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Abstract

We prove a constructive existence theorem for abelian envelopes of non-abelian monoidal categories. This establishes a new tool for the construction of tensor categories. As an example we obtain new proofs for the existence of several universal tensor categories as conjectured by Deligne. Another example constructs interesting tensor categories in positive characteristic via tilting modules for SL_2.

Introduction

Fix a field k. A k-linear symmetric rigid monoidal Karoubian category in which the endomorphisms of the tensor identity $\mathbb{1}$ constitute k will be called a 'pseudo-tensor category'. When the category is abelian, it is called a 'tensor category', following [Del90]. The canonical example of the latter is the category of algebraic representations of an affine group scheme over k. It is often easy to construct specific examples of *pseudo*-tensor categories, for instance diagrammatically or via generators and relations. On the other hand, constructing tensor categories with certain requested properties is typically more challenging. In many recent constructions of important new tensor categories (see [BE19, CEH19, CO14, Del07, EHS20]) the desired tensor categories happen to be 'abelian envelopes' of straightforward pseudo-tensor categories. We review these examples below via applications of our main result.

A tensor category is the abelian envelope of a pseudo-tensor subcategory if every faithful monoidal functor from the subcategory to a tensor category lifts to an exact monoidal functor out of the original category. Not every pseudo-tensor category admits an abelian envelope. A classical example is given in [Del07, § 5.8] and we will give an example of a different nature below. A powerful 'recognition theorem' for abelian envelopes was obtained in [EHS20]. However, the construction of abelian envelopes in [BE19, CO14, EHS20] drew from a rich variety of different methods, rather than some standard approach, and moreover at present there is no 'existence theorem' in the literature for abelian envelopes.

The latter is precisely the aim of the current paper. We derive sufficient internal conditions on a pseudo-tensor category for its abelian envelope to exist, along with a unifying construction of the envelope. We apply this to recover old and construct new abelian envelopes. To state our main theorem, we call an object X with dual $X^\vee$ in a pseudo-tensor category $\mathbf{D}$ 'strongly faithful' if the evaluation $X^\vee \otimes X \to \mathbb{1}$ is the coequaliser of the two evaluation morphisms $X^\vee \otimes X \otimes X^\vee \otimes X \rightrightarrows X^\vee \otimes X$. We show that this is equivalent to the property that $X \otimes - : \mathbf{D} \to \mathbf{D}$ reflects all kernels and cokernels in $\mathbf{D}$.

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Theorem A. *If for every morphism f in $\mathbf{D}$ there exists a strongly faithful $X \in \mathbf{D}$ for which $X \otimes f$ is split, then $\mathbf{D}$ admits an abelian envelope $\mathbf{T}$. Moreover, the ind-completion* $\mathrm{Ind}\mathbf{T}$ *is monoidally equivalent to the category* $\mathrm{Sh}\mathbf{D}$ *of all presheaves $\mathbf{D}^{\mathrm{op}} \to \mathsf{Vec}_k$ which send the sequences*

$$D \otimes X^\vee \otimes X \otimes X^\vee \otimes X \to D \otimes X^\vee \otimes X \to D \to 0,$$

for all $D \in \mathbf{D}$ and strongly faithful $X \in \mathbf{D}$, to exact sequences in Vec_k.

A slightly more general version of this is proved in Theorem 4.1.1. In the following sense Theorem A cannot be improved. In Lemma 2.3.4 we provide a category $\mathbf{D}$ where all assumptions are satisfied but with 'strongly faithful' replaced by the weaker 'faithful' in ordinary sense ($X \otimes -$ is faithful) and which does not admit an abelian envelope. We also demonstrate that the recognition theorem from [EHS20] can be derived from Theorem A. In particular, Theorem A gives an explicit construction of the abelian envelope in all cases where one might apply said recognition theorem.

Note that, under the assumptions in Theorem A, one can prove that every non-zero object in $\mathbf{D}$ is strongly faithful; in particular, the definition of $\mathrm{Sh}\mathbf{D}$ can then be adjusted. However, we demonstrate that $\mathrm{Sh}\mathbf{D}$ as defined above is always (without the splitting condition in Theorem A) the category of sheaves with respect to some k-linear Grothendieck topology on $\mathbf{D}$. This shows that $\mathrm{Sh}\mathbf{D}$ is always a symmetric closed monoidal Grothendieck category. We also observe that whenever $\mathrm{Sh}\mathbf{D}$ is the ind-completion of some tensor category, the latter must be the abelian envelope of $\mathbf{D}$. Moreover, we determine an intrinsic criterion for when $\mathrm{Sh}\mathbf{D}$ is the ind-completion of a tensor category.

Simultaneously and independently, Benson, Etingof and Ostrik have obtained related results in [BEO20]. On the one hand, the scope of their paper is more general in the sense that it does not require braidings and includes analogues of envelopes in which the subcategory is not full. On the other hand, [BEO20] is restricted to tensor categories which have enough projective objects, which for instance does not include the ones in [EHS20].

Application I: Deligne's universal monoidal categories

Let k be a field of characteristic 0. In [Del07], Deligne introduced three one-parameter families of universal pseudo-tensor categories $[\mathrm{S}_t, k]$, $[\mathrm{GL}_t, k]$ and $[\mathrm{O}_t, k]$, for $t \in k$, and embedded them into tensor categories. He also formulated conjectures about the universality of the latter. As observed in [CO14, EHS20], the conjectures can be reformulated, via the Tannakian formalism of [Del90], into the existence of abelian envelopes.

These conjectures were proved for $[\mathrm{S}_t, k]$ in [CO14] and for $[\mathrm{GL}_t, k]$ in [EHS20]. In [CO14] the envelope is constructed as the heart of a t-structure on the homotopy category $K^b([\mathrm{S}_t, k])$, and in [EHS20] the envelope of $[\mathrm{GL}_t, k]$ is realised as a limit of truncations of representation categories of general linear supergroups of growing rank.

Since Theorem A applies to $[\mathrm{S}_t, k]$, $[\mathrm{GL}_t, k]$ and $[\mathrm{O}_t, k]$, it gives a new and unifying proof and construction of all the abelian envelopes, so of all corresponding universal tensor categories. Moreover, we do not require $\bar{k} = k$, contrary to [EHS20]. The construction of the abelian envelope of $[\mathrm{O}_t, k]$ is new, although one would expect that the methods from [EHS20] can be extended to this case.

Yet another construction of the abelian envelope of $[\mathrm{GL}_t, \mathbb{C}]$, described in [Har16], realises it inside an ultraproduct $\prod_{\mathcal{U}}[\mathrm{GL}_{t_i}, \bar{\mathbb{F}}_{p_i}]$. However, recognising the tensor category inside the product as the abelian envelope requires the knowledge of the existence of the latter (as proved first in [EHS20]).

Application II: tensor categories in positive characteristic

The structure theory of tensor categories over fields of positive characteristic is in full development; see for instance [BE19, BEO20, Cou20, EG19, EO19, Ost20]. An important tool developed in [Cou20, EO19, Ost20] is the 'Frobenius twist' in arbitrary tensor categories. In [BE19] a family of tensor categories in characteristic 2 was constructed in which this functor is *not* exact. One way to interpret those categories is as the abelian envelopes of the monoidal quotients of the pseudo-tensor category $\mathsf{Tilt}\,\mathrm{SL}_2$ of tilting modules of the reductive group SL_2. We will show that these quotients also admit abelian envelopes when $p > 2$ by application of Theorem A.

These envelopes are also constructed independently in [BEO20], and studied in full detail there. In particular, they provide the first examples of tensor categories for $p > 2$ on which the Frobenius twist is not exact.

Structure of the paper

In § 1 we recall the necessary background. In § 2 we introduce and study the notions of strongly faithful objects and monoidal splitting of morphisms. As an application, we show that the conditions in Theorem A are satisfied for $[\mathrm{GL}_t, k]$ and $[\mathrm{O}_t, k]$. In § 3 we study the category $\mathrm{Sh}\mathbf{D}$. In § 4 we prove Theorem A and apply it to the above examples.

In the appendix we recall the notions of Grothendieck topologies and sheaves on k-linear sites. An alternative approach to the methods in § 3 would be to argue that our set-up allows us to apply a general theory developed in [Sch20] by Schäppi. Since our case is rather specific, it is more transparent to use a direct approach, but in order to highlight this connection we also recall some results from [Sch20] in the appendix.

1. Preliminaries

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$. Throughout the paper we let k denote an arbitrary field, unless further specified.

1.1 Exactness and split morphisms

Let $\mathbf{A}$ be a preadditive category.

1.1.1 We denote by $\Xi = \Xi(\mathbf{A})$ the class of all exact sequences

$$X_2 \xrightarrow{p} X_1 \xrightarrow{q} X_0 \to 0 \tag{1}$$

in $\mathbf{A}$, that is, all sequences (1) where q is the cokernel of p, which is equivalent to

$$0 \to \mathbf{A}(X_0, A) \xrightarrow{-\circ q} \mathbf{A}(X_1, A) \xrightarrow{-\circ p} \mathbf{A}(X_2, A)$$

being exact in Ab for each $A \in \mathbf{A}$.

1.1.2 A morphism $f : X \to Y$ in $\mathbf{A}$ is *split* if there exists $g : Y \to X$ such that $f \circ g \circ f = f$. Note that this implies that $f \circ g$ and $g \circ f$ are idempotents. If $\mathbf{A}$ is Karoubi (idempotent complete) it thus follows that f is split if and only if we have $X \simeq A \oplus X_0$ and $Y \simeq A \oplus Y_0$ and f is the composition of these isomorphisms with $(\mathrm{id}_A, 0)$.

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1.2 Symmetric monoidal categories

Let K be a commutative ring.

1.2.1 By a K-linear symmetric monoidal (K-LSM or LSM) category $(\mathbf{C}, \otimes, \mathbb{1}, \sigma)$, we mean a monoidal category $(\mathbf{C}, \otimes, \mathbb{1})$ with a symmetric braiding σ with a fixed K-linear structure on $\mathbf{C}$ for which $- \otimes -$ is K-linear in each variable. As is customary, we suppress the associativity constraints and unitors from all notation. Correspondingly we do not place brackets in iterated tensor products. Furthermore, in order to keep long expressions legible, the functor $X \otimes -$, for $X \in \mathbf{C}$, will sometimes be shortened to $X-$. So we might write XY or Xf for an object Y or morphism f in $\mathbf{C}$.

1.2.2 An LSM functor between two K-LSM categories is a K-LSM functor. Usually we will denote the LSM functor simply by the underlying functor. For two K-LSM categories $(\mathbf{C}, \otimes, \mathbb{1}, \sigma)$ and $(\mathbf{C}', \otimes', \mathbb{1}', \sigma')$, we denote by $\mathsf{LSM}(\mathbf{C}, \mathbf{C}')$ the category of LSM functors $\mathbf{C} \to \mathbf{C}'$ and use $\mathsf{LSM}^{\text{faith}}$ for the subcategory of faithful functors.

1.2.3 For a K-LSM category $(\mathbf{C}, \otimes, \mathbb{1}, \sigma)$ and $X \in \mathbf{C}$, a dual of X is a triple $(X^\vee, \mathrm{ev}_X, \mathrm{co}_X)$ of an object $X^\vee \in \mathbf{C}$ and morphisms $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$ and $\mathrm{co}_X : \mathbb{1} \to X \otimes X^\vee$, such that

$$\mathrm{id}_X = (X \otimes \mathrm{ev}_X) \circ (\mathrm{co}_X \otimes X) \quad \text{and} \quad \mathrm{id}_{X^\vee} = (\mathrm{ev}_X \otimes X^\vee) \circ (X^\vee \otimes \mathrm{co}_X). \tag{2}$$

An object which admits a dual is called *rigid*. If every object in $\mathbf{C}$ admits a dual, then $\mathbf{C}$ is called rigid. The dimension $\dim(X) \in \mathrm{End}(\mathbb{1})$ of a rigid object is given by $\mathrm{ev}_X \circ \sigma_{XX^\vee} \circ \mathrm{co}_X$.

1.2.4 A tensor ideal $\mathcal{J}$ in a K-LSM category $(\mathbf{C}, \otimes, \mathbb{1}, \sigma)$ is an assignment of K-submodules $\mathcal{J}(X, Y) \subset \mathbf{C}(X, Y)$ for each $X, Y \in \mathbf{C}$ such that the corresponding class of morphisms is closed under composing or taking the tensor product with any morphism in $\mathbf{C}$. For a tensor ideal $\mathcal{J}$, the quotient category $\mathbf{C}/\mathcal{J}$ has by definition the same objects as $\mathbf{C}$ and as morphism sets the quotient K-modules $\mathbf{C}(X, Y)/\mathcal{J}(X, Y)$. By construction, $\mathbf{C}/\mathcal{J}$ is again K-LSM, such that $\mathbf{C} \to \mathbf{C}/\mathcal{J}$ is a LSM functor. We can therefore alternatively define tensor ideals as the kernels of LSM functors.

1.3 Pseudo-tensor categories

Let k be an arbitrary field.

1.3.1 A k-LSM category $(\mathbf{D}, \otimes, \mathbb{1}, \sigma)$ is a *pseudo-tensor category over* k if:

(i) $\mathbf{D}$ is essentially small;
(ii) $k \to \mathrm{End}(\mathbb{1})$ is an isomorphism;
(iii) $(\mathbf{D}, \otimes, \mathbb{1}, \sigma)$ is rigid;
(iv) $\mathbf{D}$ is pseudo-abelian (additive and Karoubi).

A pseudo-tensor subcategory of such $\mathbf{D}$ is a full monoidal subcategory closed under taking duals, direct sums and summands. It is thus again a pseudo-tensor category. The quotient of a pseudo-tensor category with respect to a non-trivial tensor ideal is again pseudo-tensor.

Occasionally we will use categories as above except that the field k is replaced by some commutative ring R. We will use the same terminology 'tensor category over R'.

If only (i)–(iii) are satisfied, we can take the pseudo-abelian envelope (see [AK02, § 1.2]) by formally adjoining direct sums and summands, to obtain a pseudo-tensor category.

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Remark 1.3.2. Let $\mathbf{D}$ be a pseudo-tensor category and $\xi \in \Xi(\mathbf{D})$. For any $A \in \mathbf{D}$, the sequence $A \otimes \xi$ is still exact, so $A \otimes \xi \in \Xi$, since $A \otimes -$ has a right adjoint $A^\vee \otimes -$.

1.3.3 Following [Del90, Del02], a *tensor category over* k is a pseudo-tensor category which is abelian (i.e. assumption 1.3.1(iv) is strengthened). In such a category, $\mathbb{1}$ is automatically a simple object. A *tensor functor* between tensor categories is an exact LSM functor. Categories of tensor functors will be denoted by Tens. Following [CEH19, EHS20], we use the following terminology.

Definition 1.3.4. For a pseudo-tensor category $\mathbf{D}$ over k, a pair $(F, \mathbf{T})$ of a tensor category $\mathbf{T}$ over k and a faithful LSM functor $F : \mathbf{D} \to \mathbf{T}$ constitutes an *abelian envelope* of $\mathbf{D}$ if, for each tensor category $\mathbf{T}_1/k$, composition with F induces an equivalence

$$\mathsf{Tens}(\mathbf{T}, \mathbf{T}_1) \;\simeq\; \mathsf{LSM}^{\mathrm{faith}}(\mathbf{D}, \mathbf{T}_1).$$

We will indulge in the usual abuse of terminology, by referring to the tensor category $\mathbf{T}$ of a pair $(F, \mathbf{T})$ as in Definition 1.3.4 as '*the* abelian envelope of $\mathbf{D}$'. The use of the definite article is justified by obvious uniqueness up to equivalence.

Remark 1.3.5. By [Del90, Corollaire 2.10(ii)], functors in $\mathsf{Tens}(\mathbf{T}, \mathbf{T}_1)$ are automatically faithful, so composition with F in Definition 1.3.4, automatically lands in $\mathsf{LSM}^{\mathrm{faith}}(\mathbf{D}, \mathbf{T}_1)$. Furthermore, [Del90, Corollaire 2.10(i)] shows that right exact functors in $\mathsf{LSM}(\mathbf{T}, \mathbf{T}_1)$ are automatically in $\mathsf{Tens}(\mathbf{T}, \mathbf{T}_1)$.

1.3.6 For a tensor category $\mathbf{T}$, the ind-completion $\mathrm{Ind}\mathbf{T}$ has an essentially unique LSM structure such that $- \otimes -$ is exact and cocontinuous and $\mathbf{T}$ is a monoidal subcategory; see [Del90, § 7]. Since $\mathbf{T}$ is assumed to be essentially small, we can define $\mathrm{Ind}\mathbf{T}$ also as the category of left exact functors $\mathbf{T}^{\mathrm{op}} \to \mathsf{Vec}$.

Lemma 1.3.7. *The subcategory of rigid objects in* $\mathrm{Ind}\mathbf{T}$ *is equivalent to* $\mathbf{T}$.

Proof. We need to show that any rigid object $N \in \mathrm{Ind}\mathbf{T}$ is compact. Take therefore a filtered colimit $\varinjlim Y_\alpha$ in $\mathrm{Ind}\mathbf{T}$. As the tensor product in $\mathrm{Ind}\mathbf{T}$ is cocontinuous, we find

$$\mathrm{Ind}\mathbf{T}(N, \varinjlim Y_\alpha) \;\simeq\; \mathrm{Ind}\mathbf{T}(\mathbb{1}, \varinjlim (N^\vee \otimes Y_\alpha)).$$

Since $\mathbb{1} \in \mathbf{T} \subset \mathrm{Ind}\mathbf{T}$ is compact, we thus find indeed that $\mathrm{Ind}\mathbf{T}(N, -)$ commutes with filtered colimits. □

The following lemma is straightforward, but it will be useful to have it spelled out.

Lemma 1.3.8. *Consider a pseudo-tensor category* $\mathbf{D}$ *with pseudo-tensor subcategory* $\mathbf{D}_0 \subset \mathbf{D}$, *and take* $X_0 \in \mathbf{D}_0$. *The full subcategory* $\mathbf{D}_1$ *of objects* $V \in \mathbf{D}$ *for which* $V \otimes X_0 \in \mathbf{D}_0$ *is a pseudo-tensor subcategory of* $\mathbf{D}$.

Proof. That $\mathbf{D}_1$ is closed under taking direct sums and summands follows from the corresponding property of $\mathbf{D}_0$. Clearly $\mathbb{1} \in \mathbf{D}_1$. Now if $V, W \in \mathbf{D}_1$, then by definition

$$V \otimes X_0 \otimes W \otimes X_0 \in \mathbf{D}_0 \;\Rightarrow\; V \otimes W \otimes (X_0 \otimes X_0^\vee \otimes X_0) \in \mathbf{D}_0.$$

Since X_0 is a direct summand of $X_0 \otimes X_0^\vee \otimes X_0$, it follows that $V \otimes W \otimes X_0$ is a direct summand of an object in $\mathbf{D}_0$ and hence also in $\mathbf{D}_0$. In conclusion, $V \otimes W \in \mathbf{D}_1$. That $\mathbf{D}_1$ is closed under taking duals follows similarly. □

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1.3.9 Consider a pseudo-tensor category $\mathbf{D}$ over k and a field extension K/k. The naive extension of scalars of $\mathbf{D}$ (see [AK02, 5.1.1]) is the K-linear category with the same objects as $\mathbf{D}$, but with morphism sets given by $K \otimes_k \mathbf{D}(-,-)$. We define $\mathbf{D}_K$ as the Karoubi envelope of the naive extension of scalars. Note that in [AK02, § 5.3], the notation $(\mathbf{D}_K)^\sharp$ is used for what we call $\mathbf{D}_K$. Now $\mathbf{D}_K$ is canonically a pseudo-tensor category over K.

1.4 Deligne's universal monoidal categories

Fix a commutative ring R and $t \in R$.

1.4.1 Following [Del07, § 10], we have the category $[\mathrm{GL}_t, R]_0$, which is the free rigid R-LSM monoidal category on one object V_t of dimension t. Its objects are (up to isomorphism) tensor products of V_t and $V_t^\vee$.

The pseudo-abelian envelope $[\mathrm{GL}_t, R]$ of $[\mathrm{GL}_t, R]_0$ is thus a pseudo-tensor category over R. By construction, every object X in $[\mathrm{GL}_t, R]$ is a direct summand of a direct sum of objects $\otimes^i V_t \otimes \otimes^j V_t^\vee$. We denote by $\deg X$ the minimal $d \in \mathbb{N}$ such that X is a direct summand of a direct sum of $\otimes^a V_t \otimes \otimes^b V_t^\vee$ with $a + b \leq d$. By [Del07, Théorème 10.5], $[\mathrm{GL}_t, k]$ is a semisimple tensor category when $\mathrm{char}(k) = 0$ and $t \notin \mathbb{Z}$.

The following is a reformulation of [Del07, Proposition 10.3].

Lemma 1.4.2. *Consider a pseudo-tensor category $\mathbf{D}$ over R. Evaluation at V_t yields an equivalence between $\mathsf{LSM}([\mathrm{GL}_t, R], \mathbf{D})$ and the groupoid of objects of dimension t in $\mathbf{D}$ with their isomorphisms.*

1.4.3 Set $\mathbf{D} := [\mathrm{GL}_t, k]$ for an algebraically closed field k of characteristic 0, for $t \in \mathbb{Z} \subset k$. Consider the tensor category svec of finite-dimensional super vector spaces; see [Del90, § 1.4]. Let $\mathrm{GL}(m|n)$ be the affine group scheme in svec of automorphisms of the super space $k^{m|n}$ of even dimension m and odd dimension n. As in [Del02, § 0.3], we have the tensor category $\mathsf{Rep}_k \mathrm{GL}(m|n)$ of its representations in svec which restrict to the canonical $\mathbb{Z}/2$-action along the homomorphism $\mathbb{Z}/2 \to \mathrm{GL}(m|n)$ defining the grading on $\mathrm{GL}(m|n)$. As an application of Lemma 1.4.2, there exists an LSM functor

$$H_{m|n} : \mathbf{D} \to \mathsf{Rep}_k \mathrm{GL}(m|n), \quad V_t \mapsto k^{m|n},$$

for every $m, n \in \mathbb{N}$ with $m - n = t$.

Lemma 1.4.4. *Retain the notation of § 1.4.3.*

(i) *The functor $H_{m|n}$ is full.*
(ii) *For $X, Y \in \mathbf{D}$ with $\deg X + \deg Y < 2(m+1)(n+1)$, $H_{m|n}$ induces an isomorphism*

$$\mathbf{D}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}(m|n)}(H_{m|n}(X), H_{m|n}(Y)).$$

(iii) *There exists an indecomposable object Q in $\mathbf{D}$ with $\deg Q = mn$, such that $H_{m|n}(Q)$ is projective in $\mathsf{Rep}_k \mathrm{GL}(m|n)$.*

Proof. These statements are well known; see, for example, [Hei17, Ser84]. The precise statements can also be found in [Cou18, Theorem 7.2.1], the paragraph above [Cou18, Corollary 7.2.2], and [Cou18, Proposition 8.2.3(i)]. □

1.4.5 A rigid object X in a LSM category is *symmetrically self-dual* if we can take $X^\vee = X$ and we have $\mathrm{ev}_X = \mathrm{ev}_X \circ \sigma_{X,X}$. Following [Del07, § 9], we have the category $[\mathrm{O}_t, R]_0$, which is

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the free R-LSM category on one symmetrically self-dual object U_t of dimension t. Its objects are tensor powers of U_t.

The pseudo-abelian envelope $[\mathrm{O}_t, R]$ of $[\mathrm{O}_t, R]_0$ is thus a pseudo-tensor category over R. By construction, every object X in $[\mathrm{O}_t, R]$ is a direct summand of a direct sum of objects $\otimes^i U_t$. We denote by $\deg X$ the minimal $d \in \mathbb{N}$ such that X is a direct summand of a direct sum of $\otimes^i U_t$ with $i \leq d$. By [Del07, Théorème 9.7], $[\mathrm{O}_t, k]$ is a semisimple tensor category when $\mathrm{char}(k) = 0$ and $t \notin \mathbb{Z}$.

Lemma 1.4.6 [Del07, Proposition 9.4]. *Consider a pseudo-tensor category* $\mathbf{D}$ *over* R. *Evaluation at* U_t *yields an equivalence between* $\mathsf{LSM}([\mathrm{O}_t, R], \mathbf{D})$ *and the groupoid of symmetrically self-dual objects of dimension* t *in* $\mathbf{D}$.

1.4.7 Set $\mathbf{D} := [\mathrm{O}_t, k]$ for an algebraically closed field k of characteristic 0, for $t \in \mathbb{Z} \subset k$. Consider a non-degenerate (super)symmetric bilinear form on $k^{m|2n} \in \mathsf{svec}$ and let $\mathrm{OSp}(m|2n)$ be the closed subgroup of $\mathrm{GL}(m|2n)$ which preserves the form. As an application of Lemma 1.4.6, there exists an LSM functor

$$F_{m|2n} : \mathbf{D} \to \mathsf{Rep}_k \mathrm{OSp}(m|2n), \quad U_t \mapsto k^{m|2n}$$

for every $m, n \in \mathbb{N}$ with $m - 2n = t$.

Lemma 1.4.8. *Retain the notation of § 1.4.7.*

(i) *The functor* $F_{m|2n}$ *is full.*
(ii) *For* $X, Y \in \mathbf{D}$ *with* $\deg X + \deg Y < 2(m+1)(n+1)$, $F_{m|2n}$ *induces an isomorphism*

$$\mathbf{D}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{OSp}(m|2n)}(F_{m|2n}(X), F_{m|2n}(Y)).$$

(iii) *There exists an indecomposable object* Q *in* $\mathbf{D}$ *with* $\deg Q = mn$, *such that* $F_{m|2n}(Q)$ *is projective in* $\mathsf{Rep}_k \mathrm{OSp}(m|2n)$.

Proof. Claim (i) is [LZ17, Theorem 5.3]. Claim (ii) is [Cou18, 7.1.1(ii) and 8.1.3(i)] or follows from [Zha18, Theorem 5.12]. If $m \leq 1$ or $n = 0$, then $\mathsf{Rep}\mathrm{OSp}(m|2n)$ is semisimple, so claim (iii) becomes trivial. The case $m > 1$ and $n > 0$ follows from the observation in [CH17] that the objects in $\mathbf{D}$ sent to projective objects under $F_{m|2n}$ are the same ones which are sent to zero by $F_{m-2|2n-2}$, and the description of that kernel as in [Cou18, Theorem 7.1.1]. □

2. Monoidal splitting and faithfulness

We fix a field k and a pseudo-tensor category $(\mathbf{D}, \otimes, \mathbb{1}, \sigma)$ over k.

2.1 Splitting of morphisms

Definition 2.1.1. An object $X \in \mathbf{D}$ *splits a morphism* $f : A \to B$ in $\mathbf{D}$ if $X \otimes f$ is split. The category $\mathbf{D}$ is *self-splitting* if, for every morphism h in $\mathbf{D}$, there exists a non-zero object which splits h.

For an object $X \in \mathbf{D}$, we will encounter the morphism

$$\mathcal{E}_X := X^\vee \otimes X \otimes \mathrm{ev}_X - \mathrm{ev}_X \otimes X^\vee \otimes X : \;\; X^\vee \otimes X \otimes X^\vee \otimes X \to X^\vee \otimes X$$

several times, hence we give it a name.

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Lemma 2.1.2.

(i) *The morphisms* ev_X *and* co_X *are split by* X *and by* $X^\vee$.
(ii) *The morphism* $\mathcal{E}_X$ *is split by* $X \otimes X^\vee$.

Proof. It follows from (2) that $f := X \otimes \mathrm{ev}_X$ is split, with $g := \mathrm{co}_X \otimes X$, proving part (i). It follows similarly that $f := X \otimes \mathcal{E}_X \otimes X^\vee$ is split, with

$$g := X \otimes X^\vee \otimes X \otimes X^\vee \otimes \mathrm{co}_X - \mathrm{co}_X \otimes X \otimes \mathrm{ev}_X \otimes X^\vee \otimes \mathrm{co}_X,$$

which proves part (ii). □

The following lemma is well known.

Lemma 2.1.3. *Assume* $\mathbf{D}$ *is a tensor category and take* $X \in \mathbf{D}$. *The following assertions are equivalent.*

(i) X *is projective.*
(ii) X *is injective.*
(iii) $X \otimes f$ *is split for every morphism* f *in* $\mathbf{D}$.

Proof. First we show that (i) implies (iii). For a morphism $f : M \to N$ we denote the image and cokernel by A and B. By adjunction, $X \otimes D$ is projective, for every $D \in \mathbf{D}$. Consequently $X \otimes M \twoheadrightarrow X \otimes A$ and $X \otimes N \twoheadrightarrow X \otimes B$ split, from which it follows that $X \otimes f$ is split. That (ii) implies (iii) is proved similarly.

Now if (iii) is satisfied, then it follows by adjunction that $X^\vee$ is both projective and injective. Also by adjunction, the fact that $X^\vee$ is projective (respectively, injective) implies that X is injective (respectively, projective). Hence (iii) implies (i) and (ii). □

2.2 Faithfulness of objects

Definition 2.2.1. An object $X \in \mathbf{D}$ is *faithful* if one of the following two equivalent conditions is satisfied.

(i) The functor $X \otimes - : \mathbf{D} \to \mathbf{D}$ is faithful.
(ii) The evaluation $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$ is an epimorphism in $\mathbf{D}$.

Definition 2.2.2. An object $X \in \mathbf{D}$ is *strongly faithful* if one of the following two equivalent conditions is satisfied.

(i) For every $M, N \in \mathbf{D}$, the sequence

$$0 \to \mathbf{D}(M,N) \xrightarrow{X\otimes -} \mathbf{D}(XM,XN) \xrightarrow{(X\otimes -)-(s\otimes N)(X\otimes -)(s\otimes M)} \mathbf{D}(XXM,XXN),$$

with $s = \sigma_{XX}$, is exact in Vec.
(ii) The sequence

$$\gamma_X : \quad X^\vee \otimes X \otimes X^\vee \otimes X \xrightarrow{\mathcal{E}_X} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1} \to 0$$

is exact in $\mathbf{D}$, meaning $\gamma_X \in \Xi(\mathbf{D})$.
(iii) The evaluation $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$ is a strict epimorphism in $\mathbf{D}$.

By definition, (ii) implies (iii). We briefly explain why (iii) implies (ii). Assume that ev_X is a strict epimorphism and consider $f : X^\vee \otimes X \to A$ in $\mathbf{D}$ with $f \otimes \mathrm{ev}_X = \mathrm{ev}_X \otimes f$ (i.e. $f \circ \mathcal{E}_X = 0$).

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Now take an arbitrary $g : Y \to X^\vee \otimes X$ with $\mathrm{ev}_X \circ g = 0$. It follows that

$$(f \circ g) \otimes \mathrm{ev}_X = (\mathrm{ev}_X \circ g) \otimes f = 0.$$

Since ev_X is an epimorphism, it follows that $f \circ g = 0$. Since g was arbitrary with $\mathrm{ev}_X \circ g = 0$, strictness of the epimorphism ev_X implies that f factors through ev_X.

Clearly X is (strongly) faithful if and only if $X^\vee$ is (strongly) faithful.

Example 2.2.3.

- (i) The unit $\mathbb{1}$ is strongly faithful in any pseudo-tensor category $\mathbf{D}$.
- (ii) The objects V_t and U_t in $[\mathrm{GL}_t, k]$ and $[\mathrm{O}_t, k]$ are strongly faithful. This follows easily from the diagrammatic calculus.

We say that $X \in \mathbf{D}$ *reflects cokernels* when every sequence γ as in (1) is exact if and only if $X \otimes \gamma$ is exact. Note that one direction of the condition is automatic by Remark 1.3.2. Reflecting kernels is defined similarly. Remark 1.3.2 also shows that $X \otimes Y$ reflects cokernels if and only if both X and Y reflect cokernels, a fact that we will use freely.

Lemma 2.2.4. *For any $X \in \mathbf{D}$, the sequence $X^\vee \otimes X \otimes \gamma_X$ is split exact.*

Proof. By Lemma 2.1.2, the sequence $\xi_X := X^\vee \otimes X \otimes \gamma_X$ is split. Set $t = \dim X$. Then ξ_X is the image of ξ_{V_t} under the LSM functor $[\mathrm{GL}_t, k] \to \mathbf{D}$ corresponding to $V_t \mapsto X$ in Lemma 1.4.2. Since ξ_{V_t} is split exact, by Example 2.2.3(ii), and LSM functors are additive, ξ_X is also split exact. □

Proposition 2.2.5. *The following are equivalent for $X \in \mathbf{D}$.*

- (i) *X is strongly faithful.*
- (ii) *$X \otimes X^\vee$ reflects cokernels.*
- (iii) *X reflects both kernels and cokernels.*

Proof. Assume first that X is strongly faithful and consider a sequence $X_2 \to X_1 \to X_0$ in $\mathbf{D}$. Tensoring with γ_X yields a commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
X^\vee X X_2 & \longrightarrow & X^\vee X X_1 & \longrightarrow & X^\vee X X_0 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
X^\vee X X^\vee X X_2 & \longrightarrow & X^\vee X X^\vee X X_1 & \longrightarrow & X^\vee X X^\vee X X_0 & \longrightarrow & 0
\end{array}$$

with exact columns. If the second row is exact, then so is the third, by Remark 1.3.2. It then follows from elementary diagram chasing that the first row is also exact. Hence $X^\vee \otimes X$ reflects cokernels.

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Now assume that $X^\vee \otimes X$ reflects cokernels. By Lemma 2.2.4, application of the functor $X^\vee \otimes X \otimes -$ to the sequence γ_X yields an exact sequence. Hence γ_X is also exact and X is strongly faithful by definition. This already shows that (i) and (ii) are equivalent.

Claim (ii) is equivalent to the claim that both X and $X^\vee$ reflect cokernels. By adjunction, $X^\vee$ reflects cokernels if and only if X reflects kernels. Hence (ii) and (iii) are equivalent. □

PROPOSITION 2.2.6. *Let X, Y be objects in $\mathbf{D}$.*

(i) *X and Y are strongly faithful if and only if $X \otimes Y$ is strongly faithful.*
(ii) *If $\dim X \neq 0$, then X is strongly faithful.*
(iii) *If $\mathbf{D}$ is a tensor category and $X \neq 0$, then X is strongly faithful.*

Proof. Part (i) follows from Proposition 2.2.5.

If $d := \dim X$ is invertible, then consider the morphisms

$$f := \frac{1}{d}\sigma_{X,X^\vee} \circ \mathrm{co}_X : \mathbb{1} \to X^\vee \otimes X$$

and

$$((f \otimes f) \circ \mathrm{ev}_X - X^\vee \otimes X \otimes f) : X^\vee \otimes X \to X^\vee \otimes X \otimes X^\vee \otimes X.$$

It follows from direct computation that these ensure the sequence in Definition 2.2.2(ii) is split exact. This proves part (ii).

Part (iii) follows from Proposition 2.2.5, since all non-zero objects in tensor categories reflect cokernels. □

We can also prove Proposition 2.2.6(i) directly from Definition 2.2.2, using a 'monoidal analogue' of the diagram in [SGA3, IV.1.7]. The following corollary is a direct consequence of Proposition 2.2.6(iii).

COROLLARY 2.2.7. *If $\mathbf{D}$ admits a fully faithful LSM functor into a tensor category, every non-zero object in $\mathbf{D}$ is strongly faithful.*

LEMMA 2.2.8. *Consider a field extension K/k.*

(i) *If $X \in \mathbf{D}$ is strongly faithful in $\mathbf{D}_K$, it is also strongly faithful in $\mathbf{D}$.*
(ii) *If $f \in \mathbf{D}(X, Y)$ interpreted in $\mathbf{D}_K$ is split, then f is also split in $\mathbf{D}$.*

Proof. Part (i) follows from applying either version of Definition 2.2.2 and using the fact that the functor $K \otimes_k -$ from Vec_k to Vec_K is faithful and exact.

For part (ii), by assumption, there exists $g \in K \otimes_k \mathbf{D}(Y, X)$ with $f \circ g \circ f = f$. We fix a complement V in K of the canonical k-subspace $k \subset K$. We have $g = g_0 + g_1$ with $g_0 \in \mathbf{D}(Y, X)$ and $g_1 \in V \otimes_k \mathbf{D}(Y, X)$. It follows immediately that $f \circ g_0 \circ f = f$. □

LEMMA 2.2.9. *Consider $X, Y, Z \in \mathbf{D}$ such that Y is a direct summand of $X \otimes Z$. Then the sequence $\mathbf{D}(\gamma_X, Y)$ is exact in Vec.*

Proof. Since X is a direct summand of $X \otimes X^\vee \otimes X$, it follows that Y is also a direct summand of $X^\vee \otimes X \otimes Z'$, for $Z' := X \otimes Z$. By functoriality and adjunction, it therefore suffices to prove that

$$\mathbf{D}(X^\vee \otimes X \otimes \gamma_X, Z')$$

is exact. The latter is a consequence of Lemma 2.2.4. □

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2.3 Examples

Theorem 2.3.1. *If* $\mathrm{char}(k)=0$ *and* $t\in\mathbb{Z}$, *the categories* $[\mathrm{GL}_t,k]$ *and* $[\mathrm{O}_t,k]$ *are self-splitting and every non-zero object is strongly faithful.*

Proof. By Lemma 2.2.8 and the fact that $[\mathrm{GL}_t,k]_K\simeq[\mathrm{GL}_t,K]$ and $[\mathrm{O}_t,k]_K\simeq[\mathrm{O}_t,K]$ for any field extension K/k, it is sufficient to prove the theorem for algebraically closed k.

Set $\mathbf{D}:=[\mathrm{O}_t,k]$. We start by proving the claim about strong faithfulness. It follows immediately from Definition 2.2.1(ii) and the diagrammatic calculus that all objects in $\mathbf{D}$ are faithful. For $0\neq X\in\mathbf{D}$ we need to demonstrate that for a given morphism $f:X^\vee\otimes X\to Y$ in $\mathbf{D}$ with $f\circ\mathcal{E}_X=0$, there exists $g:\mathbb{1}\to Y$ such that $f=g\circ\mathrm{ev}_X$. Set $a=\deg X$ and $b=\deg Y$. Take $m,n\in\mathbb{N}$ with $m-2n=t$ and $2a+b<2(m+1)(n+1)$ and consider the LSM functor

$$F_{m|2n}:\mathbf{D}\to\mathsf{Rep}_k\mathrm{OSp}(m|2n)$$

from § 1.4.7. By Proposition 2.2.6(iii) and the fullness of $F_{m|2n}$ in Lemma 1.4.8(i), there exists a morphism $g:\mathbb{1}\to Y$ in $\mathbf{D}$ such that $F_{m|2n}(f)=F_{m|2n}(g\circ\mathrm{ev}_X)$. By Lemma 1.4.8(ii), this implies that $f=g\circ\mathrm{ev}_X$, as desired.

Now consider an arbitrary morphism $f:A\to B$ in $\mathbf{D}$ and set $a=\deg A$ and $b=\deg B$. Take $m,n\in\mathbb{N}$ with $m-2n\in t$ and $a+b\leq m+n$. The latter inequality implies

$$a+b+2mn<2(m+1)(n+1). \tag{3}$$

We consider again the functor $F_{m|2n}$. By Lemma 1.4.8(iii), there exists Q in $\mathbf{D}$, with $\deg Q=mn$, such that $F_{m|2n}(Q)$ is projective. By Lemma 2.1.3,

$$f':=F_{m|2n}(Q\otimes f)\ :\ F_{m|2n}(Q\otimes A)\to F_{m|2n}(Q\otimes B)$$

is split. Thus there exists a morphism g' in $\mathsf{Rep}\mathrm{OSp}(m|2n)$ such that $f'g'f'=f'$. By Lemma 1.4.8(ii) and the inequality (3), there thus exists a morphism $Q\otimes B\to Q\otimes A$ which ensures that $Q\otimes f$ is split. Hence $\mathbf{D}$ is self-splitting.

The claims for $[\mathrm{GL}_t,k]$ are similarly proved using Lemma 1.4.4. □

Remark 2.3.2. That non-zero objects in $[\mathrm{GL}_t,k]$ are strongly faithful when $\mathrm{char}(k)=0$, also follows from Corollary 2.2.7 and [Del07, Proposition 10.17], which states that $[\mathrm{GL}_t,k]$ admits a fully faithful LSM functor into a tensor category.

2.3.3 For a commutative k-algebra K, consider the subcategory $\mathbf{C}$ of $[\mathrm{GL}_0,K]_0$ which has the same objects and for which the inclusion functor $\mathbf{C}\to[\mathrm{GL}_0,K]_0$ is full on each morphism set, except that on $\mathbf{C}(\mathbb{1},\mathbb{1})$ it realises the unit morphism $k\to K$. That $\mathbf{C}$ constitutes a (monoidal) subcategory of $[\mathrm{GL}_0,K]_0$ follows from the fact that the collection of all morphisms in $[\mathrm{GL}_0,K]_0$ excluding the ones $\mathbb{1}\to\mathbb{1}$ form a (tensor) ideal.

Now the pseudo-abelian envelope $\mathbf{D}$ of $\mathbf{C}$ is a pseudo-tensor category over k.

Lemma 2.3.4.

- (i) *The object* V_0 *in* $\mathbf{D}$ *is faithful but not strongly faithful, unless* $K=k$.
- (ii) *If* $\mathrm{char}(k)=0$ *and* K/k *is a field extension then* $\mathbf{D}$ *is self-splitting and every non-zero object is faithful.*

Proof. Part (ii) can be derived from Theorem 2.3.1.

For part (i), we consider the sequence in Definition 2.2.2(i) for $M = N = \mathbb{1}$ and $X = V_0$, which yields

$$0 \to k \to K \to K\mathrm{S}_2,$$

where S_2 is the symmetric group on two symbols, the morphism $k \to K$ is the unit morphism and the morphism $K \to K\mathrm{S}_2$ is zero. This follows either by direct computation or from the fact that V_0 is strongly faithful in $[\mathrm{GL}_0, K]$, which shows that the kernel of the morphism $K \to K\mathrm{S}_2$ is K, the endomorphism ring of $\mathbb{1}$ in $[\mathrm{GL}_0, K]$. □

3. A closed monoidal Grothendieck category

Fix an arbitrary pseudo-tensor category $\mathbf{D}$ over a field k.

3.1 The category of sheaves

3.1.1 We consider the k-linear presheaf category $\mathrm{PSh}\mathbf{D}$ of k-linear functors $\mathbf{D}^{\mathrm{op}} \to \mathsf{Vec}_k$. Then $\mathrm{PSh}\mathbf{D}$ is symmetric closed monoidal for the Day convolution $\star$; see, for example, [Sch20, § 3.2]. The tensor product of two presheaves F, G is given by the co-end expression

$$F \star G := \int^{X,Y\in\mathbf{D}} F(X) \otimes_k G(Y) \otimes_k \mathbf{D}(-, X \otimes Y),$$

and the internal Hom is given by

$$[G, H] = \int_{Y\in\mathbf{D}} \mathrm{Hom}_k(G(Y), H(Y \otimes -)).$$

The Yoneda embedding $Y : \mathbf{D} \to \mathrm{PSh}\mathbf{D}$ is canonically LSM. By construction, the tensor product $- \star -$ is cocontinuous in each variable.

3.1.2 We define the full subcategory $\mathrm{Sh}\mathbf{D}$ of $F \in \mathrm{PSh}\mathbf{D}$ for which $F(D \otimes \gamma_X)$ is exact in Vec, for every exact sequence

$$D \otimes \gamma_X : \quad DX^\vee XX^\vee X \to DX^\vee X \to D \to 0, \tag{4}$$

with X strongly faithful and D arbitrary in $\mathbf{D}$. Since 'limits commute', it follows that the inclusion functor from $\mathrm{Sh}\mathbf{D}$ to $\mathrm{PSh}\mathbf{D}$ is continuous and hence (by Freyd's special adjoint functor theorem) admits a left adjoint

$$S \,:\, \mathrm{PSh}\mathbf{D} \to \mathrm{Sh}\mathbf{D}, \tag{5}$$

the sheafification or reflection. The restriction of S to $\mathrm{Sh}\mathbf{D}$ is the identity. If $F \in \mathrm{Sh}\mathbf{D}$, then clearly the functor $F(Z \otimes -)$ is also in $\mathrm{Sh}\mathbf{D}$, for each $Z \in \mathbf{D}$. It then follows as a direct application of Day's reflection theorem [Day72, Theorem 1.2(2)] that there is a unique closed symmetric monoidal structure on $\mathrm{Sh}\mathbf{D}$ which makes S LSM. We denote the tensor product on $\mathrm{Sh}\mathbf{D}$ again by $\otimes$. The Yoneda embedding $Y : \mathbf{D} \to \mathrm{PSh}\mathbf{D}$ factors through the embedding of the subcategory $\mathrm{Sh}\mathbf{D}$. We will denote the corresponding fully faithful functor by $Y_0 : \mathbf{D} \to \mathrm{Sh}\mathbf{D}$. It is isomorphic to the composite $S \circ Y$, so, in particular, Y_0 is LSM.

3.1.3 We refer to the appendix for the notions of sieve, Grothendieck topology, the category of sheaves with respect to a topology and localisations of Grothendieck categories.

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For each $D \in \mathbf{D}$, denote by $\mathcal{T}(D)$ the set of all sieves $R \subset \mathbf{D}(-, D)$ such that there exists a strongly faithful $X \in \mathbf{D}$ for which $D \otimes \mathrm{ev}_X \in R(DX^\vee X)$. Our notation $\mathrm{Sh}\mathbf{D}$ is justified by the following theorem.

Theorem 3.1.4.

(i) *The assignment $D \mapsto \mathcal{T}(D)$ from § 3.1.3 is a k-linear Grothendieck topology on $\mathbf{D}$ and the subcategory $\mathrm{Sh}\mathbf{D}$ of $\mathrm{PSh}\mathbf{D}$ is precisely the category of $\mathcal{T}$-sheaves $\mathrm{Sh}(\mathbf{D}, \mathcal{T})$.*

(ii) *$\mathrm{Sh}\mathbf{D}$ is a localisation of $\mathrm{PSh}\mathbf{D}$, so, in particular, a Grothendieck category.*

(iii) *Every object in $\mathbf{D}$ is compact in $\mathrm{Sh}\mathbf{D}$ and every object in $\mathrm{Sh}\mathbf{D}$ is a quotient of a (possibly infinite) coproduct of objects in $\mathbf{D}$.*

(iv) *Given a functor $F : \mathsf{J} \to \mathrm{Sh}\mathbf{D}$ out of a filtered category J, its colimit taken in $\mathrm{PSh}\mathbf{D}$ is contained in $\mathrm{Sh}\mathbf{D}$ (and hence equal to the colimit of F in there).*

(v) *The functor $Y_0 : \mathbf{D} \to \mathrm{Sh}\mathbf{D}$ sends $D \otimes \gamma_X$ to an exact sequence in $\mathrm{Sh}\mathbf{D}$, for every $D \in \mathbf{D}$ and strongly faithful $X \in \mathbf{D}$.*

Proof. Part (i) will be proved in § 3.2. We explain how (i) implies (ii) in the usual fashion. As a left adjoint, the reflection S in (5) is cocontinuous. This already implies that $\mathrm{Sh}\mathbf{D}$ is cocomplete. It also follows that the coproduct in $\mathrm{Sh}\mathbf{D}$ over the set of isomorphism classes of objects in $\mathbf{D}$,

$$G := \bigoplus_{X \in \mathrm{Ob}\mathbf{D}/\simeq} X,$$

is a generator of $\mathrm{Sh}\mathbf{D}$, meaning that $\mathrm{Sh}\mathbf{D}(G, -) : \mathrm{Sh}\mathbf{D} \to \mathsf{Vec}$ is faithful. Hence it suffices to show that $\mathrm{Sh}\mathbf{D}$ is abelian and that direct limits of short exact sequences are (left) exact. Both properties follow easily if $S : \mathrm{PSh}\mathbf{D} \to \mathrm{Sh}\mathbf{D}$ is (left) exact, that is, when $\mathrm{Sh}\mathbf{D}$ is a localisation of $\mathrm{PSh}\mathbf{D}$. Hence claim (ii) follows from claim (i) and Theorem A.1.4.

For part (v), we can observe that by definition and the Yoneda lemma,

$$\mathrm{Sh}\mathbf{D}(Y_0(D \otimes \gamma_X), F) = \mathrm{PSh}\mathbf{D}(Y(D \otimes \gamma_X), F) = F(D \otimes \gamma_X),$$

for arbitrary $F \in \mathrm{Sh}\mathbf{D}$. Hence $Y_0(D \otimes \gamma_X)$ is indeed exact.

Part (iv) follows easily from the fact that in Vec, a filtered colimit of short exact sequences is exact.

Finally, we prove part (iii). Objects in $\mathbf{D}$ are compact in $\mathrm{PSh}\mathbf{D}$, so part (iv) implies that $Y_0 : \mathbf{D} \to \mathrm{Sh}\mathbf{D}$ sends every object in $\mathbf{D}$ to a compact object in $\mathrm{Sh}\mathbf{D}$. That every object is a quotient of coproduct of objects in $\mathbf{D}$ follows from the above fact that G is a generator. □

3.2 Proof of Theorem 3.1.4(i)

Here we complete the proof of Theorem 3.1.4. As a heuristic explanation of Theorem 3.1.4(i) we also present a non-enriched but similar site in Analogy 3.2.3.

3.2.1 First we prove that $\mathcal{T}$ from § 3.1.3 constitutes a topology as in Definition A.1.2. Condition (T1) is immediate from Example 2.2.3(i). For condition (T2), consider $A \in \mathbf{D}$, $R \in \mathcal{T}(A)$ and a morphism $f : B \to A$ in $\mathbf{D}$. By definition, there exists a strongly faithful $X \in \mathbf{D}$ such that $A \otimes \mathrm{ev}_X$ is in R. It then follows that $B \otimes \mathrm{ev}_X$ is in $f^{-1}R$, so $f^{-1}R \in \mathcal{T}(B)$.

For condition (T3), consider $S \subset \mathbf{D}(-, A)$ and $R \in \mathcal{T}(A)$ as in (T3). Since there exists $f := A \otimes \mathrm{ev}_X$ in $R(AX^\vee X)$, for some strongly faithful X, there must exist a strongly faithful $Y \in \mathbf{D}$ such that $AX^\vee X\mathrm{ev}_Y$ is in $f^{-1}S(AX^\vee XY^\vee Y)$. The latter just means that $A \otimes \mathrm{ev}_X \otimes \mathrm{ev}_Y$ is in

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$S(AX^\vee XY^\vee Y)$, which means that $A \otimes \mathrm{ev}_{X\otimes Y}$ is also in $S(AY^\vee X^\vee XY)$. It then follows from Proposition 2.2.6(i) that $S \in \mathcal{T}(A)$.

3.2.2 We now prove the equality $\mathrm{Sh}\mathbf{D} = \mathrm{Sh}(\mathbf{D}, \mathcal{T})$. Take an arbitrary presheaf $F \in \mathrm{PSh}\mathbf{D}$. By a 'pair' (D, X) we mean an arbitrary $D \in \mathbf{D}$ and a strongly faithful $X \in \mathbf{D}$. For each pair (D, X), denote by R^D_X the sieve on D generated by the morphism $D \otimes \mathrm{ev}_X$. This is the minimal sieve on D containing $D \otimes \mathrm{ev}_X$, or equivalently the image of

$$\mathbf{D}(-, DX^\vee X) \xrightarrow{(D\otimes \mathrm{ev}_X)\circ -} \mathbf{D}(-, D).$$

Since the representable functors yield a set of generators for $\mathrm{PSh}\mathbf{D}$, we can complete the epimorphism $\mathbf{D}(-, DX^\vee X) \twoheadrightarrow R^D_X$ to give an exact sequence

$$\bigoplus_{g:B\to DX^\vee X, f\circ g=0} \mathbf{D}(-, B) \to \mathbf{D}(-, DX^\vee X) \to R^D_X \to 0$$

in $\mathrm{PSh}\mathbf{D}$ with $f := D \otimes \mathrm{ev}_X$. Consequently, $F(D) \to \mathrm{Nat}(R^D_X, F)$ is an isomorphism if and only if the sequence

$$0 \to F(D) \xrightarrow{F(f)} F(DX^\vee X) \to \prod_{g:B\to DX^\vee X, f\circ g=0} F(B) \tag{6}$$

is exact. On the other hand, by definition, $F \in \mathrm{Sh}\mathbf{D}$ if and only if

$$0 \to F(D) \xrightarrow{F(f)} F(DX^\vee X) \xrightarrow{F(D\mathcal{E}_X)} F(DX^\vee XX^\vee X) \tag{7}$$

is exact for every pair (D, X).

Clearly, if (7) is exact, then so is (6). On the other hand, assume that (6) is exact, for a fixed X but for every $D \in \mathbf{D}$. For any $g : B \to DX^\vee X$ with $f \circ g = 0$, we have the following commutative diagram.

$$\begin{array}{ccc} DX^\vee XX^\vee X & \xrightarrow{D\otimes\mathcal{E}_X} & DX^\vee X \\ {\scriptstyle g\otimes X^\vee X}\uparrow & & \uparrow{\scriptstyle g} \\ BX^\vee X & \xrightarrow[B\otimes \mathrm{ev}_X]{} & B \end{array}$$

Applying F yields the following commutative diagram.

$$\begin{array}{ccc} F(DX^\vee XX^\vee X) & \xleftarrow{F(D\mathcal{E}_X)} & F(DX^\vee X) \\ \downarrow & & \downarrow{\scriptstyle F(g)} \\ F(BX^\vee X) & \hookleftarrow & F(B) \end{array}$$

The fact that the lower horizontal arrow is a monomorphism follows from our assumption that (6) with D replaced by B is exact. Consider $a \in F(DX^\vee X)$ such that $F(D\mathcal{E}_X)(a) = 0$. By commutativity of the diagram (and using the monomorphism) we also find that $F(g)(a) = 0$. Exactness of (6) thus implies that a is the image of $F(f)$ and therefore (7) is exact too.

By the above two paragraphs we find that when $F \in \mathrm{Sh}(\mathbf{D}, \mathcal{T})$, it follows that (6) is exact for all pairs (D, X), so (7) is exact for all pairs and consequently $F \in \mathrm{Sh}\mathbf{D}$. Next, assume that

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$F \in \mathrm{Sh}\mathbf{D}$. Then by the reverse reasoning we find that $F(D) \to \mathrm{Nat}(R_X^D, F)$ is an isomorphism for all pairs (D, X). Now take an arbitrary $R \in \mathcal{T}(D)$. It is of the form $R_X^D \subset R \subset \mathbf{D}(-, D)$ for some strongly faithful X. We have already found that the composite

$$F(X) \to \mathrm{Nat}(R, F) \to \mathrm{Nat}(R_X^D, F)$$

is an isomorphism, so we only need to show that $\mathrm{Nat}(R, F) \to \mathrm{Nat}(R_X^D, F)$ is a monomorphism to be able to conclude that F is a $\mathcal{T}$-sheaf. Take therefore arbitrary $A \in \mathbf{D}$ and $h \in R(A) \subset \mathbf{D}(A, X)$. In $\mathrm{PSh}\mathbf{D}$, we can expand the commutative square of morphisms $AX^\vee X \rightrightarrows D$ to give the following commutative diagram.

$$\begin{array}{ccccc} & & \mathbf{D}(-,A) & \longrightarrow & R \hookrightarrow \mathbf{D}(-,D) \\ & \nearrow & & & \uparrow \\ \mathbf{D}(-,AX^\vee X) & \longrightarrow & \mathbf{D}(-,DX^\vee X) & \twoheadrightarrow & R_X^D \end{array}$$

Evaluation of $\mathrm{Nat}(-, F)$ yields the commutative diagram

$$\begin{array}{ccccc} F(A) & & \xleftarrow{\eta \mapsto \eta_A(h)} & & \mathrm{Nat}(R,F) \\ \downarrow & & & & \downarrow \\ F(AX^\vee X) & \longleftarrow & F(DX^\vee X) & \longleftarrow & \mathrm{Nat}(R_X^D,F) \end{array}$$

where the left vertical arrow is a monomorphism by exactness in (7) for the pair (A, X). A natural transformation $\eta : R \Rightarrow F$ which is sent to zero in $\mathrm{Nat}(R_X^D, F)$ therefore satisfies $\eta_A(h) = 0$ for all $A \in \mathbf{D}$ and $h \in R(A)$, or in other words $\eta = 0$. This proves the required monomorphism and thus concludes the proof of the claim $\mathrm{Sh}\mathbf{D} = \mathrm{Sh}(\mathbf{D}, \mathcal{T})$.

Analogy 3.2.3. Consider a category $\mathbf{B}$ with finite products, with terminal object $*$. By [SGA3, IV.1.3], the morphism $U \to *$ is a 'universal effective epimorphism' if the induced

$$V \times U \times U \rightrightarrows V \times U \to V \tag{8}$$

is a coequaliser for every $V \in \mathbf{B}$. By [SGA3, IV.1.8], if $U \to *$ and $U' \to *$ are universal effective epimorphisms, the same is true for $U \times U' \to *$. Take $C \subset \mathrm{Ob}\mathbf{B}$, containing $*$ and closed under products, such that $U \to *$ is a universal effective epimorphism for every $U \in C$. The corresponding collection of coverings $V \times U \to V$ forms a classical Grothendieck pretopology. The sheaves are the presheaves $F : \mathbf{B}^{\mathrm{op}} \to \mathsf{Set}$ which send the diagrams (8) to equalisers.

Remark 3.2.4. Denote by $\Sigma \subset \Xi(\mathbf{D})$ the class of all exact sequences

$$D \otimes \gamma_X \quad \text{and} \quad D \otimes (X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1} \to 0 \to 0),$$

for arbitrary $D \in \mathbf{D}$ and strongly faithful $X \in \mathbf{D}$. It follows easily, and from similar arguments to those used in § 3.2.2, that Σ constitutes an 'ind-class', as in Definition A.2.1 in the appendix. We can therefore also prove Theorem 3.1.4(i) by applying Propositions A.2.2 and A.2.3.

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3.3 Connection with abelian envelopes

Theorem 3.3.1. *If* Sh$\mathbf{D}$ *is LSM equivalent to the ind-completion of a tensor category* $\mathbf{T}$ *over* k*, then* $\mathbf{T}$ *is the abelian envelope of* $\mathbf{D}$*.*

Proof. If Sh$\mathbf{D}$ is equivalent to the ind-completion of a tensor category, we can define a tensor category $\mathbf{T}$ as the full subcategory of Sh$\mathbf{D}$ of all rigid objects, by Lemma 1.3.7. Since $\mathbf{D}$ is rigid, this means that Y_0 takes values in the subcategory $\mathbf{T}$. This allows us to construct a fully faithful LSM functor $F : \mathbf{D} \to \mathbf{T}$, which admits the following commutative diagram.

$$\begin{array}{ccc} \mathbf{T} & \hookrightarrow & \mathrm{Ind}\mathbf{T} \\ {\scriptstyle F}\uparrow & & \uparrow\sim \\ \mathbf{D} & \xhookrightarrow{Y_0} & \mathrm{Sh}\mathbf{D} \end{array} \tag{9}$$

We introduce the category $\mathsf{LSM}^{\mathrm{rex}}$ of right exact LSM functors, the category $\mathsf{LSM}^{\gamma}(\mathbf{D},-)$ of LSM functors which send every exact sequence (4), for X strongly faithful, to an exact sequence, and the category LSM^{cc} of all cocontinuous LSM functors. For each tensor category $\mathbf{T}_1$, diagram (9) (and Theorem 3.1.4(v)) induces a commutative diagram

$$\begin{array}{ccccc} \mathsf{LSM}^{\mathrm{rex}}(\mathbf{T},\mathbf{T}_1) & \longrightarrow & \mathsf{LSM}^{\mathrm{rex}}(\mathbf{T},\mathrm{Ind}\mathbf{T}_1) & \longleftarrow & \mathsf{LSM}^{cc}(\mathrm{Ind}\mathbf{T},\mathrm{Ind}\mathbf{T}_1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{LSM}^{\gamma}(\mathbf{D},\mathbf{T}_1) & \longrightarrow & \mathsf{LSM}^{\gamma}(\mathbf{D},\mathrm{Ind}\mathbf{T}_1) & \longleftarrow & \mathsf{LSM}^{cc}(\mathrm{Sh}\mathbf{D},\mathrm{Ind}\mathbf{T}_1) \end{array} \tag{10}$$

where each functor is given by composition with an LSM functor. The right vertical arrow is an equivalence, since it is induced from a LSM equivalence. Inverses of the two right horizontal arrows are given by taking left Kan extensions; see [Sch20, Theorem 3.2.4]. Consequently, the middle vertical arrow is also an equivalence. The two left horizontal arrows are equivalences since any LSM functor from a pseudo-tensor category to the ind-completion of a tensor category takes values in rigid objects. We can thus use the equivalence between $\mathbf{T}_1$ and the category of rigid objects in Ind$\mathbf{T}_1$ from Lemma 1.3.7 to construct inverses. Consequently, the left vertical arrow is also an equivalence.

We will now argue that the latter equivalence can be rewritten as the equivalence required by Definition 1.3.4. Firstly, by Remark 1.3.5, we have

$$\mathsf{LSM}^{\mathrm{rex}}(\mathbf{T},\mathbf{T}_1) = \mathsf{Tens}(\mathbf{T},\mathbf{T}_1) \subset \mathsf{LSM}^{\mathrm{faith}}(\mathbf{T},\mathbf{T}_1). \tag{11}$$

We claim that we always have an inclusion

$$\mathsf{LSM}^{\mathrm{faith}}(\mathbf{D},\mathbf{T}_1) \subset \mathsf{LSM}^{\gamma}(\mathbf{D},\mathbf{T}_1).$$

Indeed, a faithful LSM functor $H : \mathbf{D} \to \mathbf{T}_1$ maps every non-zero object in $\mathbf{D}$ to a non-zero object in $\mathbf{T}_1$. By Proposition 2.2.6(iii) every non-zero object in $\mathbf{T}_1$ is strongly faithful, from which it follows that H sends every sequence (4) to an exact sequence.

Moreover, by (11) and the left equivalence in (10), every functor H in $\mathsf{LSM}^{\gamma}(\mathbf{D},\mathbf{T}_1)$ extends to a faithful functor $\mathbf{T} \to \mathbf{T}_1$, hence H must be faithful as well. In particular, $\mathsf{LSM}^{\mathrm{faith}}(\mathbf{D},\mathbf{T}_1)$ is equal to $\mathsf{LSM}^{\gamma}(\mathbf{D},\mathbf{T}_1)$. Combining that equality with the equality in (11) and the equivalence on the left in diagram (10) completes the proof. □

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Remark 3.3.2. (i) Theorem 3.3.1 is not specific to Sh**D**. Indeed, the same statement is true, for instance, for PSh**D** itself.

(ii) If **D** is a semisimple tensor category, then Sh**D** = PSh**D** = Ind**D**.

Motivated by Theorem 3.3.1, we provide an explicit criterion for when Sh**D** is equivalent to the ind-completion of a tensor category.

Proposition 3.3.3. *The following conditions are equivalent.*

(i) Sh**D** *is LSM equivalent to the ind-completion of a tensor category over* k.
(ii) *There exists* $M \in \mathrm{Sh}\mathbf{D}$, *with* $M \otimes - : \mathrm{Sh}\mathbf{D} \to \mathrm{Sh}\mathbf{D}$ *faithful and exact, which splits every morphism in* **D**.
(iii) *For every morphism* f *in* **D**, *there exists* $M \in \mathrm{Sh}\mathbf{D}$, *with* $M \otimes -$ *faithful and exact, for which* $M \otimes f$ *is split.*

Proof. First we show that (i) implies (ii). Assume that $\mathrm{Sh}\mathbf{D} \simeq \mathrm{Ind}\mathbf{T}$ for a tensor category **T**. The category Ind**T** is a Grothendieck category and thus has enough injective objects. We take a non-zero $Y \in \mathbf{T}$ and an injective object $I \in \mathrm{Ind}\mathbf{T}$ which contains Y as a subobject. As for any object in Ind**T**, the functor $I \otimes -$ is exact. Furthermore, if $I \otimes N = 0$, for $N \in \mathrm{Ind}\mathbf{T}$, then the subobject $Y \otimes N$ is also zero. However, N is a subobject of $Y^\vee \otimes Y \otimes N$, which implies $N = 0$. Hence $I \otimes -$ is faithful. By applying adjunction, it follows that $X \otimes I$ is also injective for any rigid object X. Consider a morphism $f : X \to Y$ in $\mathbf{D} \subset \mathbf{T}$. Since **T** is an abelian subcategory of $\mathrm{Sh}\mathbf{D} \simeq \mathrm{Ind}\mathbf{T}$, the image and kernel of f, which we denote by Z and K, are in **T** and hence also rigid. Then clearly I splits $K \hookrightarrow X$ and $Z \hookrightarrow Y$, so also f.

That (ii) implies (iii) is trivial.

To prove that (iii) implies (i), we will freely use Theorem 3.1.4, which implies in particular that Sh**D** is a finitely presented Grothendieck category.

First we take an arbitrary compact object X in Sh**D**. It is the cokernel of a morphism $Y_0(f)$, for $f : B \to A$ in **D**. Consider $M \in \mathrm{Sh}\mathbf{D}$ as in part (iii) which splits f. By exactness of $M \otimes -$, it follows that $M \otimes X$ is isomorphic to a direct summand of $M \otimes A$. Hence $M \otimes X \otimes -$ is exact. Since $M \otimes -$ is faithful and exact, $X \otimes -$ must also be exact. Since $\otimes$ is cocontinuous and every object in Sh**D** is a filtered colimit of compact objects, it actually follows that $\otimes$ is exact on Sh**D**.

Now we prove that rigid and compact objects coincide and that they form an abelian subcategory of Sh**D**. That rigid objects are compact follows from as in the proof of Lemma 1.3.7. The full claim then follows if we can show that the dual of the kernel of a morphism between rigid objects is given by the cokernel of the dual morphism. The latter is a standard property in abelian monoidal categories with exact tensor product.

Hence the category of compact objects in Sh**D** is an abelian monoidal subcategory which is a tensor category. It follows that Sh**D** is the ind-completion of this tensor category. ☐

4. Main theorem and applications

4.1 Main results

Fix a pseudo-tensor category **D** over a field k

Theorem 4.1.1. *Assume that one of the following conditions is satisfied.*

(i) *For every morphism* f *in* **D**, *there exists* $M \in \mathrm{Sh}\mathbf{D}$, *with* $M \otimes - : \mathrm{Sh}\mathbf{D} \to \mathrm{Sh}\mathbf{D}$ *faithful and exact, such that* $M \otimes f$ *is split in* Sh**D**.

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(ii) *Every morphism f in $\mathbf{D}$ is split by a strongly faithful object in $\mathbf{D}$.*

Then $\mathbf{D}$ admits an abelian envelope $(F, \mathbf{T})$. Moreover, there is an LSM equivalence $\mathrm{Ind}\mathbf{T} \simeq \mathrm{Sh}\mathbf{D}$, which admits a commutative (up to isomorphism) diagram of LSM functors as follows.

$$\begin{array}{ccc} \mathrm{Sh}\mathbf{D} & \xrightarrow{\sim} & \mathrm{Ind}\mathbf{T} \\ {\scriptstyle Y_0}\uparrow & & \uparrow \\ \mathbf{D} & \overset{F}{\hookrightarrow} & \mathbf{T} \end{array}$$

Proof. We claim that condition (ii) implies condition (i). Indeed, for $X \in \mathbf{D}$, the fact that $X \otimes -$ is exact follows from bi-adjunction with $X^\vee \otimes -$. Moreover, by Theorem 3.1.4(v), if X is strongly faithful then ev_X is an epimorphism in $\mathrm{Sh}\mathbf{D}$. For every $A \in \mathrm{Sh}\mathbf{D}$ we thus have an epimorphism $X^\vee \otimes X \otimes A \twoheadrightarrow A$. So $X \otimes A = 0$ implies $A = 0$, and $X \otimes -$ is faithful.

That condition (i) implies the conclusion is an immediate consequence of Theorem 3.3.1 and Proposition 3.3.3. □

Remark 4.1.2. Theorem 4.1.1(ii) shows that a self-splitting pseudo-tensor category in which every non-zero object is strongly faithful admits an abelian envelope. We cannot relax the second assumption. Indeed, over any field k of characteristic 0, Lemma 2.3.4 provides self-splitting pseudo-tensor categories $\mathbf{D}$ where non-zero objects are faithful, but which do not admit an abelian envelope (by Corollary 2.2.7). If we do not demand that k be algebraically closed, these examples $\mathbf{D}$ can be taken to have finite-dimensional morphism spaces.

Now we show how our existence result implies the recognition result from [EHS20].

Corollary 4.1.3 [EHS20, Theorem 9.2.2]. *Consider a fully faithful LSM functor $I : \mathbf{D} \to \mathbf{V}$ to a tensor category $\mathbf{V}$, such that:*

(i) *any $X \in \mathbf{V}$ is a quotient of an object $I(A)$, with $A \in \mathbf{D}$;*
(ii) *for any epimorphism $X \to Y$ in $\mathbf{V}$ there exists a non-zero $T \in \mathbf{D}$ such that $X \otimes I(T) \twoheadrightarrow Y \otimes I(T)$ is split.*

Then $\mathbf{V}$ is the abelian envelope of $\mathbf{D}$.

Proof. By Corollary 2.2.7, every non-zero object in $\mathbf{D}$ is strongly faithful. Now consider a morphism $a : A \to B$ in $\mathbf{D}$. Denote its image in $\mathbf{V}$ by Z and its cokernel by W. By assumption, there exists $0 \neq T \in \mathbf{D}$ such that T splits the epimorphisms $A \twoheadrightarrow Z$ and $B \twoheadrightarrow W$. It follows that $T \otimes f$ is split. Hence the condition in Theorem 4.1.1(ii) is satisfied.

Thus there exists an abelian envelope $F : \mathbf{D} \to \mathbf{T}$. By definition, there exists a tensor functor $E : \mathbf{T} \to \mathbf{V}$, which extends I. By Remark 1.3.5, E is faithful. Since every object in $\mathbf{V}$ can be written as the cokernel of a morphism between objects in $\mathbf{D} \subset \mathbf{T}$, E is also essentially surjective. By applying the tensor duality, we find also that every object in $\mathbf{V}$ can be written as the kernel of a morphism between objects in $\mathbf{D} \subset \mathbf{T}$. Taking presentations and copresentations of objects in $\mathbf{V}$ by objects in $\mathbf{D}$ allows us to show that E inherits full faithfulness from I.

In conclusion, $E : \mathbf{T} \to \mathbf{V}$ is an equivalence, so $\mathbf{V}$ is the abelian envelope of $\mathbf{D}$. □

Example 4.1.4. Let $\mathbf{T}$ be a tensor category which has enough projective objects (or equivalently one non-zero projective object). Then $\mathbf{T}$ is the abelian envelope of every pseudo-tensor

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subcategory which contains the projective objects. Indeed, this follows immediately from Corollary 4.1.3 and Lemma 2.1.3.

Example 4.1.5. Let $\mathbf{T}$ be a tensor category which has enough projective objects and an object X such that every object in $\mathbf{T}$ is a subquotient of a direct sum of objects $\otimes^i X \otimes \otimes^j X^\vee$. Then $\mathbf{T}$ is the abelian envelope of every pseudo-tensor subcategory $\mathbf{D} \subset \mathbf{T}$ which contains X. Indeed, since every projective object is injective (Lemma 2.1.3), it must appear as a direct summand of a direct sum of objects $\otimes^i X \otimes \otimes^j X^\vee$ and hence be contained in $\mathbf{D}$. We can thus reduce to Example 4.1.4.

The following example is well known.

Example 4.1.6. Let k be an algebraically closed field of characteristic 0 and $m, n \in \mathbb{N}$. Denote by $\mathcal{K}$ the kernel of the LSM functor $H_{m|n} : [\mathrm{GL}_t, k] \to \mathsf{Rep}\mathrm{GL}(m|n)$ from § 1.4.3, for $t = m - n$. By Lemma 1.4.4(i), the functor $H_{m|n}$ is full. The induced functor $[\mathrm{GL}_t, k]/\mathcal{K} \to \mathsf{Rep}\mathrm{GL}(m|n)$ is thus fully faithful. Every faithful representation of an algebraic supergroup generates the representation category in the sense of Example 4.1.5; see [CH17, § 7.1]. Hence $\mathsf{Rep}_k\mathrm{GL}(m|n)$ is the abelian envelope of $[\mathrm{GL}_t, k]/\mathcal{K}$.

4.2 Deligne's categories

Fix a field k with $\mathrm{char}(k) = 0$ and $t \in \mathbb{Z} \subset k$. Our results now allow us to recover the following theorem of [EHS20].

Theorem 4.2.1.

(i) *The category* $[\mathrm{GL}_t, k]$ *has an abelian envelope* $\mathbf{V}_t$.

(ii) *Assume* $\bar{k} = k$. *Let* $\mathbf{T}$ *be a tensor category and take* $X \in \mathbf{T}$ *with* $\dim X = t$. *Either there exists a tensor functor*

$$\mathbf{V}_t \to \mathbf{T}, \quad \textit{with } V_t \mapsto X,$$

or there are unique $m, n \in \mathbb{N}$ *with* $m - n = t$ *for which there exists a tensor functor*

$$\mathsf{Rep}_k\mathrm{GL}(m|n) \to \mathbf{T}, \quad \textit{with } k^{m|n} \mapsto X.$$

Proof. Part (i) is an immediate application of Theorem 4.1.1(ii), by Theorem 2.3.1.

Set $\mathbf{D} := [\mathrm{GL}_t, k]$. For part (ii) we start from an LSM functor $F : \mathbf{D} \to \mathbf{T}$ which maps V_t to X, which is guaranteed to exist by Lemma 1.4.2. Denote by $\mathcal{J}$ the kernel of F. Since F is monoidal, this is a tensor ideal. By the classification of tensor ideals in [Cou18, Theorem 7.2.1], either '$\mathcal{J} = 0$' or $\mathcal{J}$ is equal to the kernel $\mathcal{J}_{m|n}$ of $H_{m|n}$ from § 1.4.3 for some m, n.

If $\mathcal{J} = 0$ the functor F is faithful, so by Definition 1.3.4, F extends to a tensor functor $\mathbf{V}_t \to \mathbf{T}$. If $\mathcal{J} = \mathcal{J}_{m|n}$, F yields a faithful functor $\mathbf{D}/\mathcal{J} \to \mathbf{T}$ and the tensor functor follows from the fact that $\mathsf{Rep}\mathrm{GL}(m|n)$ is an abelian envelope as in Example 4.1.6. □

Remark 4.2.2. It follows easily from the description of the tensor ideals in $[\mathrm{GL}_t, k]$ in [Cou18, § 7.2] that one can determine from which tensor category in Theorem 4.2.1(ii) the tensor functor comes by which Schur functors annihilate $X \in \mathbf{T}$. This is explained in detail in [EHS20], where it is also demonstrated that Theorem 4.2.1, together with the Tannakian formalism of [Del90], yields an affirmative answer to [Del07, Question 10.18].

The proof of Theorem 4.2.1, using the input from Lemmata 1.4.6 and 1.4.8 and [Cou18, § 7.1], also yields the following analogue.

Theorem 4.2.3.

(i) *The category* $[\mathrm{O}_t, k]$ *has an abelian envelope* $\mathbf{U}_t$.
(ii) *Assume* $\bar{k} = k$. *Let* $\mathbf{T}$ *be a tensor category and* X *a symmetrically self-dual object of dimension* t. *Either there exists a tensor functor*

$$\mathbf{U}_t \to \mathbf{T}, \quad \textit{with } U_t \mapsto X,$$

or there are unique $m, n \in \mathbb{N}$ *with* $m - 2n = t$ *for which there exists a tensor functor*

$$\mathsf{Rep}_k \mathrm{OSp}(m|2n) \to \mathbf{T}, \quad \textit{with } k^{m|2n} \mapsto X.$$

4.2.4 In [Del07, § 2], a pseudo-tensor category $[\mathrm{S}_t, k]$ is defined for every $t \in k$, which is a semisimple tensor category when $t \notin \mathbb{N}$ by [Del07, Théorème 2.18]. In [Del07, Proposition 8.18] it is shown that, for $n \in \mathbb{N}$, the pseudo-tensor category $[\mathrm{S}_n, k]$ admits a fully faithful tensor F functor into a tensor category $\mathbf{T}_n$. By Corollary 2.2.7, every non-zero object in $[\mathrm{S}_n, k]$ is strongly faithful. Furthermore, it is proved in [CO14, Lemma 3.11] that there exists a non-zero object which splits every morphism in $[\mathrm{S}_n, k]$. Theorem 4.1.1 therefore demonstrates that $[\mathrm{S}_n, k]$ admits an abelian envelope. This recovers one of the main results in [CO14].

It seems worthwhile to point out the following observations (although the equivalent properties are of course known to be true by [CO14]), which do not rely on [CO14, Lemma 3.11]. The latter lemma is one of the cornerstones in both the original and above proof that $[\mathrm{S}_n, k]$ admits an abelian envelope, but has a rather intricate proof.

Proposition 4.2.5. *For* $F : [\mathrm{S}_n, k] \to \mathbf{T}_n$ *in § 4.2.4, the following properties are equivalent.*

(i) *Every object in* $\mathbf{T}_n$ *is a quotient of an object* $F(D)$ *with* $D \in [\mathrm{S}_n, k]$.
(ii) *For every indecomposable* $D \in [\mathrm{S}_n, k]$ *with* $\dim D = 0$, $F(D)$ *is projective in* $\mathbf{T}_n$.

Each statement implies that $\mathbf{T}_n$ *is the abelian envelope of* $[\mathrm{S}_n, k]$.

Proof. We set $\mathbf{D} = [\mathrm{S}_n, k]$. First we prove that (i) implies (ii). By construction of $\mathbf{T}_n$ in [Del07, § 2], all objects have finite length and morphism spaces are finite-dimensional. It follows that every object in $\mathrm{Ind}\mathbf{T}_n$ is the union of its subobjects in $\mathbf{T}_n$ and that $\mathbb{1}$ admits an injective hull I in $\mathrm{Ind}\mathbf{T}_n$. So I is the union of objects $I_\alpha \in \mathbf{T}_n$ with socle $\mathbb{1}$. Moreover, by (i), each I_α is a subobject of an object in $\mathbf{D}$. By [Cou18, § 3.4], there is a unique indecomposable object X_0 in $\mathbf{D}$, different from $\mathbb{1}$, for which there exist non-zero morphisms $\mathbb{1} \to X_0$ and moreover $\mathbf{D}(\mathbb{1}, X_0) = k$. This shows that $X_0 = I$, so, in particular, X_0 is projective in $\mathbf{T}_n$. Also by [Cou18, § 3.4], $\dim X_0 = 0$ and every other indecomposable object in $\mathbf{D}$ of dimension 0 is a direct summand of a tensor product of X_0 with some $Z \in \mathbf{D}$. Thus (i) implies (ii).

We now prove that (ii) implies (i). By [Del07, Proposition B1], every object in $\mathbf{T}_n$ is a subquotient of an object in $\mathbf{D}$. Now consider an arbitrary $X \in \mathbf{T}_n$. It is a subquotient of $M \in \mathbf{D}$. By assumption, and existence of $D \in \mathbf{D}$ with $\dim D = 0$ as in the above paragraph, there exists a projective (and hence injective) object P in $\mathbf{T}_n$ contained in $\mathbf{D}$. It then follows that $P \otimes X$ is a direct summand of $P \otimes M$. On the other hand, X is a quotient of $P^\vee \otimes P \otimes X$, which is itself a direct summand of $P^\vee \otimes P \otimes M \in \mathbf{D}$. So (i) follows.

The combination of (i) and (ii) implies that $\mathbf{T}_n$ is the abelian envelope of $\mathbf{D}$, for instance by Corollary 4.1.3. □

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4.3 Tilting modules

Now let k be an algebraically closed field of characteristic $p > 0$.

4.3.1 We work in the tensor category $\mathsf{Rep}\,\mathrm{SL}_2$ of finite-dimensional algebraic representations of the algebraic group SL_2/k. We have the pseudo-tensor subcategory $\mathbf{D} := \mathsf{Tilt}\,\mathrm{SL}_2$ of tilting modules; see [Jan03, §II.E]. We denote the simple module and the indecomposable tilting module with highest weight $i\omega$ (with ω the fundamental weight) by L_i and T_i, for $i \in \mathbb{N}$. The Steinberg modules (see [Jan03, II.3.18]) are

$$\mathrm{St}_j = L_{p^j-1} = T_{p^j-1}, \quad \text{for } j \in \mathbb{N}.$$

For $r \in \mathbb{Z}_{>0}$, we consider the tensor ideal $\mathcal{J}_r$ in $\mathsf{Tilt}\,\mathrm{SL}_2$ of morphisms which factor through a direct sum of objects T_i, with $i \geq (p^r - 1)$. This gives a complete and irredundant list of the non-trivial tensor ideals in $\mathsf{Tilt}\,\mathrm{SL}_2$; see [Cou18, § 5.3]. Consequently, $\mathcal{J}_r$ is generated by $\mathrm{id}_{\mathrm{St}_r}$.

Theorem 4.3.2. *If $p > 2$, then $(\mathsf{Tilt}\,\mathrm{SL}_2)/\mathcal{J}_r$ admits an abelian envelope, for each $r > 0$.*

The condition $p > 2$ is not required and only reflects the limitations of the proof of Lemma 4.3.5 below. Indeed, the equivalent of Theorem 4.3.2 for $p = 2$ is already known by [BE19]. We start the proof with the following lemma.

Lemma 4.3.3. *If L_a is in the same block of $\mathsf{Rep}\,\mathrm{SL}_2$ as $\mathrm{St}_j = L_{p^j-1}$, for $a, j \in \mathbb{N}$, then either $a = p^j - 1$ or $a \geq 2p^{j+1} - p^j - 1$.*

Proof. This is an immediate consequence of [Jan03, II.7.2(3)]. □

Lemma 4.3.4. *If $i \leq p^r - 1$, then $L_i \otimes \mathrm{St}_{r-1}$ is a tilting module.*

Proof. By the Steinberg tensor product theorem [Jan03, II.3.17], for $i < p^r$ we have

$$L_i \simeq \bigotimes_{a=0}^{r-1} L_{p^a i_a}, \quad \text{with } i = \sum_{a=0}^{r-1} p^a i_a \text{ and } 0 \leq i_a < p.$$

By Lemma 1.3.8, it therefore suffices to prove that $L_{p^a b} \otimes \mathrm{St}_{r-1}$ is a tilting module for $a < r$ and $b < p$. We prove the more general claim that $L_m \otimes \mathrm{St}_{r-1}$ is a tilting module for $m \leq p^r - p^{r-1}$. By [Jan03, Proposition E.1], it then suffices to prove that

$$\mathrm{Ext}^1(\Delta_n, L_m \otimes \mathrm{St}_{r-1}) = 0, \quad \text{for } n \in \mathbb{N} \text{ and } m \leq p^r - p^{r-1}, \tag{12}$$

where Δ_n is the Weyl module with top L_n.

We divide (12) into two cases. First assume that $n \geq p^r - 1$. Then $n \geq m + p^{r-1} - 1$, so $L_m \otimes \mathrm{St}_{r-1}$ belongs to the Serre subcategory $\mathsf{Rep}\,\mathrm{SL}_2^{\leq n}$ generated by simples L_j with $j \leq n$ in which Δ_n is projective. Hence (12) is satisfied. Now assume that $n < p^r - 1$. The left-hand side of (12) can be rewritten as $\mathrm{Ext}^1(\Delta_n \otimes L_m, \mathrm{St}_{r-1})$, and by our assumption

$$n + m < 2p^r - p^{r-1} - 1.$$

By Lemma 4.3.3, this means that the direct summand of $\Delta_n \otimes L_m$ in the block of St_{r-1} is a direct sum of copies of St_{r-1}, so the extension vanishes and (12) is again satisfied. □

We set $\mathbf{D} = \mathsf{Tilt}\,\mathrm{SL}_2$ and $\mathbf{C} = \mathbf{D}/\mathcal{J}_r$.

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Lemma 4.3.5. *If $p > 2$, the object* St_{r-1} *is strongly faithful in* $\mathbf{C}$.

Proof. By definition, we need to prove that the sequence

$$\mathbf{C}(\gamma_{\mathrm{St}_{r-1}}, T_i):\ 0 \to \mathbf{C}(\mathbb{1}, T_i) \to \mathbf{C}(\otimes^2\mathrm{St}_{r-1}, T_i) \to \mathbf{C}(\otimes^4\mathrm{St}_{r-1}, T_i) \tag{13}$$

is exact, for each $0 \le i < p^r - 1$.

The structure of tensor ideals recalled in § 4.3.1 implies that, for $i \ge p^{r-1} - 1$, the module T_i is a direct summand of an object $T \otimes \mathrm{St}_{r-1}$. That (13) is exact for $i \ge p^{r-1} - 1$ is thus an example of Lemma 2.2.9.

Next, we consider T_i with $i < p^{r-1} - 1$. We claim that

$$\mathcal{J}_r(\mathbb{1}, T_i) = 0 = \mathcal{J}_r(\otimes^2\mathrm{St}_{r-1}, T_i) = \mathcal{J}_r(\otimes^4\mathrm{St}_{r-1}, T_i).$$

That the leftmost space is zero follows immediately from the description of the ideals $\mathcal{J}_l$ in [Cou18, § 3.2]. We now prove the claim for the rightmost space; the proof for the middle space is similar but easier. By adjunction, we can equivalently prove

$$\mathcal{J}_r(\mathrm{St}_{r-1}, \otimes^3\mathrm{St}_{r-1} \otimes T_i) = 0.$$

By definition of $\mathcal{J}_r$ and Lemma 4.3.3, the contrary would necessarily imply that

$$[\otimes^3\mathrm{St}_{r-1} \otimes T_i : L_a] \ne 0, \quad \text{for some } a \ge 2p^r - p^{r-1} - 1.$$

However, since we have

$$i + 3(p^{r-1} - 1) < 4p^{r-1} - 4 < 2p^r - p^{r-1} - 1,$$

under the assumption $p > 2$, this non-vanishing multiplicity is impossible. It follows that for $i < p^{r-1} - 1$ we have the following commutative diagram, with the second row given by (13).

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbf{D}(\mathbb{1}, T_i) & \longrightarrow & \mathbf{D}(\otimes^2\mathrm{St}_{r-1}, T_i) & \longrightarrow & \mathbf{D}(\otimes^4\mathrm{St}_{r-1}, T_i) \\
& \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
0 \longrightarrow & \mathbf{C}(\mathbb{1}, T_i) & \longrightarrow & \mathbf{C}(\otimes^2\mathrm{St}_{r-1}, T_i) & \longrightarrow & \mathbf{C}(\otimes^4\mathrm{St}_{r-1}, T_i)
\end{array}$$

The first row is exact by Corollary 2.2.7 and the inclusion $\mathbf{D} \subset \mathsf{Rep}\,\mathrm{SL}_2$. Hence the second row is exact. This concludes the proof. □

Proof of Theorem 4.3.2. Consider a morphism $f : T \to T'$ in $\mathbf{D} = \mathsf{Tilt}\,\mathrm{SL}_2$, where T and T' are direct sums of indecomposable tilting modules T_i with $i < p^r - 1$. By Lemma 4.3.4 the image, kernel and cokernel of f are objects $X \in \mathsf{Rep}\,\mathrm{SL}_2$ such that $\mathrm{St}_{r-1} \otimes X$ is a tilting module. Indeed, this follows from the fact that there are no first extensions between tilting modules; see [Jan03, §II.E]. The same fact then also shows that $\mathrm{St}_{r-1} \otimes f$ is split in $\mathbf{D}$; see also [CEH19]. It then follows trivially that $\mathrm{St}_{r-1} \otimes f$ is also split in $\mathbf{C} = \mathbf{D}/\mathcal{J}_r$. Any morphism in $\mathbf{C}$ can be written as above. Hence St_{r-1} splits every morphism in $\mathbf{C}$.

Since St_{r-1} is strongly faithful in $\mathbf{C}$, by Lemma 4.3.5, we can apply Theorem 4.1.1(ii). □

Remark 4.3.6. Let G be a simple simply-connected algebraic group. The category $\mathsf{Rep}G$ is self-splitting via the Steinberg modules; see [CEH19, § 3.3]. This thus gives an example of a

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self-splitting tensor category which is not a finite tensor category, and $\mathsf{Rep}G$ is its own abelian envelope by Corollary 4.1.3.

Remark 4.3.7. As proved in [CEH19, Theorem 3.3.1], in the generality of Remark 4.3.6, $\mathsf{Rep}G$ is the abelian envelope of $\mathsf{Tilt}G$. Let $\mathsf{Rep}^\infty G$ denote the category of all algebraic representations (which is equivalent to $\mathrm{Ind}\mathsf{Rep}G$). Our results can be used to prove that $\mathsf{Rep}^\infty G$ is equivalent to the category of k-linear functors $(\mathsf{Tilt}G)^{\mathrm{op}} \to \mathsf{Vec}$ which send all sequences in $\Xi(\mathsf{Tilt}G)$ (or alternatively all sequences $T \otimes \gamma_{\mathrm{St}_n}$ for tilting modules T and $n \in \mathbb{N}$) to exact sequences.

4.4 Tensor ideals

Fix a pseudo-tensor category $\mathbf{D}$ over a field k and assume that the *morphism spaces in* $\mathbf{D}$ *are finite-dimensional.*

4.4.1 A *thick tensor ideal* in $\mathbf{D}$ is a full Karoubi subcategory J of $\mathbf{D}$ such that $X \in J$ implies that $Y \otimes X \in J$ for all $Y \in \mathbf{D}$. The *decategorification map* (see [Cou18, § 4.1]) sends a tensor ideal $\mathcal{J}$ in $\mathbf{D}$ to the thick tensor ideal of objects X with $\mathrm{id}_X \in \mathcal{J}$. By [Cou18, Theorem 4.1.2], this map is always surjective.

Proposition 4.4.2. *Assume that the decategorification map is a bijection for* $\mathbf{D}$ *and that there exists a fully faithful LSM functor* $I : \mathbf{D} \to \mathbf{V}$ *to a tensor category* $\mathbf{V}$*, such that any* $X \in \mathbf{V}$ *is a quotient of an object* $I(A)$*, with* $A \in \mathbf{D}$*. Then* $\mathbf{V}$ *is the abelian envelope of* $\mathbf{D}$*.*

Proof. By Corollary 4.1.3, it suffices to prove that any epimorphism in $\mathbf{V}$ is split by a non-zero object in $\mathbf{D}$. We do this in three steps.

(1) Consider a non-zero morphism $f : D \to \mathbb{1}$ in $\mathbf{D}$. This is automatically an epimorphism in $\mathbf{V}$. By [Cou18, Proposition 4.2.2], f is unique up to composition with endomorphisms of D. Furthermore, [Cou18, Lemma 4.2.4] then implies that there exists $X \in \mathbf{D}$ such that ev_X is given by a composition

$$\mathrm{ev}_X : \; X^\vee \otimes X \to D \xrightarrow{f} \mathbb{1}.$$

By Lemma 2.1.2(i) the morphism $X \otimes \mathrm{ev}_X$, and hence also $X \otimes f$, is split.

(2) Consider an epimorphism $g : M \twoheadrightarrow \mathbb{1}$ in $\mathbf{V}$. By assumption, there exists $D \in \mathbf{D}$ such that we have an epimorphism $\pi : D \twoheadrightarrow M$. By step 1, $X \otimes (g \circ \pi)$ is split for some non-zero $X \in \mathbf{D}$ from which it follows that also $X \otimes g$ is split.

(3) Finally, we consider an arbitrary epimorphism $h : M \twoheadrightarrow N$ in $\mathbf{V}$. Tensoring with $N^\vee$ and taking a pullback yields the following commutative diagram.

$$\begin{array}{ccc} M \otimes N^\vee & \xrightarrow{h \otimes N^\vee} & N \otimes N^\vee \\ \big\uparrow & & \big\uparrow \mathrm{co}_N \\ (M \otimes N^\vee) \times_{(N \otimes N^\vee)} \mathbb{1} & \twoheadrightarrow & \mathbb{1} \end{array}$$

By step 2, there exists a non-zero $X \in \mathbf{D}$ which splits the epimorphism on the lower line. After applying $X \otimes -$, the diagram thus admits a diagonal morphism $X \to XMN^\vee$ which makes the upper triangle commute. It then follows that the associated morphism $XN \to XM$ ensures that $X \otimes h$ is split. □

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Remark 4.4.3. Let k be algebraically closed. It is proved in [Cou18] that the decategorification map is a bijection for $[\mathrm{GL}_t, k]$, $[\mathrm{O}_t, k]$ and $[\mathsf{S}_t, k]$, when $\mathrm{char}(k) = 0$, and that the same is true for $\mathsf{Tilt}\,\mathrm{SL}_2$ when $\mathrm{char}(k) > 0$.

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Appendix. Grothendieck topologies

Fix a commutative ring K and an essentially small K-linear category $\mathbf{A}$. Denote by $\mathrm{PSh}\mathbf{A}$ the category of presheaves $\mathbf{A}^{\mathrm{op}} \to K - \mathsf{Mod}$.

A.1 *K*-linear sheaves

A.1.1 For $A \in \mathbf{A}$, a *sieve on* A is a K-linear subfunctor of $\mathbf{A}(-, A) \in \mathrm{PSh}\mathbf{A}$. For a sieve R on A and a morphism $f : B \to A$ in $\mathbf{A}$, the assignment

$$\mathrm{Ob}\mathbf{A} \to K - \mathsf{Mod}, \quad C \mapsto \{g \in \mathbf{A}(C, B) \,|\, f \circ g \in R(C)\}$$

yields a sieve on B, which we denote by $f^{-1}R$. In other words, $f^{-1}R$ is the pullback of $R \to \mathbf{A}(-, A) \leftarrow \mathbf{A}(-, B)$.

The following definition is taken from [BQ96, 1.2 and 1.6].

Definition A.1.2. A K-linear Grothendieck topology $\mathcal{T}$ on $\mathbf{A}$ is an assignment to each $A \in \mathbf{A}$ of a collection $\mathcal{T}(A)$ of sieves on A such that for every $A \in \mathbf{A}$:

(T1) we have $\mathbf{A}(-, A) \in \mathcal{T}(A)$;
(T2) for $R \in \mathcal{T}(A)$ and a morphism $f : B \to A$ in $\mathbf{A}$, we have $f^{-1}R \in \mathcal{T}(B)$;
(T3) for a sieve S on A and $R \in \mathcal{T}(A)$ such that for every $B \in \mathbf{A}$ and $f \in R(B) \subset \mathbf{A}(B, A)$ we have $f^{-1}S \in \mathcal{T}(B)$, it follows that $S \in \mathcal{T}(A)$.

The following definition is taken from [BQ96, 1.3 and 1.6].

Definition A.1.3. For a K-linear Grothendieck topology $\mathcal{T}$ on $\mathbf{A}$, a presheaf $F \in \mathrm{PSh}\mathbf{A}$ is a *$\mathcal{T}$-sheaf* if for every $A \in \mathbf{A}$ and $R \in \mathcal{T}(A)$, the canonical morphism

$$F(A) \simeq \mathrm{Nat}(\mathbf{A}(-, A), F) \to \mathrm{Nat}(R, F)$$

is an isomorphism. The full subcategory of $\mathrm{PSh}\mathbf{A}$ of $\mathcal{T}$-sheaves is denoted by $\mathrm{Sh}(\mathbf{A}, \mathcal{T})$.

Our interest in Grothendieck topologies derives from [BQ96, Theorem 1.5]. Recall that a localisation of an abelian category is a full replete subcategory for which the inclusion functor has a left adjoint which is left exact (and hence exact).

Theorem A.1.4 (Borceux and Quinteiro). *The localisations of* $\mathrm{PSh}\mathbf{A}$ *are precisely the subcategories* $\mathrm{Sh}(\mathbf{A}, \mathcal{T})$*, for all Grothendieck topologies* $\mathcal{T}$ *on* $\mathbf{A}$.

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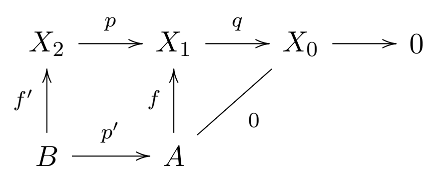
A.2 Schäppi's formalism

We start by recalling a definition from [Sch20].

Definition A.2.1. For a class $\Sigma \subset \Xi(\mathbf{A})$ of exact sequences (1), denote by $\mathcal{C}o(\Sigma)$ the set of morphisms q which appear as the cokernels in sequences in Σ. Then Σ is an *ind-class* if the following conditions hold.

(i) For every $q \in \mathcal{C}o(\Sigma)$, there is a sequence $X_1 \xrightarrow{q} X_0 \to Z \xrightarrow{\sim} 0$ in Σ.
(ii) For each sequence (1) in Σ and each morphism $f : A \to X_1$ in $\mathbf{A}$ with $q \circ f = 0$, there exist $p' : B \to A$ in $\mathcal{C}o(\Sigma)$ and $f' : B \to X_2$ in $\mathbf{A}$ yielding the following commutative diagram.



The following proposition follows immediately from [Sch20, A.1.2 and A.2.3].

Proposition A.2.2 (Schäppi). *Consider a subclass $\Sigma \subset \Xi(\mathbf{A})$. For each $A \in \mathbf{A}$, denote by $\mathcal{T}(A)$ the set of all sieves $R \subset \mathbf{A}(-, A)$ which contain a composite $r = r_1 \circ r_2 \circ \cdots \circ r_m$ (with $m \in \mathbb{N}$, where the empty composite is interpreted as id_A) of morphisms $r_i \in \mathcal{C}o(\Sigma)$.*

If Σ is an ind-class, then $\{A \mapsto \mathcal{T}(A)\}$ is a K-linear Grothendieck topology on $\mathbf{A}$.

The following proposition follows from the combination of [Sch20, A.1.4 and A.2.5].

Proposition A.2.3 (Schäppi). *For the topology $\mathcal{T}$ associated to an ind-class $\Sigma \subset \Xi(\mathbf{A})$ as in Lemma A.2.2 and $F \in \mathrm{PSh}\mathbf{A}$, the following assertions are equivalent.*

(i) *F is a $\mathcal{T}$-sheaf.*
(ii) *The sequence $F(\xi)$ is exact in $K - \mathsf{Mod}$ for each $\xi \in \Sigma$.*

References

AK02 Y. André and B. Kahn, *Nilpotence, radicaux et structures monoïdales. With an appendix by Peter O'Sullivan*, Rend. Sem. Mat. Univ. Padova **108** (2002), 107–291.

BE19 D. Benson and P. Etingof, *Symmetric tensor categories in characteristic 2*, Adv. Math. **351** (2019), 967–999.

BEO20 D. Benson, P. Etingof and V. Ostrik, *New incompressible symmetric tensor categories in positive characteristic*, Preprint (2020), arXiv:2003.10499.

BQ96 F. Borceux and C. Quinteiro, *A theory of enriched sheaves*, Cahiers Topol. Géom. Différ. Catég. **37** (1996), 145–162.

CH17 J. Comes and T. Heidersdorf, *Thick ideals in Deligne's category $\underline{Rep}(O_t)$*, J. Algebra **480** (2017), 237–265.

CO14 J. Comes and V. Ostrik, *On Deligne's category $\underline{Rep}^{ab}(S_d)$*, Algebra Number Theory **8** (2014), 473–496.

Cou18 K. Coulembier, *Tensor ideals, Deligne categories and invariant theory*, Selecta Math. (N.S.) **24** (2018), 4659–4710.

Cou20 K. Coulembier, *Tannakian categories in positive characteristic*, Duke Math. J. **169** (2020), 3167–3219.

https://doi.org/10.1112/S0010437X21007399 Published online by Cambridge University Press

CEH19 K. Coulembier, I. Entova-Aizenbud and T. Heidersdorf, *Monoidal abelian envelopes and a conjecture of Benson–Etingof* , Preprint (2019), arXiv:1911.04303.

Day72 B. Day, *A reflection theorem for closed categories*, J. Pure Appl. Algebra **2** (1972), 1–11.

Del90 P. Deligne, *Catégories tannakiennes*, in *The Grothendieck Festschrift*, vol. II, Progress in Mathematics, vol. 87 (Birkhäuser, Boston, 1990), 111–195.

Del02 P. Deligne, *Catégories tensorielles*, Mosc. Math. J. **2** (2002), 227–248.

Del07 P. Deligne, *La catégorie des représentations du groupe symétrique* S_t*, lorsque* t *n'est pas un entier naturel*, in *Algebraic groups and homogeneous spaces* (Tata Institute of Fundamental Research Studies in Mathematics, Mumbai, 2007), 209–273.

EO19 P. Etingof and V. Ostrik, *On the Frobenius functor for symmetric tensor categories in positive characteristic*, Preprint (2019), arXiv:1912.12947.

EHS20 I. Entova-Aizenbud, V. Hinich and V. Serganova, *Deligne categories and the limit of categories* $Rep(GL(m|n))$, Int. Math. Res. Not. IMRN **2020** (2020), 4602–4666.

EG19 P. Etingof and S. Gelaki, *Finite symmetric integral tensor categories with the Chevalley property*, to appear in IMRN, Preprint (2019), arXiv:1901.00528.

Har16 N. Harman, *Deligne categories as limits in rank and characteristic*, Preprint (2016), arXiv:1601.03426.

Hei17 T. Heidersdorf, *Mixed tensors of the general linear supergroup*, J. Algebra **491** (2017), 402–446.

Jan03 J. C. Jantzen, *Representations of algebraic groups*, second edition, Mathematical Surveys and Monographs, vol. 107 (American Mathematical Society, Providence, RI, 2003).

LZ17 G. I. Lehrer and R. B. Zhang, *The first fundamental theorem of invariant theory for the orthosymplectic supergroup*, Commun. Math. Phys. **349** (2017), 661–702.

Ost20 V. Ostrik, *On symmetric fusion categories in positive characteristic*, Selecta Math. (N.S.) **26** (2020), Paper No. 36.

Sch20 D. Schäppi, *Constructing colimits by gluing vector bundles*, Adv. Math. **375** (2020), 107394.

Ser84 A. N. Sergeev, *Representations of the Lie superalgebras* $gl(n,m)$ *and* $Q(n)$ *in a space of tensors*, Funktsional. Anal. i Prilozhen. **18** (1984), 80–81.

SGA3 M. Demazure and A. Grothendieck, *Schémas en groupes (SGA3)*, Tome 1, Lecture Notes in Mathematics, vol. 151 (Springer, Berlin, 1970).

Zha18 Y. Zhang, *On the second fundamental theorem of invariant theory for the orthosymplectic supergroup*, J. Algebra **501** (2018), 394–434.

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