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Connectedness of Julia sets for a quadratic random dynamical system

ZHIMIN GONG†‡§, WEIYUAN QIU‡ and YING LI§

† *Business School, Xiangtan University, Xiangtan 411105, People's Republic of China*
(e-mail: gggzm@vip.sina.com)
‡ *Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, People's Republic of China*
§ *Department of Mathematics, Xiangtan University, Xiangtan 411105, People's Republic of China*

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Abstract. For a sequence (c_n) of complex numbers, the quadratic polynomials $f_{c_n} := z^2 + c_n$ and the sequence (F_n) of iterates $F_n := f_{c_n} \circ \cdots \circ f_{c_1}$ are considered. The Fatou set $\mathcal{F}(c_n)$ is defined as the set of all $z \in \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ such that (F_n) is normal in some neighbourhood of z, while the complement $\mathcal{J}(c_n)$ of $\mathcal{F}(c_n)$ (in $\hat{\mathbb{C}}$) is called the Julia set. In this paper we discuss the conditions for $\mathcal{J}(c_n)$ to be totally disconnected. A problem posed by Brück is solved.

1. *Introduction and known results*

We keep some notations used by Brück, but for the reader's convenience, we recall some basic and important notations in [**4**] and [**6**]. For a sequence (c_n) of complex numbers we consider the quadratic polynomials $f_{c_n} := z^2 + c_n$ and the sequence (F_n) of iterates $F_n := f_{c_n} \circ \cdots \circ f_{c_1}$. The Fatou set $\mathcal{F}(c_n)$ is defined as the set of all $z \in \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ such that (F_n) is normal in some neighbourhood of z, while the complement of the Fatou set $\mathcal{F}(c_n)$ (in $\hat{\mathbb{C}}$) is called the Julia set for (c_n), denoted by $\mathcal{J}(c_n)$. A component of $\mathcal{F}(c_n)$ is called a stable domain, or a Fatou component.

Throughout this paper, we always assume that $c_n \in K_\delta := \{z \in \mathbb{C} : |z| \leq \delta\}$ $(n \in \mathbb{N})$ for some $\delta > 0$, that is, $(c_n) \in K_\delta^{\mathbb{N}}$. The filled Julia $\mathcal{K}(c_n)$ is by definition the set of all $z \in \mathbb{C}$ such that $(F_n(z))_{n=1}^{\infty}$ is bounded. From the results in [**8**], we know that both $\mathcal{J}(c_n)$ and $\mathcal{K}(c_n)$ are compact in $\mathbb{C}$ and $\partial\mathcal{K}(c_n) = \mathcal{J}(c_n)$. A simple and important fact is that the filled Julia set for (c_n) is contained in K_{R_δ}, where

$$R_\delta = \tfrac{1}{2}(1 + \sqrt{1 + 4\delta}).$$

In particular, if $c_n = c$ for all $n \in \mathbb{N}$, we write f_c^n instead of F_n. In this case, the Fatou set and Julia set are denoted by $\mathcal{F}(f_c)$ and $\mathcal{J}(f_c)$ respectively. The Fatou set and Julia set are basic objects studied in the iteration theory of a single function. For more information and details on the iteration theory of a single function, see [**1–3, 9, 10, 12, 13**]. We also refer the reader to [**4–8, 11**] for further results on the random dynamical system.

The Mandelbrot set $\mathcal{M}$ is defined as the set of all $c \in \mathbb{C}$ such that $(f_c^n(0))_{n=0}^{\infty}$ is bounded. It is well known that $\mathcal{J}(f_c)$ is either connected or totally disconnected†, depending on whether $c \in \mathcal{M}$ or not. We also know that $K_{1/4}$ is contained in $\mathcal{M}$, and it is the largest disc centred at 0 which is contained in $\mathcal{M}$.

A natural question is the following.

Question 1. What is the condition for $\mathcal{J}(c_n)$ to be totally disconnected?

Brück *et al* obtained a very interesting result in [**6**] on this problem, which is stated as follows.

THEOREM A. *Let $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta > \frac{1}{4}$, and assume that there exists a simply connected bounded domain D such that $\overline{\bigcup_{k=0}^{\infty} F_k(\mathcal{J}(c_n))} \subset D$ and $(f_{c_k} \circ \cdots \circ f_{c_{j+1}})(0) \notin D$ for all $j = 0, 1, \ldots, k-1$ and all $k \in \mathbb{N}$. Then $\mathcal{J}(c_n)$ is totally disconnected.*

In addition, they gave a number of interesting examples. Using their idea, we can get a more general form of Theorem A, and give a sufficient condition in §2 for $\mathcal{J}(c_n)$ to be totally disconnected.

Brück studied the following sets:

$$\mathcal{D} := \{(c_n) \in K_\delta^{\mathbb{N}} : \mathcal{J}(c_n) \text{ is disconnected}\}, \tag{1.1}$$

$$\mathcal{D}_N := \{(c_n) \in K_\delta^{\mathbb{N}} : \mathcal{J}(c_n) \text{ has more than } N \text{ components}\}, \tag{1.2}$$

$$\mathcal{D}_\infty := \{(c_n) \in K_\delta^{\mathbb{N}} : \mathcal{J}(c_n) \text{ has infinitely many components}\}, \tag{1.3}$$

$$\mathcal{T} := \{(c_n) \in K_\delta^{\mathbb{N}} : \mathcal{J}(c_n) \text{ is totally disconnected}\}. \tag{1.4}$$

It is obvious that $\mathcal{T} \subset \mathcal{D}_\infty \subset \mathcal{D}_N \subset \mathcal{D}$, and $\mathcal{D} = \emptyset$ for $\delta \le \frac{1}{4}$ follows from Theorem 1.1 in [**6**].

We equip $K_\delta^{\mathbb{N}}$ with the product topology, where K_δ carries the usual topology induced from $\mathbb{C}$. In other words, it has a subbase

$$\{U_1 \times U_2 \times \cdots : U_i = K_\delta \text{ for } i \ne k, \text{ and } U_k \text{ is open in } K_\delta, k = 1, 2, \ldots\}.$$

Brück discussed the topological properties of $\mathcal{D}$, $\mathcal{D}_N$, $\mathcal{D}_\infty$ and $\mathcal{T}$, and he obtained many interesting results. For example, in [**4**] he proved the following two theorems.

THEOREM B. *The set $\mathcal{T}$ defined by (1.4) is dense in $K_\delta^{\mathbb{N}}$ provided $\delta > \frac{1}{4}$.*

THEOREM C. *The set $\mathcal{D}_\infty$ defined by (1.3) is a countable intersection of dense open subsets of $K_\delta^{\mathbb{N}}$ provided $\delta > \frac{1}{4}$. In particular, $\mathcal{D}_\infty$ is of the second Baire category in $K_\delta^{\mathbb{N}}$, while the complement $K_\delta^{\mathbb{N}} \setminus \mathcal{D}_\infty$ is of the first Baire category in $K_\delta^{\mathbb{N}}$.*

Brück put forward the following question (see Question 1.1 in [**4**]).

Question 2. Is it true that the set $\mathcal{T}$ defined by (1.4) is of the second Baire category in $K_\delta^{\mathbb{N}}$ provided that $\delta > \frac{1}{4}$?

† A set S is said to be totally disconnected if each component of S contains only one point.

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We will give a positive answer to this problem in §3. In fact, it is shown that Theorem C holds true if D_∞ is replaced by $\mathcal{T}$.

2. *Conditions for the Julia set to be totally disconnected*

Let $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta > 0$, and $F_n = f_{c_n} \circ \cdots \circ f_{c_1}$. Let $\mathbb{Z} = \{0\} \cup \mathbb{N}$. Set

$$\mathcal{O}^+(c_n) = \{f_{c_{n+k}} \circ \cdots \circ f_{c_k}(0) : k \in \mathbb{N}, n \in \mathbb{Z}\},$$
$$\mathcal{K}^+(c_n) = \bigcup_{k=0}^{\infty} F_k(\mathcal{K}(c_n)),$$

where $\mathcal{K}(c_n)$ is the filled Julia set and $F_0 =$ identity. It is clear that $z \in F_k(\mathcal{K}(c_n))$ if and only if $(f_{c_{m+k+1}} \circ \cdots \circ f_{c_{k+1}}(z))_{m=1}^{\infty}$ is bounded.

Using the idea of the proof of Theorem 4.2 in [**6**], we can prove the following.

THEOREM 2.1. *Let $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta > 0$. If $\overline{\mathcal{K}^+(c_n)} \cap \overline{\mathcal{O}^+(c_n)} = \emptyset$, then $\mathcal{J}(c_n)$ is totally disconnected.*

Notes.

(1) It is easy to prove that $\mathcal{K}^+(c_n)$ is bounded. In fact, $\mathcal{K}^+(c_n) \subset K_{R_\delta}$.

(2) Under the condition of Theorem A, we have $\overline{\bigcup_{k=0}^{\infty} F_k(\mathcal{K}(c_n))} \subset D$, which implies $\overline{\mathcal{K}^+(c_n)} \cap \overline{\mathcal{O}^+(c_n)} = \emptyset$.

(3) The condition $\overline{\mathcal{K}^+(c_n)} \cap \overline{\mathcal{O}^+(c_n)} = \emptyset$ cannot be replaced by $\overline{\bigcup_{k=0}^{\infty} F_k(\mathcal{J}(c_n))} \cap \overline{\mathcal{O}^+(c_n)} = \emptyset$. This can be shown by taking $c_n = 0, n \in \mathbb{N}$, since in this case $\mathcal{J}(c_n) = \{z : |z| = 1\}$, and $\mathcal{O}^+(c_n) = \{0\}$.

Furthermore, from Theorem 2.1 we can get the following Theorem†.

THEOREM 2.2. *For every $c \notin \mathcal{M}$, there exists a neighbourhood $U(c)$ of c such that $\mathcal{J}(c_n)$ is totally disconnected if $c_n \in U(c)$ for all $n \in \mathbb{N}$.*

Proof of Theorem 2.1. First, from the assumption of the theorem, there exist a finite number of finitely connected domains Ω_i $(i = 1, 2, \dots, k)$ such that

$$\overline{\mathcal{K}^+(c_n)} \subset \Omega = \bigcup_{i=1}^{k} \Omega_i \quad \text{and} \quad \mathcal{O}^+(c_n) \subset \mathbb{C} \setminus \overline{\Omega}.$$

Then for every $z_0 \in \mathcal{J}(c_n)$ we have $z_0 \in F_m^{-1}(\Omega)$ for all $m \in \mathbb{N}$.

Let $z_0 \in \mathcal{J}(c_n)$ be fixed. Since $\Omega = \bigcup_{i=1}^{k} \Omega_i$, for each $m \in \mathbb{N}$, there is one and only one $j(m) \in \{1, 2, \dots, k\}$ such that $z_0 \in F_m^{-1}(\Omega_{j(m)})$. In the sequence $\{j(m)\}_{m=1}^{\infty}$, at least one number of $\{1, 2, \dots, k\}$, say, 1, appears infinitely many times. The corresponding m's are denoted by $\{i_n\}_{n=1}^{\infty}$. That is to say, $z_0 \in F_{i_n}^{-1}(\Omega_1)$ for all $n \in \mathbb{N}$.

Since $\overline{\Omega_1} \cap \mathcal{O}^+(c_n) = \emptyset$, it is easy to see that $F_{i_n}^{-1}(\Omega_1) = f_{c_1}^{-1} \circ \cdots \circ f_{c_{i_n}}^{-1}(\Omega_1)$ consists of exactly 2^{i_n} finitely connected components, and among these components, one and only one contains z_0. Hence, $F_{i_n} : F_{i_n}^{-1}(\Omega_1) \to \Omega_1$ has 2^{i_n} inverse functions, and only one of them, denoted by $g_{i_n} = h_n$, satisfies the condition $z_0 \in h_n(\Omega_1)$.

† The authors thank the referee for his valuable suggestion on the improvement of Theorem 2.2.

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Let $D_R = \{z : |z| < R\}$. Since Ω_1 is bounded, we can choose R large enough such that $\Omega \subset D_R$, and $|f_{c_n}(z)| > |z|$ for all $n \in \mathbb{N}$ provided $|z| \geq R$. Therefore, $h_n(\Omega_1) \subset D_R$ for all $n \in \mathbb{N}$, and consequently (h_n) is normal on Ω_1 by Montel's principle. Let $\{h_{j_n}\}$ be a subsequence of h_n such that $\{h_{j_n}\}$ converges to $I(z)$ locally uniformly on Ω_1. We can prove $I(z)$ is a constant as follows.

Choose two domains U_1, V_1 such that $\overline{V_1} \subset U_1$, $\overline{U_1} \subset \Omega_1$ and $\partial V_1 \cap \overline{\mathcal{K}^+(c_n)} = \emptyset$. Since h_{j_n} converges to $I(z)$ uniformly on $\overline{U_1}$, we have for n large enough

$$h_{j_n}(\overline{V_1}) \subset I(U_1), \tag{2.1}$$

$$I(U_1) \subset h_{j_n}(\Omega_1) \subset F_{j_n}^{-1}(\Omega_1). \tag{2.2}$$

From (2.1) and (2.2) we get

$$\overline{V_1} \subset F_{j_n}(I(U_1)) \subset \Omega_1 \tag{2.3}$$

for n large enough. Thus

$$F_m(I(U_1)) \subset D_R \tag{2.4}$$

for all $m \in \mathbb{N}$.

On the other hand, from (2.3) one can choose a natural number n_0 and a point $z_1 \in I(U_1)$ such that $z_2 = F_{j_{n_0}}(z_1) \in \partial V_1$. Since $\partial V_1 \cap \overline{\mathcal{K}^+(c_n)} = \emptyset$, $f_{c_{q+p}} \circ \cdots \circ f_{c_{q+1}}(z)$ converges to ∞ uniformly on ∂V_1 as $p \to \infty$ for any $q \in \mathbb{N}$ and hence $f_{c_{q+p}} \circ \cdots \circ f_{c_{q+1}}(z_2)$ converges to ∞ $(p \to \infty)$. In particular, $f_{c_{j_{n_0}+p}} \circ \cdots \circ f_{c_1}(z_1) = F_{j_{n_0}+p}(z_1) \to \infty$ as $p \to \infty$, which contradicts (2.4). Therefore, $I(z)$ must be a constant function.

In other words, $g_{i_{j_n}}$ converges to z_0 locally uniformly on Ω_1.

By the hypothesis of the theorem, one may choose finitely connected domains Ω_i' $(i = 1, \ldots, k)$ such that $\overline{\Omega_i'} \subset \Omega_i$ for $i = 1, 2, \ldots, k$, and

$$\overline{\mathcal{K}^+(c_n)} \subset \Omega' = \bigcup_{i=1}^{k} \Omega_i' \quad \text{and} \quad \mathcal{O}^+(c_n) \subset C \setminus \overline{\Omega'}.$$

Then the above argument remains valid for Ω_1' and hence $z_0 \in h_n(\Omega_1')$ for all $n \in \mathbb{N}$, and h_{j_n} converges to z_0 uniformly on Ω_1'. Obviously, the component of $\mathcal{J}(c_n)$ containing z_0 must be contained in $h_{j_n}(\Omega')(n \in \mathbb{N})$, so such a component of $\mathcal{J}(c_n)$ contains only one point z_0. It is evident now that $\mathcal{J}(c_n)$ is totally disconnected. □

Proof of Theorem 2.2. Since $c \notin \mathcal{M}$, $|c| > \frac{1}{4}$. Take $\delta > |c| > \frac{1}{4}$. Let $s = (1 + \sqrt{1 + 4\delta})/2$. For convenience, we suppose that $(c_n) \in K_\delta^{\mathbb{N}}$. As a matter of fact, this is satisfied if $c_n \in \overline{O(c)}$ $(O(c) := O(c, \delta - |c|))$ for all $n \in \mathbb{N}$. Let $D_s = \{z : |z| < s\}$. It is clear that $|f_{c_n}(z)| \geq |z|$ for all $|z| \geq s$ and $n \in \mathbb{N}$. In fact, it is not hard to prove that $\mathcal{K}(c_n) \subset \overline{D_s}$, and furthermore $\mathcal{K}^+(c_n) \subset \overline{D_s}$. In addition, we point out the following fact: if $|z| \geq s + 1$, then $|f_{c_n}(z)| \geq (s + 1)^2 - \delta = 3s + 1$.

It is well known that $f_c^n(0) \to \infty$ as $n \to \infty$ if $c \notin \mathcal{M}$, where $f_c(z) = z^2 + c$. There exists a natural number p such that $f_c^p(0) > s + 2$. Therefore, there exists a neighbourhood $D_0(c)$ of c such that

$$|f_{c_{m+p}} \circ \cdots \circ f_{c_{m+1}}(0)| \geq s + 1$$

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provided that $c_n \in D_0(c)$ for all $n \in \mathbb{N}$. Hence

$$|f_{c_{m+q}} \circ \cdots \circ f_{c_{m+1}}(0)| \geq s+1 \quad (q \geq p),$$

provided that $c_n \in O(c) \cap D_0(c)$ for all $n \in \mathbb{N}$.

Let

$$\begin{aligned} \mathcal{O}_p(c_n) &= \{f_{c_{m+q}} \circ \cdots \circ f_{c_{m+1}}(0) : q \geq p, m \in \mathbb{Z}\}, \\ \mathcal{O}_{p-1}(c_n) &= \{f_{c_{m+p-1}} \circ \cdots \circ f_{c_{m+1}}(0) : m \in \mathbb{Z}\}, \\ &\vdots \\ \mathcal{O}_2(c_n) &= \{f_{c_{m+2}} \circ f_{c_{m+1}}(0) : m \in \mathbb{Z}\}, \\ \mathcal{O}_1(c_n) &= \{f_{c_{m+1}}(0) : m \in \mathbb{Z}\}. \end{aligned}$$

Then we have

$$\overline{\mathcal{O}_p(c_n)} \subset \overline{\hat{\mathbb{C}} \setminus D_{s+1}} \subset \hat{C} \setminus D_{s+1/2}$$

provided $c_n \in O(c) \cap D_0(c)$, $n \in \mathbb{N}$. In other words, we have

$$\overline{\mathcal{K}^+(c_n)} \cap \overline{\mathcal{O}_p(c_n)} = \emptyset,$$

provided that $c_n \in O(c) \cap D_0(c)$, $n \in \mathbb{N}$.

For each $r = 1, 2, \ldots, p-1$, since $|f_c^{p-r}(f_c^r(0))| = |f_c^p(0)| > s+2$, there exists a neighbourhood $U_r = O(f_c^r(0), \varepsilon_r)$ of $f_c^r(0)$ and a neighbourhood $D_r(c)$ of c such that

$$|f_{c_{m+p-r}} \circ \cdots \circ f_{c_{m+1}}(z)| \geq s+1$$

for all $z \in U_r$ and $m \in \mathbb{Z}$ provided that $c_n \in D_r(c)$, $n \in \mathbb{N}$. This implies $\mathcal{K}^+(c_n) \cap U_r = \emptyset$ provided $c_n \in O(c) \cap D_r(c)$, $n \in \mathbb{N}$. Moreover, there exists a neighbourhood $D'_r(c)$ of c such that, when $c_n \in D'_r(c)$, $n \in \mathbb{N}$, we have

$$f_{c_{m+r}} \circ \cdots \circ f_{c_{m+1}}(0) \in U'_r = O\left(f_c^r(0), \frac{\varepsilon_r}{2}\right).$$

That is to say, $\mathcal{O}_r(c_n) \subset U'_r$ provided that $c_n \in D'_r(c)$, $n \in \mathbb{N}$. Therefore,

$$\overline{\mathcal{K}^+(c_n)} \cap \overline{\mathcal{O}_r(c_n)} = \emptyset, \quad r = 1, 2, \ldots, p-1$$

provided $c_n \in O(c) \cap D_r(c) \cap D'_r(c)$, $n \in \mathbb{N}$.

Let $U(c) = O(c) \cap D_0(c) \bigcap_{r=1}^{p-1} D_r(c) \cap D'_r(c)$. Using the fact that $\mathcal{O}^+(c_n) = \bigcup_{r=1}^{p} \mathcal{O}_r(c_n)$, and then $\overline{\mathcal{O}^+(c_n)} = \overline{\bigcup_{r=1}^{p} \mathcal{O}_r(c_n)}$, we obtain $\overline{\mathcal{K}^+(c_n)} \cap \overline{\mathcal{O}^+(c_n)} = \emptyset$ provided $c_n \in U(c)$, $n \in \mathbb{N}$. By Theorem 2.1, $\mathcal{J}(c_n)$ is totally disconnected as required. □

3. *Topological properties of* $\mathcal{T}$

In this section, we study the properties of $\mathcal{T}$ defined by (1.4). Here, we always assume that $\delta > \frac{1}{4}$ to avoid the trivial case $\mathcal{T} \subset \mathcal{D} = \emptyset$, where $\mathcal{D}$ is defined by (1.1). At the end of this section, we will give a positive answer to Question 2 posed by Brück.

It is necessary to recall the notion of Baire category. Let X be a topological space. A set $A \subset X$ is called *nowhere dense* in X if the closure $\overline{A}$ has empty interior, and we say A is of the *first Baire category* in X if A is a countable union of nowhere dense sets. Otherwise, A is of the *second Baire category*.

To prove our theorems we need the Hausdorff metric, defined as follows.

Let A, B be two non-empty compact subsets of $\mathbb{C}$, then the Hausdorff distance between A and B is defined as

$$H(A, B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\right\},$$

where $\operatorname{dist}(a, B) = \inf\{|a - b| : b \in B\}$.

Let $(c_n) \in K_\delta^{\mathbb{N}}$ for some $\delta > 0$. Let $\Delta_R := \{z : |z| > R\}$ and $U_m = F_m^{-1}(\Delta_R)$, then we have from [**8**] the following result.

LEMMA 3.1. *There exists a stable domain* $\mathcal{A}(c_n)(\infty)$ *which contains the point* ∞ *and wherein* $F_n \to \infty$ $(n \to \infty)$ *locally uniformly. If* $R > R_\delta$*, then* $\overline{U}_m \subset U_{m+1}$*, and* $\mathcal{A}(c_n)(\infty) = \bigcup_{m=1}^{\infty} U_m$ *and* $\mathcal{J}(c_n) = \partial\mathcal{A}(c_n)(\infty)$.

Obviously, $\partial U_m = F_m^{-1}(\partial\Delta_R)$. From Lemma 3.1 we can prove the following†.

LEMMA 3.2. $\lim_{m\to\infty} H(\partial Um, \mathcal{J}(c_n)) = 0$.

Proof. We need only to prove

$$\lim_{m\to\infty} \sup_{z\in\mathcal{J}(c_n)} \operatorname{dist}(z, \partial U_m) = 0, \tag{3.1}$$

$$\lim_{m\to\infty} \sup_{z\in\partial U_m} \operatorname{dist}(z, \mathcal{J}(c_n)) = 0. \tag{3.2}$$

Given $\varepsilon > 0$. For every $z \in \mathcal{J}(c_n)$, by Lemma 3.1, there exists a natural number m_z such that

$$O_z \cap U_{m_z} \neq \emptyset \quad\Longrightarrow\quad O_z \cap \partial U_{m_z} \neq \emptyset,$$

where $O_z = O(z, \varepsilon) = \{z' : |z' - z| < \varepsilon\}$. Since $\mathcal{J}(c_n)$ is compact, one can choose a finite number of points $z_1, z_2, \ldots, z_k$ in $\mathcal{J}(c_n)$ such that $\mathcal{J}(c_n) \subset \bigcup_{i=1}^{k} O(z_i, \varepsilon)$.

Let $m_0 = \max\{m_{z_1}, \ldots, m_{z_k}\}$. We have $\operatorname{dist}(z_1, \partial U_{m_0}) < 2\varepsilon$ and hence

$$\operatorname{dist}(z_1, \partial U_m) < 2\varepsilon$$

for all $z_1 \in \mathcal{J}(c_n)$ and $m \geq m_0$. This shows $\sup_{z\in\mathcal{J}(c_n)} \operatorname{dist}(z, \partial U_m) \leq 2\varepsilon$. Then (3.1) is proved.

To prove (3.2), it is enough to show $\rho := \limsup_{m\to\infty} \sup_{z\in\partial U_m} \operatorname{dist}(z, \mathcal{J}(c_n)) = 0$. If the contrary holds, we can choose a subsequence $\{n_\ell\}$ of $\{n\}$ and a sequence $\{z_\ell\}$ of points such that $z_\ell \in \partial U_{n_\ell}$ and $\operatorname{dist}(z_\ell, \mathcal{J}(c_n)) \geq \rho/2$, $\ell \in \mathbb{N}$. We may as well suppose that $z_\ell \to z_0 (\ell \to \infty)$, then $\operatorname{dist}(z_0, \mathcal{J}(c_n)) \geq \rho/2$, which implies $z_0 \in \mathcal{F}(c_n)$. On the other hand, it is easy to see from Lemma 3.1 that $z_0 = \lim_{\ell\to\infty} z_\ell \in \mathcal{J}(c_n) = \partial\mathcal{A}(c_n)(\infty)$. This is a contradiction and (3.2) is proved. From (3.1) and (3.2) we get $\lim_{m\to\infty} H(\partial Um, \mathcal{J}(c_n)) = 0$. □

† The authors thank the referee for his useful suggestion to simplify the proof of this Lemma.

Let $V(c_n)$ be the set composed of all the components of $\mathcal{J}(c_n)$. Denote by $d(u)$ the diameter of u for $u \in V(c_n)$, and let $d^*(c_n) = \sup_{u \in V(c_n)}\{d(u)\}$. Set

$$L_r := \{(c_n) \in K_\delta^{\mathbb{N}} : d^*(c_n) < r\}.$$

We first prove the following.

THEOREM 3.1. *The set L_r is a dense open subset of $K_\delta^{\mathbb{N}}$ provided $\delta > \frac{1}{4}$.*

Proof. Since $L_r \supset \mathcal{T}$, from Theorem B, it is enough to show that L_r is open in K_δ^N. Let $(c_n^0) \in L_r$, then $d^*(c_n^0) = r_0 < r$. Take R large enough such that $|f_{c_n}(z)| > |z|$ for all $|z| \geq R$ and all $(c_n) \in K_\delta^{\mathbb{N}}$, and let $F_m = f_{c_m^0} \circ \cdots \circ f_{c_1^0}$, $D_R = \{z : |z| < R\}$ and $B_m = {F_m}^{-1}(D_R)$. By Lemma 3.2, we have

$$\lim_{m\to\infty} H(\partial B_m, \mathcal{J}(c_n^0)) = \lim_{m\to\infty} H(F_m^{-1}(\partial D_R), \mathcal{J}(c_n^0)) = 0. \tag{3.3}$$

Let $\varepsilon = (r - r_0)/2$. Then we have the following.

ASSERTION. *The diameters of all components of $F_m^{-1}(D_R)$ are less than $r_0 + \varepsilon/2$, provided m is large enough.*

If this is not true, then for every $m \in \mathbb{N}$ there exists a natural number $j(m) > m$ and a component $u_{j(m)}$ of $F_{j(m)}^{-1}(D_R)$ such that

$$d(u_{j(m)}) \geq r_0 + \varepsilon/2. \tag{3.4}$$

Clearly, one can require that $j(m+1) > j(m)$.

It can be shown that

$$u_{j(m)} \cap \mathcal{J}(c_n^0) \neq \emptyset \tag{3.5}$$

for each $m \in \mathbb{N}$. Otherwise, we have $\overline{u_{j(m)}} \subset \mathcal{F}(c_n^0)$ since $\partial u_{j(m)} \subset \mathcal{F}(c_n^0)$. Note that $F_n(z)$ converges to ∞ uniformly on $\partial u_{j(m)}$; we have then

$$F_n(z) \to \infty \quad (k \to \infty) \tag{3.6}$$

uniformly on $\overline{u_{j(m)}}$. Take a point $\gamma \in \mathcal{J}(c_n) \subset \mathcal{K}(c_n)$. Then we have $\alpha = F_{j(m)}(\gamma) \in D_R$ and

$$F_{j(m)+n}(\gamma) = f_{c^0_{j(m)+n}} \circ \cdots \circ f_{c^0_{j(m)+1}}(\alpha) \in D_R \tag{3.7}$$

for $n \in \mathbb{N}$. On the other hand, $F_{j(m)}(u_{j(m)}) = D_R$, and there exists $\beta \in u_{j(m)}$ such that $\alpha = F_{j(m)}(\beta)$. It follows from (3.6) that $F_{j(m)+n_k}(\beta) = f_{c^0_{j(m)+n_k}} \circ \cdots \circ f_{c^0_{j(m)+1}}(\alpha) \to \infty$ as $k \to \infty$. This contradicts (3.7). So (3.5) is proved.

For each $m \in \mathbb{N}$, take a point $z_{j(m)} \in u_{j(m)} \cap \mathcal{J}(c_n^0)$. There is no harm in assuming that $z_{j(m)} \to z_0 \in \mathcal{J}(c_n^0)$ as $m \to \infty$. It is clear that for each $m \in \mathbb{N}$ that there is one and only one component $v_{j(m)}$ of $F_{j(m)}^{-1}(D_R)$ containing z_0, and $v_{j(m+1)} \subset v_{j(m)}$.

Let u_0 be the component of $\mathcal{J}(c_n^0)$ which contains z_0, then we have $u_0 \subset v_{j(m)}$ and $d(u_0) \leq r_0$ since $d^*(c_n^0) = r_0$.

From (3.3), it is easy to see $d(v_{j(m)}) \to d(u_0)$ $(m \to \infty)$ and hence $d(v_{j(m)}) < r_0 + \varepsilon/4$ for m large enough. Since each B_m consists of a finite number of Jordan domains with sectionally analytic boundaries, and $\overline{B_{m+1}} \subset B_m$ for all $m \in \mathbb{N}$ (see Lemma 3.1), from

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the fact that $z_{j(n)} \to z_0$ $(n \to \infty)$, it is not difficult to show that for a fixed m, $u_{j(n)}$ will be contained in $v_{j(m)}$ for n large enough. It follows that $d(u_{j(m)}) < r_0 + \varepsilon/4$ for m large enough. This contradicts (3.4), so the assertion is proved.

Choose a natural number p such that the diameters of all components of $F_p^{-1}(D_R)$ are less than $r_0 + \varepsilon/2$.

Let $G_m = f_{c_m} \circ \cdots \circ f_{c_1}$. By a simple continuity argument, there exists $\delta_0 > 0$ such that the diameter of all components of $G_p^{-1}(D_R)$ are less than $(r_0 + \varepsilon)$ provided that $|c_m - c_m^0| < \delta_0$ $(m = 1, 2, \ldots, p)$.

Since each component of $G_{q+1}^{-1}(D_R)$ must be contained in some component of $G_q^{-1}(D_R)$, the diameters of all components of $G_q^{-1}(D_R)$ are less than $(r_0 + \varepsilon)$, $q \geq p$ provided $|c_m - c_m^0| < \delta_0$ $(m = 1, 2, \ldots, p)$.

If $(c_n) \in K_\delta^{\mathbb{N}}$, then $\mathcal{J}(c_n)$ is clearly contained in $G_q^{-1}(D_R)$ for each $q \in \mathbb{N}$. Therefore, the diameters of all components of $\mathcal{J}(c_n)$ are less than $(r_0 + \varepsilon) < r$ provided that $(c_n) \in K_\delta^{\mathbb{N}}$ and $|c_m - c_m^0| < \delta_0$ $(m = 1, 2, \ldots, p)$. That is to say, if

$$(c_n) \in K_\delta^{\mathbb{N}}, \quad |c_m - c_m^0| < \delta_0 \quad (m = 1, 2, \ldots, p),$$

we have $d^*(c_n) \leq r_0 + \varepsilon < r$, which implies $(c_n) \in L_r$.

Let $N_m = \{z : |z - c_m^0| < \delta_0\}$; then $N_1 \times N_2 \times \cdots \times N_p \times K_\delta^{\mathbb{N}} \subset L_r$. Hence L_r is open. The proof is complete. □

Now we can obtain the following theorem, which gives a positive answer to Brück's question (see Question 2 in §1).

THEOREM 3.2. *The set $\mathcal{T}$ defined by (1.4) is a countable intersection of dense open subsets of $K_\delta^{\mathbb{N}}$ provided $\delta > \frac{1}{4}$. In particular, $\mathcal{T}$ is of the second Baire category in $K_\delta^{\mathbb{N}}$, while the complement $K_\delta^{\mathbb{N}} \setminus \mathcal{T}$ is of the first Baire category in $K_\delta^{\mathbb{N}}$.*

Proof. Since $\mathcal{T} = \bigcap_{n=1}^{\infty} L_{1/n}$, the assertion follows from Theorem 3.1. □

Remark. Theorem 3.1 and hence Theorem 3.2 remain true if K_δ is replaced by any bounded set $K \subset \mathbb{C}$ provided $K \cap (C \setminus \mathcal{M}) \neq \emptyset$.

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