

# TRANSITIONAL DYNAMICS OF OPTIMAL CAPITAL TAXATION

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A known result holds that capital taxes should be high in the short run and low or zero in the long-run steady state. This paper studies the dynamics of optimal capital taxation during the transition, when a high rate is no longer optimal but the economy is still in flux. The main result is that capital should be taxed whenever the sum of the elasticities of marginal utility with respect to consumption and labor supply are rising and subsidized whenever this sum is falling. If the utility function displays increasing relative risk aversion, this paradoxically implies that capital should be taxed when the capital stock is below the modified golden-rule level and subsidized whenever it exceeds this level. Thus, savings incentives sometimes can be more desirable when the capital stock is large than when it is small.

**Keywords:** Optimal Capital Taxation, Economic Growth, Fiscal Policy, Savings Incentives

## 1. INTRODUCTION

Our existing knowledge of how capital should be taxed is more or less limited to two contrasting results. In the short run, the capital income tax is an efficient revenue source and should be set quite high [Chamley (1986)]. In steady state, however, the optimal tax rate is zero [Chamley (1986), Judd (1985)].<sup>1</sup> We know very little about optimal taxation during the transition from the short run to the steady state. What we do know comes from a special case that Chamley (1986) analyzes. When the utility function is additively separable in consumption and labor supply and displays constant relative risk aversion in consumption, the capital income tax jumps directly from the maximum feasible rate to zero and remains there forever. Chamley (1986, p. 608) notes that this does not happen with general utility functions. However, what does occur is an open question.

This paper shows that, after the initial period of maximum taxation, capital is taxed whenever the sum of the elasticities of the marginal utility of consumption with respect to consumption and labor supply is rising (over time) and is subsidized whenever this sum is falling. To illustrate the application of this rule, consider the special case in which utility is separable in consumption and labor supply.

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In this case, the rule implies that capital should be taxed when the own-price elasticity of consumption is falling and subsidized when this elasticity is rising. In Chamley's special subcase, the own-price elasticity of consumption is a constant, so the optimum tax is zero after the initial period of maximum taxation. Without constant relative risk aversion, however, this paper's result implies that the capital tax is exactly zero *only* in steady state. This holds because consumption, and thus the own-price elasticity of consumption, is constant only in steady state.

The result also has an unexpected implication. If utility is separable in consumption and leisure and displays increasing relative risk aversion in consumption, the optimal capital income tax is positive whenever capital is below the modified golden-rule level<sup>2</sup> and negative whenever capital exceeds this level. That is, capital income should be taxed if the capital stock is relatively low and subsidized if it is relatively high. This implies that it sometimes may be more beneficial to use savings incentives when the capital stock is high than when it is low.

The intuition for the general result is as follows. The agent has two kinds of marginal rates of substitution: those between contemporaneous consumption and leisure, on the one hand, and those between consumption at different dates, on the other. (Other marginal rates of substitution are captured by various combinations of these rates.) Taxes introduce differences between these marginal rates of substitution and the corresponding marginal rates of transformation in the economy. The planner's objective is to minimize the welfare loss that comes from such differences.

The planner has a perfect tool for influencing the price of leisure: a wage tax, which can be set arbitrarily. This ensures that the marginal rate of substitution between consumption and leisure at any given date will be efficient, given the other distortions that exist. However, the planner's tool for influencing the price of consumption is imperfect: It is not a consumption tax, but rather a capital income tax. This is why the planner cannot simply raise the prices of consumption and leisure by the same proportion at all dates. If it could do so, marginal rates of transformation (being the ratios between these prices) would be unaffected, and there would be no deadweight loss. But there is a cap on the capital tax, so the planner cannot tax early consumption as much as needed to implement this efficient scheme. This is because a positive capital income tax rate at time  $t$  raises the price of consumption for all times  $s > t$  by the same proportion. Thus, the planner's ability to raise the price of consumption at time  $t$  goes to zero as  $t$  goes to zero.

Given that distortions are inevitable, the planner seeks to minimize them. This essentially consists of minimizing distortions in *quantities*. (This is why optimal taxes tend to be lower on commodities with more elastic demand or supply functions.) Because the planner has a perfect tool (the wage tax) for controlling the price of leisure, she uses the capital tax to influence the price of consumption. Note that an increase in the price of consumption at time  $t$  leads the agent to raise the marginal utility of consumption at  $t$  proportionally, which she does by changing time  $t$  consumption and labor supply. When the marginal utility of consumption

is very elastic with respect to consumption and labor supply, she does not need to change her consumption and labor supply by much for a given change in marginal utility. In this case, an increase in the price of consumption does not distort quantities very much: A small adjustment in consumption and labor supply is all that is needed. Hence, the capital tax should be used to raise the price of consumption more when the marginal utility of consumption is very elastic with respect to consumption and labor supply than when it is not so elastic.

How can the capital tax be used to accomplish this? A positive capital tax rate at time  $t$  causes a proportional increase in the prices of consumption at all times  $s > t$ . Likewise, a negative rate causes a proportional decrease in future consumption prices. Hence, capital should be taxed when the elasticity of marginal utility with respect to consumption and labor supply is growing, and subsidized when this elasticity is falling. Such a policy ensures that the price of consumption will rise while this elasticity is growing and fall while the elasticity is shrinking. In this way, the implicit tax rate on consumption will be lower when consumption and labor supply are more sensitive to the price of consumption.<sup>3</sup> This completes the intuition for the main result.

We also prove a corollary: If utility is (i) separable in consumption and labor and (ii) displays increasing relative risk aversion in consumption, then capital should be taxed whenever the capital stock is below the modified golden-rule level and subsidized when it is above. The intuition is as follows. With increasing relative risk aversion, the own-price elasticity of consumption is low when consumption is high. Accordingly, the implicit tax rate on consumption should be high when consumption is high. This requires the capital tax rate to be positive when consumption is rising and negative when it is falling. To finish, we need only show that optimality requires consumption to rise when capital is below the modified golden-rule level and fall otherwise. Suppose, for instance, that capital is below the modified golden-rule level. This means that the pretax interest rate exceeds the discount rate. Can consumption decline over time? For the private sector to choose declining consumption, the after-tax interest rate must be less than the discount rate. Because the pretax interest rate exceeds the discount rate, the capital tax must be positive. However, as argued above, a positive capital tax can be optimal only if consumption is *increasing* over time. Hence, consumption *cannot* decline over time. A similar argument shows that consumption must fall whenever capital exceeds the modified golden-rule level. This completes the intuition for the corollary.

It can be shown that, if a consumption tax is available, the efficient policy is to impose a constant consumption tax and wage subsidy. This is equivalent to a capital levy because it causes an equiproportionate decrease in the amounts of consumption and leisure that the private sector's initial stock of savings can purchase at any date. The first-best consumption tax is depicted in Figure 1. If only a capital tax is available, the highest *possible* consumption tax begins at zero and increases exponentially. In the optimal policy, the implicit tax on consumption initially coincides with this exponential path. At some point, it diverges. Suppose

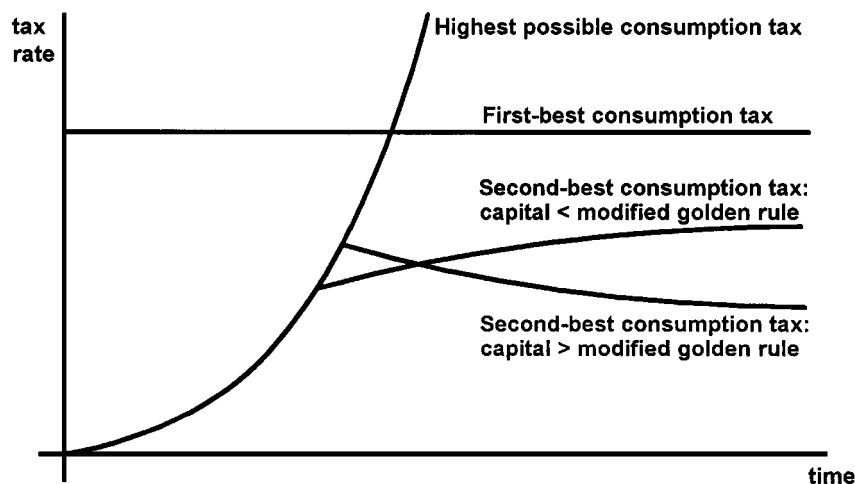


FIGURE 1. Implicit consumption tax rates with separable utility and increasing relative risk aversion.

now that utility is such that the corollary applies. In a growing economy, the implicit tax on consumption continues to rise, but at a slower rate. This corresponds to a positive capital tax. If the economy is shrinking because there is too much capital, the implicit consumption tax instead declines. This corresponds to a capital subsidy. Both cases are depicted in Figure 1.

This paper assumes that the government can commit to a path of tax rates. The case of no commitment is studied by Frankel (1993). The result there is disastrous: Because capital is always an efficient revenue source in the short run, the capital tax rate must be at its maximum until no further revenue is needed. This means that the implicit tax on consumption must continue to follow the exponential curve of Figure 1 until all of the government's future spending plans can be funded out of existing government savings. This policy has severe effects on private saving and growth because it creates a powerful incentive for early consumption. This result suggests that an ability to commit to future tax rates can be quite beneficial. The relatively low rates of capital taxation that we observe in the real world are consistent with the existence of some commitment power. Persson and Tabellini (1990) discuss several ways that governments may attain this power.

This paper builds on the important contributions of several other authors not yet mentioned. Judd (1985) shows that there is no capital tax in the long run in a variety of deterministic models with exogenous growth, fixed government spending plans, and perfect markets. Lucas (1990) finds no capital income taxation in the long run under endogenous growth. Jones et al. (1993) show that there can be a positive limiting capital income tax when government spending is endogenous.

Judd (1990) and Chari et al. (1990) have examined optimal taxation in stochastic economies. Their results show that the optimal long-run tax rate on capital income

is zero *in expectation*. Aiyagari (1995) finds that credit market imperfections can give rise to a positive capital income tax in the long run.

This paper uses a dynastic model, which is equivalent to an overlapping-generations model with bequests. Diamond (1973) was the first to study optimal capital taxation in a variety of overlapping-generations models without bequests. Atkinson and Sandmo (1980) showed that the limiting capital income tax rate can be either positive or negative in such models.

## 2. MODEL

The private sector is represented by a single, infinitely lived agent. At time  $t$ , the agent supplies labor  $l_t \in [0, 1]$  and consumes  $c_t \geq 0$  units of a homogeneous good. The economy has the per-capita, net production function  $f(k_t, l_t)$ , where  $k_t$  is per-capita national capital. The government can borrow or lend. Thus, national capital,  $k_t$ , equals private wealth,  $k_t^p$ , less government bonds.<sup>4</sup> The government must finance time  $t$  per-capita public consumption at an exogenous rate,  $g_t$ .<sup>5</sup>

The government raises revenue by levying linear taxes  $\tau_t^w$  and  $\tau_t^r$  on wages and capital income, respectively. We assume perfect competition, so that

$$r_t = (1 - \tau_t^r) f_k(k_t, l_t), \quad (1)$$

$$w_t = (1 - \tau_t^w) f_l(k_t, l_t), \quad (2)$$

where  $r_t$  and  $w_t$  are the after-tax interest rate and wage, respectively. We assume that the government has a commitment technology.

In optimal capital taxation, one must take account of firms' and individuals' ability to avoid and evade capital income taxes. Otherwise, the only tax is an arbitrarily high capital income tax at time zero [e.g., Chamley (1986)]. Following the usual method, we assume that there is an arbitrary lower bound  $\underline{r}$  on the after-tax interest rate,  $r_t$ . Let us refer to the capital income tax that gives  $r_t = \underline{r}$  as the maximum capital income tax.

Because we focus on the period in which this constraint is not binding, the exact form of the constraint on the capital tax is unimportant. The same results hold, for example, with a general constraint such as  $\tau_t^r \leq \phi_t(k_t, l_t, \tau_t^w)$ . A more realistic model would include convex evasion costs in the utility function. One effect of such a specification undoubtedly would be to smooth the tax path at the point where the constraint  $r \geq \underline{r}$  ceases to bind, because the threat of evasion would have a continuing influence on optimal policy rather than an influence that is present only when the constraint binds. In other respects, few qualitative changes in the results would be expected.

Both the private and the social objective functions are given by

$$\int_{t=0}^{\infty} e^{-\theta t} u(c_t, l_t) dt. \quad (3)$$

Derivatives have the usual signs:  $u_c > 0$ ,  $u_{cc} < 0$ ,  $u_l < 0$ , and  $u_{ll} < 0$ . We also assume that the utility function is strictly concave. This implies that the determinant of its Hessian is positive:

$$u_{cc}u_{ll} - (u_{cl})^2 > 0. \tag{4}$$

In addition, we assume that both leisure and consumption are normal goods. For consumption, this holds if

$$u_{cl}u_l - u_{ll}u_c > 0. \tag{5}$$

For leisure, normality holds if

$$u_{cc}u_l - u_{cl}u_c > 0 \tag{6}$$

[Killingsworth (1988, p. 223)]. Throughout, we assume an interior solution for consumption ( $c_t > 0$ ) and labor [ $l_t \in (0, 1)$ ]. This can be guaranteed by assuming that  $f$  and  $u$  satisfy the usual Inada conditions.

At time zero, the agent uses information about the equilibrium strategies<sup>6</sup> to forecast the future path  $\{w_t, r_t; t \geq 0\}$ . Given this forecast, the agent chooses  $\{c_t, l_t; t \geq s\}$  to maximize the objective function (3) subject to

$$k_0^p, \text{ given} \tag{7}$$

$$\dot{k}_t^p = r_t k_t^p + w_t l_t - c_t, \tag{8}$$

$$\lim_{t \rightarrow \infty} \exp\left(-\int_0^t r_v dv\right) k_t^p \geq 0, \tag{9}$$

$$c_t \geq 0; \quad 0 \leq l_t \leq 1.$$

Lemma 1 characterizes the agent’s behavior.

LEMMA 1. *For any  $t > 0$ , define*

$$\mu_t = u_c(c_t, l_t). \tag{10}$$

*The agent’s optimal strategy is characterized by*

$$u_l(c_t, l_t) = -\mu_t w_t, \tag{11}$$

$$\dot{\mu}_t = \mu_t[\theta - r_t]. \tag{12}$$

*The agent selects  $\mu_0$  so that, given (11) and (12), the budget constraint [(7), (8), and (9)] is satisfied with equality.*

Proof. This is a standard application of optimal control theory [Pontryagin et al. (1962), Seierstad and Sydsæter (1987)].

The government chooses tax rates  $\{\tau_t^r, \tau_t^w: t \geq 0\}$  to maximize (3) subject to the social budget constraint<sup>7</sup>

$$k_0, \text{ given}$$

$$\dot{k}_t = f(k_t, l_t) - c_t - g_t,$$

$$\lim_{t \rightarrow \infty} \exp\left(-\int_0^t f_k(k_v, l_v) dv\right) k_t \geq 0;$$

the conditions (1) and (2) for market clearance; the tax evasion constraint,

$$r_t \geq r;$$

and the solution to the private problem, which comprises (7)–(12).

### 3. OPTIMAL POLICY

Chamley (1986) and Judd (1985) show that the capital tax is at its maximum at time zero and is zero in steady state. If the utility function is separable ( $u_{cl} = 0$ ) and CRRA in consumption, Chamley (1986) shows that, when the capital income tax rate falls below the maximum, it jumps directly to zero and remains there forever. In Theorem 1, we show what happens for general utility functions.

**THEOREM 1.** *The following properties hold after the cap on the capital income tax ceases to bind.*

- (i) *The capital income tax is positive (negative) whenever the sum of the elasticities of marginal utility with respect to consumption and labor supply,*

$$\left(-\frac{c}{u_c} \frac{\partial u_c}{\partial c}\right) + \left(-\frac{l}{u_c} \frac{\partial u_c}{\partial l}\right) = -\left(\frac{cu_{cc} + lu_{cl}}{u_c}\right), \quad (13)$$

*is increasing (decreasing) over time.*<sup>8</sup>

- (ii) *If the utility function is separable ( $u_{cl} = 0$ ) and utility displays increasing relative risk aversion (IRRA) in consumption,*

$$\frac{d}{dc} \left(-\frac{cu_{cc}}{u_c}\right) > 0,$$

*then capital is taxed if the capital stock exceeds the modified golden-rule level (at which  $f_k = \theta$ ) and negative if the capital stock is less than this level.*

Proof. See Appendix.

### NOTES

1. The presence of borrowing constraints [Aiyagari (1995)] or endogenous government spending [Jones et al. (1993)] can alter the second conclusion.

2. The modified golden-rule level is the level at which the pretax marginal product of capital equals the discount rate. In the absence of capital taxation, this is the limit to which the capital stock converges.
3. By the implicit tax rate on consumption, we mean the percentage difference between the private and social prices of the consumption good at time  $t$  relative to their price at time zero. For example, suppose the marginal product of capital is constant at  $R$  and the capital tax rate is the constant  $\tau^f$ . Then, the private cost of time  $t$  consumption in units of time zero consumption is  $e^{-(1-\tau^f)Rt}$  whereas the social cost is  $e^{-Rt}$ . The implicit tax on consumption at time  $t$  is thus  $e^{\tau^f Rt} - 1$ .
4. Government bonds may be negative: The government may be a net creditor. We assume that production depends only on  $k$ , and not on the government's debt or credit position.
5. The government purchases the consumption good only. Government spending affects neither marginal utility nor marginal factor productivity.
6. These are the government's policy  $\{\tau_t^w, \tau_t^r: t \geq 0\}$  and the strategies of the other agents, which in equilibrium must be the same as the representative agent's strategy.
7. Because government debt is  $k^P - k$ , the usual government budget constraint, stated in terms of debt, is implied by the private budget constraint [(7), (8), and (9)] combined with the social budget constraint.
8. Because  $cu_{cc}/u_c < 0$ , we define the elasticity of  $u_c$  with respect to  $c$  to be  $-cu_{cc}/u_c$  to make it positive. Although  $u_{cl}$  may be positive or negative, the same sign convention is used in defining the elasticity of  $u_c$  with respect to  $l$  as  $-lu_{cl}/u_c$ .
9. Our use of the term semielasticity is the opposite of the usual meaning.
10. The abbreviations  $C$  and  $L$  stand for  $C(\mu, w)$  and  $L(\mu, w)$ , respectively.
11. We are forced to rely on informal arguments because the signs of  $v_0$  and  $\eta_0$  cannot be inferred from first-order conditions alone. Chamley (1986, pp. 611, 616) also uses an intuitive approach in such cases.

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## APPENDIX: PROOFS

Let us first solve for the optimal behavior of the agent. Define  $\Delta = u_{cc}u_{ll} - (u_{cl})^2$ ;  $\Delta$  is positive by (4).

LEMMA A.1. For any  $t$ , equations (10) and (11) give  $c_t$  and  $l_t$  as functions of  $\mu_t$  and  $w_t$ :

$$\begin{aligned} c_t &= C(\mu_t, w_t), \\ l_t &= L(\mu_t, w_t). \end{aligned} \tag{A.1}$$

These functions satisfy

$$\frac{\partial C}{\partial \mu} = \frac{1}{\mu} Y^c, \tag{A.2}$$

$$\frac{\partial C}{\partial w} = \frac{1}{w} S^{cl}, \tag{A.3}$$

$$\frac{\partial L}{\partial \mu} = \frac{1}{\mu} Y^l, \tag{A.4}$$

$$\frac{\partial L}{\partial w} = \frac{1}{w} S^{ll}, \tag{A.5}$$

where

$$Y^c = \frac{u_{ll}u_c - u_{cl}u_l}{\Delta} < 0, \tag{A.6}$$

$$S^{cl} = -\frac{u_{cl}u_l}{\Delta}, \tag{A.7}$$

$$Y^l = \frac{u_{cc}u_l - u_{cl}u_c}{\Delta} > 0, \tag{A.8}$$

$$S^{ll} = \frac{u_{cc}u_l}{\Delta} > 0. \tag{A.9}$$

**Proof.** One totally differentiates (10) and (11) to show (A.2) through (A.5). The signs of  $Y^c$  and  $Y^l$  are given by (4) through (6).

The quantities  $-Y^c$  and  $-Y^l$  are the wealth semielasticities<sup>9</sup> of consumption and labor supply, because an increase in  $\mu$  corresponds to a decrease in wealth.  $S^{cl}$  and  $S^{ll}$  are the price semielasticities. The sign of the cross-price semielasticity,  $S^{cl}$ , depends on whether leisure and consumption are net substitutes or net complements.

**Proof of Theorem 1.** One can reformulate the problem so that the control variables are  $r$  and  $w$ . (Time subscripts sometimes are omitted for the sake of brevity.) The planner chooses  $\{r_t, w_t: t \geq 0\}$  to maximize

$$\int_{t=0}^{\infty} e^{-\theta t} u(c_t, l_t) dt \tag{A.10}$$

subject to the constraints

$$k_0, k_0^p, \text{ given} \tag{A.11}$$

$$\lim_{t \rightarrow \infty} \exp\left(-\int_0^t f_k(k_v, l_v) dv\right) k_t = 0, \tag{A.12}$$

$$\dot{k} = f(k, l) - c - g, \tag{A.13}$$

$$\dot{k}^p = rk^p + wl - c, \tag{A.14}$$

$$\dot{\mu} = (\theta - r)\mu \quad \text{where} \quad \mu = u_c, \tag{A.15}$$

$$c = C(\mu, w), \tag{A.16}$$

$$l = L(\mu, w), \tag{A.17}$$

$$r \geq \underline{r}. \tag{A.18}$$

The present-value Hamiltonian is<sup>10</sup>

$$H = e^{-\theta t} [u(C, L) + v(f(k, L) - C - g) + \eta(rk^p + wL - C) + \zeta\mu(\theta - r) + \gamma(r - \underline{r})]. \tag{A.19}$$

First-order conditions are

$w$ :

$$0 = u_c \frac{\partial C}{\partial w} + u_l \frac{\partial L}{\partial w} + \eta \left( l + w \frac{\partial L}{\partial w} - \frac{\partial C}{\partial w} \right) + v \left( f_l \frac{\partial L}{\partial w} - \frac{\partial C}{\partial w} \right), \tag{A.20}$$

$r$ :

$$0 = \eta k^p - \zeta \mu + \gamma, \tag{A.21}$$

$k^p$ :

$$\dot{\eta} - \eta\theta = -\eta r, \tag{A.22}$$

$k$ :

$$\dot{v} - v\theta = -vf_k, \tag{A.23}$$

$\mu$ :

$$\dot{\zeta} - \zeta\theta = -u_c \frac{\partial C}{\partial \mu} - u_l \frac{\partial L}{\partial \mu} - \eta \left( w \frac{\partial L}{\partial \mu} - \frac{\partial C}{\partial \mu} \right) - v \left( f_l \frac{\partial L}{\partial \mu} - \frac{\partial C}{\partial \mu} \right) - \zeta(\theta - r). \tag{A.24}$$

Equations (A.20) and (A.24) can be simplified. By (A.3) and (A.5), (A.20) can be written

$$0 = -\mu(wS^{ll} - S^{cl}) + \eta(wl + wS^{ll} - S^{cl}) + v(f_l S^{ll} - S^{cl}). \tag{A.25}$$

By (A.2) and (A.4), (A.24) becomes

$$\dot{\zeta}\mu - \zeta\mu r = (\mu - \eta)(wY^l - Y^c) - v(f_l Y^l - Y^c). \tag{A.26}$$

When  $r > \underline{r}$  in an interval of time,  $\dot{\gamma} = \gamma = 0$  in the interior of that interval, but, by (A.21), it always holds that

$$\begin{aligned} \dot{\gamma} &= \zeta\dot{\mu} + \dot{\zeta}\mu - \eta\dot{k}^p - \dot{\eta}k^p \\ &= \theta\gamma + (\mu - v - \eta)(wY^l - Y^c) - v\tau^w f_l Y^l - \eta(wl - c). \end{aligned} \tag{A.27}$$

This uses (A.22), (A.15), (A.21), (A.26), and (A.14). Thus, when  $\dot{\gamma} = \gamma = 0$ , (A.27) implies that

$$\tau^w f_l = \frac{(\mu - v - \eta)(wY^l - Y^c) - \eta(wl - c)}{vY^l}, \tag{A.28}$$

but, by (A.25), it always holds that

$$\tau^w f_l = \frac{(\mu - v - \eta)wY^l - \eta wl}{vS^{ll}}. \tag{A.29}$$

This uses the fact that

$$wS^{ll} - S^{cl} = wY^l, \tag{A.30}$$

which follows directly from (11), (A.6), (A.7), and (A.8). Equate the right-hand sides of (A.28) and (A.29) to obtain

$$(\mu - v - \eta)[wY^l(Y^l - S^{ll}) + Y^c S^{ll}] = \eta wl(Y^l - S^{ll}) + \eta c S^{ll}.$$

Now, using (A.30) to eliminate the  $Y^l - S^{ll}$  terms, one obtains

$$(\mu - v - \eta) = \frac{\eta(lS^{cl} - cS^{ll})}{Y^l S^{cl} - Y^c S^{ll}}. \tag{A.31}$$

When  $r > \underline{r}$ , equation (A.31) implies that

$$\frac{\mu - \eta - v}{\eta} = \frac{lS^{cl} - cS^{ll}}{S^{cl}Y^l - S^{ll}Y^c} = \frac{cu_{cc} + lu_{cl}}{u_c}. \tag{A.32}$$

This also uses (A.7), (A.9), and

$$S^{cl}Y^l - S^{ll}Y^c = -\frac{u_c u_l}{\Delta} > 0.$$

Differentiate (A.32) with respect to time. By (A.22) and (A.23), this gives

$$\frac{v_l}{\eta_l} [f_k(k_t, l_t) - r_t] = \frac{d}{dt} \left( \frac{cu_{cc} + lu_{cl}}{u_c} \right). \tag{A.33}$$

We now need to know the sign of  $v_t/\eta_t$ . By (A.22) and (A.23), this equals the sign of  $v_0/\eta_0$ . By (A.19),  $\eta_0$  is the shadow value of an increase in private savings  $k_0^p$  at time zero, holding national capital  $k_0$  constant, but this is just a transfer from the government to the private sector. Any such transfer increases the amount of distortionary taxation that is needed, so  $\eta_0 < 0$ . On the other hand,  $v_0$  is the shadow value of an increase in national capital at time zero, holding private capital constant. Because this reduces the amount of taxation needed while increasing the national capital stock, its effect is positive:  $v_0 > 0$ .<sup>11</sup>

This shows that  $v_t/\eta_t < 0$ . By (A.33),  $\tau^r = (f_k - r)/f_k$  is positive, negative, or zero, depending on whether the quantity in (13) is, respectively, increasing, decreasing, or unchanging (over time). This establishes part (i) of Theorem 1.

If  $u_{cl} \equiv 0$ , the right-hand side of (A.33) equals  $-(d/dt)R(c)$ , where  $R(c)$  is the coefficient of relative risk aversion:

$$R(c) = -\frac{cu_{cc}}{u_c}.$$

Thus,  $\tau^r$  is positive, negative, or zero, depending on whether  $R(c_t)$  is, respectively, increasing, decreasing, or unchanging. With CRRA utility,  $R(c_t)$  must be constant over time, so  $\tau^r \equiv 0$ . This is Chamley's (1986) result.

Part (ii) of Theorem 1 now can be proved. By (A.7) and because  $u_{cl} = 0$ ,  $S^{cl} = 0$ . Thus, by (A.1), (A.2), and (A.3),  $\dot{c} = Y^c(\theta - r)$ . By (A.6),  $Y^c < 0$ . Thus, consumption is increasing, decreasing, or constant if  $r - \theta$  is, respectively, positive, negative, or zero.

Now suppose that  $u$  is IRRRA in consumption [ $R'(c) > 0$ ]. If  $f_k \neq \theta$ , then there are six possible orders for  $r$ ,  $f_k$ , and  $\theta$ . Four of these can be ruled out:

1. If  $r \leq \theta < f_k$ , then  $\dot{c} \leq 0$ , so  $\dot{R} \leq 0$ ; hence  $\tau^r \leq 0$ .
2. If  $\theta < f_k \leq r$ , then  $\dot{c} > 0$ , so  $\dot{R} > 0$ ; hence  $\tau^r > 0$ .
3. If  $\theta < r < f_k$ , then  $\dot{c} > 0$ , so  $\dot{R} > 0$ ; hence  $\tau^r > 0$ .
4. If  $r \leq f_k < \theta$ , then  $\dot{c} < 0$ , so  $\dot{R} < 0$ ; hence  $\tau^r < 0$ .
5. If  $f_k < \theta \leq r$ , then  $\dot{c} \geq 0$ , so  $\dot{R} \geq 0$ ; hence  $\tau^r \geq 0$ .
6. If  $f_k < r < \theta$ , then  $\dot{c} < 0$ , so  $\dot{R} < 0$ ; hence  $\tau^r < 0$ .

Only cases 3 and 6 are internally consistent. If  $f_k = \theta$ , then  $r = f_k = \theta$  is the only possibility. This confirms that  $f_k - \theta$  and  $\tau^r$  are of the same sign if  $r > \underline{r}$ , which establishes part (ii) of Theorem 1.