z_V, \zeta_V)$. Since G^* determines G uniquely up to weak isomorphism, Corollary 4.4 tells us $b^{G^*}(\delta^*)$ depends only on G^*.

As in § 2, it is instructive to introduce more manageable domains for the new coefficients. We shall construct discrete subsets $\Gamma^{\mathcal{E}}(G, V, \zeta)$ and $\Delta^{\mathcal{E}}(G, V, \zeta)$ of $\Gamma(G^Z_V, \zeta_V)$ and $\Delta^{\mathcal{E}}(G^{\mathcal{E}}_V, \zeta_V)$, respectively, that contain the supports of $a^{G,\mathcal{E}}(\gamma)$ and $b^G(\delta)$.

We are first to introduce 'elliptic' subsets $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G, V, \zeta)$, $\Delta_{\mathrm{ell}}(G, V, \zeta)$, $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G, V, \zeta)$ of $\Delta^{\mathcal{E}}(G^Z_V, \zeta_V)$, $\Delta(G^Z_V, \zeta_V)$, $\Gamma^{\mathcal{E}}(G^Z_V, \zeta_V)$, respectively. As we might expect, we have to give an inductive definition that is based on the sets $\Delta_{\mathrm{ell}}(\tilde{G}', V, \tilde{\zeta}')$ attached to groups $G' \in \mathcal{E}^0_{\mathrm{ell}}(G, V)$. We define the first set $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G, V, \zeta)$ to be the collection of δ in $\Delta^{\mathcal{E}}(G^Z_V, \delta_V)$ such that either $\Delta_G(\gamma, \delta) \neq 0$, for some $\gamma \in \Gamma_{\mathrm{ell}}(G, V, \zeta)$, or δ is the image in $\Delta^{\mathcal{E}}(G^Z_V, \zeta_V)$ of an element δ' in the subset $\Delta_{\mathrm{ell}}(\tilde{G}', V, \tilde{\zeta})$ of $\Delta((\tilde{G}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$, for some $G' \in \mathcal{E}^0_{\mathrm{ell}}(G, V)$. The second set is then just the intersection

$$\Delta_{\mathrm{ell}}(G, V, \zeta) = \Delta^{\mathcal{E}}_{\mathrm{ell}}(G, V, \zeta) \cap \Delta(G^Z_V, \zeta_V).$$

We define the third set $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G, V, \zeta)$ to be the collection of γ in $\Gamma(G^Z_V, \zeta_V)$ such that $\Delta_G(\delta, \gamma) \neq 0$, for some $\delta \in \Delta^{\mathcal{E}}_{\mathrm{ell}}(G, V, \zeta)$. Then $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G, V, \zeta)$ contains $\Gamma_{\mathrm{ell}}(G, V, \zeta)$. Having constructed the elliptic sets, we can then define the larger subsets $\Delta^{\mathcal{E}}(G, V, \zeta)$, $\Delta(G, V, \zeta)$, $\Gamma^{\mathcal{E}}(G, V, \zeta)$ of $\Delta^{\mathcal{E}}(G^Z_V, \zeta_V)$, $\Delta(G^Z_V, \zeta_V)$, $\Gamma(G^Z_V, \zeta_V)$, respectively, exactly as in § 2. For

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example, $\Gamma^{\mathcal{E}}(G,V,\zeta)$ is the set of elements γ in $\Gamma(G_V^Z,\zeta_V)$ that are constituents of induced classes μ^G, where μ belongs to $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(M,V,\zeta)$, for some $M\in\mathcal{L}$. It is easy to see that the coefficient $a^{G,\mathcal{E}}(\gamma)$ is supported on $\Gamma^{\mathcal{E}}(G,V,\zeta)$, and in case G is quasisplit, that $b^G(\delta)$ is supported on $\Delta^{\mathcal{E}}(G,V,\zeta)$. Moreover, the sums over δ' and δ in the definition (7.1) can be taken over the smaller sets $\Delta(\tilde{G}',V,\tilde{\zeta}')$ and $\Delta^{\mathcal{E}}(G,V,\zeta)$, respectively.

We shall also need to consider endoscopic and stable analogues of the more fundamental elliptic coefficients $a^G_{\mathrm{ell}}(\dot{\gamma}_S)$. Recall that $a^G_{\mathrm{ell}}(\dot{\gamma}_S)$ was defined in (2.6) for any large finite set of valuations $S\supset V_{\mathrm{ram}}(G,\zeta)$, and for any admissible element $\dot{\gamma}_S$ in $\Gamma_{\mathrm{ell}}(G,S,\zeta)$. We construct parallel coefficients $a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S)$ and $b^G_{\mathrm{ell}}(\dot{\delta}_S)$, for admissible elements $\dot{\gamma}_S\in\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$ and $\dot{\delta}_S\in\Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$, by the natural variants of the inductive definitions (7.1) and (7.2). In other words, we set

$$a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S)=\sum_{G'}\sum_{\dot{\delta}'_S}\iota(G,G')b^{\tilde{G}'}_{\mathrm{ell}}(\dot{\delta}'_S)\Delta_G(\dot{\delta}'_S,\dot{\gamma}_S)+\varepsilon(G)\sum_{\dot{\delta}_S}b^G_{\mathrm{ell}}(\dot{\delta}_S)\Delta_G(\dot{\delta}_S,\dot{\gamma}_S), \tag{7.3}$$

with G', $\dot{\delta}'_S$ and $\dot{\delta}_S$ summed over $\mathcal{E}^0_{\mathrm{ell}}(G,S)$, $\Delta_{\mathrm{ell}}(\tilde{G}',S,\tilde{\zeta}')$ and $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$, respectively, and with the coefficients $b^{\tilde{G}'}_{\mathrm{ell}}(\dot{\delta}'_S)$ defined inductively by the requirements that

$$a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S)=a^G_{\mathrm{ell}}(\dot{\gamma}_S) \tag{7.4}$$

and

$$b^{G^*}_{\mathrm{ell}}(\dot{\delta}^*_S)=b^G_{\mathrm{ell}}(\dot{\delta}_S),$$

in case G is quasisplit. As in the earlier case, we know from Corollary 4.4 that the last coefficient depends only on G^*.

Global Theorem 1.

(a) *For any G, we have*

$$a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S)=a^G_{\mathrm{ell}}(\dot{\gamma}_S),$$

for any admissible element $\dot{\gamma}_S$ in $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$.

(b) *If G is quasisplit, $b^G_{\mathrm{ell}}(\dot{\delta}_S)$ vanishes for any admissible element $\dot{\delta}_S$ in the complement of $\Delta_{\mathrm{ell}}(G,S,\zeta)$ in $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)$.*

Global Theorem 1 pertains to the basic elliptic coefficients that are the global foundation of the geometric expansion in § 2. For the actual comparison of trace formulae, we also require the corresponding theorem for the general coefficients.

Global Theorem 1′.

(a) *For any G, we have*

$$a^{G,\mathcal{E}}(\gamma)=a^G(\gamma),\quad \gamma\in\Gamma^{\mathcal{E}}(G,V,\zeta).$$

(b) *If G is quasisplit, $b^G(\delta)$ vanishes for any δ in the complement of $\Delta(G,V,\zeta)$ in $\Delta^{\mathcal{E}}(G,V,\zeta)$.*

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Global Theorems 1 and 1′ describe the basic identities satisfied by the geometric coefficients. As the original definition (2.8) suggests, they are closely related. In § 10, we shall show that Global Theorem 1 implies Global Theorem 1′ by establishing endoscopic and stable analogues of the expansion (2.8).

It is useful to note that Global Theorems 1 and 1′ can be reformulated in terms of endoscopic and stable analogues of the linear forms $I_{\mathrm{ell}}(\dot{f}_S)$ and $I_{\mathrm{orb}}(f)$ of § 2. We define invariant linear forms $I^{\mathcal{E}}_{\mathrm{ell}}(\dot{f}_S) = I^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{f}_S)$ and $I^{\mathcal{E}}_{\mathrm{orb}}(f) = I^{G,\mathcal{E}}_{\mathrm{orb}}(f)$, for functions $\dot{f}_S \in \mathcal{H}_{\mathrm{adm}}(G,S,\zeta)$ and $f \in \mathcal{H}(G,V,\zeta)$, by setting

$$I^{\mathcal{E}}_{\mathrm{ell}}(\dot{f}_S) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,S)} \iota(G,G')\hat{S}^{\tilde{G}'}_{\mathrm{ell}}(\dot{f}'_S) + \varepsilon(G)S^G_{\mathrm{ell}}(\dot{f}_S) \tag{7.5}$$

and

$$I^{\mathcal{E}}_{\mathrm{orb}}(f) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}^{\tilde{G}'}_{\mathrm{orb}}(f') + \varepsilon(G)S^G_{\mathrm{orb}}(f). \tag{7.6}$$

The terms $\hat{S}^{\tilde{G}'}_{\mathrm{ell}}$ and $\hat{S}^{\tilde{G}'}_{\mathrm{orb}}$ on the right are linear forms on the spaces $S\mathcal{I}_{\mathrm{adm}}(\tilde{G}',S,\tilde{\zeta}')$ and $S\mathcal{I}(\tilde{G}',V,\tilde{\zeta}')$, respectively, which are defined inductively by the requirements that $I^{\mathcal{E}}_{\mathrm{ell}}(\dot{f}_S) = I_{\mathrm{ell}}(\dot{f}_S)$ and $I^{\mathcal{E}}_{\mathrm{orb}}(f) = I_{\mathrm{orb}}(f)$ in the case that G is quasisplit. The definition includes the induction hypothesis that $S^{\tilde{G}'}_{\mathrm{ell}}$ and $S^{\tilde{G}'}_{\mathrm{orb}}$ are stable, for data G' in $\mathcal{E}^0_{\mathrm{ell}}(G,S)$ and $\mathcal{E}^0_{\mathrm{ell}}(G,V)$, respectively.

Lemma 7.2.

(a) *If G is arbitrary,*

$$I^{\mathcal{E}}_{\mathrm{ell}}(\dot{f}_S) = \sum_{\dot{\gamma}_S \in \Gamma^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)} a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S)\dot{f}_{S,G}(\dot{\gamma}_S)$$

and

$$I^{\mathcal{E}}_{\mathrm{orb}}(f) = \sum_{\gamma \in \Gamma^{\mathcal{E}}(G,V,\zeta)} a^{G,\mathcal{E}}(\gamma)f_G(\gamma).$$

In particular, the statements (a) of Global Theorems 1 and 1′ are equivalent to the general identities $I^{\mathcal{E}}_{\mathrm{ell}}(\dot{f}_S) = I_{\mathrm{ell}}(\dot{f}_S)$ and $I^{\mathcal{E}}_{\mathrm{orb}}(f) = I_{\mathrm{orb}}(f)$.

(b) *If G is quasisplit,*

$$S^G_{\mathrm{ell}}(\dot{f}_S) = \sum_{\dot{\delta}_S \in \Delta^{\mathcal{E}}_{\mathrm{ell}}(G,S,\zeta)} b^G_{\mathrm{ell}}(\dot{\delta}_S)\dot{f}^{\mathcal{E}}_{S,G}(\dot{\delta}_S)$$

and

$$S^G_{\mathrm{orb}}(f) = \sum_{\delta \in \Delta^{\mathcal{E}}(G,V,\zeta)} b^G(\delta)f^{\mathcal{E}}_G(\delta).$$

In particular, the statements (b) of Global Theorems 1 and 1′ are equivalent to the assertions that $S^G_{\mathrm{ell}}(\dot{f}_S)$ and $S^G_{\mathrm{orb}}(f)$ are stable.

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Proof. By varying the test functions $\dot{f}_S$ and f, we see immediately that the given expansions imply that Global Theorems 1 and 1′ can be reformulated as claimed. It is enough, then, to establish the expansions.

Consider the case of $I^{\mathcal{E}}_{\mathrm{orb}}(f)$ and $S^G_{\mathrm{orb}}(f)$. We can assume inductively that the expansions in (b) hold for quasisplit inner K-forms of groups $\tilde{G}'$, with $G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)$. We are also assuming that $\hat{S}^{\tilde{G}'}_{\mathrm{orb}}$ comes from a stable distribution on a quasisplit inner K-form of $\tilde{G}'$, from which it follows that $b^{\tilde{G}'}$ is supported on the subset $\Delta(\tilde{G}',V,\tilde{\zeta}')$ of $\Delta^{\mathcal{E}}(\tilde{G}',V,\tilde{\zeta}')$. Therefore,

$$\hat{S}^{\tilde{G}'}_{\mathrm{orb}}(f') = \sum_{\delta' \in \Delta(\tilde{G}',V,\tilde{\zeta}')} b^{\tilde{G}'}(\delta')f'(\delta'),$$

for any $f \in \mathcal{H}(G,V,\zeta)$. It follows from the various definitions that

$$\begin{aligned} I^{\mathcal{E}}_{\mathrm{orb}}(f) - \varepsilon(G)S^G_{\mathrm{orb}}(f) &= \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}^{\tilde{G}'}_{\mathrm{orb}}(f') \\ &= \sum_{G'} \iota(G,G') \sum_{\delta' \in \Delta(\tilde{G}',V,\tilde{\zeta}')} b^{\tilde{G}'}(\delta')f'(\delta') \\ &= \sum_{G',\delta'} \iota(G,G')b^{\tilde{G}'}(\delta') \sum_{\gamma \in \Gamma^{\mathcal{E}}(G,V,\zeta)} \Delta_G(\delta',\gamma)f_G(\gamma) \\ &= \sum_{\gamma}\Big(\sum_{G',\delta'} \iota(G,G')b^{\tilde{G}'}(\delta')\Delta_G(\delta',\gamma)\Big)f_G(\gamma) \\ &= \sum_{\gamma}\Big(a^{G,\mathcal{E}}(\gamma) - \varepsilon(G)\sum_{\delta \in \Delta^{\mathcal{E}}(G,V,\zeta)} b^G(\delta)\Delta_G(\delta,\gamma)\Big)f_G(\gamma) \\ &= \sum_{\gamma} a^{G,\mathcal{E}}(\gamma)f_G(\gamma) - \varepsilon(G)\sum_{\delta} b^G(\delta)f^{\mathcal{E}}_G(\delta). \end{aligned}$$

If $\varepsilon(G)=0$, the required expansion for $I^{\mathcal{E}}_{\mathrm{orb}}(f)$ follows. If $\varepsilon(G)=1$, the expansion for $I^{\mathcal{E}}_{\mathrm{orb}}(f)$ is just part of the definition, so we also obtain the required expansion for $S^G_{\mathrm{orb}}(f)$.

The expansions for $I^{\mathcal{E}}_{\mathrm{ell}}(\dot{f}_S)$ and $S^G_{\mathrm{ell}}(\dot{f}_S)$ follow in the same way. □

We turn now to the spectral coefficients. The discussion will depend on a fixed non-negative number t, as in § 3, but will otherwise be parallel to that above. In particular, we shall use the global information at hand to construct suitable domains for the coefficients we are about to define.

We first construct discrete subsets $\Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$, $\Phi_{t,\mathrm{disc}}(G,\zeta)$ and $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$ of the respective adelic sets $\Phi^{\mathcal{E}}(G(\mathbb{A})^Z,\zeta)$, $\Phi(G(\mathbb{A})^Z,\zeta)$ and $\Pi(G(\mathbb{A})^Z,\zeta)$. The inductive definition is similar to that of the 'elliptic' sets above. Thus, $\Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$ is the set of $\dot{\phi}$ in $\Phi^{\mathcal{E}}(G(\mathbb{A})^Z,\zeta)$ such that either $\Delta_G(\dot{\pi},\dot{\phi}) \neq 0$, for some $\dot{\pi}$ in the set $\Pi_{t,\mathrm{disc}}(G,\zeta)$, or $\dot{\phi}$ is the image of an element $\dot{\phi}'$ in the subset $\Phi_{t,\mathrm{disc}}(\tilde{G}',\tilde{\zeta}')$ of $\Phi(\tilde{G}'(\mathbb{A})^{\tilde{Z}'},\tilde{\zeta}')$, for some

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$G' \in \mathcal{E}^0_{\mathrm{ell}}(G)$. The second set is defined by

$$\Phi_{t,\mathrm{disc}}(G,\zeta) = \Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta) \cap \Phi(G(\mathbb{A})^Z,\zeta),$$

while the third set $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$ is the set of $\dot{\pi}$ in $\Pi(G(\mathbb{A})^Z,\zeta)$ such that $\Delta_G(\dot{\phi},\dot{\pi}) \neq 0$, for some $\dot{\phi} \in \Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$. Then $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$ contains $\Pi_{t,\mathrm{disc}}(G,\zeta)$.

Now suppose that $V \supset V_{\mathrm{ram}}(G,\zeta)$ is a finite set of valuations as above. We define discrete subsets $\Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,V,\zeta)$, $\Phi_{t,\mathrm{disc}}(G,V,\zeta)$ and $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,V,\zeta)$ of $\Phi^{\mathcal{E}}(G^Z_V,\zeta_V)$, $\Phi(G^Z_V,\zeta_V)$ and $\Pi(G^Z_V,\zeta_V)$ in exactly the same way, except with $\Pi_{t,\mathrm{disc}}(G,\zeta)$ replaced by its analogue $\Pi_{t,\mathrm{disc}}(G,V,\zeta)$ for V. From these discrete subsets, we construct larger subsets $\Phi^{\mathcal{E}}_t(G,V,\zeta)$, $\Phi_t(G,V,\zeta)$ and $\Pi^{\mathcal{E}}_t(G,V,\zeta)$, with corresponding Borel measures, as in § 3. Thus, if M belongs to $\mathcal{L}$, $\Pi^{G,\mathcal{E}}_{t,\mathrm{disc}}(M,V,\zeta)$ stands for the set of $i\mathfrak{a}^*_{G,Z}$-orbits in the preimage $\tilde{\Pi}^{\mathcal{E}}_{t,\mathrm{disc}}(M,V,\zeta)$ of $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(M,V,\zeta)$ in $\Pi(M_V,\zeta_V)$. We define $\Pi^{\mathcal{E}}_t(G,V,\zeta)$ to be the union, over $M \in \mathcal{L}$ and $\rho \in \Pi^{G,\mathcal{E}}_{t,\mathrm{disc}}(M,V,\zeta)$, of the irreducible constituents of the induced representations ρ^G. The Borel measure $\mathrm{d}\pi$ on $\Pi^{\mathcal{E}}_t(G,V,\zeta)$ is defined by setting

$$\int_{\Pi^{\mathcal{E}}_t(G,V,\zeta)} h(\pi)\,\mathrm{d}\pi = \sum_{M\in\mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\rho\in\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(M,V,\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} h(\rho^G_\lambda)\,\mathrm{d}\lambda,$$

for any $h \in C_c(\Pi^{\mathcal{E}}_t(G,V,\zeta))$. The sets

$$\Phi_t(G,V,\zeta) \subset \Phi^{\mathcal{E}}_t(G,V,\zeta)$$

are defined in exactly the same way. For example, $\Phi_t(G,V,\zeta)$ is the union, over $M \in \mathcal{L}$ and $\chi \in \Phi^G_{t,\mathrm{disc}}(M,V,\zeta)$, of the induced elements χ^G. (By construction [**10**], the induced elements χ^G are irreducible, in the sense that they lie in the basis $\Phi(G^Z_V,\zeta_V)$.) The Borel measure $\mathrm{d}\phi$ is defined by

$$\int_{\Phi_t(G,V,\zeta)} h(\phi)\,\mathrm{d}\phi = \sum_{M\in\mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\chi\in\Phi_{t,\mathrm{disc}}(M,V,\zeta)} \int_{i\mathfrak{a}^*_{M,Z}/i\mathfrak{a}^*_{G,Z}} h(\chi^G_\lambda)\,\mathrm{d}\lambda,$$

for any $h \in C_c(\Phi_t(G,V,\zeta))$.

We now construct global coefficients $a^{G,\mathcal{E}}(\pi)$ and $h^G(\phi)$ inductively on the domains $\Pi^{\mathcal{E}}_t(G,V,\zeta)$ and $\Phi^{\mathcal{E}}_t(G,V,\zeta)$. If π belongs to $\Pi^{\mathcal{E}}_t(G,V,\zeta)$, we set

$$a^{G,\mathcal{E}}(\pi) = \sum_{G'}\sum_{\phi'} \iota(G,G')b^{\tilde{G}'}(\phi')\Delta_G(\phi',\pi) + \varepsilon(G)\sum_{\phi} b^G(\phi)\Delta_G(\phi,\pi), \tag{7.7}$$

with G', ϕ' and ϕ summed over $\mathcal{E}^0_{\mathrm{ell}}(G,V)$, $\Phi_t(\tilde{G}',V,\tilde{\zeta}')$ and $\Phi^{\mathcal{E}}_t(G,V,\zeta)$, respectively, and with coefficients $b^{\tilde{G}'}(\phi')$ defined inductively by the requirement that

$$a^{G,\mathcal{E}}(\pi) = a^G(\pi), \tag{7.8}$$

in the case that G is quasisplit. The construction is obviously identical to that of the geometric coefficients. In particular, $b^G(\phi)$ exists only when G is quasisplit, in which case

$$b^{G^*}(\phi^*) = b^G(\phi), \quad \phi \in \Phi^{\mathcal{E}}_t(G,V,\zeta),$$

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is defined by the relations (7.7) and (7.8), together with the local inversion formulae (5.8).

The arithmetic information in the trace formula is concentrated in the fundamental adelic coefficients

$$a^G_{\mathrm{disc}}(\dot\pi), \quad \dot\pi \in \Pi_{t,\mathrm{disc}}(G,\zeta).$$

We shall therefore want to consider endoscopic and stable analogues of these objects. We construct coefficients $a^{G,\mathcal{E}}_{\mathrm{disc}}(\dot\pi)$ and $b^G_{\mathrm{disc}}(\dot\phi)$, for elements $\dot\pi \in \Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$ and $\dot\phi \in \Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$, by the natural variants of the inductive definitions (7.7) and (7.8). We set

$$a^{G,\mathcal{E}}_{\mathrm{disc}}(\dot\pi) = \sum_{G'}\sum_{\dot\phi'} \iota(G,G') b^{\tilde G'}_{\mathrm{disc}}(\dot\phi')\Delta_G(\dot\phi',\dot\pi) + \varepsilon(G)\sum_{\dot\phi} b^G_{\mathrm{disc}}(\dot\phi)\Delta_G(\dot\phi,\dot\pi), \tag{7.9}$$

with G', $\dot\phi'$ and $\dot\phi$ summed over $\mathcal{E}^0_{\mathrm{ell}}(G)$, $\Phi_{t,\mathrm{disc}}(\tilde G',\tilde\zeta')$ and $\Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$, respectively, and with coefficients $b^{\tilde G'}_{\mathrm{disc}}(\dot\phi')$ defined inductively by the requirements that

$$a^{G,\mathcal{E}}_{\mathrm{disc}}(\dot\pi) = a^G_{\mathrm{disc}}(\dot\pi) \tag{7.10}$$

and

$$b^{G^*}_{\mathrm{disc}}(\dot\phi^*) = b^G_{\mathrm{disc}}(\dot\phi),$$

in case G is quasisplit.

Global Theorem 2.

(a) *For any G, we have*

$$a^{G,\mathcal{E}}_{\mathrm{disc}}(\dot\pi) = a^G_{\mathrm{disc}}(\dot\pi), \quad \dot\pi \in \Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta).$$

(b) *If G is quasisplit, $b^G_{\mathrm{disc}}(\dot\phi)$ vanishes for any $\dot\phi$ in the complement of $\Phi_{t,\mathrm{disc}}(G,\zeta)$ in $\Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)$.*

Global Theorem 2 will be the most important result from a purely arithmetic standpoint. It can be regarded as a reciprocity law between automorphic spectra on different groups. For the comparison of trace formulae, however, we again require the corresponding theorem for the general coefficients.

Global Theorem 2′.

(a) *For any G, we have*

$$a^{G,\mathcal{E}}(\pi) = a^G(\pi), \quad \pi \in \Pi^{\mathcal{E}}_t(G,V,\zeta).$$

(b) *If G is quasisplit, $b^G(\phi)$ vanishes for any ϕ in the complement of $\Phi_t(G,V,\zeta)$ in $\Phi^{\mathcal{E}}_t(G,V,\zeta)$.*

In § 10 we shall show that Global Theorem 2 implies Global Theorem 2′ by establishing endoscopic and stable analogues of the expansion (3.12).

Following the discussion of the geometric coefficients, we shall reformulate Global Theorems 2 and 2′ in terms of the basic distributions $I_{t,\mathrm{disc}}(\dot{f})$ and $I_{t,\mathrm{unit}}(f)$ of § 3. The transfer mapping entails a shift in archimedean infinitesimal characters that must be reflected in the norms $t = \|\operatorname{Im}(\nu)\|$. For any $G' \in \mathcal{E}(G)$, there is a canonical embedding of $\mathfrak{h}^*_{\mathbb{C}}$ into the associated space $(\tilde{\mathfrak{h}}')^*_{\mathbb{C}}$ for $\tilde{G}'$. This gives an embedding

$$\mathfrak{h}^*_{\mathbb{C}}/\mathfrak{a}^*_{G,Z,\mathbb{C}} \to (\tilde{\mathfrak{h}}')^*_{\mathbb{C}}/\mathfrak{a}^*_{\tilde{G}',\tilde{Z}',\mathbb{C}}$$

of the quotient spaces whose Weyl orbits parametrize infinitesimal characters. We write

$$\nu \to \nu' = \nu + d\tilde{\eta}'_{\infty},$$

as on p. 561 of [**10**], for the transfer of infinitesimal characters, and

$$t = \|\operatorname{Im}(\nu)\| \to t' = \|\operatorname{Im}(\nu')\| = t + \|\operatorname{Im}(d\tilde{\eta}'_{\infty})\|, \quad t \geqslant 0,$$

for the corresponding shift in norms. With this notation, we define invariant linear forms $I^{\mathcal{E}}_{t,\mathrm{disc}}(\dot{f}) = I^{G,\mathcal{E}}_{t,\mathrm{disc}}(\dot{f})$ and $I^{\mathcal{E}}_{t,\mathrm{unit}}(f) = I^{G,\mathcal{E}}_{t,\mathrm{unit}}(f)$, for functions $\dot{f} \in \mathcal{H}(G,\zeta)$ and $f \in \mathcal{H}(G,V,\zeta)$, by setting

$$I^{\mathcal{E}}_{t,\mathrm{disc}}(\dot{f}) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G)} \iota(G,G')\hat{S}^{\tilde{G}'}_{t',\mathrm{disc}}(\dot{f}') + \varepsilon(G)S^G_{t,\mathrm{disc}}(\dot{f}) \tag{7.11}$$

and

$$I^{\mathcal{E}}_{t,\mathrm{unit}}(f) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}^{\tilde{G}'}_{t',\mathrm{unit}}(f') + \varepsilon(G)S^G_{t,\mathrm{unit}}(f). \tag{7.12}$$

The terms $\hat{S}^{\tilde{G}'}_{t',\mathrm{disc}}$ and $\hat{S}^{\tilde{G}'}_{t',\mathrm{unit}}$ on the right are linear forms on the respective spaces $S\mathcal{I}(\tilde{G}',\tilde{\zeta}')$ and $S\mathcal{I}(\tilde{G}',V,\tilde{\zeta}')$, which are defined inductively by the requirements that $I^{\mathcal{E}}_{t,\mathrm{disc}}(\dot{f}) = I_{t,\mathrm{disc}}(\dot{f})$ and $I^{\mathcal{E}}_{t,\mathrm{unit}}(f) = I_{t,\mathrm{unit}}(f)$ in the case that G is quasisplit. The definition depends on the induction hypothesis that $S^{\tilde{G}'}_{t',\mathrm{disc}}$ and $S^{\tilde{G}'}_{t',\mathrm{unit}}$ are stable, for data G' in $\mathcal{E}^0_{\mathrm{ell}}(G)$ and $\mathcal{E}^0_{\mathrm{ell}}(G,V)$, respectively.

The next lemma is obviously parallel to Lemma 7.2, and is proved the same way.

Lemma 7.3.

(a) *If G is arbitrary,*

$$I^{\mathcal{E}}_{t,\mathrm{disc}}(\dot{f}) = \sum_{\dot{\pi} \in \Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,\zeta)} a^{G,\mathcal{E}}_{\mathrm{disc}}(\dot{\pi})\dot{f}_G(\dot{\pi})$$

and

$$I^{\mathcal{E}}_{t,\mathrm{unit}}(f) = \int_{\Pi^{\mathcal{E}}_t(G,V,\zeta)} a^{G,\mathcal{E}}(\pi)f_G(\pi)\,\mathrm{d}\pi.$$

In particular, the statements (a) of Global Theorems 2 and 2′ are equivalent to the general identities $I^{\mathcal{E}}_{t,\mathrm{disc}}(\dot{f}) = I_{t,\mathrm{disc}}(\dot{f})$ and $I^{\mathcal{E}}_{t,\mathrm{unit}}(f) = I_{t,\mathrm{unit}}(f)$.

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(b) *If G is quasisplit,*

$$S^G_{t,\mathrm{disc}}(\dot f) = \sum_{\dot\phi \in \Phi^{\mathcal{E}}_t(G,\zeta)} b^G_{\mathrm{disc}}(\dot\phi) \dot f^{\mathcal{E}}_G(\dot\phi)$$

and

$$S^G_{t,\mathrm{unit}}(f) = \int_{\Phi^{\mathcal{E}}_t(G,V,\zeta)} b^G(\phi) f^{\mathcal{E}}_G(\phi)\,\mathrm{d}\phi.$$

In particular, the statements (b) of Global Theorems 2 and 2′ are equivalent to the assertions that $S^G_{t,\mathrm{disc}}(\dot f)$ and $S^G_{t,\mathrm{unit}}(f)$ are stable.

We have now stated all the main local and global theorems. We shall prove them, for any K-group G that satisfies Assumption 5.2, in a subsequent article. In the meantime, we shall have to carry some implicit induction assumptions, in order that the inductive definitions of the last two sections make sense. To be precise, we assume that assertions (b) of the various local and global theorems are valid if G is replaced by a group $\tilde G'$ attached to any endoscopic datum G' in $\mathcal{E}^0_{\mathrm{ell}}(G)$.

8. Stabilization of the unramified terms

We have spent the last two sections stating a series of closely related theorems. Statements (a) of the theorems represent identities between terms in the trace formula of §§ 2 and 3, and corresponding terms of an endoscopic trace formula. The statements (b) describe properties of terms in a stable trace formula. In the remaining part of the paper, we shall derive the expansions that will form the endoscopic and stable trace formulae. This can be regarded as the first stage of a long process, which will end in a subsequent paper with a proof of the theorems.

In this section, we shall deal with the unramified terms. These are the functions $r^G_M(k)$ in (2.8), and the functions $r^G_M(c)$ in (3.12). The stabilization of the geometric terms $r^G_M(k)$ will be a consequence of the generalized fundamental lemma we have taken on as an assumption. The stabilization of the spectral terms $r^G_M(c)$ is essentially the main result of [**13**].

We fix the global field F. We shall consider triplets (G, M, ζ) over F, as in earlier sections. Then G is a global K-group over F, M is a Levi subgroup of G, and ζ is an automorphic character of a central induced torus Z in G. Remember that, unless stated otherwise, (G, F) is supposed to satisfy Assumption 5.2. In particular, the generalized fundamental lemma is assumed to hold at every place v outside some finite set of valuations $V_{\mathrm{fund}}(G)$ that contains $V_{\mathrm{ram}}(G)$.

For the geometric terms, we fix finite sets $V \subset S$ of valuations of F. We take S to be suitably large, as before, but at this point we assume only that V contains the set V_∞ of archimedean places. Given G and ζ, we write $\mathcal{K}(\bar G^V_S)$ as in § 2 for the set of conjugacy classes in $\bar G^V_S = G^V_S / Z^V_S$ that are bounded. The places v in $S - V$ are non-archimedean. We can therefore arrange that any distribution in the basis

$$\Gamma(G^V_S, \zeta^V_S) = \prod_{v \in S - V} \Gamma(G_v, \zeta_v)$$

is defined by a (signed) measure on the preimage in G_S^V of a conjugacy class in $\bar{G}_S^V$. We have not assumed that V contains $V_{\mathrm{ram}}(G,\zeta)$, but for simplicity, let us suppose that the subset $\Gamma_{\mathrm{ss}}(\bar{G}_S^V,\zeta_S^V)$ of $\Gamma_{\mathrm{ss}}(\bar{G}_V^S)$ defined in § 1 actually equals $\Gamma_{\mathrm{ss}}(\bar{G}_V^S)$. There is then an injection $k \to \gamma_V^S(k)$ from $\mathcal{K}(\bar{G}_S^V)$ into $\Gamma(G_S^V,\zeta_S^V)$. We also attach a set $\mathcal{L}(\bar{G}_S^V)$ to the 'bounded' elements in $\Delta(G_S^V,\zeta_S^V)$. It is the family of formal linear combinations of classes in $\mathcal{K}(\bar{G}_S^V)$ that correspond to distributions in $\Delta(G_S^V,\zeta_S^V)$ under the linear extension of the map γ_S^V, and can be identified with a subset of the corresponding family $\mathcal{L}((\bar{G}^*)_S^V)$ for G^* by means of a canonical embedding $\ell \to \ell^*$. More generally, we attach a set $\mathcal{L}^{\mathcal{E}}(\bar{G}_S^V)$ to the 'bounded' elements in $\Delta^{\mathcal{E}}(G_S^V,\zeta_S^V)$. It is a quotient of the set of G-relevant pairs in

$$\{(G',\ell') : G' \in \mathcal{E}(G_S^V),\ \ell' \in \mathcal{L}((\bar{\tilde{G}}')_S^V) = \mathcal{L}((\bar{G}')_S^V)\},$$

and comes with an injection $\ell \to \delta_S^V(\ell)$ into $\Delta^{\mathcal{E}}(G_S^V,\zeta_S^V)$ that takes the subset $\mathcal{L}(\bar{G}_S^V)$ into $\Delta(G_S^V,\zeta_S^V)$. The sets $\mathcal{L}(\bar{G}_S^V)$ and $\mathcal{L}^{\mathcal{E}}(\bar{G}_S^V)$ are independent of ζ_S^V. In other words, they are equal to the corresponding sets in which ζ_S^V is the trivial character on Z_S^V. This amounts to a compatibility condition on the original choice of bases that we are taking for granted.

Suppose that (G,M,ζ) is given, and that V now contains $V_{\mathrm{ram}}(G,\zeta)$. As in § 2, we form the function

$$r_M^G(k) = J_M(\gamma_S^V(k), u_S^V), \quad k \in \mathcal{K}(\bar{M}_S^V),$$

on $\mathcal{K}(\bar{M}_S^V)$. This function depends on a choice of hyperspecial maximal compact subgroup

$$K_S^V = \prod_{v \in S-V} K_v$$

of G_S^V, which we assume is in good position relative to M_S^V. The intersection

$$K_S^V \cap M_S^V = \prod_{v\in S-V} (K_v \cap M_v)$$

is a hyperspecial maximal compact subgroup of M_S^V, which we use to form the normalized transfer factor

$$\Delta_{K_S^V,M}(\ell,k) = \Delta_{K_S^V\cap M_S^V}(\delta_S^V(\ell),\gamma_S^V(k)) = \prod_v \Delta_{K_v\cap M_v}(\delta_v(\ell_v),\gamma_v(k_v)),$$

for elements k and ℓ in $\mathcal{K}(\bar{M}_S^V)$ and $\mathcal{L}^{\mathcal{E}}(\bar{M}_S^V)$, respectively. This provides us in turn with a function

$$r_M^G(\ell) = \sum_{k\in\mathcal{K}(\bar{M}_S^V)} \Delta_{K_S^V,M}(\ell,k) r_M^G(k), \quad \ell \in \mathcal{L}^{\mathcal{E}}(\bar{M}_S^V), \tag{8.1}$$

on $\mathcal{L}^{\mathcal{E}}(\bar{M}_S^V)$, which is easily seen to be independent of K_S^V.

The functions $r_M^G(k)$, $\Delta_{K_V^S,M}(\ell,k)$ and $r_M^G(\ell)$ are all independent of ζ_S^V, so ζ plays no role in the following proposition. We include it in the assertion only to emphasize that the proposition is a special case of Local Theorem 1. Similarly, we include the extra structure on elements in $\mathcal{E}_{M'}(G)$ implicit in the notation $\tilde{M}'$ and $\tilde{G}'$, even though it also plays no role.

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Proposition 8.1. *For each triple (G, M, ζ) with G quasisplit, there is a function*

$$s_M^G(\ell) = s_{M^*}^{G^*}(\ell^*), \quad \ell \in \mathcal{L}(\bar{M}_S^V),$$

which vanishes unless V contains $V_{\mathrm{ram}}(G)$, and satisfies the following condition. For any triplet (G, M, ζ) with $V_{\mathrm{fund}}(G) \subset V$, any elliptic endoscopic datum M' for M, and any element $\ell' \in \mathcal{L}((\bar{M}')_S^V)$ with image ℓ in $\mathcal{L}^{\mathcal{E}}(\bar{M}_S^V)$, the identity

$$r_M^G(\ell) = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') s_{\tilde{M}'}^{\tilde{G}'}(\ell') \tag{8.2}$$

holds.

Proof. As we recall from § 5 and elsewhere, the required function is uniquely determined by (8.2). To be precise, suppose that G is quasisplit, and that ℓ belongs to $\mathcal{L}(\bar{M}_S^V)$. If V does not contain $V_{\mathrm{ram}}(G)$, we set $s_M^G(\ell) = 0$. If V does contain $V_{\mathrm{ram}}(G)$, the function $r_M^G(k)$ is defined, and we define $s_M^G(\ell)$ inductively by setting

$$s_M^G(\ell) = r_M^G(\ell) - \sum_{G' \in \mathcal{E}_{M^*}^0(G)} \iota_{M^*}(G, G') s_{M^*}^{\tilde{G}'}(\ell^*).$$

Since the coefficient $\iota_{M'}(G, G')$ vanishes unless G' is elliptic, the sum can be taken over a finite set.

We have then to establish (8.2). We fix the data (G, M, ζ), M', ℓ' and ℓ. Our task is to show that $r_M^G(\ell)$ equals the endoscopic expression

$$r_M^{G,\mathcal{E}}(\ell') = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') s_{\tilde{M}'}^{\tilde{G}'}(\ell')$$

on the right-hand side.

The problem separates into two cases, according to whether V contains $V_{\mathrm{ram}}(M, \zeta)$ or not. Our assumption that S is suitably large means in this context that S contains $V_{\mathrm{ram}}(M')$. If V does not contain $V_{\mathrm{ram}}(M')$, then M' ramifies at some place v in $S - V$. In this case, the functions

$$s_{\tilde{M}'}^{\tilde{G}'}(\ell'), \quad G' \in \mathcal{E}_{M'}(G),$$

all vanish by definition, and the problem is to show that $r_M^G(\ell)$ vanishes.

Suppose that $S - V$ is a union of two disjoint subsets F_1 and F_2. By construction, $\mathcal{L}((\bar{M}')_S^V)$ is a Cartesian product of sets attached to the places in $S - V$, and the transfer factors decompose accordingly. In particular, we can write

$$\ell' = \ell_1' \times \ell_2', \quad \ell_i' \in \mathcal{L}(\bar{M}_{F_i}').$$

Then ℓ equals a product $\ell_1 \times \ell_2$, where ℓ_i is the image of ℓ_i' in $\mathcal{L}^{\mathcal{E}}(\bar{M}_{F_i})$. Since $r_M^G(\ell)$ comes from the weighted orbital integral $r_M^G(k)$, it satisfies a splitting formula. It follows from [**7**, Corollary 7.4] that

$$r_M^G(\ell) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) r_M^{L_1}(\ell_1) r_M^{L_2}(\ell_2).$$

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The right-hand side of (8.2) also satisfies a splitting formula. By a simple variant of [**12**, Theorem 6.1], we obtain

$$r_M^{G,\mathcal{E}}(\ell') = \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) r_M^{L_1,\mathcal{E}}(\ell'_1) r_M^{L_2,\mathcal{E}}(\ell'_2).$$

The two formulae together reduce the problem to the case that $S-V$ contains one element.

We can therefore assume that $S-V=\{v\}$. We have to show that $r_M^G(\ell)$ vanishes if M' ramifies at v, and that $r_M^G(\ell)$ equals $r_M^{G,\mathcal{E}}(\ell')$ if M' is unramified at v. One purpose of the paper [**14**] is to reduce such questions to a simpler situation. The germ expansions of [**14**] reduce the problem to the case that ℓ is semisimple and strongly G-regular. The descent formulae of [**14**] reduce the problem further to the case that ℓ is elliptic, and that G, M and M' are replaced by the corresponding local objects G_v, M_v and M'_v. If M'_v is unramified, the required identity (8.2) becomes the formula (5.2) of our basic assumption. If M'_v is ramified, the corresponding vanishing assertion is just a variant of Proposition 7.5 of [**22**]. We leave the reader to extend the proof of this proposition from the special case in [**22**] that $M=G$ and G_{der} is simply connected. (For the extension to arbitrary M, one chooses the element g_1 at the top of p. 389 of [**22**] so that it normalizes M_v as well as K_v. From the symmetry of $J_{M_v}(\cdot,\cdot)$ under the automorphism $\theta=\mathrm{Int}(g_1)$ [**11**, Lemma 3.3], one deduces that

$$\begin{aligned} r_{M_v}^{G_v}(k) &= J_{M_v}(\gamma_v(k),u_v) = J_{\theta M_v}(\theta\gamma_v(k),\theta u_v)\\ &= J_{M_v}(\gamma_v(\theta k),u_v) = \chi(c)J_{M_v}(\gamma_v(k),u_v)\\ &= \chi(c) r_{M_v}^{G_v}(k), \end{aligned}$$

for the complex number $\chi(c)\neq 1$ defined on p. 389 of [**22**]. Therefore, $r_{M_v}^{G_v}(k)=0$.) The proposition follows. □

In the special case that $M=G$, the proposition is essentially the assertion that the map $f\to\dot f_S = f\times u_S^V$ commutes with transfer. To state this precisely, let $u_S^{V,G}$ be the image of the unit u_S^V in $S\mathcal{I}(G_S^V,\zeta_S^V)$, for $S\supset V\supset V_{\mathrm{ram}}(G,\zeta)$. Assuming only that V contains V_∞, we define a map $a^G\to(a^G)^{\cdot}_S$ from $S\mathcal{I}(G_V,\zeta_V)$ to $S\mathcal{I}(G_S,\zeta_S)$ by setting

$$(a^G)^{\cdot}_S = \begin{cases} a^G\times u_S^{V,G}, & \text{if } V\supset V_{\mathrm{ram}}(G,\zeta),\\ 0, & \text{otherwise,}\end{cases}$$

for any $a^G\in S\mathcal{I}(G_V,\zeta_V)$. Setting $M=G$ in the proposition, we obtain the following corollary.

Corollary 8.2. *For any pair (G,ζ) such that V contains both $V_{\mathrm{fund}}(G)$ and $V_{\mathrm{ram}}(G,\zeta)$, any endoscopic datum $G'\in\mathcal{E}_{\mathrm{ell}}(G)$, and any function $f\in\mathcal{H}(G_V,\zeta_V)$, we have*

$$\dot f'_S = (f')^{\cdot}_S. \tag{8.3}$$

In particular, the function $\dot f'_S=(f\times u_S^V)'$ vanishes unless G' belongs to $\mathcal{E}_{\mathrm{ell}}(G,V)$.

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Proposition 8.1 is really just a restatement of the generalized fundamental lemma in a form we can apply. Its role will become clear in § 10, where we will derive endoscopic and stable analogues of the expansion (2.8). Actually, for reasons of induction, the term with $M = G$ in the expansion is best treated separately. We may as well take care of it now.

We write $a^G_{\mathrm{ell}}(\gamma, S)$ for the term with $M = G$ in the expansion (2.8) for $a^G(\gamma)$. That is,

$$a^G_{\mathrm{ell}}(\gamma, S) = \sum_{k \in \mathcal{K}^V_{\mathrm{ell}}(\bar{G},S)} a^G_{\mathrm{ell}}(\gamma \times k) r_G(k). \tag{8.4}$$

Here, V is a finite set of valuations that contains $V_{\mathrm{ram}}(G, \zeta)$, and S is a large finite set of valuations that contains V. The associated linear form

$$I_{\mathrm{ell}}(f, S) = \sum_{\gamma \in \Gamma_{\mathrm{ell}}(G,V,\zeta)} a^G_{\mathrm{ell}}(\gamma, S) f_G(\gamma) \tag{8.5}$$

can be regarded as the 'elliptic' part of the linear form $I(f)$ of § 2. As the notation suggests, it depends on the choice of S. One sees directly from the definitions that

$$I_{\mathrm{ell}}(f, S) = I_{\mathrm{ell}}(\dot{f}_S), \quad \dot{f}_S = f \times u^V_S,$$

for any S large enough such that $\dot{f}_S$ belongs to $\mathcal{H}_{\mathrm{adm}}(G, S, \zeta)$.

To define endoscopic and stable analogues of the coefficients (8.4), we write

$$\delta \times \ell = \delta \times \delta^V_S(\ell),$$

for the element in $\Delta^{\mathcal{E}}(G^Z_S, \zeta_S)$ associated to a pair $\delta \in \Delta^{\mathcal{E}}(G^Z_V, \zeta_V)$ and $\ell \in \mathcal{L}^{\mathcal{E}}(\bar{G}^V_S)$. Following § 2 further, we may as well write $\mathcal{L}^{V,\mathcal{E}}_{\mathrm{ell}}(\bar{G}, S)$ for the set of $\ell \in \mathcal{L}^{\mathcal{E}}(\bar{G}^V_S)$ such that $\delta \times \ell$ belongs to $\Delta^{\mathcal{E}}_{\mathrm{ell}}(G, S, \zeta)$ for some δ, and $\mathcal{L}^V_{\mathrm{ell}}(\bar{G}, S)$ for the intersection of $\mathcal{L}^{V,\mathcal{E}}_{\mathrm{ell}}(\bar{G}, S)$ with $\mathcal{L}(\bar{G}^V_S)$. We also write $\mathcal{K}^{V,\mathcal{E}}_{\mathrm{ell}}(\bar{G}, S)$ for the set of k in $\mathcal{K}(\bar{G}^V_S)$ such that $\gamma \times k$ belongs to $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(G, S, \zeta)$ for some γ. We then define the analogues of (8.4) by setting

$$a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma, S) = \sum_{k \in \mathcal{K}^{V,\mathcal{E}}_{\mathrm{ell}}(\bar{G},S)} a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma \times k) r_G(k), \tag{8.6}$$

for G arbitrary and $\gamma \in \Gamma^{\mathcal{E}}_{\mathrm{ell}}(G, V, \zeta)$, and

$$b^G_{\mathrm{ell}}(\delta, S) = \sum_{\ell \in \mathcal{L}^{V,\mathcal{E}}_{\mathrm{ell}}(\bar{G},S)} b^G_{\mathrm{ell}}(\delta \times \ell) r_G(\ell), \tag{8.7}$$

for G quasisplit and $\delta \in \Delta^{\mathcal{E}}_{\mathrm{ell}}(G, V, \zeta)$. We can also define endoscopic and stable analogues of the linear form $I_{\mathrm{ell}}(f, S)$. Following (7.5) and (7.6), we set

$$I^{\mathcal{E}}_{\mathrm{ell}}(f, S) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G, G') \hat{S}^{\tilde{G}'}_{\mathrm{ell}}(f', S) + \varepsilon(G) S^G_{\mathrm{ell}}(f, S),$$

for linear forms $\hat{S}^{\tilde{G}'}_{\mathrm{ell}}(\cdot, S)$ on $S\mathcal{I}(\tilde{G}', V, \tilde{\zeta}')$, which are defined inductively by requiring that $I^{\mathcal{E}}_{\mathrm{ell}}(f, S) = I_{\mathrm{ell}}(f, S)$ in case G is quasisplit. Suppose that V contains $V_{\mathrm{fund}}(G)$, and that

S is large enough that the function $\dot{f}_S = f \times u_S^V$ belongs to $\mathcal{H}_{\mathrm{adm}}(G, S, \zeta)$. It then follows inductively from Corollary 8.2 and the definition (7.5) that

$$I_{\mathrm{ell}}^{\mathcal{E}}(f, S) = I_{\mathrm{ell}}^{\mathcal{E}}(\dot{f}_S),$$

and

$$S_{\mathrm{ell}}^{G}(f, S) = S_{\mathrm{ell}}^{G}(\dot{f}_S),$$

with $\dot{f}_S = f \times u_S^V$. From the expansions of $I_{\mathrm{ell}}^{\mathcal{E}}(\dot{f}_S)$ and $S_{\mathrm{ell}}^{G}(\dot{f}_S)$ in Lemma 7.2, we conclude that

$$I_{\mathrm{ell}}^{\mathcal{E}}(f, S) = \sum_{\gamma \in \Gamma_{\mathrm{ell}}^{\mathcal{E}}(G, V, \zeta)} a_{\mathrm{ell}}^{G, \mathcal{E}}(\gamma, S) f_G(\gamma) \tag{8.8}$$

and

$$S_{\mathrm{ell}}^{G}(f, S) = \sum_{\delta \in \Delta_{\mathrm{ell}}^{\mathcal{E}}(G, V, \zeta)} b_{\mathrm{ell}}^{G}(\delta, S) f_G^{\mathcal{E}}(\delta). \tag{8.9}$$

These formulae represent a stabilization of the term with $M = G$ in the original expansion (2.8).

We turn now to the unramified spectral terms. The stabilization of these terms does not rely on Assumption 5.2. It was proved unconditionally in [**13**]. We have only to state the main result of [**13**] in the form we shall use.

The spectral terms require only one set of valuations V. We fix V, and consider a pair (G, ζ) with $V \supset V_{\mathrm{ram}}(G, \zeta)$. We can then form the general set $\mathcal{C}(G^V, \zeta^V)$ of families of conjugacy classes, and the subset $\mathcal{C}_{\mathrm{disc}}^{V}(G, \zeta)$ of families associated with the discrete part of the trace formula. These sets were actually defined only for connected groups in § 3. For the K-group G, here, we simply take the union of the corresponding sets attached to the components G_α of G. As in [**13**], it is necessary to construct a possibly larger subset $\mathcal{C}_{\mathrm{disc}}^{V, \mathcal{E}}(G, \zeta)$ of $\mathcal{C}(G^V, \zeta^V)$, in order to accommodate induction arguments. We define $\mathcal{C}_{\mathrm{disc}}^{V, \mathcal{E}}(G, \zeta) = \mathcal{C}_{\mathrm{disc}}^{V, \mathcal{E}}(G^*, \zeta^*)$ inductively as the union, over all inner K-forms G_1 of G^*, of the sets $\mathcal{C}_{\mathrm{disc}}^{V}(G_1, \zeta_1)$, together with the union, over all elliptic endoscopic data $G' \in \mathcal{E}_{\mathrm{ell}}^{0}(G^*, V)$, of the images in $\mathcal{C}(G^V, \zeta^V)$ of the sets $\mathcal{C}_{\mathrm{disc}}^{V, \mathcal{E}}(\tilde{G}', \tilde{\zeta}')$. We recall here that there is a canonical map from $\mathcal{C}((\tilde{G}')^V, (\tilde{\zeta}')^V)$ to $\mathcal{C}(G^V, \zeta^V)$ for any G', which by the definition takes $\mathcal{C}_{\mathrm{disc}}^{V, \mathcal{E}}(\tilde{G}', \tilde{\zeta}')$ to $\mathcal{C}_{\mathrm{disc}}^{V, \mathcal{E}}(G, \zeta)$. We refer the reader to [**13**, § 2] for more detail. The construction of [**13**, § 2] actually uses larger sets $\mathcal{C}_{\mathrm{aut}}^{V}(\cdot)$ in place of $\mathcal{C}_{\mathrm{disc}}^{V}(\cdot)$. It provides a subset $\mathcal{C}_{\mathrm{aut}}^{V, \mathcal{E}}(G, \zeta)$ of $\mathcal{C}(G^V, \zeta^V)$, which was denoted $\mathcal{C}_{+}^{V}(G, \zeta)$ in [**13**], and which properly contains $\mathcal{C}_{\mathrm{disc}}^{V, \mathcal{E}}(G, \zeta)$.

Suppose that M is a Levi subgroup of G. The discussion prior to Lemma 3.2 of this paper was carried out in greater generality at the beginning of § 4 of [**13**]. It applies to any element c in the larger set $\mathcal{C}_{\mathrm{aut}}^{V, \mathcal{E}}(M, \zeta)$. In particular, we obtain a meromorphic function

$$r_M^G(c_\lambda), \quad \lambda \in \mathfrak{a}_{M, Z, \mathbb{C}}^*,$$

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for any c in $\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(M,\zeta)$.

The next proposition is clearly parallel to Proposition 8.1. As stated, it is a special case of [**13**, Theorem 5]. It applies to triplets (G,M,ζ) with $V_{\mathrm{ram}}(G,\zeta)\subset V$, and with (G,F) is not being required to satisfy Assumption 5.2.

Proposition 8.3. *For each triplet (G,M,ζ) with G quasisplit, and each $c\in\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(M,\zeta)$, there is a meromorphic function*

$$s^G_M(c_\lambda)=s^{G^*}_{M^*}(c_\lambda),\quad \lambda\in\mathfrak{a}^*_{M,Z,\mathbb{C}},$$

with the property that for any triplet (G,M,ζ), any endoscopic datum $M'\in\mathcal{E}_{\mathrm{ell}}(M,V)$, and any element $c'\in\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(\tilde{M}',\tilde{\zeta}')$ with image c in $\mathcal{C}^V(M,\zeta)$, the identity

$$r^G_M(c_\lambda)=\sum_{G'\in\mathcal{E}_{M'}(G)}\iota_{M'}(G,G')s^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda)\tag{8.10}$$

holds.

Corollary 8.4. *Suppose that c belongs to $\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(M,\zeta)$. Then $r^G_M(c_\lambda)$ is an analytic function of $\lambda\in i\mathfrak{a}^*_{M,Z}$ that satisfies an estimate of the form (3.9). Similar assertions apply to the function $s^G_M(c_\lambda)$, in the case that G is quasisplit.*

Proof. If c belongs to $\mathcal{C}^V_{\mathrm{disc}}(M_1,\zeta_1)$, for an inner K-form M_1 of M^*, the required properties of $r^G_M(c_\lambda)$ follow from Lemma 3.2. We can therefore assume that c is the image of an element $c'\in\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(\tilde{M}',\tilde{\zeta}')$, for some $M'\in\mathcal{E}^0_{\mathrm{ell}}(M^*,V)$, Since $M'\neq M^*$, the datum G^* does not belong to $\mathcal{E}_{M'}(G)$. We can assume inductively that for any $G'\in\mathcal{E}_{M'}(G)$, the function $s^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda)$ has the required properties. The properties for $r^G_M(c_\lambda)$ then follow from (8.10).

If G is quasisplit, consider the identity (8.10), with $M'=M^*$. We obtain

$$s^G_M(c_\lambda)=s^{G^*}_{M^*}(c_\lambda)=r^G_M(c_\lambda)-\sum_{G'\in\mathcal{E}^0_{M^*}(G)}\iota_{M^*}(G,G')s^{\tilde{G}'}_{M^*}(c^*_\lambda).$$

The required properties for $s^G_M(c_\lambda)$ then follow by induction, and what we have just established for $r^G_M(c_\lambda)$. □

If c belongs to $\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(M,\zeta)$, the corollary allows us to write $r^G_M(c)$ and $s^G_M(c)$ for the values of $r^G_M(c_\lambda)$ and $s^G_M(c_\lambda)$ at $\lambda=0$. The identity (8.10) then takes the form

$$r^G_M(c)=\sum_{G'\in\mathcal{E}_{M'}(G)}\iota_{M'}(G,G')s^{\tilde{G}'}_{\tilde{M}'}(c').\tag{8.11}$$

Proposition 8.3 is a kind of spectral analogue of the generalized fundamental lemma. It is of course much easier. It is actually a tautology if $M=G$, since in this case

$$s_G(c_\lambda)=r_G(c_\lambda)=1,$$

and there is nothing to prove. We shall use Proposition 8.3 in § 10 to derive endoscopic and stable analogues of the expansion (3.12). As before, it will be best to separate the term with $M = G$ from the rest of the expansion.

Following the discussion of the geometric terms, we write $a^G_{\mathrm{disc}}(\pi)$ for the term with $M = G$ in the definition (3.12) of $a^G(\pi)$. In other words,

$$a^G_{\mathrm{disc}}(\pi) = \sum_{c \in \mathcal{C}^V_{\mathrm{disc}}(G,\zeta)} a^G_{\mathrm{disc}}(\pi \times c), \tag{8.12}$$

since $r_G(c) = 1$. The associated linear form

$$I_{t,\mathrm{disc}}(f) = \sum_{\pi \in \Pi_{t,\mathrm{disc}}(G,V,\zeta)} a^G_{\mathrm{disc}}(\pi) f_G(\pi) \tag{8.13}$$

can be regarded as the 'discrete' part of the linear form $I_t(f)$ in § 3. It follows directly from the definitions that

$$I_{t,\mathrm{disc}}(f) = I_{t,\mathrm{disc}}(\dot{f}),$$

where $\dot{f} = f \times u^V$. Note the contrast with the geometric side, in that $I_{t,\mathrm{disc}}(f)$ does not depend on a choice of a finite set $S \supset V$.

If c belongs to $\mathcal{C}(G^V, \zeta^V)$, let $\phi^V(c)$ be the corresponding product of unramified Langlands parameters

$$\phi_v(c_v) : W_{F_v} \to {}^L G_v, \quad v \notin V.$$

We also write

$$\phi \times c = \phi \times \phi^V(c)$$

for the element in $\Phi^{\mathcal{E}}(G(\mathbb{A}), \zeta)$ associated to a pair $\phi \in \Phi^{\mathcal{E}}(G^Z_V, \zeta_V)$ and $c \in \mathcal{C}(G^V, \zeta^V)$. We then define endoscopic and stable analogues of the coefficients (8.12) by setting

$$a^{G,\mathcal{E}}_{\mathrm{disc}}(\pi) = \sum_{c \in \mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(G,\zeta)} a^G_{\mathrm{disc}}(\pi \times c), \tag{8.14}$$

for G arbitrary and $\pi \in \Pi^{\mathcal{E}}_{\mathrm{disc}}(G, V, \zeta)$, and

$$b^G_{\mathrm{disc}}(\phi) = \sum_{c \in \mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(G,\zeta)} b^G_{\mathrm{disc}}(\phi \times c), \tag{8.15}$$

for G quasisplit and $\phi \in \Phi^{\mathcal{E}}_{\mathrm{disc}}(G, V, \zeta)$. We can also define endoscopic and stable analogues $I^{\mathcal{E}}_{t,\mathrm{disc}}(f)$ and $S^G_{t,\mathrm{disc}}(f)$ of $I_{t,\mathrm{disc}}(f)$. Following (7.11) and (7.12), we set

$$I^{\mathcal{E}}_{t,\mathrm{disc}}(f) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G, G') \hat{S}^{\tilde{G}'}_{t',\mathrm{disc}}(f') + \varepsilon(G) S^G_{t,\mathrm{disc}}(f),$$

for linear forms $\hat{S}^{\tilde{G}'}_{t',\mathrm{disc}}$ on $S\mathcal{I}(\tilde{G}', V, \tilde{\zeta}')$, which are defined inductively by requiring that $I^{\mathcal{E}}_{t,\mathrm{disc}}(f) = I_{t,\mathrm{disc}}(f)$ in case G is quasisplit. It follows inductively from Corollary 8.2 and the definition (7.11) that

$$I^{\mathcal{E}}_{t,\mathrm{disc}}(f) = I^{\mathcal{E}}_{t,\mathrm{disc}}(\dot{f})$$

and

$$S^G_{t,\mathrm{disc}}(f) = S^G_{t,\mathrm{disc}}(\dot{f}),$$

for $\dot{f} = f \times u^V$. From the expansions of $I^{\mathcal{E}}_{t,\mathrm{disc}}(\dot{f})$ and $S^G_{t,\mathrm{disc}}(\dot{f})$ in Lemma 7.3, we conclude that

$$I^{\mathcal{E}}_{t,\mathrm{disc}}(f) = \sum_{\pi \in \Pi^{\mathcal{E}}_{t,\mathrm{disc}}(G,V,\zeta)} a^{G,\mathcal{E}}_{\mathrm{disc}}(\pi) f_G(\pi) \tag{8.16}$$

and

$$S^G_{t,\mathrm{disc}}(f) = \sum_{\phi \in \Phi^{\mathcal{E}}_{t,\mathrm{disc}}(G,V,\zeta)} b^G_{\mathrm{disc}}(\phi) f^{\mathcal{E}}_G(\phi). \tag{8.17}$$

These formulae represent a stabilization of the term with $M = G$ in (3.12).

9. A global vanishing theorem

In preparation for the global expansions of the next section, we shall prove a vanishing theorem. It is the general form of the global vanishing property [**2**, Theorem 14.2] for inner twists of $\mathrm{GL}(n)$, which was used in the proof of [**16**, Proposition 2.5.1]. It can also be regarded as a global analogue of Theorem 8.3 of [**12**]. The proof has the same general structure as that of the local theorem in [**12**]. However, there are additional considerations that come from the global transfer factors.

We fix the global field F, and objects G, Z ζ, G^*, Z^*, ζ^* and V, as in earlier sections. Then G is a global K-group with central character data Z and ζ, G^* is a quasisplit inner twist of G with corresponding central character data Z^* and ζ^*, and V is a finite set of valuations that contains $V_{\mathrm{ram}}(G,\zeta)$. In this section, we fix a Levi subgroup R of G^*, with dual Levi subgroup $\hat{R} \subset \hat{G}$. (As always, $\hat{G}$ represents a dual group for both G and G^*.) We shall say that R comes from G if R corresponds to some Levi subgroup M of G, in the sense of the statement of Corollary 4.2 and the definitions of [**12**, § 1]. This means that $M \amalg R$ has the structure of a Levi subgroup of the multiple group $G \amalg G^*$, and in particular, that R is a quasisplit inner twist of M.

Let R' be a fixed element in $\mathcal{E}_{\mathrm{ell}}(R,V)$, or, more precisely, a suitable representative $(R', \mathcal{R}', s'_R, \xi'_R)$ of such an element. Suppose that σ' belongs to $\Delta((\tilde{R}'_V)^{\tilde{Z}'}, \tilde{\zeta}'_V)$. The results of this section will be trivial if G is quasisplit, so we assume that $\varepsilon(G) = 0$. The linear form

$$I^{\mathcal{E}}_R(\sigma', f) = \sum_{G' \in \mathcal{E}_{R'}(G^*)} \iota_{R'}(G^*, G) \hat{S}^{\tilde{G}'}_{\tilde{R}'}(\sigma', f'), \quad f \in \mathcal{H}(G,V,\zeta), \tag{9.1}$$

is then defined, according to our implicit induction assumption that the linear forms on the right have been defined. The transfer mappings $f \to f'$ are defined by the canonical transfer factors for G'. However, R' need not come from G. Therefore, (9.1) is really a hybrid for G and G^* of the linear form (6.2).

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Suppose for a moment that R does come from G. Then R corresponds to a Levi subgroup M of G, and

$$I_R^{\mathcal{E}}(\sigma', f) = I_M^{\mathcal{E}}(\delta', f),$$

for the element δ' as in (6.2) that is defined by σ'. The local vanishing property in Proposition 6.2 tells us that this linear form vanishes unless δ' itself actually comes from M.

The following theorem applies to the other case.

Theorem 9.1. *Suppose that R does not come from G. Then*

$$I_R^{\mathcal{E}}(\sigma', f) = 0,$$

for R' and σ' as above.

Proof. Recall that σ' depends on auxiliary data $\tilde{R}'$ and $\tilde{\xi}'_R$ attached to R. We shall assume that for any endoscopic datum $G' \in \mathcal{E}(G^*)$, the extension $\tilde{G}'$ is the associated endoscopic datum in $\mathcal{E}(\tilde{G}^*)$ [**31**, (4.4)], for a fixed z-extension $\tilde{G}$ of G. This condition is purely for simplicity, and entails no loss of generality. It allows us to assume that the extensions $\tilde{R}' \to R'$ attached to elements $G' \in \mathcal{E}_{R'}(G^*)$ are all equal.

The notation will also be simpler if we write

$$\hat{S}_R^{G^*}(\sigma', f') = \iota_{R'}(G^*, G')\hat{S}_{\tilde{R}'}^{\tilde{G}'}(\sigma', f'), \quad G' \in \mathcal{E}_{R'}(G^*).$$

(See [**12**, Corollary 7.2].) This was the notation used to state the basic splitting formula of [**12**, Theorem 6.1]. To exploit the formula, we order the valuations $v_1, \dots, v_n$ in V, and agree to replace any subscript v_i simply by i.

We can assume that

$$f = \prod_{i=1}^{n} f_i,$$

where f_i belongs to the local Hecke algebra

$$\begin{aligned}\mathcal{H}(G_i, \zeta_i) &= \mathcal{H}(G_{v_i}, \zeta_{v_i}) \\ &= \mathcal{H}(G(F_{v_i}), \zeta_{v_i}).\end{aligned}$$

We apply the splitting formula [**12**, (6.3)] (or rather, its singular analogue in [**14**]) recursively to the summands on the right-hand side of (9.1). For a given $G' \in \mathcal{E}_{R'}(G^*)$, we obtain

$$\hat{S}_R^{G^*}(\sigma', f') = \sum_{L} \mathrm{e}_R^{G^*}(L)\hat{S}_R^L(\sigma', (f')^{L'}),$$

where L ranges over n-tuples $(L_1, \dots, L_n)$ of Levi subgroups in $\mathcal{L}(R)$, $\mathrm{e}_R^{G^*}(L)$ is a constant that vanishes unless the natural map

$$\mathfrak{a}_R^{L_1} \oplus \cdots \oplus \mathfrak{a}_R^{L_n} \to \mathfrak{a}_R^{G^*}$$

is an isomorphism, and

$$\hat{S}_R^L(\sigma', (f')^{L'}) = \prod_{i=1}^{n} \hat{S}_R^{L_i}(\sigma'_i, (f'_i)^{L'_i}).$$

For a given G' and L, $L' = (L'_1, \dots, L'_n)$ is the element in

$$\mathcal{E}_1 \times \cdots \times \mathcal{E}_n, \quad \mathcal{E}_i = \mathcal{E}_{R'_i}(L^*_i) = \mathcal{E}_{R'_{v_i}}(L^*_{v_i}),$$

obtained by projecting the point $s' \in s'_R Z(\hat{R})^\Gamma / Z(\hat{G})^\Gamma$ that defines G' onto

$$s'_R Z(\hat{R})^{\Gamma_1} / Z(\hat{L}_1)^{\Gamma_1} \times \cdots \times s'_R Z(\hat{R})^{\Gamma_n} / Z(\hat{L}_n)^{\Gamma_n}.$$

Observe that for any G', L and i, L'_i can be canonically identified with a Levi subgroup of G'_i that contains the image of R', and that the preimage $\tilde{L}'_i$ of L'_i in $\tilde{G}'_i$ is a Levi subgroup of $\tilde{G}'_i$. We shall interchange the sums over G' and L in the expression

$$I^{\mathcal{E}}_R(\sigma', f) = \sum_{G' \in \mathcal{E}_{R'}(G^*)} \sum_L \mathrm{e}^{G^*}_R(L) \hat{S}^L_R(\sigma', (f')^{L'}) \tag{9.2}$$

obtained from (9.1).

We fix an element $L = (L_1, \dots, L_n)$, and an associated endoscopic datum $L' = (L'_1, \dots, L'_n)$ in $\mathcal{E}_1 \times \cdots \times \mathcal{E}_n$. We can then consider the sum over those G' in (9.2) that map to L'. We are free to choose the L-embedding

$$\tilde{\xi}'_L = \prod_i \tilde{\xi}'_{L_i} : \mathcal{L}' = \prod_i \mathcal{L}'_i \to {}^L\tilde{L}' = \prod_i \tilde{L}'_i$$

independently of G'. In particular, $\tilde{\xi}'_L$ does not have to be inherited from the embedding $\tilde{\xi}'$: $\mathcal{G}' \to {}^L\tilde{G}'$ attached to G', so long as the map $f' \to (f')^{L'}$ is interpreted as a transfer from $\tilde{G}'_V$ to the Levi subgroup $\tilde{L}'_V$ that is taken with respect to a non-standard embedding of ${}^L\tilde{L}'_V$ into ${}^L\tilde{G}'_V$. We assume the coefficient $\mathrm{e}^{G^*}_R(L)$ is non-zero. The group

$$Z(\hat{L}_1)^\Gamma \cap \cdots \cap Z(\hat{L}_n)^\Gamma / Z(\hat{G})^\Gamma$$

is then finite, and has an action

$$s : G' \to G'_s$$

on the set of G' that map to L'. We shall actually restrict our attention to a certain subgroup. Recall that $\hat{L}_{i,\mathrm{sc}}$ stands for the preimage of the Levi subgroup $\hat{L}_i$ in $\hat{G}_{\mathrm{sc}}$. The group

$$Z(\hat{L}_{1,\mathrm{sc}})^\Gamma \cap \cdots \cap Z(\hat{L}_{n,\mathrm{sc}})^\Gamma / \hat{Z}^\Gamma_{\mathrm{sc}} \tag{9.3}$$

then maps injectively into $Z(\hat{L}_1)^\Gamma \cap \cdots \cap Z(\hat{L}_n)^\Gamma / Z(\hat{G})^\Gamma$, and hence acts on the set of G' that map to L'. We shall consider the orbit under (9.3) of a given G'. The contribution of the orbit to (9.2) equals the product of $\mathrm{e}^{G^*}_R(L)$ with

$$\sum_s \hat{S}^L_R(\sigma', (f'_s)^{L'}), \tag{9.4}$$

where s is summed over the group (9.3), and where

$$f'_s = f^{G'_s} = \prod_i (f_i)^{G'_{s,i}} = \prod_i (f'_{i,s}).$$

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It will be enough to show that (9.4) vanishes.

We can assume that there is an s such that the function

$$(f'_s)^{L'} = \prod_i (f'_{i,s})^{L'_i}$$

in (9.4) does not vanish. This means that the local endoscopic data L'_i for G^*_i each contain points that are images of elements in $G_i = G_{v_i}$. In particular, the Levi subgroup L_i of G^*_i corresponds to a Levi subgroup $M_i = M_{v_i}$ of the local K-group $G_i = G_{v_i}$. As we noted earlier, the choice of M_i includes structure that makes L_i a quasisplit inner twist of M_i. This allows us to identify $\hat{L}_i$ with the dual group of M_i. We obtain a character

$$\hat{\zeta}_i = \hat{\zeta}^{M_i}_{G_i}$$

on $\pi_0(Z(\hat{L}_{i,\mathrm{sc}})^{\Gamma_i})$ that is independent of the choice of M_i. (See [**12**, Corollary 2.2] and the remark following Corollary 4.2.) We can of course restrict $\hat{\zeta}_i$ to the subgroup $Z(\hat{L}_{i,\mathrm{sc}})^{\Gamma}$ of $Z(\hat{L}_{i,\mathrm{sc}})^{\Gamma_i}$. The product

$$\hat{\zeta}_V(s) = \hat{\zeta}_1(s) \dots \hat{\zeta}_n(s),$$

is then defined for any s in the intersection of the groups $Z(\hat{L}_{i,\mathrm{sc}})^{\Gamma}$. It follows from [**22**, Proposition 2.6 and Theorem 2.2], and the fact that V contains $V_{\mathrm{ram}}(G)$, that the character $\hat{\zeta}_V$ is trivial on $\hat{Z}_{\mathrm{sc}}$. It can therefore be identified with a character on the group (9.3).

To study (9.4), we have to investigate the canonical transfer map

$$f \to (f'_s)^{L'}.$$

It will be enough to deal with the normalized Langlands–Shelstad transfer factors, described in § 4, that are attached to strongly G-regular conjugacy classes. Let

$$\Delta_s(\delta', \gamma), \quad \delta' \in \Delta_{G\text{-}\mathrm{reg}}(\tilde{L}'_V), \quad \gamma \in \Gamma_{\mathrm{reg}}(G_V),$$

be the restriction to $\tilde{L}'_V$ of the canonical transfer factor for G_V and $G'_{s,V}$, modified to the extent that the $\Delta_2(\delta', \gamma)$ term in [**31**, (3.5)] is taken relative to an embedding $\tilde{\xi}'_L : \mathcal{L}' \to {}^L\tilde{L}'$ that is independent of s. Then

$$(f'_s)^{L'}(\delta') = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(G_V)} \Delta_s(\delta', \gamma) f_G(\gamma).$$

We would like to compare $\Delta_s(\delta', \gamma)$ with the corresponding transfer factor $\Delta(\delta', \gamma)$ for G_V and G'_V.

Lemma 9.2. *The transfer factors satisfy*

$$\Delta_s(\delta', \gamma) = \hat{\zeta}_V(s) \Delta(\delta', \gamma),$$

for any $\delta' \in \Delta_{G\text{-}\mathrm{reg}}(\tilde{L}'_V)$ *and* $\gamma \in \Gamma_{G\text{-}\mathrm{reg}}(G_V)$, *and any* s *in the group (9.3).*

Proof. Replacing G by a z-extension if necessary, we can assume that $\tilde{G}'_s = G'_s$ for each s. (See [**31**, (4.4)].) In so doing, we suppress the embedding ξ'_s from the notation, and simply identify $\mathcal{G}'_s$ with an L-group ${}^L G'_s$ of G'_s. We can also assume that δ' in an image of γ, in the usual sense of [**31**, (1.3)].

We fix δ' and γ. We also choose a global base point

$$(\bar{\delta}', \bar{\gamma}), \quad \bar{\delta}' \in G'_s(F), \quad \bar{\gamma} \in G(\mathbb{A}),$$

for (G, G'_s), as in § 4. The absolute transfer factor is then a product

$$\Delta_s(\delta', \gamma) = \Delta_s(\delta', \gamma; \bar{\delta}'_V, \bar{\gamma}_V)\Delta_s(\bar{\delta}'_V, \bar{\gamma}_V),$$

of the corresponding relative transfer factor with its preassigned value at the image $(\bar{\delta}'_V, \bar{\gamma}_V)$ of $(\bar{\delta}', \bar{\gamma})$ in $G'_{s,V} \times G_V$. The preassigned value $\Delta_s(\bar{\delta}'_V, \bar{\gamma}_V)$ equals the canonical product

$$d_s(\bar{\delta}', \bar{\gamma})^{-1} \prod_{v \notin V} (\Delta_{s,K_v}(\bar{\delta}'_v, \bar{\gamma}_v))^{-1} \tag{9.5}$$

defined by (4.3). The relative transfer factor $\Delta_s(\delta', \gamma; \bar{\delta}'_V, \bar{\gamma}_V)$ is of course also canonical. It is defined as a product of the relative local transfer factors of [**31**, (3.7)] (modified for local K-groups as in [**12**, § 2].) As such, it can be written as a product

$$\frac{\Delta_{\mathrm{I},s}(\delta', \gamma)}{\Delta_{\mathrm{I},s}(\bar{\delta}'_V, \bar{\gamma}_V)} \cdot \frac{\Delta_{\mathrm{II},s}(\delta', \gamma)}{\Delta_{\mathrm{II},s}(\bar{\delta}'_V, \bar{\gamma}_V)} \cdot \frac{\Delta_{2,s}(\delta', \gamma)}{\Delta_{2,s}(\bar{\delta}'_V, \bar{\gamma}_V)} \cdot \Delta_{1,s}(\delta', \gamma, \bar{\delta}'_V, \bar{\gamma}_V) \tag{9.6}$$

of four factors, corresponding to the local factors in [**31**, (3.2), (3.3), (3.5) and (3.4)]. (We continue to index the various terms by s, to emphasize their dependence on G'_s.)

Consider the three quotients in (9.6). If $\Delta_{*,s}(\delta', \gamma)$ is one of the three numerators, we have a decomposition

$$\Delta_{*,s}(\delta', \gamma) = \prod_{i=1}^{n} \Delta_{*,s}(\delta'_i, \gamma_i)$$

into local terms attached to the valuations v_i in V. For each i, δ'_i is a class in $\tilde{L}'_i(F_i)$, and s belongs to $Z(\hat{L}_{i,\mathrm{sc}})^{\Gamma_i}$. It follows easily from the definitions in the relevant sections (3.2), (3.3) or (3.5) of [**31**] that $\Delta_{*,s}(\delta'_i, \gamma_i)$ is independent of s, and can be ignored. In the case of the numerator $\Delta_{2,s}(\delta'_i, \gamma_i)$, we are relying on the fact that the embedding $\tilde{\xi}'_{L,i} : \mathcal{L}'_i \to {}^L\tilde{L}'_i$ is independent of s. If $\Delta_{*,s}(\bar{\delta}'_V, \bar{\gamma}_V)$ is one of the three denominators in (9.6), we write

$$\Delta_{*,s}(\bar{\delta}'_V, \bar{\gamma}_V) = \Delta_{*,s}(\bar{\delta}', \bar{\gamma}) \cdot \prod_{v \notin V} \Delta_{*,s}(\bar{\delta}'_v, \bar{\gamma}_v)^{-1}.$$

Since $\bar{\delta}'$ is defined over F, and since $\Delta_{*,s}(\bar{\delta}', \bar{\gamma})$ depends only on $\bar{\delta}'$, we deduce that $\Delta_{*,s}(\bar{\delta}', \bar{\gamma}) = 1$, as in the proof of [**31**, Theorem 6.4.A(ii)]. Now G_v is quasisplit for any $v \notin V$, and therefore has an absolute local transfer factor

$$\Delta_s(\bar{\delta}'_v, \bar{\gamma}_v) = \Delta_{\mathrm{I},s}(\bar{\delta}'_v, \bar{\gamma}_v)\Delta_{\mathrm{II},s}(\bar{\delta}'_v, \bar{\gamma}_v)\Delta_{1,s}(\bar{\delta}'_v, \bar{\gamma}_v)\Delta_{2,s}(\bar{\delta}'_v, \bar{\gamma}_v).$$

The extra term $\Delta_{1,s}(\bar{\delta}'_v, \bar{\gamma}_v)$ here is defined in [**31**, (3.4)], while the splitting for G_v on which the absolute factor depends is assumed to come from a fixed splitting of G^* over F. We have thus far shown that (9.6) equals the product of the expression

$$\Delta_{1,s}(\delta', \gamma; \bar{\delta}'_V, \bar{\gamma}_V)\biggl(\prod_{v\notin V} \Delta_{1,s}(\bar{\delta}'_v, \bar{\gamma}_v)^{-1}\biggr)\biggl(\prod_{v\notin V} \Delta_s(\bar{\delta}'_v, \bar{\gamma}_v)\biggr) \tag{9.7}$$

with a factor that is independent of s.

Consider the product of (9.5) with (9.7). The terms $\Delta_{s,K_v}(\bar{\delta}'_v, \bar{\gamma}_v)$ and $\Delta_s(\bar{\delta}'_v, \bar{\gamma}_v)$ in these two expressions both represent transfer factors for G_v. They differ only insofar as they are defined with respect to two different splittings of the quasisplit group G^*_v. Recall that it is the term $\Delta_{\rm I}$, defined in [**31**, (3.2)], that depends on the splitting. However, the dependence is mild, and is uniform in $(\bar{\delta}'_v, \bar{\gamma}_v)$. More precisely, if $(\bar{\bar{\delta}}'_v, \bar{\bar{\gamma}}_v)$ is any other base point for $(G'_{s,v}, G_v)$, we obtain

$$\begin{aligned}\Delta_{s,K_v}(\bar{\delta}'_v, \bar{\gamma}_v)^{-1}\Delta_s(\bar{\delta}'_v, \bar{\gamma}_v) &= \Delta_{{\rm I},s,K_v}(\bar{\delta}'_v, \bar{\gamma}_v)^{-1}\Delta_{{\rm I},s}(\bar{\delta}'_v, \bar{\gamma}_v)\\ &= \Delta_{{\rm I},s,K_v}(\bar{\bar{\delta}}'_v, \bar{\bar{\gamma}}_v)^{-1}\Delta_{{\rm I},s}(\bar{\bar{\delta}}'_v, \bar{\bar{\gamma}}_v),\end{aligned}$$

from [**31**, Lemma 3.2.A]. The second base point $(\bar{\bar{\delta}}'_v, \bar{\bar{\gamma}}_v)$ need only be local. We can choose $\bar{\bar{\delta}}'_v$ to lie in R'_v, and still be an image of a point $\bar{\bar{\gamma}}_v$ in G_v. Since s belongs to $Z(\hat{R}_{\rm sc})^{\Gamma_v}$, the definitions in [**31**, (3.2)] imply immediately that $\Delta_{{\rm I},s}(\bar{\bar{\delta}}'_v, \bar{\bar{\gamma}}_v)$ and $\Delta_{{\rm I},s,K_v}(\bar{\bar{\delta}}'_v, \bar{\bar{\gamma}}_v)$ are both independent of s. The term

$$\Delta_{s,K_v}(\bar{\delta}'_v, \bar{\gamma}_v)^{-1}\Delta_s(\bar{\delta}'_v, \bar{\gamma}_v)$$

in the product of (9.5) with (9.7) can therefore be ignored. We conclude that the original transfer factor $\Delta_s(\delta', \gamma)$ can be written as the product of

$$\Delta_{1,s}(\delta', \gamma; \bar{\delta}'_V, \bar{\gamma}_V)\biggl(\prod_{v\notin V} \Delta_{1,s}(\bar{\delta}'_v, \bar{\gamma}_v)\biggr)^{-1} d_s(\bar{\delta}', \bar{\gamma})^{-1} \tag{9.8}$$

with a factor that is independent of s.

We turn now to the relative term $\Delta_{1,s}(\delta, \gamma; \bar{\delta}'_V, \bar{\gamma}_V)$. It of course depends on s through the semisimple element

$$s'_s = s's$$

in $\hat{G}$ that is part of the endoscopic datum G'_s. As usual, $s' = s'_1$ denotes the corresponding element attached to the fixed endoscopic datum $G' = G'_1$. The term also depends on s through the global base point $(\bar{\delta}', \bar{\gamma})$. To keep track of this secondary dependence, we shall write

$$(\bar{\delta}', \bar{\gamma}) = (\bar{\delta}'_s, \bar{\gamma}_s).$$

The relative term is defined as a product

$$\Delta_{1,s}(\delta', \gamma; \bar{\delta}'_{s,V}, \bar{\gamma}_{s,V}) = \prod_{i=1}^{n} \Delta_{1,s}(\delta'_i, \gamma_i; \bar{\delta}'_{s,i}, \bar{\gamma}_{s,i})$$

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of local factors for the valuations v_i in V. As in § 4, we write $\bar{T}'_s$ for the centralizer of $\bar{\delta}'_s$ in G'_s, and we fix an admissible embedding of $\bar{T}'_s$ into a maximal torus $\bar{T}_s$ of G^*. Then $\bar{T}_s$ depends on s, but has the compensation of being defined over the global field F. For each i, we write T'_i for the centralizer of $\delta'_i = \delta'_{v_i}$ in L'_i, and we fix an admissible embedding of T'_i into a maximal torus T_i of the Levi subgroup L_i of G^*. Then T_i is defined only over the local field F_i, but is independent of s. Following [**31**, (3.4)], we form the torus

$$U_{i,s} = T_{i,\mathrm{sc}} \times \bar{T}_{s,\mathrm{sc}}/\{(z^{-1}, z) : z \in Z(G^*_{\mathrm{sc}})\}$$

over F_i. The local factor $\Delta_{1,s}(\delta'_i, \gamma_i; \bar{\delta}'_{s,i}, \bar{\gamma}_{s,i})$ is then defined by the pairing of an element

$$\mathrm{inv}\left(\frac{\delta'_i, \gamma_i}{\bar{\delta}'_{s,i}, \bar{\gamma}_{s,i}}\right) = v_i^{-1} \times \bar{v}_{s,i} \tag{9.9}$$

in $H^1(F_i, U_{i,s})$ with a point

$$(s'_s)_{U_{i,s}} = (\tilde{s}'s)_{T_i} \times (\tilde{s}'s)_{\bar{T}_s} \tag{9.10}$$

in $(\hat{U}_{i,s})^{\Gamma_i}$. The element $\tilde{s}'$ here is any point in $\hat{G}_{\mathrm{sc}}$ whose image in $\hat{G}_{\mathrm{ad}}$ coincides with that of s'.

The element s ranges over the group (9.3). In particular, we have objects $\bar{\delta}'_1$, $\bar{\gamma}_1$, $\bar{T}'_1$, $\bar{T}_1$ and $U_{i,1}$ corresponding to the element $s = 1$. To compare the local relative factor $\Delta_{1,s}$ with its specialization $\Delta_{1,1}$ at $s = 1$, we introduce a torus

$$U_s = \bar{T}_{1,\mathrm{sc}} \times \bar{T}_{s,\mathrm{sc}}/\{(z^{-1}, z) : z \in Z(G^*_{\mathrm{sc}})\},$$

which is defined over F. Now there is no simple relation between the points $(\tilde{s}')_{\bar{T}_1}$ and $(\tilde{s}'s)_{\bar{T}_s}$ in $\hat{\bar{T}}_{1,\mathrm{sc}}$ and $\hat{\bar{T}}_{s,\mathrm{sc}}$. However, since s is a Γ-invariant element $\hat{G}_{\mathrm{sc}}$, Γ acts on the two points by translating each of them by identical elements in $\hat{Z}_{\mathrm{sc}}$. The product

$$(\tilde{s}')_{\bar{T}_1} \times (\tilde{s}'s)_{\bar{T}_s}$$

therefore represents a Γ-invariant point u_s in the dual torus

$$\hat{U}_s = \hat{\bar{T}}_{1,\mathrm{sc}} \times \hat{\bar{T}}_{s,\mathrm{sc}}/\{(z, z) : z \in \hat{Z}_{\mathrm{sc}}\}.$$

We write $\Delta_1(\bar{\delta}'_{1,i}, \bar{\gamma}_{1,i}, \bar{\delta}'_{s,i}, \bar{\gamma}_{s,i})$ for the factor obtained by pairing the element

$$\mathrm{inv}\left(\frac{\bar{\delta}'_{1,i}, \bar{\gamma}_{1,i}}{\bar{\delta}'_{s,i}, \bar{\gamma}_{s,i}}\right) = \bar{v}_{1,i}^{-1} \times \bar{v}_{s,i}$$

in $H^1(F_i, U_s)$ with u_s.

We can write

$$(\tilde{s}'s)_{T_i} = (\tilde{s}')_{T_i} s,$$

since s lies in the centre of $\hat{L}_{i,\mathrm{sc}}$. The point (9.10) therefore equals the product of a point

$$(\tilde{s}')_{T_i} \times (\tilde{s}'s)_{\bar{T}_s} \tag{9.11}$$

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with $s \times 1$. There is a canonical map of $(\hat{T}_{i,\mathrm{sc}})^{\Gamma_i}$ into $(\hat{U}_{i,s})^{\Gamma_i}$, and $s \times 1$ is by definition the image of the point $s \in (\hat{T}_{i,\mathrm{sc}})^{\Gamma_i}$. The dual map of $H^1(F_i, U_{i,s})$ into $H^1(F_i, T_{i,\mathrm{ad}})$ takes the class (9.9) to an element $\mathrm{inv}(\delta'_i, \gamma_i)$ in $H^1(F_i, T_{i,\mathrm{ad}})$ that depends only on (δ'_i, γ_i). The pairing of (9.9) with $s \times 1$ is then equal to the pairing of $\mathrm{inv}(\delta'_i, \gamma_i)$ with s. It follows that $\Delta_{1,s}(\delta'_i, \gamma_i; \bar{\delta}'_{s,i}, \bar{\gamma}_{s,i})$ equals the product of

$$\langle \mathrm{inv}(\delta'_i, \gamma_i), s \rangle$$

with the value of the pairing of (9.9) with (9.11). The value of this last pairing can be rewritten according to a general transitivity property [**31**, (4.1)]. It equals the product of $\Delta_{1,1}(\delta'_i, \gamma_i; \bar{\delta}'_{1,i}, \bar{\gamma}_{1,i})$ with $\Delta_1(\bar{\delta}'_{1,i}, \bar{\gamma}_{1,i}; \bar{\delta}'_{s,i}, \bar{\gamma}_{s,i})$, by a natural variant of the proof of [**31**, Lemma 4.1.A], and the definitions above. (The transitivity property does apply here, even though the endoscopic groups G'_s and G' are distinct. The point is that they share the Levi subgroup L'_i that contains δ'_i.) We conclude that $\Delta_{1,s}(\delta'_i, \gamma_i; \bar{\delta}'_{s,i}, \bar{\gamma}_{s,i})$ equals

$$\langle \mathrm{inv}(\delta'_i, \gamma_i), s \rangle \Delta_1(\bar{\delta}'_{1,i}, \bar{\gamma}_{1,i}; \bar{\delta}'_{s,i}, \bar{\gamma}_{s,i}) \Delta_{1,1}(\delta'_i, \gamma_i; \bar{\delta}'_{1,i}, \bar{\gamma}_{1,i}).$$

Observe that the third term in the product is independent of s, and can be ignored.

We have obtained a decomposition of the first term in the product (9.8). Combining this with the other two terms in (9.8), to which we add subscripts s, we see that $\Delta_s(\delta', \gamma)$ equals the product of

$$\prod_{i=1}^{n} \langle \mathrm{inv}(\delta'_i, \gamma_i), s \rangle, \tag{9.12}$$

$$\biggl(\prod_{i=1}^{n} \Delta_1(\bar{\delta}'_{1,i}, \bar{\gamma}_{1,i}; \bar{\delta}'_{s,i}, \bar{\gamma}_{s,i})\biggr) \biggl(\prod_{v \notin V} \Delta_{1,s}(\bar{\delta}'_{s,v}, \bar{\gamma}_{s,v})\biggr)^{-1} d_s(\bar{\delta}'_s, \bar{\gamma}_s)^{-1}, \tag{9.13}$$

and a factor that is independent of s. To complete the proof of the lemma, we shall show that (9.13) is independent of s, and that (9.12) equals $\hat{\zeta}_V(s)$.

Consider the expression (9.13). The relative pairings that constitute the factors of the first product have been defined for valuations v_i in V, but they make sense for any v. Suppose that v is not in V. Since G_v is quasisplit, the relative pairing for v splits into a product of two absolute pairings. We obtain

$$\Delta_1(\bar{\delta}'_{1,v}, \bar{\gamma}_{1,v}; \bar{\delta}'_{s,v}, \bar{\gamma}_{s,v}) = \Delta_{1,1}(\bar{\delta}'_{1,v}, \bar{\gamma}_{1,v}) \Delta_{1,s}(\bar{\delta}'_{s,v}, \bar{\gamma}_{s,v})^{-1},$$

by a simple variant of the formula stated prior to Lemma 3.4.A of [**31**]. The adelic relative pairing

$$\Delta_1(\bar{\delta}'_1, \bar{\gamma}_1; \bar{\delta}'_s, \bar{\gamma}_s) = \prod_v \Delta_1(\bar{\delta}'_{1,v}, \bar{\gamma}_{1,v}; \bar{\delta}'_{s,v}, \bar{\gamma}_{s,v}),$$

defined by a product over all v, also splits. This is because it factors through the image of the map

$$\bigoplus_v H^1(F_v, U_s) \to H^1(F, U_s(\overline{\mathbb{A}})/U_s(\bar{F})).$$

Recalling the definition of $d_s(\bar{\delta}_s, \bar{\gamma}_s)$ in § 4, we obtain

$$\Delta_1(\bar{\delta}'_1, \bar{\gamma}_1; \bar{\delta}'_s, \bar{\gamma}_s) = d_1(\bar{\delta}'_1, \bar{\gamma}_1)^{-1} d_s(\bar{\delta}'_s, \bar{\gamma}_s),$$

from the natural variant of the formula [**31**, (6.3.2)]. It follows that (9.13) equals

$$\biggl(\prod_{v \notin V} \Delta_{1,1}(\bar{\delta}'_{1,v}, \bar{\gamma}_{1,v})^{-1}\biggr)^{-1} d_1(\bar{\delta}'_1, \bar{\gamma}_1)^{-1},$$

and, in particular, is independent of s.

To deal with (9.12), we have to evaluate the complex number obtained by pairing the class $\mathrm{inv}(\delta'_i, \gamma_i)$ in $H^1(F_i, T_{i,\mathrm{ad}})$ with the point s in $(\hat{T}_{i,\mathrm{sc}})^{\Gamma_i}$. Since s lies in the group (9.3), it can be represented by an element in $Z(\hat{L}_{i,\mathrm{sc}})^{\Gamma_i}$. It follows from [**22**, Theorem 1.2] that the pairing factors through the image of $\mathrm{inv}(\delta'_i, \gamma_i)$ in $H^1(F_i, L_{i,\mathrm{ad}})$. Now $\mathrm{inv}(\delta'_i, \gamma_i)$ is the class of the cocycle obtained by composing a function

$$v_i : \Gamma_i \to T_{i,\mathrm{sc}}$$

with the projection of $T_{i,\mathrm{sc}}$ onto $T_{i,\mathrm{ad}}$. There is no harm in letting γ_i stand for a representative in

$$M_{i,\alpha_i}(F_i), \quad \alpha_i \in \pi_0(M_i),$$

of the given conjugacy class, of which δ'_i is an image relative to M_i. The function v_i is then defined as

$$v_i(\tau) = h_i u_{i,\alpha_i}(\tau)\tau(h_i)^{-1}, \quad \tau \in \Gamma_i,$$

where (ψ_i, u_i) is a frame for the multiple group $M_i \amalg L_i$, and h_i is a point in $L_{i,\mathrm{sc}}$ such that

$$h_i \psi_{i,\alpha_i}(\gamma_i) h_i^{-1}$$

equals the image of δ'_i in T_i. (See [**31**, (3.4)] and [**12**, § 2].) The image of $\mathrm{inv}(\delta'_i, \gamma_i)$ in $H^1(F_i, L_{i,\mathrm{ad}})$ is actually independent of h_i. It is just the class of the cocycle obtained by composing the function u_{i,α_i} with the projection of $L_{i,\mathrm{sc}}$ onto $L_{i,\mathrm{ad}}$. This class is the element in $H^1(F_i, L_{i,\mathrm{ad}})$ that defines M_{i,α_i} as an inner twist of L_i. Its pairing with s equals $\hat{\zeta}_i(s)$, by definition. Taking the product over i, we conclude that (9.12) equals the product

$$\hat{\zeta}_V(s) = \hat{\zeta}_1(s) \dots \hat{\zeta}_n(s).$$

We have shown that $\Delta_s(\delta', \gamma)$ equals the product of $\hat{\zeta}_V(s)$ with a factor that is independent of s. Since $\Delta(\delta', \gamma)$ is the value of $\Delta_s(\delta', \gamma)$ at $s = 1$, we obtain

$$\Delta_s(\delta', \gamma) = \zeta_V(s)\Delta(\delta', \gamma).$$

This completes the proof of Lemma 9.2. □

Lemma 9.3. *The character*

$$\hat{\zeta}_V(s), \quad s \in Z(\hat{L}_{1,\mathrm{sc}})^{\Gamma} \cap \dots \cap Z(\hat{L}_{n,\mathrm{sc}})^{\Gamma} / \hat{Z}_{\mathrm{sc}}^{\Gamma},$$

is non-trivial.

Proof. Set $A = (Z(\hat{R}_{\mathrm{sc}})^{\Gamma})^0$. We shall also write $A_i = (Z(\hat{L}_{i,\mathrm{sc}})^{\Gamma})^0$, for each i. Since the constant $\mathrm{e}_R^{G^*}(L)$ is non-zero, the vector space $\mathfrak{a}_R$ equals the sum of the two subspaces $\mathfrak{a}_{L_i}$ and $\bigcap_{j \neq i} \mathfrak{a}_{L_j}$. This implies that A can be written as the product of A_i with the group

$$A^i = \bigcap_{j \neq i} A_j,$$

for any i between 1 and n. Set

$$Z = \hat{Z}_{\mathrm{sc}} \cap A = \hat{Z}_{\mathrm{sc}}^{\Gamma} \cap A.$$

Then ZA_i is a subgroup of $Z(\hat{L}_{i,\mathrm{sc}})^{\Gamma}$, and we have an injection

$$ZA_1 \cap \cdots \cap ZA_n / Z \to Z(\hat{L}_{1,\mathrm{sc}})^{\Gamma} \cap \cdots \cap Z(\hat{L}_{n,\mathrm{sc}})^{\Gamma} / \hat{Z}_{\mathrm{sc}}^{\Gamma}.$$

It is enough to show that the character

$$z \to \hat{\zeta}_V(z), \quad z \in ZA_1 \cap \cdots \cap ZA_n,$$

is non-trivial.

Let

$$z \to (z_1, \ldots, z_n)$$

be the composition of the maps

$$ZA_1 \cap \cdots \cap ZA_n \to \prod_i (ZA_i/A_i) \to \prod_i (Z/Z \cap A_i).$$

Suppose that $(z_1, \ldots, z_n)$ is any point in Z^n. For each i, we can write

$$z_i = a_i a^i, \quad a_i \in A_i, \quad a^i \in A^i.$$

Then $a^i = z_i a_i^{-1}$ belongs to ZA_i. Since it also belongs to A_j, for each $j \neq i$, a^i lies in the domain $ZA_1 \cap \cdots \cap ZA_n$. The image of a^i in ZA_j/A_j equals 1 if $j \neq i$, and equals z_i if $j = 1$. Therefore,

$$z = a^1 \ldots a^n$$

is an element in $ZA_1 \cap \cdots \cap ZA_n$ whose image in $\prod_i (Z/Z \cap A_i)$ is $(z_1, \ldots, z_n)$. For any i, we observe that

$$\hat{\zeta}_i(z) = \prod_j \hat{\zeta}_i(a^j) = \hat{\zeta}_i(a^i) = \hat{\zeta}_i(z_i),$$

since a^j belongs to the subgroup A_i of the kernel of $\hat{\zeta}_i$, for any $j \neq i$, and since a_i also lies in A_i. We conclude that

$$\hat{\zeta}_V(z) = \hat{\zeta}_1(z_1) \ldots \hat{\zeta}_n(z_n), \tag{9.14}$$

for any point $(z_1, \ldots, z_n)$ in Z^n.

The assumption from Theorem 9.1 was that the Levi subgroup R of G^* does not come from G. It follows from Corollary 4.2 that there is an i such that $\hat{\zeta}_i$ is non-trivial on the group

$$Z = \hat{Z}_{\mathrm{sc}}^{\Gamma_i} \cap (Z(\hat{R}_{\mathrm{sc}})^{\Gamma})^0.$$

The character (9.14) is therefore non-trivial. The proof of Lemma 9.3 follows. □

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We can now return to the proof of the theorem. It suffices to show that the sum (9.4) equals zero. Consider the function $(f'_s)^{L'}$ in the summand. If δ' belongs to $\Delta_{G\text{-}\mathrm{reg}}(\tilde{L}'_V)$, we have

$$\begin{aligned}(f'_s)^{L'}(\delta') &= \sum_{\gamma\in\Gamma_{\mathrm{reg}}(G_V)} \Delta_s(\delta',\gamma) f_G(\gamma) \\ &= \sum_{\gamma} \hat{\zeta}_V(s)\Delta(\delta',\gamma) f_G(\gamma) \\ &= \hat{\zeta}_V(s)(f')^{L'}(\delta'),\end{aligned}$$

by Lemma 9.2. It follows that

$$(f'_s)^{L'} = \hat{\zeta}_V(s)(f')^{L'}.$$

The expression (9.4) then factors as a product

$$\Big(\sum_s \hat{\zeta}_V(s)\Big)\hat{S}^L_R(\sigma',(f')^{L'}).$$

We recall that the sum here is over elements s in the group (9.3). Since the character $\hat{\zeta}_V$ is non-trivial on this group by Lemma 9.2, the sum vanishes. As we have seen, this implies that the original expression (9.2) for $I^{\mathcal{E}}_R(\sigma',f)$ also vanishes. We have completed the proof of Theorem 9.1. □

Theorem 9.1 has a natural spectral analogue, which we can state as a corollary. With R and R' fixed as at the beginning of the section, we choose an element $\psi'\in\Phi((\tilde{R}'_V)^{\tilde{Z}'},\tilde{\zeta}'_V)$. The linear form

$$I^{\mathcal{E}}_R(\psi',f) = \sum_{G'\in\mathcal{E}_{R'}(G^*)} \iota_{R'}(G^*,G')\hat{S}^{\tilde{G}'}_{\tilde{R}'}(\psi',f'), \quad f\in\mathcal{H}(G,V,\zeta), \tag{9.15}$$

is then defined according to our implicit induction hypotheses. If R corresponds to a Levi subgroup M of G, we have

$$I^{\mathcal{E}}_R(\psi',f) = I^{\mathcal{E}}_M(\phi',f),$$

for the element ϕ' as in the spectral analogue of (6.2) that is defined by ψ'. The local vanishing property in Proposition 6.4 tells that this linear form vanishes unless ϕ' itself comes from M.

Corollary 9.4. *Suppose that R does not come from G. Then*

$$I^{\mathcal{E}}_R(\psi',f) = 0,$$

for R' and ψ' as above.

Proof. The proof is identical to that of the theorem. The spectral analogue of the stable splitting formula quickly reduces the problem to showing that any sum

$$\sum_s \hat{S}^L_R(\psi',(f'_s)^{L'}),$$

over s in the group (9.3), vanishes. This follows from Lemmas 9.2 and 9.3, as in the final stage of the proof of the theorem. □

10. Endoscopic and stable expansions

We now take a significant step towards a stabilization of the trace formula. We shall establish endoscopic and stable analogues of the expansions in §§ 2 and 3. The results of this section include global analogues of the local expansions in [**12**, Theorem 9.1]. The proofs will follow the same general pattern. In particular, they will depend on the vanishing theorem we have just established. We note that our use of the term 'stable' is somewhat premature. We will not be able to prove the stability of the appropriate linear forms until a subsequent paper. Only then will we be able to claim to have stabilized the trace formula.

We fix our global objects G, Z, ζ, G^*, Z^*, ζ^*, and V, as at the beginning of § 9. In this section, we also fix a minimal Levi subgroup M_0 of the K-group G, with a corresponding Levi subgroup $M_0^* \subset G^*$, as well as a minimal Levi subgroup R_0 of G^* that is contained in M_0^*. We then have the Weyl group $W_0^* = W_0^{G^*} = W^{G^*}(R_0)$ of (G^*, R_0), as well as the Weyl group $W_0 = W_0^G = W^G(M_0)$ for (G, M_0). The former acts on the set $\mathcal{L}^* = \mathcal{L}^{G^*}$ of Levi subgroups of G^* that contain R_0, while the latter acts on the set $\mathcal{L} = \mathcal{L}^G$ of (M_0-equivalence classes of) Levi subgroups of G that contain M_0. (See [**12**, § 1].) The image of $\mathcal{L}$ under the natural map $M \to M^*$ is the subset $\mathcal{L}(M_0^*)$ of $\mathcal{L}^*$. A general element $R \in \mathcal{L}^*$ comes from G, in the sense of the last section, if and only if its W_0^*-orbit meets $\mathcal{L}(M_0^*)$. If G' is any endoscopic group for G, we can also form the set $\mathcal{L}' = \mathcal{L}^{G'}$ and the Weyl group $W_0' = W_0^{G'}$. Both of these of course depend on a fixed minimal Levi subgroup of G'.

The expansions of §§ 2 and 3 were established for connected groups, but they carry over verbatim to the K-group G. For example, we obtain an expansion for the linear form

$$I(f) = \sum_{\alpha \in \pi_0(G)} I(f_\alpha), \quad f = \bigoplus_\alpha f_\alpha,$$

on $\mathcal{H}(G, V, \zeta)$ by taking a sum of the expansions given by Proposition 2.2. We leave the reader to check that Proposition 2.2 and the other results of §§ 2 and 3 apply to G as stated, even though the sets $\mathcal{L}$, W_0^M, W_0^G, etc., now have a slightly different meaning. We shall freely quote them as results on G.

We begin by applying a formal construction to the linear form I. As in the construction preceding Lemmas 7.2 and 7.3, we set

$$I^{\mathcal{E}}(f) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G, G')\hat{S}'(f') + \varepsilon(G) S^G(f), \quad f \in \mathcal{H}(G, V, \zeta), \tag{10.1}$$

for linear forms $\hat{S}' = \hat{S}^{\tilde{G}'}$ on the spaces $S\mathcal{I}(\tilde{G}', V, \tilde{\zeta}')$, which are defined inductively by the supplementary requirement that

$$I^{\mathcal{E}}(f) = I(f), \tag{10.2}$$

in case G is quasisplit. As usual, we have to know that for any $G' \in \mathcal{E}^0_{\mathrm{ell}}(G, V)$, the symbol $\hat{S}'$ makes sense. We assume inductively that if G is replaced by a quasisplit inner K-form

https://doi.org/10.1017/S1474748002000051 Published online by Cambridge University Press

of $\tilde{G}'$, the corresponding analogue of S^G is defined and stable. We propose to establish expansions of these new distributions in terms of the objects constructed in §§ 6 and 7.

For the geometric expansions, we first recall the linear forms $I^{\mathcal{E}}_{\mathrm{orb}}(f)$ and $S^G_{\mathrm{orb}}(f)$ defined in § 7. The expansions of Lemma 7.2 for these forms are to be regarded as the purely 'orbital' terms of larger geometric expansions for $I^{\mathcal{E}}(f)$ and $S^G(f)$. To construct the remaining terms, we consider the differences $I^{\mathcal{E}}(f) - I^{\mathcal{E}}_{\mathrm{orb}}(f)$ and $S^G(f) - S^G_{\mathrm{orb}}(f)$, for a fixed function $f \in \mathcal{H}(G, V, \zeta)$.

Theorem 10.1.

(a) *If G is arbitrary,*

$$I^{\mathcal{E}}(f) - I^{\mathcal{E}}_{\mathrm{orb}}(f) = \sum_{M \in \mathcal{L}^0} |W^M_0||W^G_0| \sum_{\gamma \in \Gamma^{\mathcal{E}}(M,V,\zeta)} a^{M,\mathcal{E}}(\gamma) I^{\mathcal{E}}_M(\gamma, f). \tag{10.3}$$

(b) *If G is quasisplit,*

$$\begin{aligned} &S^G(f) - S^G_{\mathrm{orb}}(f) \\ &\quad = \sum_{M \in \mathcal{L}^0} |W^M_0||W^G_0|^{-1} \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M, M') \sum_{\delta' \in \Delta(\tilde{M}', V, \tilde{\zeta}')} b^{\tilde{M}'}(\delta') S^G_M(M', \delta', f). \end{aligned} \tag{10.4}$$

Remark. If G is quasisplit, Global Theorem $1'$(b) of § 7 asserts that the distributions $S^G_M(M', \delta')$ are stable, and vanish unless $M' = M^*$. If this is so, the formula (10.4) reduces to

$$S^G(f) - S^G_{\mathrm{orb}}(f) = \sum_{M \in \mathcal{L}^0} |W^M_0||W^G_0|^{-1} \sum_{\delta \in \Delta(M,V,\zeta)} b^M(\delta) S^G_M(\delta, f). \tag{10.5}$$

Proof. Let us write $I^{\mathcal{E},0}(f)$ and $S^{G,0}(f)$ for the right-hand sides of (10.3) and (10.4), respectively. The main step is to show that the difference

$$(I^{\mathcal{E}}(f) - I^{\mathcal{E}}_{\mathrm{orb}}(f)) - \varepsilon(G)(S^G(f) - S^G_{\mathrm{orb}}(f)) \tag{10.6}$$

of the left-hand sides of the two formulae equals the corresponding difference

$$I^{\mathcal{E},0}(f) - \varepsilon(G) S^{G,0}(f) \tag{10.7}$$

of right-hand sides.

According to the definitions (10.1) and (7.6), the difference (10.6) equals

$$\sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G, G')(\hat{S}^{\tilde{G}'}(f') - \hat{S}^{\tilde{G}'}_{\mathrm{orb}}(f')).$$

We assume inductively that for any $G' \in \mathcal{E}^0_{\mathrm{ell}}(G, V)$, part (b) of the theorem holds for any quasisplit inner K-form of $\tilde{G}'$. We are also carrying the general induction hypothesis

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that part (b) of Global Theorem 1′ holds for any such $\tilde{G}'$, so we can assume that the analogue of (10.5) holds for $\tilde{G}'$. It follows that (10.6) equals

$$\sum_{G'\in\mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G') \sum_{R'\in(\mathcal{L}')^0} |W_0^{R'}||W_0^{G'}|^{-1} S_{R'}(G'),$$

where

$$S_{R'}(G') = \sum_{\sigma'\in\Delta(\tilde{R}',V,\tilde{\zeta}')} b^{\tilde{R}'}(\sigma')\hat{S}^{\tilde{G}'}_{\tilde{R}'}(\sigma',f').$$

Lemma 10.2. *Suppose that*

$$S_{R'}(G'), \quad G'\in\mathcal{E}(G^*), \quad R'\in\mathcal{L}',$$

is a family of complex numbers that depend only on the $\mathrm{Aut}_{G^*}(G')$*-orbit of* R'*, and that vanish for all but finitely many* G'*. Then*

$$\sum_{G'\in\mathcal{E}_{\mathrm{ell}}(G^*)} \iota(G^*,G') \sum_{R'\in\mathcal{L}'} |W_0^{R'}||W_0^{G'}|^{-1} S_{R'}(G')$$

equals

$$\sum_{R\in\mathcal{L}^*} |W_0^{R}||W_0^{G^*}|^{-1} I_R(G^*)$$

where

$$I_R(G^*) = \sum_{R'\in\mathcal{E}_{\mathrm{ell}}(R)} \iota(R,R') \sum_{G'\in\mathcal{E}_{R'}(G^*)} \iota_{R'}(G^*,G') S_{R'}(G').$$

Proof. The statement of this rearrangement lemma matches that of its local counterpart [**12**, Lemma 9.2]. The proof is also the same, apart from the fact that the global coefficient $\iota(G^*,G')$ is slightly more complicated than the corresponding local coefficient. According to [**21**, Theorem 8.3.1 and (5.1.1)], we can write

$$\iota(G^*,G') = |\,\mathrm{Out}_{G^*}(G')|^{-1}|Z(\hat{G}')^{\Gamma}/Z(\hat{G}^*)^{\Gamma}||\ker^1(F,Z(\hat{G}'))||\ker^1(F,Z(\hat{G}^*))|^{-1},$$

where

$$\mathrm{Out}_{G^*}(G') = \mathrm{Aut}_{G^*}(G')/\hat{G}^*,$$

and $\ker^1(F,Z(\hat{G}^*))$ is the subgroup of locally trivial elements in $H^1(F,Z(\hat{G}^*))$. It is the factor

$$|\ker^1(F,Z(\hat{G}'))||\ker^1(F,Z(\hat{G}^*))|^{-1}$$

that is the extra global ingredient. However, this factor equals the corresponding factor

$$|\ker^1(F,Z(\hat{R}'))||\ker^1(F,Z(\hat{R}))|^{-1}$$

in the formula for the coefficient $\iota(R,R')$. (See, for example, [**13**, Lemma 2].) In other words, the extra factors do not contribute to the quotient

$$\iota(G^*,G')\iota(R,R')^{-1}.$$

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At the end of the proof of Lemma 9.2 of [**12**], various constants were seen to cancel. The contribution of the local analogues of $\iota(G^*, G')$ and $\iota(R, R')$ to the argument was as a quotient of one by the other. The argument therefore carries over to the global situation at hand. The proof of the global lemma here thus reduces to that of the local lemma in [**12**]. □

Returning to the proof of Theorem 10.1, we apply the lemma to the last expression we obtained from (10.6). We set $S_{R'}(G')$ equal to zero if G' belongs to the complement of $\mathcal{E}^0_{\mathrm{ell}}(G,V)$ in $\mathcal{E}(G^*)$, or if R' equals G'. Then $S_{R'}(G')$ is supported on a finite collection of groups G'. The lemma provides an expansion

$$\sum_{R\in\mathcal{L}^*} |W_0^R||W_0^{G^*}|^{-1} \sum_{R'\in\mathcal{E}_{\mathrm{ell}}(R)} \iota(R,R') \sum_{G'\in\mathcal{E}_{R'}(G^*)} \iota_{R'}(G,G')S_{R'}(G'),$$

for (10.6). Substituting for $S_{R'}(G')$, we conclude that (10.6) equals

$$\sum_{R\in(\mathcal{L}^*)^0} |W_0^R||W_0^{G^*}|^{-1} \sum_{R'\in\mathcal{E}_{\mathrm{ell}}(R,V)} \iota(R,R') \sum_{\sigma'\in\Delta(\tilde{R}',V,\tilde{\zeta}')} b^{\tilde{R}'}(\sigma')B_{R'}(\sigma',f), \qquad (10.8)$$

where

$$B_{R'}(\sigma',f) = \sum_{G'\in\mathcal{E}^0_{R'}(G)} \iota_{R'}(G,G')\hat{S}^{\tilde{G}'}_{\tilde{R}'}(\sigma',f').$$

We claim that

$$B_{R'}(\sigma',f) = I^{\mathcal{E}}_R(\sigma',f) - \varepsilon(G)S^G_R(R',\sigma',f).$$

If $\varepsilon(G) = 1$, the map from $\mathcal{L}$ to $\mathcal{L}^*$ is onto, and R is the image of a group in $\mathcal{L}$. The formula in this case is just the definition (6.2). If $\varepsilon(G) = 0$, the image of $\mathcal{L}$ in $\mathcal{L}^*$ is proper, and need not contain R. However, $\mathcal{E}^0_{R'}(G)$ equals $\mathcal{E}_{R'}(G)$ in this case, and the formula follows from the definition (9.1). Having justified the claim, we consider the contribution to (10.8) of the two terms in the formula for $B_{R'}(\sigma',f)$. The contribution of the second term is just the product of $(-\varepsilon(G))$ with the right-hand side $S^{G,0}(f)$ of (10.4). For the contribution of the first term $I^{\mathcal{E}}_R(\sigma',f)$, we appeal to Theorem 9.1. According to this theorem, $I^{\mathcal{E}}_R(\sigma',f)$ vanishes unless R comes from G, which in the present context means that (R,R',σ') lies in the W_0^*-orbit of a triplet

$$(M,M',\delta'), \quad M\in\mathcal{L}^0, \quad M'\in\mathcal{E}_{\mathrm{ell}}(M,V), \quad \delta'\in\Delta(\tilde{M}',V,\tilde{\zeta}').$$

If (R,R',σ') does have this property, we can write $I^{\mathcal{E}}_R(\sigma',f) = I^{\mathcal{E}}_M(\delta',f)$, $\iota(R,R') = \iota(M,M')$ and $b^{\tilde{R}'}(\sigma') = b^{\tilde{M}'}(\delta')$. The contribution of $I^{\mathcal{E}}_M(\sigma',f)$ to (10.8) can therefore be expressed in terms of a sum over $M\in\mathcal{L}^0$. By familiar counting arguments, we have only to replace the coefficient $|W_0^R||W_0^{G^*}|^{-1}$ in (10.8) by $|W_0^M||W_0^G|^{-1}$. We conclude that (10.8) equals the difference between the expression

$$\sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M') \sum_{\delta'\in\Delta(\tilde{M}',V,\tilde{\zeta}')} b^{\tilde{M}'}(\delta')I^{\mathcal{E}}_M(\delta',f) \qquad (10.9)$$

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and

$$\varepsilon(G)S^{G,0}(f).$$

According to (6.4) and Proposition 6.2, the term $I_M^{\mathcal{E}}(\delta', f)$ in (10.9) has an expansion

$$I_M^{\mathcal{E}}(\delta', f) = \sum_{\gamma \in \Gamma^{\mathcal{E}}(M,V,\zeta)} \Delta_M(\delta', \gamma) I_M^{\mathcal{E}}(\gamma, f).$$

We substitute this expansion into (10.9), and then take the sum over γ outside the sums over M' and δ'. In the special case that G is quasisplit, our general induction assumption implies that part (b) of Global Theorem 1′ holds for any M in (10.9), since M is summed over proper Levi subgroups of G. The definition (7.1) therefore takes the form

$$a^{M,\mathcal{E}}(\gamma) = \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M, M') \sum_{\delta' \in \Delta(\tilde{M}', V, \tilde{\zeta}')} b^{\tilde{M}'}(\delta') \Delta_M(\delta', \gamma),$$

in general. Substituting this into (10.9), we obtain an expression

$$\sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathcal{E}}(M,V,\zeta)} a^{M,\mathcal{E}}(\gamma) I_M^{\mathcal{E}}(\gamma, f)$$

that is just the right-hand side $I^{\mathcal{E},0}(f)$ of (10.3). We have shown that (10.8) equals the difference (10.7). But (10.8) equals the original expression (10.6), so we have completed the task of showing that (10.6) equals (10.7).

We can now finish the proof in the usual way from the equality of (10.6) and (10.7). If $\varepsilon(G) = 0$, we see immediately that

$$I^{\mathcal{E}}(f) - I_{\mathrm{orb}}^{\mathcal{E}}(f) = I^{\mathcal{E},0}(f),$$

which is the required identity (10.3). Suppose that $\varepsilon(G) = 1$. Then $I^{\mathcal{E}}(f) - I_{\mathrm{orb}}^{\mathcal{E}}(f)$ equals $I(f) - I_{\mathrm{orb}}(f)$ by definition. Moreover, $I^{\mathcal{E},0}(f)$ equals the expression

$$I^0(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in \Gamma(M,V,\zeta)} a^M(\gamma) I_M(\gamma, f),$$

according to the definitions (6.3) and (7.2) (along with (6.1) and (6.5)) of the original terms in the two expansions. Since $I(f) - I_{\mathrm{orb}}(f)$ equals $I^0(f)$, by the original expansion (2.9) and the definition (2.11), we again have the required identity (10.3). The remaining terms in (10.6) and (10.7) then give the identity

$$S^G(f) - S_{\mathrm{orb}}^G(f) = S^{G,0}(f),$$

which is just (10.4).

Observe that we have established the absolute convergence of the expansions (10.3) and (10.4). This is a consequence of the inductive proof of the theorem, and the absolute convergence of (2.9). One could also apply the argument of [**8**, § 3], together with the appropriate splitting and descent formulae, to show directly that the summands in (10.3) and (10.4) have finite support. □

Theorem 10.1 provides expansions with which we can investigate the distributions $I^{\mathcal{E}}(f) - I^{\mathcal{E}}_{\mathrm{orb}}(f)$ and $S^G(f) - S^G_{\mathrm{orb}}(f)$. We shall also need expansions of a similar form for the more elementary distributions $I^{\mathcal{E}}_{\mathrm{orb}}(f) - I^{\mathcal{E}}_{\mathrm{ell}}(f,S)$ and $S^G_{\mathrm{orb}}(f) - S^G_{\mathrm{ell}}(f,S)$. This is essentially the question of establishing formulae for the coefficients $a^{G,\mathcal{E}}(\gamma)$ and $b^G(\delta)$ that are parallel to (2.8). The expressions (8.6) and (8.7) for $a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma,S)$ and $b^G_{\mathrm{ell}}(\delta,S)$ can be regarded as the terms with $M = G$ in such expansions. It is therefore enough to study the differences $a^{G,\mathcal{E}}(\gamma) - a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma,S)$ and $b^G(\delta) - b^G_{\mathrm{ell}}(\delta,S)$.

For the next proposition and its corollary, we assume that V contains the finite set $V_{\mathrm{fund}}(G)$ of Assumption 5.2.

Proposition 10.3.

(a) *Suppose that γ belongs to $\Gamma^{\mathcal{E}}(G,V,\zeta)$, and that $S \supset V$ is a large finite set. Then*

$$a^{G,\mathcal{E}}(\gamma) - a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma,S) = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{k\in\mathcal{K}^{V,\mathcal{E}}_{\mathrm{ell}}(\bar{M},S)} a^{M,\mathcal{E}}_{\mathrm{ell}}(\gamma_M \times k) r^G_M(k). \tag{10.10}$$

(b) *Suppose that G is quasisplit, that δ belongs to $\Delta^{\mathcal{E}}(G,V,\zeta)$, and that $S \supset V$ is again a large finite set. Then*

$$b^G(\delta) - b^G_{\mathrm{ell}}(\delta,S) = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\ell\in\mathcal{L}^V_{\mathrm{ell}}(\bar{M},S)} b^M_{\mathrm{ell}}(\delta_M \times \ell) s^G_M(\ell), \tag{10.11}$$

if ℓ lies in the subset $\Delta(G,V,\zeta)$ of $\Delta^{\mathcal{E}}(G,V,\zeta)$, while $b^G(\delta) - b^G_{\mathrm{ell}}(\delta,S)$ vanishes if δ lies in the complement of $\Delta(G,V,\zeta)$.

Proof. The required expansions clearly have the same general structure as those of Theorem 10.1. The connection becomes more obvious if we reformulate the proposition in terms of the distributions

$$I^{\mathcal{E}}_{\mathrm{orb}}(f) - I^{\mathcal{E}}_{\mathrm{ell}}(f,S) = \sum_{\gamma\in\Gamma^{\mathcal{E}}(G,V,\zeta)} (a^{G,\mathcal{E}}(\gamma) - a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma,S)) f_G(\gamma)$$

and

$$S^G_{\mathrm{orb}}(f) - S^G_{\mathrm{ell}}(f,S) = \sum_{\delta\in\Delta^{\mathcal{E}}(G,V,\zeta)} (b^G(\delta) - b^G_{\mathrm{ell}}(\delta,S)) f^{\mathcal{E}}_G(\delta).$$

The two expansions are provided by Lemma 7.2, (8.8) and (8.9).

Set

$$I^{\mathcal{E},0}_{\mathrm{orb}}(f,S) = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{\dot{\gamma}_S} a^{M,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S) r^G_M(\dot{\gamma}_S, f),$$

where $\dot{\gamma}_S = \gamma \times k$ is summed over the product of $\Gamma^{\mathcal{E}}_{\mathrm{ell}}(M,V,\zeta)$ with $\mathcal{K}^{V,\mathcal{E}}_{\mathrm{ell}}(\bar{M},S)$, and

$$r^G_M(\dot{\gamma}_S, f) = r^G_M(k) f_M(\gamma).$$

Then $I^{\mathcal{E},0}_{\mathrm{orb}}(f,S)$ is the linear form obtained by replacing the coefficient $a^{G,\mathcal{E}}(\gamma) - a^{G,\mathcal{E}}_{\mathrm{ell}}(\gamma,S)$ in the first expression above by the corresponding right-hand side of (10.10), and then

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changing variables in the sum over γ. The function $f \in \mathcal{H}(G, V, \zeta)$ can be allowed to vary freely without affecting S, as long as the support of f remains bounded. Part (a) of the proposition is therefore equivalent to the identity

$$I^{\mathcal{E}}_{\mathrm{orb}}(f) - I^{\mathcal{E}}_{\mathrm{ell}}(f, S) = I^{\mathcal{E},0}_{\mathrm{orb}}(f, S).$$

Similarly, if G is quasisplit, set

$$S^{G,0}_{\mathrm{orb}}(f, S) = \sum_{M \in \mathcal{L}^0} |W^M_0||W^G_0|^{-1} \sum_{\dot{\delta}_S} b^M_{\mathrm{ell}}(\dot{\delta}_S) s^G_M(\dot{\delta}_S, f),$$

where $\dot{\delta}_S = \delta \times \ell$ is summed over the product of $\Delta_{\mathrm{ell}}(M, V, \zeta)$ with $\mathcal{L}^V_{\mathrm{ell}}(\bar{M}, S)$, and

$$s^G_M(\dot{\delta}_S, f) = s^G_M(\ell) f^M(\delta).$$

Then $S^{G,0}_{\mathrm{orb}}(f, S)$ is the linear form obtained by replacing the coefficient $b^G(\delta) - b^G_{\mathrm{ell}}(\delta, S)$ in the second expression above by 0 if δ does not lie in $\Delta(G, V, \zeta)$, and by the right-hand side of (10.11) if δ does lie in $\Delta(G, V, \zeta)$. Part (b) of the proposition is then equivalent to the identity

$$S^G_{\mathrm{orb}}(f) - S^G_{\mathrm{ell}}(f, S) = S^{G,0}_{\mathrm{orb}}(f, S).$$

We now proceed as in the proof of Theorem 10.1. According to the definitions in §§ 7 and 8, the difference

$$(I^{\mathcal{E}}_{\mathrm{orb}}(f) - I^{\mathcal{E}}_{\mathrm{ell}}(f, S)) - \varepsilon(G)(S^G_{\mathrm{orb}}(f) - S^G_{\mathrm{ell}}(f, S)) \tag{10.12}$$

equals

$$\sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G, G')(\hat{S}^{\tilde{G}'}_{\mathrm{orb}}(f') - \hat{S}^{\tilde{G}'}_{\mathrm{ell}}(f', S)).$$

We can assume inductively that

$$\hat{S}^{\tilde{G}'}_{\mathrm{orb}}(f') - \hat{S}^{\tilde{G}'}_{\mathrm{ell}}(f', S) = \hat{S}^{\tilde{G}',0}_{\mathrm{orb}}(f', S),$$

for any $G' \in \mathcal{E}^0_{\mathrm{ell}}(G, V)$. After making the appropriate substitution, we apply Lemma 10.2, as in the proof of Theorem 10.1. We find that (10.12) equals

$$\sum_{R \in (\mathcal{L}^*)^0} |W^R_0||W^{G^*}_0|^{-1} \sum_{R' \in \mathcal{E}_{\mathrm{ell}}(R,V)} \iota(R, R') \sum_{\dot{\sigma}'_S} b^{\tilde{R}'}_{\mathrm{ell}}(\dot{\sigma}'_S) B_{R'}(\dot{\sigma}'_S, f),$$

where $\dot{\sigma}'_S$ is summed over the product of $\Delta_{\mathrm{ell}}(\tilde{R}', V, \tilde{\zeta}')$ with $\mathcal{L}^V_{\mathrm{ell}}(\bar{R}', S)$, and

$$B_{R'}(\dot{\sigma}'_S, f) = \sum_{G' \in \mathcal{E}^0_{R'}(G)} \iota_{R'}(G, G') \hat{s}^{\tilde{G}'}_{\tilde{R}'}(\dot{\sigma}'_S, f').$$

As in the proof of Theorem 10.1, we shall break $B_{R'}(\dot{\sigma}'_S, f)$ into a sum of two terms. In this case, the decomposition takes the form

$$B_{R'}(\dot{\sigma}'_S, f) = \sum_{G' \in \mathcal{E}_{R'}(G)} \iota_{R'}(G, G') \hat{s}^{\tilde{G}'}_{\tilde{R}'}(\dot{\sigma}'_S, f') - \varepsilon(G) s^G_R(R', \dot{\sigma}'_S, f),$$

where

$$s_R^G(R', \dot\sigma'_S, f) = \begin{cases} s_R^{G^*}(\dot\sigma_S, f), & \text{if } R' = R \text{ and } \dot\sigma'_S = \dot\sigma_S, \\ 0, & \text{otherwise.} \end{cases}$$

The contribution of the second term to (10.12) is just the product of $(-\varepsilon(G))$ with $S_{\mathrm{orb}}^{G,0}(f, S)$. The contribution of the first term will be given by the generalized fundamental lemma, or rather its formulation in Proposition 8.1. If $\dot\sigma'_S = \sigma' \times \ell'$, the first term equals

$$\sum_{G' \in \mathcal{E}_{R'}(G)} \iota_{R'}(G, G')\hat{s}_{\tilde{R}'}^{\tilde{G}'}(\dot\sigma'_S, f') = \Big(\sum_{G' \in \mathcal{E}_{R'}(G)} \iota_{R'}(G, G') s_{\tilde{R}'}^{\tilde{G}'}(\ell')\Big) f^{R'}(\sigma').$$

This vanishes if R does not come from G, by the definition of $f^{R'}(\sigma')$. On the other hand, if (R, R', σ') lies in the $W_0^{G^*}$-orbit of a triplet (M, M', δ') that comes from G, Proposition 8.1 tells us that the first term equals

$$r_M^G(\ell) f^{M'}(\delta') = r_M^G(\ell) f_M^{\mathcal{E}}(\delta),$$

where $\delta \times \ell$ is the image of $\delta' \times \ell'$ in the product of $\Delta_{\mathrm{ell}}^{\mathcal{E}}(M, V, \zeta)$ with $\mathcal{L}^{V,\mathcal{E}}(\bar{M}, S)$. By the usual counting arguments, we can therefore write the contribution of the first term to (10.12) as

$$\sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M, M') \sum_{\dot\delta'_S} b_{\mathrm{ell}}^{\tilde{M}'}(\dot\delta'_S) r_M^G(\dot\delta'_S, f), \tag{10.13}$$

where $\dot\delta'_S = \delta' \times \ell'$ is summed over the product of $\Delta_{\mathrm{ell}}(\tilde{M}', V, \tilde\zeta')$ with $\mathcal{L}_{\mathrm{ell}}^V(\bar{M}', V)$, and

$$r_M^G(\dot\delta'_S, f) = r_M^G(\ell) f_M^{\mathcal{E}}(\delta).$$

We have shown that (10.12) equals the difference between (10.13) and $\varepsilon(G) S_{\mathrm{orb}}^{G,0}(f, S)$.

Consider the expression (10.13). We can actually sum M' over the larger set $\mathcal{E}_{\mathrm{ell}}(M, S)$, since Proposition 8.1 asserts that the factor $r_M^G(\ell)$ in the corresponding summand vanishes if M' lies in the complement of $\mathcal{E}_{\mathrm{ell}}(M, V)$ in $\mathcal{E}_{\mathrm{ell}}(M, S)$. Moreover, the term $r_M^G(\dot\delta'_S, f)$ can be expanded as a sum over $\dot\gamma_S$. It follows from (8.1) and the definitions above that

$$r_M^G(\dot\delta'_S, f) = \sum_{\dot\gamma_S} \Delta_M(\dot\delta'_S, \dot\gamma_S) r_M^G(\dot\gamma_S, f),$$

where $\dot\gamma_S$ is summed over the product of $\Gamma_{\mathrm{ell}}^{\mathcal{E}}(M, V, \zeta)$ with $\mathcal{K}_{\mathrm{ell}}^{V,\mathcal{E}}(\bar{M}, S)$. Finally, in the special case that G is quasisplit, our general induction hypothesis implies that part (b) of Global Theorem 1 holds for any M in (10.13). The definition (7.3) therefore takes the form

$$a_{\mathrm{ell}}^{M,\mathcal{E}}(\dot\gamma_S) = \sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,S)} \iota(M, M') \sum_{\dot\delta'_S} b^{\tilde{M}'}(\dot\delta'_S) \Delta_M(\dot\delta'_S, \dot\gamma_S),$$

in general. We conclude that the inner sum over M' in (10.13) equals

$$\sum_{\dot\gamma_S} a_{\mathrm{ell}}^{M,\mathcal{E}}(\dot\gamma_S) r_M^G(\dot\gamma_S, f).$$

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The expression (10.13) itself is then equal to $I^{\mathcal{E},0}_{\mathrm{ell}}(f,S)$.

We have shown that the original expression (10.12) is equal to

$$I^{\mathcal{E},0}_{\mathrm{orb}}(f,S) - \varepsilon(G)S^{G,0}_{\mathrm{orb}}(f,S).$$

The proof of the proposition can now be completed in the usual way, exactly as in Theorem 10.1. □

Proposition 10.3, together with (8.6) and (8.7), reduces the study of the coefficients $a^{G,\mathcal{E}}(\gamma)$ and $b^G(\delta)$ to that of the elliptic coefficients $a^{G,\mathcal{E}}_{\mathrm{ell}}(\dot{\gamma}_S)$ and $b^G_{\mathrm{ell}}(\dot{\delta}_S)$. In particular, we obtain the following corollary.

Corollary 10.4. *Global Theorem 1 implies Global Theorem 1′.*

The spectral expansions depend on a non-negative number t, but will otherwise be parallel to those above. The starting point is the linear form I_t of § 3. Given t, we set

$$I^{\mathcal{E}}_t(f) = \sum_{G'\in\mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}'_{t'}(f') + \varepsilon(G)S^G_t(f), \quad f\in\mathcal{H}(G,V,\zeta), \tag{10.14}$$

for linear forms $\hat{S}'_{t'} = \hat{S}^{\tilde{G}'}_{t'}$ on the spaces $S\mathcal{I}(\tilde{G}',V,\tilde{\zeta}')$, which are defined inductively by the supplementary requirement that

$$I^{\mathcal{E}}_t(f) = I_t(f), \tag{10.15}$$

in case G is quasisplit. We assume inductively that if G is replaced by a quasisplit inner K-form of $\tilde{G}'$, for any $G'\in\mathcal{E}^0_{\mathrm{ell}}(G,V)$, the corresponding analogue of $S^{\tilde{G}'}_{t'}$ is defined and stable.

The link between the geometric and spectral expansions is provided by the following analogue of Proposition 3.1.

Proposition 10.5.

(a) *If G is arbitrary, the linear forms*

$$I^{\mathcal{E}}_t(f), \quad f\in\mathcal{H}(G,V,\zeta), \quad t\geqslant 0,$$

satisfy the multiplier convergence estimate (3.3), and the formula

$$I^{\mathcal{E}}(f) = \sum_t I^{\mathcal{E}}_t(f).$$

(b) *If G is quasisplit, the linear forms*

$$S^G_t(f), \quad f\in\mathcal{H}(G,V,\zeta), \quad t\geqslant 0,$$

also satisfy the estimate (3.3), as well as the formula

$$S^G(f) = \sum_t S^G_t(f).$$

Proof. We assume inductively that (b) holds if G is replaced by a quasisplit inner K-form of $\tilde{G}'$, for any $G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)$. If $\alpha \in C_c^\infty(\mathfrak{h}^Z)^{W_\infty}$ is a multiplier for G, there is a multiplier $\alpha' \in C_c^\infty(\tilde{\mathfrak{h}}^{\tilde{Z}'})^{W'_\infty}$ for $\tilde{G}'$ such that

$$\hat{\alpha}'(\nu + d\tilde{\eta}'_\infty) = \hat{\alpha}(\nu), \quad \nu \in \mathfrak{h}^*_{\mathbb{C}}/\mathfrak{a}^*_{G,Z,\mathbb{C}},$$

and

$$(f_\alpha)' = f'_{\alpha'}, \quad f \in \mathcal{H}(G,V,\zeta).$$

(See [**10**, (7.9)].) It follows that the linear forms

$$I_t^{\mathcal{E}}(f) - \varepsilon(G)S_t^G(f) = \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}'_{t'}(f'), \quad f \in \mathcal{H}(G,V,\zeta), \quad t \geqslant 0,$$

satisfy the estimate (3.3). Furthermore, we observe that

$$\begin{aligned} I^{\mathcal{E}}(f) - \varepsilon(G)S^G(f) &= \sum_{G' \in \mathcal{E}^0_{\mathrm{ell}}(G,V)} \iota(G,G')\hat{S}'(f') \\ &= \sum_{G'} \iota(G,G') \sum_t \hat{S}'_{t'}(f') \\ &= \sum_t (I_t^{\mathcal{E}}(f) - \varepsilon(G)S_t^G(f)). \end{aligned}$$

The proposition follows from Proposition 3.1, (10.2) and (10.15). □

Suppose now that t is fixed. The expansions of the linear forms $I^{\mathcal{E}}_{t,\mathrm{unit}}(f)$ and $S^G_{t,\mathrm{unit}}(f)$ in Lemma 7.3 are to be regarded as purely 'unitary' terms of larger spectral expansions of $I_t^{\mathcal{E}}(f)$ and $S_t^G(f)$. For the remaining terms, we consider the differences $I_t^{\mathcal{E}}(f) - I^{\mathcal{E}}_{t,\mathrm{unit}}(f)$ and $S_t^G(f) - S^G_{t,\mathrm{unit}}(f)$, for a fixed function $f \in \mathcal{H}(G,V,\zeta)$.

Theorem 10.6.

(a) *If G is arbitrary,*

$$I_t^{\mathcal{E}}(f) - I^{\mathcal{E}}_{t,\mathrm{unit}}(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} \int_{\Pi_t^{\mathcal{E}}(M,V,\zeta)} a^{M,\mathcal{E}}(\pi) I_M^{\mathcal{E}}(\pi,f)\, \mathrm{d}\pi. \quad (10.16)$$

(b) *If G is quasisplit,*

$$\begin{aligned} S_t^G(f) - S^G_{t,\mathrm{unit}}(f) = \sum_{M \in \mathcal{L}^0} |W_0^M||W_0^G|^{-1} &\sum_{M' \in \mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M') \\ &\times \int_{\Phi_{t'}(\tilde{M}',V,\tilde{\zeta}')} b^{\tilde{M}'}(\phi') S_M^G(M',\phi',f)\, \mathrm{d}\phi'. \quad (10.17) \end{aligned}$$

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Remark. If G is quasisplit, and Local Theorem $2'$(b) holds for G, the formula (10.17) simplifies to

$$S_t^G(f) - S_{t,\mathrm{unit}}^G(f) = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1}\int_{\Phi_t(M,V,\zeta)} b^M(\phi)S_M^G(\phi,f)\,\mathrm{d}\phi. \tag{10.18}$$

This is obviously parallel to (10.5).

Proof. The proof is essentially the same as that of Theorem 10.1. We write $I_t^{\mathcal{E},0}(f)$ and $S_t^{G,0}(f)$ for the right-hand sides of (10.16) and (10.17), respectively. The main step is to show that the difference

$$(I_t^{\mathcal{E}}(f) - I_{t,\mathrm{unit}}^{\mathcal{E}}(f)) - \varepsilon(G)(S_t^G(f) - S_{t,\mathrm{unit}}^G(f)) \tag{10.19}$$

of the left-hand sides of the two formulae equals the corresponding difference

$$I_t^{\mathcal{E},0}(f) - \varepsilon(G)S_t^{G,0}(f) \tag{10.20}$$

of right-hand sides.

We have only to follow the earlier inductive argument, using Corollary 9.4 in place of Theorem 9.1. The role of (10.9) is taken by the parallel spectral expansion

$$\sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1}\sum_{M'\in\mathcal{E}_{\mathrm{ell}}(M,V)} \iota(M,M')\int_{\Phi_{t'}(\tilde{M}',V,\tilde{\zeta}')} b^{\tilde{M}'}(\phi')I_M^{\mathcal{E}}(\phi',f)\,\mathrm{d}\phi',$$

into which we substitute the inversion formula

$$I_M^{\mathcal{E}}(\phi',f) = \sum_{\pi\in\Pi_t^{\mathcal{E}}(M,V,\zeta)} \Delta_M(\phi',\pi)I_M^{\mathcal{E}}(\pi,f)$$

that is provided by (6.6) and Proposition 6.4. At this point there is a minor difference in the argument. Using the property

$$\Delta_M(\phi'_\lambda,\pi_\lambda) = \Delta_M(\phi',\pi), \quad \lambda\in i\mathfrak{a}_{M,Z}^*/i\mathfrak{a}_{G,Z}^*,$$

and the definitions of the measures $\mathrm{d}\phi'$ and $\mathrm{d}\pi$ in § 7, we transform the resulting integral over $\Phi_{t'}(\tilde{M}',V,\tilde{\zeta}')$ and sum over $\Pi_t^{\mathcal{E}}(M,V,\zeta)$ to a sum over $\Phi_{t'}(\tilde{M}',V,\tilde{\zeta}')$ and integral over $\Pi_t^{\mathcal{E}}(M,V,\zeta)$. The remaining discussion from the proof of Theorem 10.1 carries over verbatim. It confirms the equality of (10.19) with (10.20), from which the proof of the proposition follows. In particular, the absolute convergence of the integrals in (10.16) and (10.17) follows inductively from the absolute convergence of the integral in (3.13). □

Finally, we shall derive expansions for the more elementary spectral distributions $I_{t,\mathrm{unit}}^{\mathcal{E}}(f) - I_{t,\mathrm{disc}}^{\mathcal{E}}(f)$ and $S_{t,\mathrm{unit}}^G(f) - S_{t,\mathrm{disc}}^G(f)$. This is essentially the question of establishing formulae for the coefficients $a^{G,\mathcal{E}}(\pi)$ and $b^G(\phi)$ that are parallel to (3.12). Since the expressions (8.14) and (8.15) for $a_{\mathrm{disc}}^{G,\mathcal{E}}(\pi)$ and $b_{\mathrm{disc}}^G(\phi)$ can be regarded as the terms with $M = G$ in such formulae, it will be enough to study the differences $a^{G,\mathcal{E}}(\pi) - a_{\mathrm{disc}}^{G,\mathcal{E}}(\pi)$ and $b^G(\phi) - b_{\mathrm{disc}}^G(\phi)$.

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Proposition 10.7.

(a) *Suppose that π belongs to $\Pi_t^{\mathcal{E}}(G,V,\zeta)$. Then*

$$a^{G,\mathcal{E}}(\pi) - a^{G,\mathcal{E}}_{\mathrm{disc}}(\pi) = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{c\in\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(M,\zeta)} a^{M,\mathcal{E}}_{\mathrm{disc}}(\pi_M\times c) r^G_M(c). \quad (10.21)$$

(b) *Suppose that G is quasisplit, and that ϕ belongs to $\Phi_t^{\mathcal{E}}(G,V,\zeta)$. Then*

$$b^G(\phi) - b^G_{\mathrm{disc}}(\phi) = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \sum_{c\in\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(M,\zeta)} b^M_{\mathrm{disc}}(\phi_M\times c) s^G_M(c), \quad (10.22)$$

if c lies in the subset $\Phi_t(G,V,\zeta)$ of $\Phi_t^{\mathcal{E}}(G,V,\zeta)$, while $b^G(\phi) - b^G_{\mathrm{disc}}(\phi)$ vanishes if ϕ lies in the complement of $\Phi_t(G,V,\zeta)$.

Proof. The proof is parallel to that of Proposition 10.3. In particular, we reformulate the proposition in terms of the distributions

$$I^{\mathcal{E}}_{t,\mathrm{unit}}(f) - I^{\mathcal{E}}_{t,\mathrm{disc}}(f) = \int_{\Pi^{\mathcal{E}}_t(G,V,\zeta)} (a^{G,\mathcal{E}}(\pi) - a^{G,\mathcal{E}}_{\mathrm{disc}}(\pi)) f_G(\pi)\,\mathrm{d}\pi$$

and

$$S^G_{t,\mathrm{unit}}(f) - S^G_{t,\mathrm{disc}}(f) = \int_{\Phi^{\mathcal{E}}_t(G,V,\zeta)} (b^G(\phi) - b^G_{\mathrm{disc}}(\phi)) f^{\mathcal{E}}_G(\phi)\,\mathrm{d}\phi,$$

with expansions given by Lemma 7.3, (8.16) and (8.17). We set

$$I^{\mathcal{E},0}_{t,\mathrm{unit}}(f) = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \int a^{M,\mathcal{E}}_{\mathrm{disc}}(\dot\pi) r^G_M(\dot\pi, f)\,\mathrm{d}\dot\pi$$

and

$$S^{G,0}_{t,\mathrm{unit}}(f) = \sum_{M\in\mathcal{L}^0} |W_0^M||W_0^G|^{-1} \int b^M_{\mathrm{disc}}(\dot\phi) s^G_M(\dot\phi, f)\,\mathrm{d}\dot\phi.$$

The integrals are taken over elements $\dot\pi = \pi\times c$ in the product of $\Pi^{\mathcal{E}}_{t,\mathrm{disc}}(M,V,\zeta)$ with $\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(M,\zeta)$ and elements $\dot\phi = \phi\times c$ in the product of $\Phi_{t,\mathrm{disc}}(M,V,\zeta)$ with $\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(M,\zeta)$, relative to the natural measures, while

$$r^G_M(\dot\pi, f) = r^G_M(c) f_M(\pi)$$

and

$$s^G_M(\dot\phi, f) = s^G_M(c) f^M(\phi).$$

Part (a) of the proposition is equivalent to the identity

$$I^{\mathcal{E}}_{t,\mathrm{unit}}(f) - I^{\mathcal{E}}_{t,\mathrm{disc}}(f) = I^{\mathcal{E},0}_{t,\mathrm{unit}}(f),$$

while (b) is equivalent to the identity

$$S^G_{t,\mathrm{unit}}(f) - S^G_{t,\mathrm{disc}}(f) = S^{G,0}_{t,\mathrm{unit}}(f).$$

The proof of these identities proceeds as in Proposition 10.3. The role of Proposition 8.1 is taken by its spectral analogue Proposition 8.3. If $\dot\phi' = \phi' \times c'$ belongs to the product of $\Delta_{\mathrm{disc}}(\tilde M', V, \tilde\zeta')$ with $\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(\tilde M', \tilde\zeta')$, and has image $\phi \times c$ in the product of $\Delta^{\mathcal{E}}_{\mathrm{disc}}(M, V, \zeta)$ with $\mathcal{C}^{V,\mathcal{E}}_{\mathrm{disc}}(M, \zeta)$, Proposition 8.3 tells us that the expression

$$\sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') \hat s^{\tilde G'}_{\tilde M'}(\dot\phi', f') = \left(\sum_{G'} \iota_{M'}(G, G') s^{\tilde G'}_{\tilde M'}(c')\right) f^{M'}(\phi')$$

equals

$$r^G_M(c) f^{\mathcal{E}}_M(\phi).$$

Combining this formula with the other steps in the proof of Proposition 10.3, we deduce that the required identities are valid. □

Corollary 10.8. *Global Theorem 2 implies Global Theorem $2'$.*

Acknowledgements. The author was supported in part by NSERC Operating grant A3483.

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