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FACTORING VARIANTS OF CHEBYSHEV POLYNOMIALS WITH MINIMAL POLYNOMIALS OF $\cos(2\pi/d)$

D. A. WOLFRAM

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Abstract

We solve the problem of factoring polynomials $V_n(x) \pm 1$ and $W_n(x) \pm 1$, where $V_n(x)$ and $W_n(x)$ are Chebyshev polynomials of the third and fourth kinds, in terms of the minimal polynomials of $\cos(2\pi/d)$. The method of proof is based on earlier work, D. A. Wolfram, ['Factoring variants of Chebyshev polynomials of the first and second kinds with minimal polynomials of $\cos(2\pi/d)$', *Amer. Math. Monthly* **129** (2022), 172–176] for factoring variants of Chebyshev polynomials of the first and second kinds. We extend this to show that, in general, similar variants of Chebyshev polynomials of the fifth and sixth kinds, $X_n(x) \pm 1$ and $Y_n(x) \pm 1$, do not have factors that are minimal polynomials of $\cos(2\pi/d)$.

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1. Introduction

The significance of the widespread applications of Chebyshev and related polynomials in mathematics, engineering and numerical modelling motivates the study of the properties of these polynomials.

In earlier work, Wolfram [17] solved an open factorisation problem for Chebyshev polynomials of the second kind $U_n(x) \pm 1$ and gave a more direct proof of the result for Chebyshev polynomials of the first kind, $T_n(x) \pm 1$. We apply this method to solve the analogous factorisation problems for Chebyshev polynomials of the third and fourth kinds. These factorisations are also expressed in terms of the minimal polynomials of $\cos(2\pi/d)$. We then show that, in general, there are no factorisations of variants of the Chebyshev polynomials of the fifth and sixth kinds, $X_n(x) \pm 1$ and $Y_n(x) \pm 1$, with these minimal polynomials. Additionally, we give an equation that relates $U_n(x)$ to the monic form of $Y_n(x)$ in Theorem 5.1.

Chebyshev polynomials of the first and second kinds, $T_n(x)$ and $U_n(x)$, were introduced by Pafnuty Chebyshev (1821–1894) in 1854. Gautschi [8] named the Chebyshev polynomials of the third and fourth kinds $V_n(x)$ and $W_n(x)$ in 1992 [13]. They are also called airfoil polynomials [7, 13]. They are used in areas such as solving differential equations [1], numerical integration [5, 7, 13], approximations [13], interpolation [8] and combinatorics [6].

In 2006, Masjed-Jamei [12] defined orthogonal polynomials called Chebyshev polynomials of the fifth and sixth kinds, $X_n(x)$ and $Y_n(x)$, that satisfy a generalised recurrence equation for monic Chebyshev polynomials (see [3, Equations (6)–(7)]) with an exception for $\bar{T}_2(x)$ where $A_{1,-1,0,-1,1}$ is indeterminate and should equal $-\frac{1}{2}$. Abd-Elhameed and Youssri [3] specifically related $V_n(x)$ and $W_n(x)$ to the monic form of $X_n(x)$.

Chebyshev polynomials of the fifth and sixth kinds are active research areas. In 2018, shifted Chebyshev polynomials of the fifth kind were used to solve problems involving fractional-order differential equations [3]. In 2021, connection formulas and other properties of the polynomials of the fifth [2] and sixth [4] kinds were given. Also, Sadri and Aminikhah defined two-variable shifted Chebyshev polynomials of the sixth kind of the form $\bar{Y}_i(2x-1)\bar{Y}_j(2t-1)$ in [14, Equation (4.7)], and used them to solve fractional-order partial differential equations numerically.

1.1. Chebyshev polynomials of the second kind. Chebyshev polynomials of the second kind can be defined by

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)}, \tag{1.1}$$

where $x = \cos\theta$ and $n \geq 0$ (see [13, Equation (1.4)]). It follows that

$$U_n(x)^2 - 1 = U_{n-1}(x)U_{n+1}(x), \tag{1.2}$$

where $n \geq 1$, by applying the trigonometric identity

$$\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B) \quad \text{with } A = (n+1)\theta \text{ and } B = \theta.$$

These polynomials satisfy the recurrence (see [13, Equations (1.6a)–(1.6b)])

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad \text{for } n > 1. \tag{1.3}$$

DEFINITION 1.1. The polynomials $\Psi_d(x)$ are

$$\Psi_d(x) = \prod_{k \in S_{d/2}} 2\Big(x - \cos\Big(2\pi\frac{k}{d}\Big)\Big), \tag{1.4}$$

where $S_{d/2} = \{k \mid (k,d) = 1, 1 \leq k < d/2\}$ and $d > 2$. They have degree $\phi(d)/2$ where ϕ is Euler's totient function [9]. The polynomials $\Psi_1(x) = 2(x-1)$ and $\Psi_2(x) = 2(x+1)$ were defined in Wolfram [17, Definition 1]. These are polynomials with roots $\cos(2\pi)$ and $\cos(\pi)$, respectively.

Gürtaş [9] showed that

$$U_{n-1}(x) = \prod_{\substack{d \mid 2n \\ d>2}} \Psi_d(x), \quad \text{for } n \geq 1. \tag{1.5}$$

LEMMA 1.2. *The minimal polynomial in* $\mathbb{Q}[x]$ *of* $\cos(2\pi/d)$ *is* $\bar{\Psi}_d(x) = 2^{-\phi(d)/2}\Psi_d(x)$ *where* $d > 2$. *We also have* $\bar{\Psi}_1(x) = 2^{-1}\Psi_1(x)$ *and* $\bar{\Psi}_2(x) = 2^{-1}\Psi_2(x)$.

PROOF. This follows from the proof of [11, Theorem 1] and the definition of minimal polynomial. □

1.2. Chebyshev polynomials of the third and fourth kinds. Chebyshev polynomials of the third kind can be defined by

$$V_n(x) = \frac{\cos(n + 1/2)\theta}{\cos\theta/2}, \tag{1.6}$$

and of the fourth kind by

$$W_n(x) = \frac{\sin(n + 1/2)\theta}{\sin\theta/2}, \tag{1.7}$$

where $x = \cos\theta$ and $n \geq 0$. They can also be defined with respect to Chebyshev polynomials of the second kind by

$$V_n(x) = U_n(x) - U_{n-1}(x) \tag{1.8}$$

and

$$W_n(x) = U_n(x) + U_{n-1}(x), \tag{1.9}$$

where $n \geq 1$ (see [13, Equations (1.17)–(1.18)]).

2. Solution

The method of solution follows that by Wolfram [17]. The first step is to express $V_n(x)^2 - 1$ and $W_n(x)^2 - 1$ in terms of the polynomials $\Psi_d(x)$ where $d \geq 1$.

LEMMA 2.1. *For* $n \geq 1$,

$$V_n(x)^2 - 1 = \Psi_1(x) \prod_{\substack{d|2n\\ d>2}} \Psi_d(x) \prod_{\substack{d|2n+2\\ d>2}} \Psi_d(x), \tag{2.1}$$

$$W_n(x)^2 - 1 = \Psi_2(x) \prod_{\substack{d|2n\\ d>2}} \Psi_d(x) \prod_{\substack{d|2n+2\\ d>2}} \Psi_d(x).$$

PROOF. From (1.8),

$$\begin{aligned}
V_n(x)^2 - 1 &= (U_n(x) - U_{n-1}(x) + 1)(U_n(x) - U_{n-1}(x) - 1)\\
&= U_n(x)^2 - 1 - 2U_n(x)U_{n-1}(x) + U_{n-1}(x)^2\\
&= U_{n+1}(x)U_{n-1}(x) - 2U_n(x)U_{n-1}(x) + U_{n-1}(x)^2 \quad \text{from (1.2)}\\
&= U_{n-1}(x)(U_{n+1}(x) - 2U_n(x) + U_{n-1}(x))\\
&= U_{n-1}(x)(2xU_n(x) - 2U_n(x)) \quad \text{from (1.3)}
\end{aligned}$$

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$$= \Psi_1(x)U_{n-1}(x)U_n(x) \qquad \text{from Definition 1.1}$$

$$= \Psi_1(x)\prod_{\substack{d|2n\\ d>2}}\Psi_d(x)\prod_{\substack{d|2n+2\\ d>2}}\Psi_d(x) \qquad \text{from (1.5).}$$

Similarly, from (1.9),

$$\begin{aligned}
W_n(x)^2 - 1 &= (U_n(x) + U_{n-1}(x) + 1)(U_n(x) + U_{n-1}(x) - 1)\\
&= U_n(x)^2 - 1 + 2U_n(x)U_{n-1}(x) + U_{n-1}(x)^2\\
&= U_{n+1}(x)U_{n-1}(x) + 2U_n(x)U_{n-1}(x) + U_{n-1}(x)^2 && \text{from (1.2)}\\
&= U_{n-1}(x)(U_{n+1}(x) + 2U_n(x) + U_{n-1}(x))\\
&= U_{n-1}(x)(2xU_n(x) + 2U_n(x)) && \text{from (1.3)}\\
&= \Psi_2(x)U_{n-1}(x)U_n(x) && \text{from Definition 1.1}\\
&= \Psi_2(x)\prod_{\substack{d|2n\\ d>2}}\Psi_d(x)\prod_{\substack{d|2n+2\\ d>2}}\Psi_d(x) && \text{from (1.5)}
\end{aligned}$$

as required. □

The following theorem solves the factorisation problem for $V_n(x)^2 - 1$. The second step of the method involves defining the mapping that splits the $2n$ factors of $V_n(x)^2 - 1$ into the n factors of $V_n(x) + 1$ and the other n factors of $V_n(x) - 1$. The factorisations are unique up to associativity and commutativity of multiplication.

THEOREM 2.2. *If $n \geq 1$, then*

$$V_n(x) + 1 = \prod_{\substack{d|2n\\ d>2\\ 2n/d \text{ odd}}}\Psi_d(x) \prod_{\substack{d|2n+2\\ d>2\\ (2n+2)/d \text{ odd}}}\Psi_d(x) \tag{2.2}$$

and

$$V_n(x) - 1 = \Psi_1(x)\prod_{\substack{d|2n\\ d>2\\ 2n/d \text{ even}}}\Psi_d(x) \prod_{\substack{d|2n+2\\ d>2\\ (2n+2)/d \text{ even}}}\Psi_d(x). \tag{2.3}$$

PROOF. The polynomial $\Psi_1(x) = 2(x-1)$ is a factor of $V_n(x)^2 - 1$ from (2.1), and $\Psi_1(\cos(2\pi)) = 0$. It follows from (1.6) that $V_n(\cos(2\pi)) = 1$ and so $\Psi_1(x)$ is a factor of $V_n(x) - 1$.

If $d \mid 2n$ and $d > 2$, let $\theta = 2\pi k/d$ where $(k, d) = 1$, $1 \leq k < d/2$ and $a = 2n/d$. We have $\theta = \pi ak/n$ and $\Psi_d(\cos(\theta)) = 0$. From (1.6),

$$V_n(\cos(\theta)) = \frac{\cos((n+1/2)\pi ak/n)}{\cos(\theta/2)} = \frac{\cos(\pi ak)\cos(\theta/2) - \sin(\pi ak)\sin(\theta/2)}{\cos(\theta/2)}.$$

The denominator $\cos(\theta/2) \neq 0$ because $\theta/2 = \pi k/d$ cannot equal $\pi/2$ when $d > 2$. The numbers ak and a have the same parity. This is immediate when a is even. If a is odd,

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it follows that d is even and k is odd because $(k, d) = 1$. We have $\cos(\pi ak) = \cos(\pi a)$ and $V_n(\cos(\theta)) = \cos(\pi a)$.

Hence, if a is even, then $V_n(\cos(\theta)) = 1$ and $\Psi_d(x)$ is a factor of $V_n(x) - 1$. Similarly, if a is odd, then $V_n(\cos(\theta)) = -1$ and $\Psi_d(x)$ is a factor of $V_n(x) + 1$.

If $d \mid 2n + 2$ and $d > 2$, let $b = (2n + 2)/d$. We have $\theta = \pi bk/(n + 1)$ where k is such that $(k, d) = 1$ and $1 \le k < d/2$. From (1.6),

$$V_n(\cos(\theta)) = \frac{\cos((n + 1/2)\theta)}{\cos(\theta/2)} = \frac{\cos((n + 1)\theta - \theta/2)}{\cos(\theta/2)} = \frac{\cos(\pi bk)\cos(\theta/2) + \sin(\pi bk)\sin(\theta/2)}{\cos(\theta/2)}.$$

Similarly to the previous case, the denominator $\cos(\theta/2) \neq 0$, the numbers bk and b have the same parity and $V_n(\cos(\theta)) = \cos(\pi b)$. It follows that if b is odd then $\Psi_d(x)$ is a factor of $V_n(x) + 1$ and if b is even then $\Psi_d(x)$ is a factor of $V_n(x) - 1$.

From (1.3) and (1.8), $V_n(x)$ has degree n. It follows that the right-hand side of (2.1) of the factorisation of $V_n(x)^2 - 1$ has degree $2n$. It has $2n$ factors of the form $2(x - \cos(\theta))$ from (1.4) and Definition 1.1, half of which are the factors of $V_n(x) + 1$ and the other half are the factors of $V_n(x) - 1$. The mapping defined above maps every such factor of $V_n(x)^2 - 1$ to either $V_n(x) + 1$ or $V_n(x) - 1$ depending on whether $\cos(\theta)$ is a root of $V_n(x) + 1$ or $V_n(x) - 1$, respectively. The right-hand sides of (2.2) and (2.3) are the products of these mapped factors and so both have degree equal to n.

From (1.3) and (1.8), the leading coefficients of $V_n(x) \pm 1$ are 2^n. The expansions of the factorisations on the right-hand sides of (2.2) and (2.3) both have 2^n as leading coefficients also. Each is a product of n factors of the form $2(x - \cos(\theta))$. □

COROLLARY 2.3. *If $n \ge 1$, then the factorisations of $V_n(x) \pm 1$ in terms of the minimal polynomials of $\cos(2\pi/d)$ are*

$$V_n(x) + 1 = 2^n \prod_{\substack{d\mid 2n \\ d>2 \\ 2n/d \text{ odd}}} \bar{\Psi}_d(x) \prod_{\substack{d\mid 2n+2 \\ d>2 \\ (2n+2)/d \text{ odd}}} \bar{\Psi}_d(x)$$

and

$$V_n(x) - 1 = 2^n\, \bar{\Psi}_1(x) \prod_{\substack{d\mid 2n \\ d>2 \\ 2n/d \text{ even}}} \bar{\Psi}_d(x) \prod_{\substack{d\mid 2n+2 \\ d>2 \\ (2n+2)/d \text{ even}}} \bar{\Psi}_d(x).$$

The following theorem solves the factorisation problem for $W_n(x)^2 - 1$. These factorisations are also unique up to associativity and commutativity of multiplication.

THEOREM 2.4. *If $n \ge 1$, then*

$$W_n(x) + 1 = \prod_{\substack{d\mid 2n \\ d>1 \\ 2n/d \text{ odd}}} \Psi_d(x) \prod_{\substack{d\mid 2n+2 \\ d>2 \\ (2n+2)/d \text{ even}}} \Psi_d(x) \qquad (2.4)$$

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and

$$W_n(x) - 1 = \prod_{\substack{d|2n \\ d>1 \\ 2n/d \text{ even}}} \Psi_d(x) \prod_{\substack{d|2n+2 \\ d>2 \\ (2n+2)/d \text{ odd}}} \Psi_d(x). \tag{2.5}$$

PROOF. The structure of the proof is similar to that of Theorem 2.2. If $d \mid 2n$ and $d > 1$, let $a = 2n/d$ and k be such that $(k, d) = 1$ where $1 \le k < d/2$. From (1.7),

$$W_n(\cos(\theta)) = \frac{\sin((n + 1/2)\pi ak/n)}{\sin(\theta/2)} = \frac{\cos(\pi ak)\sin(\theta/2) + \sin(\pi ak)\cos(\theta/2)}{\sin(\theta/2)}.$$

The denominator $\sin(\theta/2) \neq 0$ because $\theta/2 = \pi k/d$ cannot equal π when $d > 1$. Similarly, ak and a have the same parity and $W_n(\cos(\theta)) = \cos(\pi a)$. Hence, if a is even, then $W_n(\cos(\theta)) = 1$ and $\Psi_d(x)$ is a factor of $W_n(x) - 1$. If a is odd, then $W_n(\cos(\theta)) = -1$ and $\Psi_d(x)$ is a factor of $W_n(x) + 1$.

If $d \mid 2n + 2$ and $d > 2$, let $b = (2n + 2)/d$ and k be such that $(k, d) = 1$ where $1 \le k < d/2$. We have $\theta = \pi bk/(n + 1)$ and $\theta/2 = \pi k/d$. From (1.7),

$$\begin{aligned} W_n(\cos(\theta)) &= \frac{\sin((n + 1/2)\theta)}{\sin(\theta/2)} = \frac{\sin((n + 1)\theta - \theta/2)}{\sin(\theta/2)} \\ &= \frac{-\cos(\pi bk)\sin(\theta/2) + \sin(\pi bk)\cos(\theta/2)}{\sin(\theta/2)}. \end{aligned}$$

The denominator $\sin(\theta/2) \neq 0$ and b and bk have the same parity, as above. Hence, if b is even, then $W_n(\cos(\theta)) = -1$ and $\Psi_d(x)$ is a factor of $W_n(x) + 1$. If b is odd, then $W_n(\cos(\theta)) = 1$ and $\Psi_d(x)$ is a factor of $W_n(x) - 1$.

It is straightforward to show that the degrees of the right-hand sides of (2.4) and (2.5) are both n and the leading coefficients of both sides of these equations are 2^n. □

COROLLARY 2.5. *If* $n \ge 1$, *then the factorisations of* $W_n(x) \pm 1$ *in terms of the minimal polynomials of* $\cos(2\pi/d)$ *are*

$$W_n(x) + 1 = 2^n \prod_{\substack{d|2n \\ d>1 \\ 2n/d \text{ odd}}} \bar{\Psi}_d(x) \prod_{\substack{d|2n+2 \\ d>2 \\ (2n+2)/d \text{ even}}} \bar{\Psi}_d(x)$$

and

$$W_n(x) - 1 = 2^n \prod_{\substack{d|2n \\ d>1 \\ 2n/d \text{ even}}} \bar{\Psi}_d(x) \prod_{\substack{d|2n+2 \\ d>2 \\ (2n+2)/d \text{ odd}}} \bar{\Psi}_d(x).$$

3. Examples with V

The polynomial $V_{12}(x)^2 - 1$ has 24 factors, and $V_{12}(x) + 1$ and $V_{12}(x) - 1$ each are the products of half of these factors. The mapping in the proof of

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Theorem 2.2 gives

$$\begin{aligned} V_{12}(x)+1 &= \Psi_8(x)\Psi_{24}(x)\Psi_{26}(x),\\ V_{12}(x)-1 &= \Psi_1(x)\Psi_3(x)\Psi_4(x)\Psi_6(x)\Psi_{12}(x)\Psi_{13}(x)\\ &= (2(x-1))(2x+1)(2x)(2x-1)(4x^2-3)\\ &\quad \cdot(64x^6+32x^5-80x^4-32x^3+24x^2+6x-1)\\ &= 2^{12}(x-1)(x+\tfrac{1}{2})x(x-\tfrac{1}{2})(x^2-\tfrac{3}{4})\bar{\Psi}_{13}(x). \end{aligned}$$

4. Examples with W

The polynomial $W_{12}(x)^2-1$ has 24 factors, and $W_{12}(x)+1$ and $W_{12}(x)-1$ each are the products of half of these factors. The mapping in the proof of Theorem 2.4 gives

$$\begin{aligned} W_{12}(x)+1 &= \Psi_8(x)\Psi_{24}(x)\Psi_{13}(x),\\ W_{12}(x)-1 &= \Psi_2(x)\Psi_3(x)\Psi_4(x)\Psi_6(x)\Psi_{12}(x)\Psi_{26}(x)\\ &= (2(x+1))(2x+1)(2x)(2x-1)(4x^2-3)\\ &\quad \cdot(64x^6-32x^5-80x^4+32x^3+24x^2-6x-1)\\ &= 2^{12}(x+1)(x+\tfrac{1}{2})x(x-\tfrac{1}{2})(x^2-\tfrac{3}{4})\bar{\Psi}_{26}(x). \end{aligned}$$

When n is odd, $\Psi_2(x)$ is a factor of $W_n(x)+1$:

$$W_{11}(x)+1 = \Psi_2(x)\Psi_{22}(x)\Psi_3(x)\Psi_4(x)\Psi_6(x)\Psi_{12}(x).$$

5. Chebyshev polynomials of the fifth and sixth kinds

Masjed-Jamei [12] defined the Chebyshev polynomials of the fifth kind, $X_n(x)$, and sixth kind, $Y_n(x)$. Similarly to the other four kinds of Chebyshev polynomials, they are orthogonal polynomials with integer coefficients. The polynomials $X_n(x)$ and $Y_n(x)$ have degree n where $n \geq 0$ [3, 4], and have definite parity, that is, they have the form

$$\sum_{v=0}^{\lfloor n/2 \rfloor} a_v x^{n-2v}.$$

Monic Chebyshev polynomials of the fifth and sixth kinds, $\bar{X}_n(x)$ and $\bar{Y}_n(x)$, can be defined by the following recurrences which we simplify from [3]:

$$\begin{aligned} G_{0,m}(x) &= 1,\\ G_{1,m}(x) &= x,\\ G_{n,m}(x) &= xG_{n-1,m}(x)+A_{n-1,m}\, G_{n-2,m}(x), \quad n>1, \end{aligned}$$

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where

$$
\begin{aligned}
A_{n,m} &= \frac{(2n+m-2)(-1)^n + (2n-(m-2)) - nm - n^2}{(2n+m-1)(2n+m-3)},\\
\bar{X}_n(x) &= G_{n,3}(x),\\
\bar{Y}_n(x) &= G_{n,5}(x).
\end{aligned}
$$

The first seven Chebyshev polynomials of the fifth kind over $\mathbb{Z}$ are

$$
\begin{aligned}
X_0(x) &= 1,\\
X_1(x) &= x,\\
X_2(x) &= 4x^2 - 3,\\
X_3(x) &= 6x^3 - 5x,\\
X_4(x) &= 16x^4 - 20x^2 + 5,\\
X_5(x) &= 80x^5 - 112x^3 + 35x,\\
X_6(x) &= 64x^6 - 112x^4 + 56x^2 - 7.
\end{aligned}
$$

They are orthogonal over $[-1, 1]$ with weight function $x^2/\sqrt{1-x^2}$ [3]. An interesting property is

$$
V_n(x) = 2^{2n}\bar{X}_{2n}\Big(\sqrt{\frac{1+x}{2}}\Big), \quad W_n(x) = (-1)^n 2^{2n}\bar{X}_{2n}\Big(\sqrt{\frac{1-x}{2}}\Big)
$$

where $n \geq 0$ [3, Section 2.2]. The term $(-1)^n$ is missing in Abd-Elhameed and Youssri [3], but it follows because $V_n(-x) = (-1)^n W_n(x)$ where $n \geq 0$ (see [13, Equation (1.19)]).

The first seven Chebyshev polynomials of the sixth kind over $\mathbb{Z}$ are

$$
\begin{aligned}
Y_0(x) &= 1,\\
Y_1(x) &= x,\\
Y_2(x) &= 2x^2 - 1,\\
Y_3(x) &= 8x^3 - 5x,\\
Y_4(x) &= 16x^4 - 16x^2 + 3,\\
Y_5(x) &= 24x^5 - 28x^3 + 7x,\\
Y_6(x) &= 16x^6 - 24x^4 + 10x^2 - 1.
\end{aligned}
$$

These polynomials are orthogonal over $[-1, 1]$ with weight function $x^2\sqrt{1-x^2}$ [4]. Their monic forms have been expressed explicitly as sums [4, Equations (3) and (4)].

The monic Chebyshev polynomials of the sixth kind can be related to Chebyshev polynomials of the second kind.

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THEOREM 5.1. *We have*

$$U_n(x) = 2^{2n}\bar{Y}_{2n}\Big(\sqrt{\frac{1+x}{2}}\Big), \quad \textit{for } n \geq 0.$$

PROOF. From [4, Equation (3)], the monic form of $\bar{Y}_{2n}(x)$ is

$$\bar{Y}_{2n}(x) = \frac{\Gamma(\frac{3}{2}+n)}{(2n+1)!}\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{n-k}(1+2n-k)!}{\Gamma(\frac{3}{2}+n-k)}x^{2n-2k}$$

where $n \geq 0$. After simplifying $2^{2n}\bar{Y}_{2n}(\sqrt{(1+x)/2})$ with a computer algebra program, and then substituting $\cos\theta$ for x and simplifying manually, we find

$$2^{2n}\bar{Y}_{2n}\Big(\sqrt{\frac{1+x}{2}}\Big) = \cos(n\theta) + \cot\theta\,(\sin(n\theta)) = U_n(\cos\theta).$$

The last step follows from the identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$, with $A = n\theta$ and $B = \theta$, and (1.1). □

THEOREM 5.2. *In general, the polynomials $X_n(x) \pm 1$ and $Y_n(x) \pm 1$ do not have factorisations using the minimal polynomials of* $\cos(2\pi/d)$.

PROOF. The polynomials $X_5(x) \pm 1$ and $Y_5(x) \pm 1$ are irreducible over $\mathbb{Z}$. We can check this by using a computer algebra program.

It suffices to show that none of them is a polynomial $\Psi_d(x)$ where $\Psi_d(x)$ has degree 5. This is because, similarly to $\Psi_d(x)$, their coefficients are integers, the greatest common factor of the coefficients in each polynomial is 1 and the leading coefficients are positive.

The degree of $\Psi_d(x) = \phi(d)/2$ where $d > 2$ from (1.4), and from Definition 1.1, $\phi(d) = 1$ when $d = 1$ or $d = 2$.

From Vaidya [16], $\phi(n) \geq \sqrt{n}$ except for $n = 2$ and $n = 6$. Therefore, the values of d such that $\phi(d) = 10$ are in the interval $3 \leq d \leq 100$. They are $d = 11$ and $d = 22$ by checking an enumeration of ϕ. We have

$$\Psi_{11}(x) = 32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1,$$
$$\Psi_{22}(x) = 32x^5 - 16x^4 - 32x^3 + 12x^2 + 6x - 1.$$

The result follows from these counterexamples. □

REMARK 5.3. The original paper by Vaidya [16] is difficult to find. The result appears in Sándor *et al.* [15, Section 1.1, Equation (1)]. A proof of the lower bound depends on finding the conditions for $(1 - 1/p)p^{\alpha/2} \geq 1$ where p is a prime factor of n with multiplicity α.

Another lower bound is $\phi(n) > n^{2/3}$ when $n > 30$ in Kendall and Osborn [10]. This bound also appears in Sándor *et al.* [15, Section 1.1, Equation (2)]. It would have reduced the upper bound of the interval from 100 to 31. However, it is not generally correct: $\phi(42) = 12$ and $42^{2/3} > 12$.

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D. A. WOLFRAM, College of Engineering and Computer Science,
The Australian National University, Canberra, ACT 0200, Australia
e-mail: David.Wolfram@anu.edu.au