Macroeconomic Dynamics, **3**, 1999, 293–310. Printed in the United States of America.

ARTICLES

MODEL OF OPTIMAL ECONOMIC GROWTH WITH ENDOGENOUS BIAS

RYUZO SATO AND RAMA V. RAMACHANDRAN
Economics Department and Center for Japan-U.S. Business and Economic Studies, Stern School of Business

CHENG PING LIAN
Council of Policy & Strategy

The objective of the paper is to develop a model of optimal endogenous technological progress that will exhibit two properties sought in growth models: (1) The bias will depend on the parameters of the model—particularly those affecting the cost of inputs—instead of being constrained to be Harrod neutral; (2) factor shares will be constant in steady state. Using previously derived sufficient conditions, we show the conditions under which such a model can be constructed.

Keywords: Technological Progress, Endogenous Bias, Steady State

1. INTRODUCTION

Economists have always accorded technological change a crucial role in the interpretation of long-term trends. Theoretical considerations have shown that a clear distinction has to be made between movements along the production function due to changing relative scarcity of inputs and movements of the function itself due to technological change. Further research has been aimed at explaining the rate of shift of the function, the continued growth of wages in the absence of any persistent trend for interest rate, and the constancy of the factor shares.[1] The resulting discussion, instead of leading to a commonly accepted paradigm, has generated three distinct trends in the literature.

The first of these can be traced to Hicks' (1963) *Theory of Wages* in which he applied the analysis of neoclasical theory to the examination of the factor incomes without assuming, as in some of the earlier works, homogenous output and inputs. In the process of developing the analysis, he argued that technological change will be biased toward saving the input whose relative price was increasing; this bias and changes in the elasticity of substitution were used to explain the relative constancy of factor shares in the long run.

We wish to acknowledge helpful comments from Jess Benhabib, William A. Barnett, Paul A. Samuelson, and Andrew Schotter. Address correspondence to: Ryuzo Sato, Director, The Center for Japan-U.S. Business and Economic Studies, Stern School of Business, 44 West Fourth Street, New York, NY 10012-1126, USA; e-mail: rsato@stern.nyu.edu.

 1365-1005/99 $9.50

If firms were price takers, Fellner (1961) noted that some type of learning process had to be postulated to justify such firms investing in biased technological innovations. Salter (1960) argued that firms should minimize total cost and not the cost of any input. Such cost minimization does not imply the Hicksian process of biased technological progress but Kennedy (1964) responded by postulating a static innovation-possibility frontier to reinstate the Hicksian hypothesis. Kennedy, however, took his theory to imply a rejection of the marginal productivity theory, but Samuelson (1965) showed that it is consistent with the neoclassical theory of production.

Nordhaus (1967) and Kamien and Schwartz (1969) generalized the model and made the position of the frontier dependent on the level of research expenditure. Although these papers provided a microeconomic foundation for investment in endogenous technological progress, they did not consider biased technological progress. An attempt to simultaneously determine the levels of and the bias of technological progress was made by Sato and Ramachandran (1987). They assumed that a monopolistic firm facing differential growth in input prices responds to the cost increases by increasing factor efficiency; using an optimal control model, they showed that both the rate and the bias of technological progress were such as to counterbalance the increases in input prices.

The second tradition can be traced back to the development of neoclassical theory of growth by Solow and Swan. Solow (1957), using the observable factor shares, estimated the rate of growth of total factor productivity and showed that more than half of the growth rate in per capita output should be attributed to technological change. In his empirical analysis, Solow had assumed Hicks neutral technological progress while his theoretical model has steady state if and only if technological progress is Harrod-neutral. Other forms of bias recieved less prominence in the voluminous literature on neoclassical growth models.

One of the reasons for this neglect is that there are some well-known problems in estimating biased technological progress. If the production function is Cobb-Douglas, then all factor-augmenting technological progress reduces to the Hicks-neutral form. If the production function has constant but nonunitary elasticity, then the Diamond-McFadden impossibility theorem states that the elasticity of substitution and bias cannot be estimated simultaneously. The theoretical explanation for the constancy of the factor shares seems to be beyond empirical verification.

To break this impasse, Sato (1970) derived the constant elasticity of derived demand (CEDD) production functions which had the convenient property that the elasticity of substitution was proportional to the factor share. It then was shown that the elasticities of derived demand for the factors are constant so that a simple regression analysis can be applied to estimate these elasticities. This property made it particularly useful for the estimation of the biased technological progress. The data from the U.S. nonfarm sector for 1909 to 1960 was used to test how well this function fitted relative to Cobb-Douglas and other forms. Rejection of the Cobb-Douglas form, which precluded any meaningful discussion of bias, set the stage for the estimation of biased technological progress using CEDD functions and comparing it with estimates using CES function. It is shown that a function with variable elasticity of substitution [$E_x(\alpha) = 1.558\alpha$, where x is the capital/labor ratio and α is the share of capital] had more explanatory power than the CES functions with reasonable values for the elasticity of substitution. Further, technological progress is shown to be biased to labor savings.

But the analysis by Sato (1970) was not embedded in a model of economic growth. We construct a model with endogenous technological progress that has two properties sought in growth models:

1. It shows the possibility of a steady state with biased technological progress, where the bias will depend on the parameters of the model. This constrasts with the earlier neoclassical growth models which have steady state only if technological progess is Harrod-neutral.
2. The motivation for introducing biased technological progress in growth models was to explain the constancy of factor share and, in this model, the factor shares are constant in the steady state.

2. APPLYING HETEROGENEOUS CAPITAL GOODS MODEL TO TECHNOLOGICAL PROGRESS

Samuelson and Solow (1956) extended the Ramsey model to the case of many capital goods. They assumed that there are n goods that can be used for production or consumption and that the stock of these goods determines the output through a general transformation relation:

$$C_1 + \dot{S}_1 = f(S_1 \ldots S_n; C_2 + \dot{S}_2 \ldots C_n + \dot{S}_n),$$

where S_i, $i = 1 \ldots n$, are stocks of n goods and C_i is the rate of consumption of the goods in a period and $\dot{S}_i$ is the additions to the stock of the goods.

The utility $U(C_1 \ldots C_n)$ is a concave function in the consumption of n goods and the problem is to maximize, using calculus of variations,

$$\int_0^\infty U(C_1 \ldots C_n)\, dt \quad \text{subject to} \quad f(S_1 \ldots S_n; C_2 + \dot{S}_2 \ldots C_n + \dot{S}_n)$$
$$- C_1 - \dot{S}_1 = 0 \quad \text{and} \quad S_i(0) = \text{ initial stock}, i = 1 \ldots n.$$

They assumed the Legendre condition that $((\partial^2 f)/(\partial C_i \partial C_j))$ is negative definite but did not explicitly discuss the Jacobi and Weierstrauss conditions [Gelfand and Fomin (1967, pp. 97–149)].

In applying their analysis to the problem of technological progress, we specialize it to the case of four variables. One represents the stock of a capital good, K. The second is labor that grows at a constant rate, n. The third and fourth variables are the effciency levels, A and B, of K and L, respectively. We now write the transformation function, using their suggestion (1967, p. 285) as

$$C_1 = \tilde{f}(K, L; A, B; \dot{K}, \dot{L}, \dot{A}, \dot{B}),$$

https://doi.org/10.1017/S1365100599012018 Published online by Cambridge University Press

and the problem is to maximize $\int_0^\infty U(C)\,dt$ subject to the transformation function and initial conditions.

As with the Samuelson-Solow model, various specializations are possible. We can rewrite the transformation function as $\tilde{F}\,[(AK), (BL), (A{\cdot}K), (B{\cdot}L)]$ and take $\tilde{F}$ to be homogeneous of degree one in *AK* and *BL* or of two in A, K, B, and L. It can be shown that the Legendre condition is satisfied for this problem where *AK* (*BL*) is treated as one variable but, because the increasing returns already noticed in the case where A, K, B, and L are considered as separate variables, it is not possible to establish the sufficiency conditions in the general case. We reformulate the problem in terms of optimal control with a specialized version of the transformation function and establish the concavity of the maximized Hamiltonian (known to be a weaker condition than those stated above) to establish the sufficiency condition.

3. FORMULATION OF THE PROBLEM USING OPTIMAL CONTROL

The transformation function is now written as

$$C_1 = F(AK, BL) - AK\phi_1\left(\frac{\dot{A}}{A}\right) - BL\phi_2\left(\frac{\dot{B}}{B}\right) - AK\phi_3\left(\frac{\dot{K}}{K}\right). \qquad \mathbf{(1)}$$

This brings the model in line with the production-function approach common in microeconomic models and neoclassical growth theory. $F(\cdot)$ is the production function of the numeraire commodity that can be either consumed or used for increasing the stock of the inputs other than labor.

The functions ϕ_i, $i = 1$ or 2, can be thought of as inverses of the technological progress functions common in endogenous growth models [Sato and Ramachandran (1987)]; if we fix the total resources devoted to increasing A and B, then they will jointly define the Kennedy-Weizacker-Samuelson technological progress frontier.

If the expenditure on research intended to increase the efficiency of a factor is ϕ_i, $i = 1, 2$, then the rate of technological progress, $\dot{A}/A$ or $\dot{B}/B$, is taken as a function of ϕ_i. If current expenditures on research increase the efficiency of an input, and if it retains this efficiency level into the future, then an expenditure today leads to an incremental stream of income that extends to the planning horizon of the firm. The net discounted return from the investment may be so high that there is no interior solution to the allocation problem; the system moves to the corner with all of the output devoted to research as has been noticed in earlier models of endogenous technological progress. The economically uninteresting corner solution is avoided by assuming that the generation of technological progress has counterbalancing diminishing returns. This is partly achieved by assuming that the technological progress functions are concave; increases in the rate of expenditure bring about smaller increases in the rate of growth of efficiency, but, in models of optimal technological progress with infinite horizon, an additional assumption that strengthens the diminishing returns is necessary for the system to have an interior solution. Sato (1996) showed that the necessary condition for the existence of an interior steady state is that C_1 is (at least asymptotically) homogeneous of degree 2 with respect to its variables, A, K, B, L, $\dot{A}$, $\dot{K}$, $\dot{B}$, and $\dot{L}$.

We assume that the efficiency factor increases through investment in applied research and that it takes resources to transfer the new technology to the physical units such as labor and capital. As the quantity of an input in efficiency units increases, more resources are needed for generating incremental increases in efficiency; in other words, the expenditures, on entering the technological progress functions, are deflated by the quantity of inputs in efficiency units.

Let the numeraire good, $Y = F(AK, BL)$, allocated to increasing the effciency of K and L be

$$M_i = m_i Y, \qquad 0 < m_i < 1, \qquad i = 1, 2. \tag{2}$$

Rates of growth of the efficiency of inputs are given by technological progress functions as linear homogeneous functions of the expenditure per unit of input and the level of technological progress of that factor:

$$\dot{A} = H_1\left[\frac{m_1 Y}{K}, A\right] = A h_1\left[\frac{m_1 Y}{AK}\right]$$

and

$$\dot{B} = H_2\left[\frac{m_2 Y}{L}, B\right] = B h_2\left[\frac{m_1 Y}{BL}\right].$$

The economics assumption behind these equations is that rate of growth of efficiency increases with expenditure per unit of input. Higher levels of efficiency would lead to a higher rate of growth but the proportionate growth rate of efficiency will decrease as the level of factor augmentation increases. These equations can be rewritten as

$$\frac{\dot{A}}{A} = h_1\left(\frac{m_1 Y}{AK}\right) \tag{3a}$$

$$\frac{\dot{B}}{B} = h_2\left(\frac{m_2 Y}{BL}\right) \tag{3b}$$

where, because of the concavity assumption, $h_i'' < 0 < h_i'$, $i = 1, 2$.

Capital in efficiency terms, AK, increases either because of the increase in A or in K. Following Ramsey (1928), we assume that the saving rate is determined by the economy as a solution to the optimization problem. Under the assumption of one malleable output, most growth models of the Solow-Swan type assume that a unit of savings and investment will lead to a unit increase in capital; this assumes a linear transformation curve between consumption goods and capital goods.

Combining the traditional assumption with technological progress functions implies that an allocation of a unit of resources would bring about a unit increase in K whereas additional increases in research expenditures will bring about only smaller and smaller increases in A. These models have an inherent bias against investment in capital augmenting technological progress. It is uneconomic to enhance AK by increasing A, and the model leads to Harrod-neutral technological progress in equilibrium. Following Liviatan and Samuelson (1969), we assume that the transformation of consumption goods into capital goods is nonlinear, with the function taking a form similar to that of technological progress functions:

$$\dot{K} = H_3\left[\frac{M_3}{A}, K\right] = H_3\left[\frac{m_3 Y}{A}, K\right] = K h_3\left[\frac{m_3 Y}{AK}\right]$$

or

$$\frac{\dot{K}}{K} = h_3\left[\frac{m_3 Y}{AK}\right]. \quad \textbf{(4)}$$

In this model, the increase in K and A are both subject to diminishing returns and the bias against capital-augmenting technological progress is eliminated. Notice that the forms of the technological progress functions are such that it satisfies the necessary condition stated by Sato (1996).

Labor grows at a constant proportionate rate,

$$\dot{L}/L = n. \quad \textbf{(5)}$$

Output can grow if A, K, B, or L increases. Increase in L is taken to be exogenous whereas increases in the other three require allocation of current output, and so, Y is allocated between consumption, savings, and expenditures on research:

$$Y = C + (m_1 + m_2 + m_3)Y. \quad \textbf{(6)}$$

We assume a linear utility function with per-capita consumption as the argument:

$$c(t) = \frac{C(t)}{L(t)} = \frac{Y(t)}{L(t)}(1 - m_1 - m_2 - m_3), \quad \textbf{(7)}$$

where t is the time and the second equality follows from (6).

The society chooses the three control variables, m_1, m_2, and m_3 so as to maximize the discounted value of the sum of utility over an infinite time. After substituting $L(t) = L_0 e^{nt} = e^{nt}$ in (7) (with L_0 equal to unity) and using (1), (3), and (4), the maximization problem can be written as

$$\underset{\{m_1, m_2, m_3\}}{\text{Maximize}} \int_0^{\infty} e^{-(\rho+n)t} F(AK, BL)(1 - m_1 - m_2 - m_3)\, dt \quad \textbf{(8)}$$

subject to

$$\frac{\dot{A}}{A} + \frac{\dot{K}}{K} = h_1\left[\frac{m_1 F(AK, BL)}{AK}\right] + h_3\left[\frac{m_3 F(AK, BL)}{AK}\right]$$

https://doi.org/10.1017/S1365100599012018 Published online by Cambridge University Press

and

$$\frac{\dot{B}}{B} + \frac{\dot{L}}{L} = h_2\left[\frac{m_2 F(AK, BL)}{BL}\right] + n.$$

The current-value Hamiltonian can be written as

$$\tilde{H} = F(AK, BL)(1 - m_1 - m_2 - m_3) + P_1 AK[h_1() + h_3()] + P_2 BL[h_2() + n]. \quad \textbf{(9)}$$

Notice that we are taking AK and BL to be the two state variables and P_1 and P_2 to be the two costate variables. In analyzing this model, we first establish that a steady state with constant AK/BL is consistent with this model and then we use the steady-state growth rate as an auxiliary variable to transform the model into one that is more tractable for considering stability.

The instantaneous values of the control variables m_1, m_2, and m_3 are those that maximize the value of H; differentiating H partially with respect to these variables and the first-order conditions can be written as

$$P_1 h_1'[m_1 F(1, BL/AK)] = 1, \quad \textbf{(10a)}$$

$$P_2 h_2'[m_2 F(AK/BL, 1)] = 1, \quad \textbf{(10b)}$$

and

$$P_1 h_3'[m_3 F(1, BL/AK)] = 1. \quad \textbf{(10c)}$$

See Appendix A for details.

The values of m_i, $i = 1, 2$, and 3, that are obtained by solving (10) are written as $\tilde{m}_i$. Notice that $\tilde{m}_1$ and $\tilde{m}_3$ are functions of P_1 and AK/BL only and m_2 of P_2 and AK/BL; this separability will be of use in drawing phase diagrams for the model. The equations of motion of the two costate variables, obtained from the partial differential of the Hamiltonian with respect to the corresponding state variables, are given by

$$\dot{P}_1 = -F_1 + P_1(\rho + n - h_1 - h_3) + (\tilde{m}_1 + \tilde{m}_3)(F/AK), \quad \textbf{(11a)}$$

$$\dot{P}_2 = -F_2 + P_2(\rho - h_2) + (\tilde{m}_2 F/BL). \quad \textbf{(11b)}$$

If AK/BL is a constant at the steady state, then

$$\left(\frac{\dot{A}}{A}\right)^* + \left(\frac{\dot{K}}{K}\right)^* = h_1^* + h_3^* = \left(\frac{\dot{B}}{B}\right)^* + \left(\frac{\dot{L}}{L}\right)^* = h_2^* + n = \left(\frac{\dot{Y}}{Y}\right)^*, \quad \textbf{(12)}$$

where * indicates the values at steady state. Substituting (12) into (11) and setting $\dot{P}_i = 0$, $i = 1, 2$, we get

$$P_1^*\left(\rho - h_2^*\right) = F_1 - \left(\tilde{m}_1^* + \tilde{m}_3^*\right)\frac{F}{AK} \quad \textbf{(13a)}$$

and

$$P_2^*\left(\rho - h_2^*\right) = F_2 - \tilde{m}_2^* \frac{F}{BL}. \tag{13b}$$

Multiplying (13a) by AK and (13b) by BL, adding the two equations, and dividing both sides of the equality by Y and by $(\rho - h_2^*)$, we get

$$\frac{P_1^* \cdot AK + P_2^* \cdot BL}{Y} = \frac{1 - \tilde{m}_1^* - \tilde{m}_2^* - \tilde{m}_3^*}{\rho - h_2^*}. \tag{13c}$$

Because of (12), AK/Y and BL/Y are constants and so are the values of the control variables. Therefore, the equation is consistent with the assumption that AK/BL, P_1^*, and P_2^* are constants.

For an economic interpretation of (13c), note that the numerator on the right-hand side is the steady-state consumption out of a unit of output, c^* for example. Then, per-capita consumption is c^*Y/L. Because Y increases at a rate $h_2^* + n$ while labor increases at a rate n, the per-capita income increases at a rate h_2^* and can be written as $c^* \cdot e^{h_2^* t}$. This consumption stream discounted from zero to infinity at a social discount rate of ρ has a present value $c^*/(\rho - h_2^*)$. The numerator on the right-hand side is the stocks of physical and human capital in efficiency units evaluated at their shadow prices. Hence (13c) states that the asset per unit of output equals the discounted value of consumption per person.

4. ANALYSIS OF STABILITY

To study stability, we introduce an auxiliary variable E whose rate of growth is equal to that of the steady-state growth rate of Y,

$$\dot{E}/E = \dot{Y}^*/Y^* = \varepsilon, \tag{14}$$

and redefine the state variables as

$$g_1 = AK/E, \qquad g_2 = BL/E. \tag{15}$$

The production function and per-capita consumption can be written as functions of g_1 and g_2:

$$Y = EF\left(\frac{AK}{E}, \frac{BL}{E}\right) = EF(g_1, g_2)$$

and

$$c = \frac{C}{L} = \frac{E}{L} F(g_1, g_2)(1 - m_1 - m_2 - m_3).$$

Further, the technological progress functions can be expressed as

$$\frac{\dot{A}}{A} = h_1\left[\frac{m_1 F(g_1, g_2)}{g_1}\right], \tag{16a}$$

https://doi.org/10.1017/S1365100599012018 Published online by Cambridge University Press

$$\frac{\dot{B}}{B} = h_2\left[\frac{m_2 F(g_1, g_2)}{g_2}\right], \tag{16b}$$

and

$$\frac{\dot{K}}{K} = h_3\left[\frac{m_3 F(g_1, g_2)}{g_1}\right]. \tag{16c}$$

The current-value Hamiltonian now can be written as

$$H = F(g_1, g_2)(1 - m_1 - m_2 - m_3) + P_1 g_1(h_1 + h_3 - \varepsilon) + P_2 g_2(h_2 + n - \varepsilon). \tag{17}$$

(See Appendix B.) The first-order conditions are given by equations similar in form to (10) with the appropriate transformation of variables. The equations of motion of the state and costate variables are given by

$$\dot{g}_1 = (h_1 + h_3 - \varepsilon)g_1, \tag{18a}$$

$$\dot{g}_2 = (h_2 + n - \varepsilon)g_2, \tag{18b}$$

$$\dot{P}_1 = P_1(\rho + n - h_1 - h_3) - F_{g_1} + (\tilde{m}_1 + \tilde{m}_3)\frac{F}{g_1}, \tag{18c}$$

$$\dot{P}_2 = P_2(\rho - h_2) - F_{g_2} + \tilde{m}_2\frac{F}{g_2}. \tag{18d}$$

Except for a change of variables, the model is the same as the one in Section 2 and so, the properties of the model including that of the steady state are the same. The stability of the system can be examined by evaluating the Jacobian of the system of differential equations (18) at the steady state and calculating the characteristic roots. Direct calculation shows that one of the characteristic roots is zero, indicating that the dimensions of the system can be reduced once more.

We define a new variable $\theta = (g_1/g_2) = (AK/BL)$. The equations (18a–d) are replaced by

$$\frac{\dot{\theta}}{\theta} = h_1\left[\frac{\tilde{m}_1 f(\theta)}{\theta}\right] + h_3\left[\frac{\tilde{m}_3 f(\theta)}{\theta}\right] - h_2[\tilde{m}_2 f(\theta)] - n \tag{19a}$$

where

$$\frac{F(g_1, g_2)}{g_2} = F(g_1/g_2, 1) = F(\theta, 1) = f(\theta), \text{ and}$$

$$\dot{P}_1 = -f'(\theta) + P_1\{\rho + n - h_1 - h_3\} + (\tilde{m}_1 + \tilde{m}_3)\frac{f(\theta)}{\theta} \tag{19b}$$

$$\dot{P}_2 = -\{f(\theta) - \theta f'(\theta)\} + P_2\{\rho - h_2\} + \tilde{m}_2 f(\theta) \tag{19c}$$

The Jacobian of the system evaluated at the steady state is

$$\begin{bmatrix} 0 & v_1 & -v_2 \\ -f'' & \rho - h_2^* & 0 \\ \theta^* f' & 0 & \rho - h_2^* \end{bmatrix}$$

https://doi.org/10.1017/S1365100599012018 Published online by Cambridge University Press

where

$$v_1 = -\frac{\theta^*}{P_1^*}\left[\frac{(h_1')^2}{h_1''} + \frac{(h_3')^2}{h_3''}\right]^* > 0$$

and

$$v_2 = -\frac{\theta^*}{P_2^*}\left[\frac{(h_3')^2}{h_2''}\right]^* > 0.$$

(See Appendix C for details of calculation.)

The characteristic equation can be written as

$$\left(\rho - h_2^* - \lambda\right)\left\{-\lambda\left(\rho - h_2^* - \lambda\right) + v_1 f'' + v_2 f''\theta\right\} = 0. \tag{20}$$

The three characteristic roots are

$$\lambda_1 = \rho - h_2^*,$$

$$\lambda_2 = \frac{\rho - h_2^* + \sqrt{\left(\rho - h_2^*\right)^2 - 4f''(v_1 + v_2\theta)}}{2},$$

$$\lambda_3 = \frac{\rho - h_2^* - \sqrt{\left(\rho - h_2^*\right)^2 - 4f''(v_1 + v_2\theta)}}{2}.$$

We assume that $\rho > h_2^*$; hence λ_1 and λ_2 are positive and λ_3 is negative. We have a saddle point of type 2.

The stability analysis can be illustrated using the phase diagram (Figure 1). Even though the diagram in its full generality is three-dimensional—corresponding to the variables θ, P_1, and P_2—it can be reduced to a series of two-dimensional cross sections.

First, as noted in the discussion of (10), $\tilde{m}_1$ and $\tilde{m}_3$ are functions of θ and P_1 only; hence $\dot{P}_1$ is a function of θ and P_1 only. Next, $\tilde{m}_2$ is a function of P_2 and θ; for given value of P_2, it is a function of θ. Hence, from equations (19), we see that for a given value of P_2, $\dot{\theta}$ are functions of P_1 and θ. Hence, we can fix the value of P_2 and study the motion of the system in the cross section of the phase diagram.[2]

The Jacobian of the reduced system is

$$\begin{bmatrix} 0 & v_1 \\ -f'' & \rho - \beta \end{bmatrix},$$

and the roots of the characteristic equation is given by

$$\lambda = \frac{(\rho - \beta) \pm \sqrt{(\rho - \beta)^2 - 4f''v_1}}{2}.$$

One of the roots is positive and the other is negative, showing saddle-point stability in the cross section. As stated in note 2, the value of θ corresponding to the singular points of these equations will vary from cross section to cross section. The value of

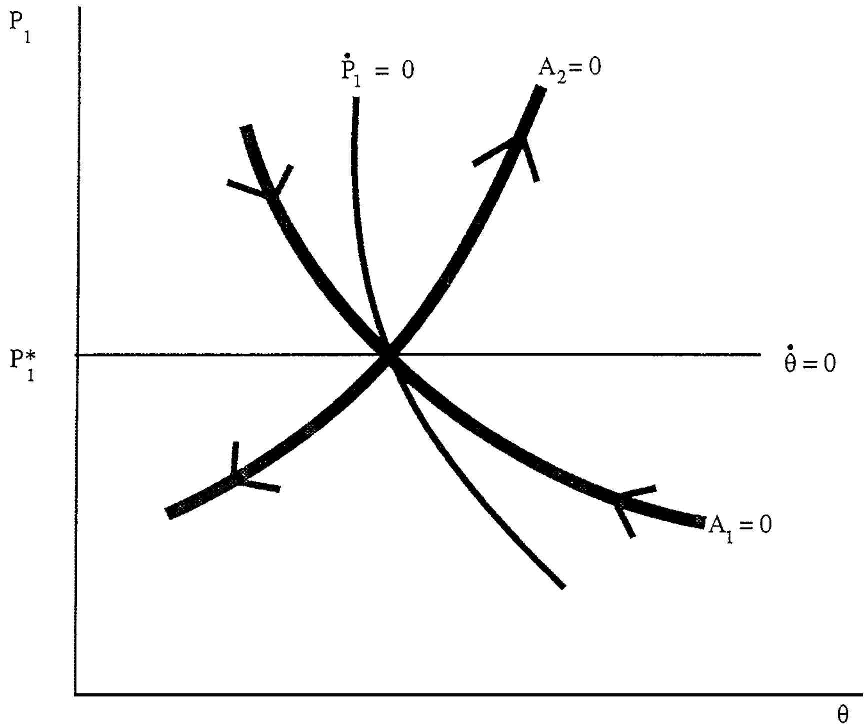

FIGURE 1. Phase diagram in $P_1 - \theta$ plane.

P_2 corresponding to the steady state of the model is already derived. For that value of P_2, the system will show the convergence to the steady state in the cross-sectional diagram.

5. CONCLUSION

One of the paradoxes of the neoclassical model of growth is that it has a steady state only if technological progress is Harrod-neutral; a positive rate of labor augmentation has the same net consequence as a higher rate of growth of labor except that the per-capita output in steady state increases at the rate of increase in labor efficiency.

In contrast, this model has an interior solution and a steady state in which there is positive accumulation of capital and increases in the efficiency of both capital and labor. A comparison of the two models seems to indicate that the crucial difference is in the assumption about the effect of expenditure on investment and in increasing the efficiency of capital. Models of optimal endogenous technological progress assume that the expenditure on research has diminishing returns; without this assumption the models will explode. However, investment is assumed to increase capital in a linear manner. Under these assumptions, it is uneconomic for the system to increase AK by increasing A instead of K. The bias of the model toward Harrod-neutral technological progress is obvious.

We followed Liviatan and Samuelson (1969) in assuming a nonlinear transformation of the output to capital good. The output is taken as malleable, capable of being used both as a consumption good and a capital good but there is an implicit transformation process in converting the output into capital good and this transformation function is concave and nonlinear (as in the traditional neoclassical models). Further, we assumed, as in other models of endogeneous technological progress, that the technical progress functions are concave.

At steady state, the ratio of inputs measured in efficiency terms will remain constant and with it the factor shares. If one assumes a CEDD production function, then the elasticity of substitution is also a constant at steady state. However, as the system converges to the steady state, both factor share and the elasticity of substitution will vary. Thus, it provides a growth theoretical model that is consistent with the empirical work of Sato (1970).

NOTES

1. Ironically, the most recent concern is to explain the observed fall in the wages of relatively unskilled workers in the industrial nations. As a survey of the literature shows, some attribute it to the working of the factor price equalization theorem whereas others attribute it to shifts in technology [Burtless (1995)].

2. Note that θ is a function of $\tilde{m}(\theta, P_2)$ so that the singular point of the reduced system is dependent on P_2. As we move over the cross sections, the $\dot{P}_1 = 0$ curve does not shift but the $\dot{\theta} = 0$ curve will. Similarly, we can take cross sections of the phase diagram, keeping P_1 constant, and draw the $\dot{\theta} = 0$ and $\dot{P}_2 = 0$ curves.

REFERENCES

Burtless, G. (1995) International trade and the rise in earnings inequality. *Journal of Economic Literature* 32, 800–816.

Fellner, W. (1961) Two propositions in the theory of induced innovations. *Economic Journal* 71, 305–308.

Gelfand, I.M. & S.V. Fomin (1967) *Calculus of Variations*. Englewood Cliffs, NJ: Prentice-Hall.

Hicks, J.R. (1963) *The Theory of Wages*. London: Macmillan.

Kamien, M.I. & N.L. Schwartz (1969) Induced factor augmenting technical progress from a microeconomic viewpoint. *Econometrica* 37, 668–684.

Kennedy, C. (1964) Induced bias in innovation and the theory of distribution. *Economic Journal* 37, 541–547.

Liviatan, N. & P.A. Samuelson (1969) Notes on turnpikes: Stable and unstable. *Journal of Economic Theory* 1, 454–475.

Nordhouse, W.D. (1967) The optimal rate and direction of technical change. In K. Shell (ed.), *Essays in the Theory of Optimal Economic Growth*, pp. 53–66. Cambridge, MA: MIT Press.

Ramsey, F.P. (1928) A mathematical theory of saving. *Economic Journal* 38, 543–559.

Salter, W.E.G. (1960) *Productivity and Technical Change*. Cambridge, England: Cambridge University Press.

Samuelson, P.A. (1965) A theory of induced innovations along Kennedy-Wizaker lines. *Review of Economics and Statistics* 47, 343–356.

Samuelson, P.A. & R.M. Solow (1956) A complete capital model involving heterogenoeus capital goods. *Quarterly Journal of Economics* 70, 537–562.

Sato, R. (1970) The estimation of biased technical progress and the production function. *International Economic Review* 11, 179–208.
Sato, R (1996) A note on modeling endogenous growth. *Keio Economic Studies* 39, 343–356.
Sato, R. & R. Ramachandran (1987) Factor price variation and the Hicksian hypothesis: A microeconomic model, *Oxford Economic Papers* 39, 343–356.
Solow, R. (1957) Technical change and the aggregate production function. *Review of Economics and Statistics* 39, 312–320.

APPENDIX A: DERIVATION OF BASIC MODEL

From equation (8), the Hamiltonian of the system can be written as

$$\tilde{H} = e^{-(\rho+n)t} F(AK, BL)(1 - m_1 - m_2 - m_3) + q_2 BL \left\{ h_2 \left[\frac{m_2 F(AK, BL)}{BL} \right] + n \right\} + q_1 AK \left\{ h_1 \left[\frac{m_1 F(AK, BL)}{AK} \right] + h_3 \left[\frac{m_3 F(AK, BL)}{AK} \right] \right\}. \quad \textbf{(A.1)}$$

The first-order conditions are as follows:

$$\tilde{H}_{m_1} = 0 \Rightarrow q_1 e^{(\rho+n)t} h_1' = 1, \quad \textbf{(A.2a)}$$

$$\tilde{H}_{m_2} = 0 \Rightarrow q_2 e^{(\rho+n)t} h_2' = 1, \quad \textbf{(A.2b)}$$

$$\tilde{H}_{m_3} = 0 \Rightarrow q_1 e^{(\rho+n)t} h_3' = 1, \quad \textbf{(A.2c)}$$

$$\dot{q}_1 = -\frac{\partial \tilde{H}}{\partial (AK)} = -e^{-(\rho+n)t} F_1 (1 - m_1 - m_2 - m_3) - q_1 (h_1' m_1 + h_3' m_3) \frac{F_1 AK - F}{AK} - q_1 (h_1 + h_3) - q_2 h_2' m_2 F_1, \quad \textbf{(A.2d)}$$

$$\dot{q_2} = -\frac{\partial \tilde{H}}{\partial (BL)} = -e^{-(\rho+n)t} F_2 (1 - m_1 - m_2 - m_3) - q_1 (h_1' m_1 + h_3' m_3) F_2 - q_2 (h_2 + n) - q_2 h_2' m_2 \frac{F_2 BL - F}{BL}. \quad \textbf{(A.2e)}$$

Let

$$P_1 = q_1 e^{(\rho+n)t}, \quad \textbf{(A.3a)}$$

$$P_2 = q_2 e^{(\rho+n)t}; \quad \textbf{(A.3b)}$$

then,

$$\dot{P}_1 = \dot{q}_1 e^{(\rho+n)t} + (\rho + n) P_1, \quad \textbf{(A.4a)}$$

$$\dot{P}_2 = \dot{q}_2 e^{(\rho+n)t} + (\rho + n) P_2. \quad \textbf{(A.4b)}$$

Using (A.3), from (A.2a), (A.2b), and (A.2c), we get

$$P_1 h_1' \left[\frac{m_1 F(AK, BL)}{AK}\right] = 1, \quad \textbf{(A.5a)}$$

$$P_2 h_2' \left[\frac{m_2 F(AK, BL)}{BL}\right] = 1, \quad \textbf{(A.5b)}$$

$$P_1 h_3' \left[\frac{m_3 F(AK, BL)}{BL}\right] = 1. \quad \textbf{(A.5c)}$$

Multiplying (A.2d) and (A.2e) by $e^{(\rho+n)t}$ and substituting the two equations into (A.4a) and (A.4b), respectively, we get

$$\dot{P}_1 = -F_1 + P_1 \left\{\rho + n + h_1\left[\frac{\tilde{m}_1 F(AK, BL)}{AK}\right] - h_3\left[\frac{\tilde{m}_3 F(AK, BL)}{AK}\right]\right\}$$
$$+ (\tilde{m}_1 + \tilde{m}_3)\frac{F(AK, BL)}{AK} \quad \textbf{(A.5d)}$$

$$\dot{P}_2 = -F_2 + P_2 \left\{\rho - h_2\left[\frac{\tilde{m}_2 F(AK, BL)}{BL}\right]\right\} + \tilde{m}_2 \frac{F(AK, BL)}{BL}. \quad \textbf{(A.5e)}$$

APPENDIX B: MODEL WITH AUXILIARY VARIABLE

After the transformations [see equations (14)–(16)], the original problem [equation (8)] can be written as

$$\underset{\{m_1, m_2, m_3\}}{\text{Maximize}} \int_0^\infty e^{-(\rho+n-\varepsilon)t} F(g_1, g_2)(1 - m_1 - m_2 - m_3)\, dt$$

subject to

$$\frac{\dot{g}_1}{g_1} = h_1\left[\frac{m_1 F(g_1, g_2)}{g_1}\right] + h_3\left[\frac{m_3 F(g_1, g_2)}{g_1}\right] - \varepsilon, \quad \textbf{(B.1a)}$$

$$\frac{\dot{g}_2}{g_2} = h_2\left[\frac{m_2 F(g_1, g_2)}{g_2}\right] + n - \varepsilon. \quad \textbf{(B.1b)}$$

The current-value Hamiltonian is

$$H = F(g_1, g_2)(1 - m_1 - m_2 - m_3) + P_2 g_2 \left\{h_2\left[\frac{m_2 F(g_1, g_2)}{g_2}\right] + n - \varepsilon\right\}$$
$$+ P_1 g_1 \left\{h_1\left[\frac{m_1 F(g_1, g_2)}{g_1}\right] + h_3\left[\frac{m_3 F(g_1, g_2)}{g_1}\right] - \varepsilon\right\}. \quad \textbf{(B.2)}$$

The first-order conditions are

$$P_1 h_1' \left[\frac{m_1 F(g_1, g_2)}{g_1} \right] = 1, \tag{B.3a}$$

$$P_2 h_2' \left[\frac{m_2 F(g_1, g_2)}{g_2} \right] = 1, \tag{B.3b}$$

$$P_1 h_3' \left[\frac{m_3 F(g_1, g_2)}{g_1} \right] = 1. \tag{B.3c}$$

The equation of motion of the costate variables can be written as

$$\begin{aligned} \dot{P}_1 &= P_1(\rho + n - \varepsilon) - \frac{\partial H}{\partial g_1} \\ &= P_1 \left\{ \rho + n - h_1 \left[\frac{\tilde{m}_1 F(g_1, g_2)}{g_1} \right] - h_3 \left[\frac{\tilde{m}_3 F(g_1, g_2)}{g_1} \right] \right\} \\ &\quad - F_{g_1} + (\tilde{m}_1 + \tilde{m}_3) \frac{F(g_1, g_2)}{g_1}, \end{aligned} \tag{B.3d}$$

$$\begin{aligned} \dot{P}_2 &= P_2(\rho + n - \varepsilon) - \frac{\partial H}{\partial g_2} \\ &= P_2 \left\{ \rho - h_2 \left[\frac{\tilde{m}_2 F(g_1, g_2)}{g_2} \right] \right\} - F_{g_2} + \tilde{m}_2 \frac{F(g_1, g_2)}{g_2}. \end{aligned} \tag{B.3e}$$

APPENDIX C: STABILITY

Define $\theta = (g_1/g_2) = (AK/BL)$ and remember that

$$\frac{F(AK, BL)}{AK} = \frac{F(g_1, g_2)}{g_1} = \frac{F(\theta, 1)}{\theta} = \frac{f(\theta)}{\theta},$$

$$\frac{F(AK, BL)}{BL} = \frac{F(g_1, g_2)}{g_2} = F(\theta, 1) = f(\theta),$$

$$F_1 = F_{g_1} = f'(\theta),$$

and

$$F_2 = F_{g_2} = f(\theta) - \theta f'(\theta).$$

https://doi.org/10.1017/S1365100599012018 Published online by Cambridge University Press

Then, from either (A.5) or (B.3), we can get

$$P_1 h_1'\left[\frac{m_1 f(\theta)}{\theta}\right] = 1, \tag{C.1a}$$

$$P_2 h_2'[m_2 f(\theta)] = 1, \tag{C.1b}$$

$$P_1 h_3'\left[\frac{m_3 f(\theta)}{\theta}\right] = 1. \tag{C.1c}$$

Solving (C.1a), (C.1b), and (C.1c), we get $\tilde{m}_i$ $(i = 1, 2, 3)$ and substitute into the equations of motion for costate variables:

$$\dot{P}_1 = -f'(\theta) + P_1\left\{\rho + n - h_1\left[\frac{\tilde{m}_1 f(\theta)}{\theta}\right] - h_3\left[\frac{\tilde{m}_3 f(\theta)}{\theta}\right]\right\} + (\tilde{m}_1 + \tilde{m}_3)\frac{f(\theta)}{\theta}, \tag{C.1d}$$

$$\dot{P}_2 = -[f(\theta) - \theta f'(\theta)] + P_2\{\rho - h_2[\tilde{m}_2 f(\theta)]\} + \tilde{m}_2 f(\theta). \tag{C.1e}$$

From the definition of $\theta = (AK/BL)$, we can get

$$\frac{\dot{\theta}}{\theta} = h_1\left[\frac{\tilde{m}_1 f(\theta)}{\theta}\right] + h_3\left[\frac{\tilde{m}_3 f(\theta)}{\theta}\right] - h_2[\tilde{m}_2 f(\theta)] - n. \tag{C.1f}$$

The stability conditions of the system in the neighborhood of the steady state can be determined by the characteristic roots of the Jacobian matrix,

$$J = \frac{D(\dot{\theta}, \dot{P}_1, \dot{P}_2)}{D(\theta, P_1, P_2)} = \begin{bmatrix} \frac{\partial\dot{\theta}}{\partial\theta} & \frac{\partial\dot{\theta}}{\partial P_1} & \frac{\partial\dot{\theta}}{\partial P_2} \\ \frac{\partial\dot{P}_1}{\partial\theta} & \frac{\partial\dot{P}_1}{\partial P_1} & \frac{\partial\dot{P}_1}{\partial P_2} \\ \frac{\partial\dot{P}_2}{\partial\theta} & \frac{\partial\dot{P}_2}{\partial P_1} & \frac{\partial\dot{P}_2}{\partial P_2} \end{bmatrix}. \tag{C.2}$$

Using equations (C.1a), (C.1b), and (C.1c), we get

$$\frac{\partial\tilde{m}_1}{\partial\theta} = -\frac{P_1 h_1''\tilde{m}_1(\theta f' - f)/\theta^2}{P_1 h_1'' f/\theta} = -\frac{\tilde{m}_1}{\theta f}(\theta f' - f), \tag{C.3a}$$

$$\frac{\partial\tilde{m}_1}{\partial P_1} = -h'/(P_1 h_1'' f/\theta) = -\theta h_1'/(P_1 h_1'' f), \tag{C.3b}$$

$$\frac{\partial\tilde{m}_1}{\partial P_2} = 0, \tag{C.3c}$$

$$\frac{\partial\tilde{m}_2}{\partial\theta} = -\frac{P_2 h_2''\tilde{m}_2 f}{P_2 h_2'' f} = -\frac{\tilde{m}_2 f'}{f}, \tag{C.3d}$$

$$\frac{\partial\tilde{m}_2}{\partial P_1} = 0, \tag{C.3e}$$

https://doi.org/10.1017/S1365100599012018 Published online by Cambridge University Press

$$\frac{\partial \tilde{m}_2}{\partial P_2} = -h_2'/P_2 h_2'' f, \tag{C.3f}$$

$$\frac{\partial \tilde{m}_3}{\partial \theta} = -\frac{P_1 h_3'' \tilde{m}_3 (\theta f' - f)/\theta^2}{P_1 h_3'' f/\theta} = -\frac{\tilde{m}_3}{\theta f}(\theta f' - f), \tag{C.3g}$$

$$\frac{\partial \tilde{m}_3}{\partial P_1} = -h_3'/(P_1 h_3'' f/\theta) = -\theta h_3'/(P_1 h_3'' f), \tag{C.3h}$$

$$\frac{\partial \tilde{m}_3}{\partial P_2} = 0. \tag{C.3i}$$

Then,

$$\frac{\partial(\tilde{m}_1 f/\theta)}{\partial \theta} = \frac{\partial \tilde{m}_1}{\partial \theta}\frac{f}{\theta} + \tilde{m}_1(\theta f' - f)/\theta^2 = 0, \tag{C.4a}$$

$$\frac{\partial(\tilde{m}_1 f/\theta)}{\partial P_1} = \frac{\partial \tilde{m}_1}{\partial P_1}\frac{f}{\theta} = -h_1'/P_1 h_1'', \tag{C.4b}$$

$$\frac{\partial(\tilde{m}_1 f/\theta)}{\partial P_2} = 0, \tag{C.4c}$$

$$\frac{\partial(\tilde{m}_2 f)}{\partial \theta} = \frac{\partial \tilde{m}_2}{\partial \theta} f + \tilde{m}_2 f' = -\tilde{m}_2 f' + \tilde{m}_2 f' = 0, \tag{C.4d}$$

$$\frac{\partial(\tilde{m}_2 f)}{\partial P_1} = 0, \tag{C.4e}$$

$$\frac{\partial(\tilde{m}_2 f)}{\partial P_2} = \frac{\partial \tilde{m}_2}{\partial P_2} f = -h_2'/P_2 h_2'', \tag{C.4f}$$

$$\frac{\partial(\tilde{m}_3 f/\theta)}{\partial \theta} = \frac{\partial \tilde{m}_3}{\partial \theta}\frac{f}{\theta} + \tilde{m}_3(\theta f' - f)/\theta^2 = 0, \tag{C.4g}$$

$$\frac{\partial(\tilde{m}_3 f/\theta)}{\partial P_1} = \frac{\partial \tilde{m}_3}{\partial P_1}\frac{f}{\theta} = -h_3'/P_1 h_3'', \tag{C.4h}$$

$$\frac{\partial(\tilde{m}_3 f/\theta)}{\partial P_2} = 0, \tag{C.4i}$$

The elements of the Jacobian matrix, which are evaluated at the steady state, are as follows:

$$\left(\frac{\partial \dot{\theta}}{\partial \theta}\right)^* = \theta^* \left[h_1' \frac{\partial(\tilde{m}_1 f/\theta)}{\partial \theta} + h_3' \frac{\partial(\tilde{m}_3 f/\theta)}{\partial \theta} - h_2' \frac{\partial(\tilde{m}_2 f)}{\partial \theta}\right]^* = 0, \tag{C.5a}$$

$$\begin{aligned}\left(\frac{\partial \dot{\theta}}{\partial P_1}\right)^* &= \theta^* \left[h_1' \frac{\partial(\tilde{m}_1 f/\theta)}{\partial P_1} + h_3' \frac{\partial(\tilde{m}_3 f/\theta)}{\partial P_1} - h_2' \frac{\partial(\tilde{m}_2 f)}{\partial P_1}\right]^* \\ &= -\frac{\theta^*}{P_1^*}\left[\frac{(h_1')^2}{h_1''} + \frac{(h_3')^2}{h_3''}\right]^* = v_1, \end{aligned} \tag{C.5b}$$

$$\begin{aligned}\left(\frac{\partial \dot{\theta}}{\partial P_2}\right)^* &= \theta^* \left[h_1' \frac{\partial(\tilde{m}_1 f/\theta)}{\partial P_2} + h_3' \frac{\partial(\tilde{m}_3 f/\theta)}{\partial P_2} - h_2' \frac{\partial(\tilde{m}_2 f)}{\partial P_2}\right]^* \\ &= -\frac{\theta^*}{P_2^*}\left[\frac{(h_2')^2}{h_2''}\right]^* = -v_2, \end{aligned} \tag{C.5c}$$

https://doi.org/10.1017/S1365100599012018 Published online by Cambridge University Press

$$\left(\frac{\partial \dot{P}_1}{\partial \theta}\right)^* = -f'' + P_1^*\left[-h_1'\frac{\partial(\tilde{m}_1 f/\theta)}{\partial\theta} - h_3'\frac{\partial(\tilde{m}_3 f/\theta)}{\partial\theta}\right]^*$$

$$+\frac{\partial(\tilde{m}_1 f/\theta)}{\partial\theta} + \frac{\partial(\tilde{m}_3 f/\theta)}{\partial\theta} = -f'' \qquad \textbf{(C.5d)}$$

(since $P_1 h_i' - 1 = 0$, $i = 1$ & 3),

$$\left(\frac{\partial \dot{P}_1}{\partial P_1}\right)^* = \left(\rho + n - h_1^* - h_3^*\right) - P_1^*\left[h_1'\frac{\partial(\tilde{m}_1 f/\theta)}{\partial P_1} + h_3'\frac{\partial(\tilde{m}_3 f/\theta)}{\partial P_1}\right]^*$$

$$+\frac{\partial(\tilde{m}_1 f/\theta)}{\partial P_1} + \frac{\partial(\tilde{m}_3 f/\theta)}{\partial P_1} = \rho - h_2^* \qquad \textbf{(C.5e)}$$

(at the steady state, $h_1^* + h_3^* = h_2^* + n$),

$$\left(\frac{\partial \dot{P}_1}{\partial P_2}\right)^* = -P_1^*\left[h_1'\frac{\partial(\tilde{m}_1 f/\theta)}{\partial P_2} + h_3'\frac{\partial(\tilde{m}_3 f/\theta)}{\partial P_2}\right]^*$$

$$+\frac{\partial(\tilde{m}_1 f/\theta)}{\partial P_2} + \frac{\partial(\tilde{m}_3 f/\theta)}{\partial P_2} = 0, \qquad \textbf{(C.5f)}$$

$$\left(\frac{\partial \dot{P}_2}{\partial \theta}\right)^* = \theta^* f'' - \left[(P_2 h_2')^* - 1\right]\frac{\partial(\tilde{m}_2 f)}{\partial\theta} = \theta^* f'', \qquad \textbf{(C.5g)}$$

$$\left(\frac{\partial \dot{P}_2}{\partial P_1}\right)^* = -\left[(P_2 h_2')^* - 1\right]\frac{\partial(\tilde{m}_2 f)}{\partial P_1} = 0, \qquad \textbf{(C.5h)}$$

$$\left(\frac{\partial \dot{P}_2}{\partial P_2}\right)^* = \rho - h_2^* - \left[(P_2 h_2')^* - 1\right]\frac{\partial(\tilde{m}_2 f)}{\partial P_2} = \rho - h_2^*. \qquad \textbf{(C.5i)}$$

So, substituting (C.5) into (C.2), we get

$$J^* = \frac{D(\dot{\theta}, \dot{P}_1, \dot{P}_2)^*}{D(\theta, P_1, P_2)^*} = \begin{pmatrix} \frac{\partial\dot{\theta}}{\partial\theta} & \frac{\partial\dot{\theta}}{\partial P_1} & \frac{\partial\dot{\theta}}{\partial P_2} \\ \frac{\partial\dot{P}_1}{\partial\theta} & \frac{\partial\dot{P}_1}{\partial P_1} & \frac{\partial\dot{P}_1}{\partial P_2} \\ \frac{\partial\dot{P}_2}{\partial\theta} & \frac{\partial\dot{P}_2}{\partial P_1} & \frac{\partial\dot{P}_2}{\partial P_2} \end{pmatrix}^* = \begin{pmatrix} 0 & v_1 & -v_2 \\ -f'' & \rho - h_2^* & 0 \\ \theta^* f'' & 0 & \rho - h_2^* \end{pmatrix}$$

$$\textbf{(C.2}'\textbf{)}$$

where

$$v_1 = -\frac{\theta^*}{P_1^*}\left[\frac{(h_1')^2}{h_1''} + \frac{(h_3')^2}{h_3''}\right]^* > 0,$$

and

$$v_2 = -\frac{\theta^*}{P_2^*}\left[\frac{(h_2')^2}{h_2''}\right]^* > 0.$$

https://doi.org/10.1017/S1365100599012018 Published online by Cambridge University Press