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# Non-trivial solutions of local and non-local Neumann boundary-value problems

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We prove new results on the existence, non-existence, localization and multiplicity of non-trivial solutions for perturbed Hammerstein integral equations. Our approach is topological and relies on the classical fixed-point index. Some of the criteria involve a comparison with the spectral radius of some related linear operators. We apply our results to some boundary-value problems with local and non-local boundary conditions of Neumann type. We illustrate in some examples the methodologies used.

*Keywords:* fixed-point index; cone; non-trivial solution; Neumann conditions

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## 1. Introduction

In this paper we discuss the existence, localization, multiplicity and non-existence of non-trivial solutions of the second-order differential equation

$$u''(t) + h(t, u(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

subject to (local) Neumann boundary conditions (BCs)

$$u'(0) = u'(1) = 0 \tag{1.2}$$

or to non-local BCs of Neumann type:

$$u'(0) = \alpha[u], \qquad u'(1) = \beta[u], \tag{1.3}$$

where $\alpha[\cdot]$, $\beta[\cdot]$ are linear functionals given by Stieltjes integrals, namely

$$\alpha[u] = \int_0^1 u(s)\,\mathrm{d}A(s), \qquad \beta[u] = \int_0^1 u(s)\,\mathrm{d}B(s).$$

The local boundary-value problem (BVP) (1.1), (1.2) has been studied by Miciano and Shivaji [35], who proved the existence of multiple positive solutions by means of the quadrature technique; using Morse theory, Li [32] proved the existence of positive solutions and Li *et al*. [33] continued the study in [32] and proved the existence of multiple solutions. Multiple positive solutions were also investigated by Boscaggin [4] via shooting-type arguments.

Note that, since $\lambda = 0$ is an eigenvalue of the associated linear problem

$$u''(t) + \lambda u(t) = 0, \quad u'(0) = u'(1) = 0,$$

the corresponding Green function does not exist. Therefore, we use a shift argument similar to those in [16, 44, 56] and we study two related BVPs for which the Green function can be constructed, namely

$$-u''(t) - \omega^2 u(t) = f(t, u(t)) := h(t, u(t)) - \omega^2 u(t), \quad u'(0) = u'(1) = 0, \tag{1.4}$$

and (with an abuse of notation)

$$-u''(t) + \omega^2 u(t) = f(t, u(t)) := h(t, u(t)) + \omega^2 u(t), \quad u'(0) = u'(1) = 0. \tag{1.5}$$

The BVPs (1.4) and (1.5) have recently been of interest to a number of authors (see, for example, [3, 9, 12, 41, 42, 45–47, 58–62]). In § 5 we study in detail the properties of the associated Green functions and we improve and complement some estimates that occur in earlier papers (see remark 5.3).

The formulation of the non-local BCs in terms of linear functionals is fairly general and includes as special cases multi-point and integral conditions, namely

$$\alpha[u] = \sum_{j=1}^{m} \alpha_j u(\eta_j) \quad \text{or} \quad \alpha[u] = \int_0^1 \phi(s) u(s)\, \mathrm{d}s.$$

Multi-point and integral BCs have been widely studied. The study of multi-point BCs was, as far as we know, initiated in 1908 by Picone [39]. Reviews on differential equations with BCs involving Stieltjes measures were written in 1942 by Whyburn [57] and in 1967 by Conti [8]. More recent reviews are given in [34, 37, 40] and the papers by Karakostas and Tsamatos [27, 28] and Webb and Infante [53].

One motivation for studying non-local problems in the context of Neumann problems is that they occur naturally when modelling heat-flow problems.

For example, the four-point BVP

$$u''(t) + h(t, u(t)) = 0, \quad u'(0) = \alpha u(\xi), \; u'(1) = \beta u(\eta), \; \xi, \eta \in [0, 1],$$

models a thermostat where controllers at $t = 0$ and $t = 1$ add or remove heat according to the temperatures detected by two sensors at $t = \xi$ and $t = \eta$. Thermostat models of this type were studied in a number of papers (see, for example, [7, 10, 18, 19, 25, 29, 38, 48, 49, 51] and the references therein). In particular Webb [51] studied the existence of *positive* solutions of the BVP

$$u''(t) + h(t, u(t)) = 0, \quad u'(0) = \alpha[u], \; u'(1) = -\beta[u].$$

The methodology in [51] is somewhat different from ours and relies on a careful rewriting of the associated Green function, due to the presence of the term $-\beta[u]$

in the BCs. The existence of solutions that *change sign* has been investigated in [10] in the case of the BVP

$$u''(t) + h(t, u(t)) = 0, \quad u'(0) = \alpha u(\xi),\ u'(1) = -\beta u(\eta),\ \xi, \eta \in [0, 1],$$

and in [7, 21, 25] for the BVP

$$u''(t) + h(t, u(t)) = 0, \quad u'(0) = -\alpha[u],\ u'(1) = -\beta u(\eta),\ \eta \in [0, 1].$$

A common feature of the papers [7, 10, 21, 25] is that a direct construction of a Green function is possible due to the term $-\beta u(\eta)$.

In § 2 we develop a fairly general theory for the existence and multiplicity of non-trivial solutions of the perturbed Hammerstein integral equation of the form

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t, s)g(s)f(s, u(s))\,\mathrm{d}s \tag{1.6}$$

that covers as *special cases* the BVP (1.1), (1.3) and the BVP (1.1), (1.2) when $\alpha$ and $\beta$ are the trivial functionals. We recall that the existence of *positive solutions* of this type of integral equation has been investigated by Webb and Infante in [53], under a non-negativity assumption on the terms $\gamma$, $\delta$, $k$, by working on a suitable cone of positive functions that takes into account the functionals $\alpha$, $\beta$.

In § 3 we provide some sufficient conditions on the nonlinearity $f$ for the non-existence of solutions of (1.6). This is achieved via an associated Hammerstein integral equation

$$u(t) = \int_0^1 k_S(t, s)g(s)f(s, u(s))\,\mathrm{d}s,$$

whose kernel $k_S$ is allowed to change sign and is constructed along the lines of [53], which deals with positive kernels.

In § 4 we provide a number of results that link the existence of non-trivial solutions of (1.6) with the spectral radius of some associated linear integral operators. The main tool here is the celebrated Krein–Rutman theorem, combined with some ideas from [55]; here, due to the non-constant sign of the Green function, the situation is more delicate than that in [55], and we introduce a number of different linear operators that yield different growth restrictions on the nonlinearity $f$.

In § 6 we illustrate the applications of our theory in three examples, two of which deal with solutions that change sign. The third example is taken from an interesting paper by Bonanno and Pizzimenti [3], who proved the existence, with respect to the parameter $\lambda$, of positive solutions of the following BVP:

$$-u''(t) + u(t) = \lambda t \mathrm{e}^{u(t)}, \quad u'(0) = u'(1) = 0.$$

The methodology used in [3] relies on a critical-point theorem of Bonanno [2]. Here we enlarge the range of the parameters and provide a sharper localization result. We also prove a non-existence result for this BVP.

Our results complement those of [53], focusing on the existence of solutions that are allowed to *change sign*, in the spirit of the earlier works [22,23,25]. The approach that we use is topological and relies on classical fixed-point index theory, and we make use of the ideas from [7, 23, 52, 53, 55].

## 2. Non-zero solutions of perturbed Hammerstein integral equations

In this section we study the existence of solutions of the perturbed Hammerstein equations of the type

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s := Tu(t), \tag{2.1}$$

where

$$\alpha[u] = \int_0^1 u(s)\,\mathrm{d}A(s), \qquad \beta[u] = \int_0^1 u(s)\,\mathrm{d}B(s),$$

and $A$ and $B$ are functions of bounded variation. If we set

$$Fu(t) := \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s,$$

we can write

$$Tu(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + Fu(t);$$

i.e. we consider $T$ as a perturbation of the simpler operator $F$.

We work in the space $C[0,1]$ of the continuous functions on $[0,1]$ endowed with the usual norm $\|w\| := \max\{|w(t)|,\ t \in [0,1]\}$.

We make the following assumptions on the terms that occur in (2.1).

(C1) $k\colon [0,1]\times[0,1] \to \mathbb{R}$ is measurable, and for every $\tau \in [0,1]$ we have

$$\lim_{t\to\tau} |k(t,s) - k(\tau,s)| = 0 \quad \text{for almost every } s \in [0,1].$$

(C2) There exist a subinterval $[a,b] \subseteq [0,1]$, a function $\Phi \in L^\infty[0,1]$ and a constant $c_1 \in (0,1]$ such that

$$\begin{aligned} |k(t,s)| &\leqslant \Phi(s) && \text{for } t \in [0,1] \text{ and almost every } s \in [0,1],\\ k(t,s) &\geqslant c_1\Phi(s) && \text{for } t \in [a,b] \text{ and almost every } s \in [0,1]. \end{aligned}$$

(C3) $g\Phi \in L^1[0,1]$, $g(s) \geqslant 0$ for almost every $s \in [0,1]$ and

$$\int_a^b \Phi(s)g(s)\,\mathrm{d}s > 0.$$

(C4) The nonlinearity $f\colon [0,1]\times(-\infty,\infty) \to [0,\infty)$ satisfies Carathéodory conditions, i.e. $f(\cdot,u)$ is measurable for each fixed $u \in (-\infty,\infty)$, $f(t,\cdot)$ is continuous for almost every $t \in [0,1]$ and for each $r > 0$ there exists $\phi_r \in L^\infty[0,1]$ such that

$$f(t,u) \leqslant \phi_r(t) \quad \text{for all } u \in [-r,r] \text{ and almost every } t \in [0,1].$$

(C5) $A$, $B$ are functions of bounded variation and $\mathcal{K}_A(s), \mathcal{K}_B(s) \geqslant 0$ for almost every $s \in [0,1]$, where

$$\mathcal{K}_A(s) := \int_0^1 k(t,s)\,\mathrm{d}A(t) \quad \text{and} \quad \mathcal{K}_B(s) := \int_0^1 k(t,s)\,\mathrm{d}B(t).$$

(C6) $\gamma \in C[0,1]$, $0 \leqslant \alpha[\gamma] < 1$, $\beta[\gamma] \geqslant 0$.

There exists $c_2 \in (0,1]$ such that $\gamma(t) \geqslant c_2 \|\gamma\|$ for $t \in [a,b]$.

(C7) $\delta \in C[0,1]$, $0 \leqslant \beta[\delta] < 1$, $\alpha[\delta] \geqslant 0$.

There exists $c_3 \in (0,1]$ such that $\delta(t) \geqslant c_3 \|\delta\|$ for $t \in [a,b]$.

(C8) $D := (1-\alpha[\gamma])(1-\beta[\delta]) - \alpha[\delta]\beta[\gamma] > 0$.

From (C6)–(C8) it follows that, for $\lambda \geqslant 1$,

$$D_\lambda := (\lambda - \alpha[\gamma])(\lambda - \beta[\delta]) - \alpha[\delta]\beta[\gamma] \geqslant D > 0.$$

We recall that a *cone* $K$ in a Banach space $X$ is a closed convex set such that $\lambda x \in K$ for $x \in K$ and $\lambda \geqslant 0$ and $K \cap (-K) = \{0\}$. The assumptions above allow us to work in the cone

$$K := \Big\{ u \in C[0,1] \colon \min_{t \in [a,b]} u(t) \geqslant c\|u\|,\ \alpha[u], \beta[u] \geqslant 0 \Big\},$$

where $c = \min\{c_1, c_2, c_3\}$.

Note that we have

$$K = K_0 \cap \{u \in C[0,1] \colon \alpha[u] \geqslant 0\} \cap \{u \in C[0,1] \colon \beta[u] \geqslant 0\},$$

where

$$K_0 := \Big\{ u \in C[0,1] \colon \min_{t \in [a,b]} u(t) \geqslant c\|u\| \Big\}.$$

The functions in $K_0$ are positive on the subset $[a,b]$ but are allowed to change sign in $[0,1]$. The cone $K_0$ is similar to a cone of *non-negative* functions first used by Krasnosel′skiĭ (see, for example, [30]) and Guo (see, for example, [15]). $K_0$ was introduced by Infante and Webb in [23] and later used in [6, 7, 10, 13, 14, 17, 20–22, 24, 25, 36]. The cone $K$ allows the use of signed measures, taking into account two functionals. In the case of one functional, this has been done in [7], which also dealt with non-trivial solutions of the perturbed integral equation

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s. \tag{2.2}$$

In [7] Cabada *et al.* work in the cone $K_0 \cap \{u \in C[0,1] \colon \alpha[u] \geqslant 0\}$, extending the earlier results in [25] to the case of signed measures and those from [54] to the context of non-trivial solutions. Clearly, (2.2) is a special case of (2.1) and, by considering $\beta$ the trivial functional, we have $K = K_0 \cap \{u \in C[0,1] \colon \alpha[u] \geqslant 0\}$.

A similar observation holds for the Hammerstein case,

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \tag{2.3}$$

studied in [17, 22, 23] by means of the cone $K = K_0$. We mention that multiple solutions of (2.3) were investigated in the case of symmetric, sign changing kernels by Faraci and Moroz [11] by variational methods.

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We also stress that, if we denote by $P$ the cone of positive functions, namely

$$P := \{u \in C[0,1] \colon u(t) \geqslant 0,\ t \in [0,1]\},$$

and consider $K \cap P$, we regain the cone of positive functions introduced by Webb and Infante in [53].

First of all we prove that $T$ leaves $K$ invariant and is compact.

Lemma 2.1. *The operator (2.1) maps $K$ into $K$ and is compact.*

*Proof.* Take $u \in K$ such that $\|u\| \leqslant r$. First of all, we observe that $Tu(t) \geqslant 0$ for $t \in [a,b]$. We have, for $t \in [0,1]$,

$$|Tu(t)| \leqslant |\gamma(t)|\alpha[u] + |\delta(t)|\beta[u] + \int_0^1 |k(t,s)|g(s)f(s,u(s))\,\mathrm{d}s.$$

Therefore, taking the supremum on $t \in [0,1]$, we get

$$\|Tu\| \leqslant \|\gamma\|\alpha[u] + \|\delta\|\beta[u] + \int_0^1 \varPhi(s)g(s)f(s,u(s))\,\mathrm{d}s,$$

and combining this with (C2), (C6) and (C7) yields

$$\begin{aligned}\min_{t\in[a,b]} Tu(t) &\geqslant c_2\|\gamma\|\alpha[u] + c_3\|\delta\|\beta[u] + c_1\int_0^1 \varPhi(s)g(s)f(s,u(s))\,\mathrm{d}s \\ &\geqslant c\|Tu\|.\end{aligned}$$

Furthermore, by (C3) and (C5)–(C7),

$$\alpha[Tu] = \alpha[\gamma]\alpha[u] + \alpha[\delta]\beta[u] + \int_0^1 \mathcal{K}_A(s)g(s)f(s,u(s))\,\mathrm{d}s \geqslant 0$$

and

$$\beta[Tu] = \beta[\gamma]\alpha[u] + \beta[\delta]\beta[u] + \int_0^1 \mathcal{K}_B(s)g(s)f(s,u(s))\,\mathrm{d}s \geqslant 0.$$

Hence, we have $Tu \in K$.

Moreover, the map $T$ is compact since it is sum of three compact maps: the compactness of $F$ is well known and, since $\gamma$ and $\delta$ are continuous, the perturbation $\gamma(t)\alpha[u] + \delta(t)\beta[u]$ maps bounded sets into bounded subsets of a finite-dimensional space. □

For $\rho > 0$ we define the following open subsets of $K$:

$$K_\rho := \{u \in K \colon \|u\| < \rho\}, \qquad V_\rho := \Bigl\{u \in K \colon \min_{t\in[a,b]} u(t) < \rho\Bigr\}.$$

We have $K_\rho \subset V_\rho \subset K_{\rho/c}$.

We recall some useful facts concerning real $2 \times 2$ matrices.

Definition 2.2 (Webb and Infante [53]). A $2 \times 2$ matrix $\mathcal{Q}$ is said to be order preserving (or non-negative) if $p_1 \geqslant p_0$, $q_1 \geqslant q_0$ imply

$$\mathcal{Q}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \geqslant \mathcal{Q}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix},$$

in the sense of components.

We have the following property, as stated in [53], whose proof is straightforward.

LEMMA 2.3. *Let*

$$\mathcal{Q} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

*with* $a, b, c, d \geqslant 0$ *and* $\det \mathcal{Q} > 0$. *Then* $\mathcal{Q}^{-1}$ *is order preserving.*

REMARK 2.4. It is a consequence of lemma 2.3 that if

$$\mathcal{N} = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}$$

satisfies the hypothesis of lemma 2.3, $p \geqslant 0$, $q \geqslant 0$ and $\mu > 1$, then

$$\mathcal{N}_\mu^{-1} \begin{pmatrix} p \\ q \end{pmatrix} \leqslant \mathcal{N}^{-1} \begin{pmatrix} p \\ q \end{pmatrix},$$

where

$$\mathcal{N}_\mu = \begin{pmatrix} \mu-a & -b \\ -c & \mu-d \end{pmatrix}.$$

The next lemma summarizes some classical results regarding the fixed-point index (for more details see [1,15]). If $\Omega$ is a open bounded subset of a cone $K$ (in the relative topology), we denote by $\bar{\Omega}$ and $\partial\Omega$ respectively the closure and the boundary relative to $K$. When $\Omega$ is an open bounded subset of $X$ we write $\Omega_K = \Omega \cap K$, an open subset of $K$.

LEMMA 2.5. *Let* $\Omega$ *be an open bounded set with* $0 \in \Omega_K$ *and* $\bar{\Omega}_K \neq K$. *Assume that* $F\colon \bar{\Omega}_K \to K$ *is a compact map such that* $x \neq Fx$ *for all* $x \in \partial\Omega_K$. *Then the fixed-point index* $i_K(F, \Omega_K)$ *has the following properties.*

(1) *If there exists* $e \in K \setminus \{0\}$ *such that* $x \neq Fx + \lambda e$ *for all* $x \in \partial\Omega_K$ *and all* $\lambda > 0$, *then* $i_K(F, \Omega_K) = 0$.

(2) *If* $\mu x \neq Fx$ *for all* $x \in \partial\Omega_K$ *and for every* $\mu \geqslant 1$, *then* $i_K(F, \Omega_K) = 1$.

(3) *If* $i_K(F, \Omega_K) \neq 0$, *then* $F$ *has a fixed point in* $\Omega_K$.

(4) *Let* $\Omega^1$ *be open in* $X$ *with* $\overline{\Omega^1} \subset \underline{\Omega_K}$. *If* $i_K(F, \Omega_K) = 1$ *and* $i_K(F, \Omega_K^1) = 0$, *then* $F$ *has a fixed point in* $\Omega_K \setminus \overline{\Omega_K^1}$. *The same result holds if* $i_K(F, \Omega_K) = 0$ *and* $i_K(F, \Omega_K^1) = 1$.

The following proposition will be useful later. We give the proof for completeness.

PROPOSITION 2.6. *Let* $\omega \in L^1[0,1]$ *and denote*

$$\omega^+(s) = \max\{\omega(s), 0\}, \qquad \omega^-(s) = \max\{-\omega(s), 0\}.$$

*Then we have*

$$\left| \int_0^1 \omega(s)\, \mathrm{d}s \right| \leqslant \max\left\{ \int_0^1 \omega^+(s)\, \mathrm{d}s, \int_0^1 \omega^-(s)\, \mathrm{d}s \right\} \leqslant \int_0^1 |\omega(s)|\, \mathrm{d}s.$$

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*Proof.* Observing that, since $\omega = \omega^+ - \omega^-$,

$$\int_0^1 \omega(s)\,\mathrm{d}s = \int_0^1 \omega^+(s)\,\mathrm{d}s - \int_0^1 \omega^-(s)\,\mathrm{d}s \leqslant \int_0^1 \omega^+(s)\,\mathrm{d}s,$$
$$-\int_0^1 \omega(s)\,\mathrm{d}s = \int_0^1 \omega^-(s)\,\mathrm{d}s - \int_0^1 \omega^+(s)\,\mathrm{d}s \leqslant \int_0^1 \omega^-(s)\,\mathrm{d}s,$$

we get the first inequality; the second comes from the fact that $|\omega| = \omega^+ + \omega^-$. □

We now give a sufficient condition on the growth of the nonlinearity that provides that the index is 1 on $K_\rho$.

Lemma 2.7. *Assume that*

$(I_\rho^1)$ *there exists $\rho > 0$ such that*

$$f^{-\rho,\rho}\Bigg(\sup_{t\in[0,1]}\Bigg\{\bigg(\frac{|\gamma(t)|}{D}(1-\beta[\delta]) + \frac{|\delta(t)|}{D}\beta[\gamma]\bigg)\int_0^1 \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \bigg(\frac{|\gamma(t)|}{D}\alpha[\delta] + \frac{|\delta(t)|}{D}(1-\alpha[\gamma])\bigg)\int_0^1 \mathcal{K}_B(s)g(s)\,\mathrm{d}s + \max\bigg\{\int_0^1 k^+(t,s)g(s)\,\mathrm{d}s, \int_0^1 k^-(t,s)g(s)\,\mathrm{d}s\bigg\}\Bigg\}\Bigg) < 1, \tag{2.4}$$

*where*

$$f^{-\rho,\rho} := \operatorname{ess\,sup}\bigg\{\frac{f(t,u)}{\rho} : (t,u) \in [0,1]\times[-\rho,\rho]\bigg\}. \tag{2.5}$$

*Then we have $i_K(T, K_\rho) = 1$.*

*Proof.* We show that $Tu \neq \lambda u$ for all $\lambda \geqslant 1$ when $u \in \partial K_\rho$, which implies that $i_K(T, K_\rho) = 1$. In fact, if this does not happen, then there exist $u$ with $\|u\| = \rho$ and $\lambda \geqslant 1$ such that $\lambda u(t) = Tu(t)$, i.e.

$$\lambda u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + Fu(t). \tag{2.6}$$

Therefore, we obtain

$$\lambda\alpha[u] = \alpha[\gamma]\alpha[u] + \alpha[\delta]\beta[u] + \alpha[Fu]$$

and

$$\lambda\beta[u] = \beta[\gamma]\alpha[u] + \beta[\delta]\beta[u] + \beta[Fu].$$

Thus, we have

$$\begin{pmatrix}\lambda-\alpha[\gamma] & -\alpha[\delta]\\ -\beta[\gamma] & \lambda-\beta[\delta]\end{pmatrix}\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix} = \begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}. \tag{2.7}$$

Note that the matrix that occurs in (2.7) satisfies the hypothesis of lemma 2.3, so its inverse is order preserving. Then, applying its inverse matrix to both sides

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of (2.7), we have

$$\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix} = \frac{1}{D_\lambda}\begin{pmatrix}\lambda-\beta[\delta] & \alpha[\delta]\\ \beta[\gamma] & \lambda-\alpha[\gamma]\end{pmatrix}\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}.$$

By remark 2.4, we obtain that

$$\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix} \leqslant \frac{1}{D}\begin{pmatrix}1-\beta[\delta] & \alpha[\delta]\\ \beta[\gamma] & 1-\alpha[\gamma]\end{pmatrix}\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}. \tag{2.8}$$

Hence, from (2.6) and (2.8) we get

$$\lambda|u(t)| \leqslant \frac{|\gamma(t)|}{D}((1-\beta[\delta])\alpha[Fu] + \alpha[\delta]\beta[Fu]) \\ + \frac{|\delta(t)|}{D}((1-\alpha[\gamma])\beta[Fu] + \beta[\gamma]\alpha[Fu]) + |Fu(t)|.$$

Taking the supremum over $[0,1]$ gives

$$\lambda\rho \leqslant \rho f^{-\rho,\rho}\Bigg(\sup_{t\in[0,1]}\Bigg\{\Bigg(\frac{|\gamma(t)|}{D}(1-\beta[\delta]) + \frac{|\delta(t)|}{D}\beta[\gamma]\Bigg)\int_0^1 \mathcal{K}_A(s)g(s)\,\mathrm{d}s \\ + \Bigg(\frac{|\gamma(t)|}{D}\alpha[\delta] + \frac{|\delta(t)|}{D}(1-\alpha[\gamma])\Bigg)\int_0^1 \mathcal{K}_B(s)g(s)\,\mathrm{d}s \\ + \max\Bigg\{\int_0^1 k^+(t,s)g(s)\,\mathrm{d}s, \int_0^1 k^-(t,s)g(s)\,\mathrm{d}s\Bigg\}\Bigg\}\Bigg).$$

From (2.4) we obtain that $\lambda\rho < \rho$, contradicting the fact that $\lambda \geqslant 1$. □

Remark 2.8. In similar way as in [53] (where the positive case was studied) we point out that a stronger (but easier to check) condition than $(I_\rho^1)$ is given by the following:

$$f^{-\rho,\rho}\Bigg[\Bigg(\frac{\|\gamma\|}{D}(1-\beta[\delta]) + \frac{\|\delta\|}{D}\beta[\gamma]\Bigg)\int_0^1 \mathcal{K}_A(s)g(s)\,\mathrm{d}s \\ + \Bigg(\frac{\|\gamma\|}{D}\alpha[\delta] + \frac{\|\delta\|}{D}(1-\alpha[\gamma])\Bigg)\int_0^1 \mathcal{K}_B(s)g(s)\,\mathrm{d}s + \frac{1}{m}\Bigg] < 1,$$

where

$$\frac{1}{m} := \sup_{t\in[0,1]}\Bigg\{\max\Bigg\{\int_0^1 k^+(t,s)g(s)\,\mathrm{d}s, \int_0^1 k^-(t,s)g(s)\,\mathrm{d}s\Bigg\}\Bigg\}.$$

Note that, since $\max\{k^+, k^-\} \leqslant |k|$, the constant $m$ provides a better estimate on the growth of the nonlinearity $f$ than the constant

$$\sup_{t\in[0,1]}\int_0^1 |k(t,s)|g(s)\,\mathrm{d}s$$

used in [6, 7, 10, 13, 14, 17, 20–22, 24, 25, 36].

Remark 2.9. If the functions $\gamma$, $\delta$, $k$ are non-negative, we can work within the cone $K \cap P$, regaining the condition given in [53], namely

$$f^{0,\rho}\biggl(\sup_{t\in[0,1]}\biggl\{\biggl(\frac{\gamma(t)}{D}(1-\beta[\delta])+\frac{\delta(t)}{D}\beta[\gamma]\biggr)\int_0^1 \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \biggl(\frac{\gamma(t)}{D}\alpha[\delta]+\frac{\delta(t)}{D}(1-\alpha[\gamma])\biggr)\int_0^1 \mathcal{K}_B(s)g(s)\,\mathrm{d}s + \int_0^1 k(t,s)g(s)\,\mathrm{d}s\biggr\}\biggr) < 1,$$

where

$$f^{0,\rho} := \operatorname{ess\,sup}\biggl\{\frac{f(t,u)}{\rho}\colon (t,u)\in[0,1]\times[0,\rho]\biggr\}.$$

Lemma 2.10. *Assume that*

$(I^0_\rho)$ *there exists* $\rho > 0$ *such that*

$$f_{\rho,\rho/c}\biggl(\inf_{t\in[a,b]}\biggl\{\biggl(\frac{\gamma(t)}{D}(1-\beta[\delta])+\frac{\delta(t)}{D}\beta[\gamma]\biggr)\int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \biggl(\frac{\gamma(t)}{D}\alpha[\delta]+\frac{\delta(t)}{D}(1-\alpha[\gamma])\biggr)\int_a^b \mathcal{K}_B(s)g(s)\,\mathrm{d}s + \int_a^b k(t,s)g(s)\,\mathrm{d}s\biggr\}\biggr) > 1, \tag{2.9}$$

*where*

$$f_{\rho,\rho/c} := \operatorname{ess\,inf}\biggl\{\frac{f(t,u)}{\rho}\colon (t,u)\in[a,b]\times[\rho,\rho/c]\biggr\}. \tag{2.10}$$

*Then we have* $i_K(T,V_\rho)=0$.

*Proof.* Let

$$e(t)=\int_0^1 k(t,s)\,\mathrm{d}s \quad \text{for } t\in[0,1].$$

Then, according to (C2), (C3) and (C5), we have $e \in K\setminus\{0\}$. We show that $u \neq Tu+\lambda e$ for all $\lambda \geqslant 0$ and $u \in \partial V_\rho$, which implies that $i_K(T,V_\rho)=0$. In fact, if this does not happen, there exist $u\in\partial V_\rho$ (and so for $t\in[a,b]$ we have $\min u(t)=\rho$ and $\rho \leqslant u(t) \leqslant \rho/c$) and $\lambda \geqslant 0$ with

$$u(t)=Tu(t)+\lambda e(t)=\gamma(t)\alpha[u]+\delta(t)\beta[u]+Fu(t)+\lambda e(t).$$

Applying $\alpha$ and $\beta$ to both sides of the previous equation, we get

$$\begin{pmatrix}1-\alpha[\gamma] & -\alpha[\delta]\\ -\beta[\gamma] & 1-\beta[\delta]\end{pmatrix}\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix}=\begin{pmatrix}\alpha[Fu]+\lambda\alpha[e]\\ \beta[Fu]+\lambda\beta[e]\end{pmatrix}\geqslant\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}. \tag{2.11}$$

Note that the matrix that occurs in (2.11) satisfies the hypothesis of lemma 2.3, so its inverse is order preserving. Then, applying the inverse matrix to both sides of (2.11), we have

$$\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix} \geqslant \frac{1}{D}\begin{pmatrix}1-\beta[\delta] & \alpha[\delta]\\ \beta[\gamma] & 1-\alpha[\gamma]\end{pmatrix}\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}.$$

Therefore, for $t \in [a,b]$, we obtain

$$\begin{aligned} u(t) &\geqslant \left(\frac{\gamma(t)}{D}(1-\beta[\delta]) + \frac{\delta(t)}{D}\beta[\gamma]\right)\alpha[Fu] \\ &\quad + \left(\frac{\gamma(t)}{D}\alpha[\delta] + \frac{\delta(t)}{D}(1-\alpha[\gamma])\right)\beta[Fu] + Fu(t) + \lambda e(t) \\ &\geqslant \left(\frac{\gamma(t)}{D}(1-\beta[\delta]) + \frac{\delta(t)}{D}\beta[\gamma]\right)\int_a^b \mathcal{K}_A(s)g(s)f(s,u(s))\,\mathrm{d}s \\ &\quad + \left(\frac{\gamma(t)}{D}\alpha[\delta] + \frac{\delta(t)}{D}(1-\alpha[\gamma])\right)\int_a^b \mathcal{K}_B(s)g(s)f(s,u(s))\,\mathrm{d}s \\ &\quad + \int_a^b k(t,s)g(s)f(s,u(s))\,\mathrm{d}s. \end{aligned}$$

Taking the infimum for $t \in [a,b]$ then gives

$$\begin{aligned} \rho &= \min u(t) \\ &\geqslant \rho f_{\rho,\rho/c}\Bigg(\inf_{t\in[a,b]}\Bigg\{\left(\frac{\gamma(t)}{D}(1-\beta[\delta]) + \frac{\delta(t)}{D}\beta[\gamma]\right)\int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s \\ &\qquad + \left(\frac{\gamma(t)}{D}\alpha[\delta] + \frac{\delta(t)}{D}(1-\alpha[\gamma])\right)\int_a^b \mathcal{K}_B(s)g(s)\,\mathrm{d}s \\ &\qquad + \int_a^b k(t,s)g(s)\,\mathrm{d}s\Bigg\}\Bigg), \end{aligned}$$

contradicting (2.9). □

Remark 2.11. We point out, in similar way as in [53], that a stronger (but easier to check) condition than $(I_\rho^0)$ is given by the following:

$$\begin{aligned} f_{\rho,\rho/c}\Bigg(&\left(\frac{c_2\|\gamma\|}{D}(1-\beta[\delta]) + \frac{c_3\|\delta\|}{D}\beta[\gamma]\right)\int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s \\ &+ \left(\frac{c_2\|\gamma\|}{D}\alpha[\delta] + \frac{c_3\|\delta\|}{D}(1-\alpha[\gamma])\right)\int_a^b \mathcal{K}_B(s)g(s)\,\mathrm{d}s + \frac{1}{M(a,b)}\Bigg) > 1, \end{aligned} \tag{2.12}$$

where

$$\frac{1}{M(a,b)} := \inf_{t\in[a,b]}\int_a^b k(t,s)g(s)\,\mathrm{d}s.$$

We now combine the results above in order to prove a theorem regarding the existence of one, two or three non-trivial solutions. The proof is a direct consequence of the properties of the fixed-point index and is omitted. It is possible to state a result for the existence of four or more solutions; we refer the reader to [31] for similar statements.

Theorem 2.12. *The integral equation (2.1) has at least one non-zero solution in* $K$ *if one of the following conditions holds:*

(S1) *there exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1/c < \rho_2$ *such that* $(I^0_{\rho_1})$ *and* $(I^1_{\rho_2})$ *hold;*

(S2) *there exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \rho_2$ *such that* $(I^1_{\rho_1})$ *and* $(I^0_{\rho_2})$ *hold.*

*The integral equation (2.1) has at least two non-zero solutions in* $K$ *if one of the following conditions holds:*

(S3) *there exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_1/c < \rho_2 < \rho_3$ *such that* $(I^0_{\rho_1})$, $(I^1_{\rho_2})$ *and* $(I^0_{\rho_3})$ *hold;*

(S4) *there exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ *with* $\rho_1 < \rho_2$ *and* $\rho_2/c < \rho_3$ *such that* $(I^1_{\rho_1}), (I^0_{\rho_2})$ *and* $(I^1_{\rho_3})$ *hold.*

*The integral equation (2.1) has at least three non-zero solutions in* $K$ *if one of the following conditions holds:*

(S5) *there exist* $\rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty)$ *with* $\rho_1/c < \rho_2 < \rho_3$ *and* $\rho_3/c < \rho_4$ *such that* $(I^0_{\rho_1})$, $(I^1_{\rho_2})$, $(I^0_{\rho_3})$ *and* $(I^1_{\rho_4})$ *hold;*

(S6) *there exist* $\rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty)$ *with* $\rho_1 < \rho_2$ *and* $\rho_2/c < \rho_3 < \rho_4$ *such that* $(I^1_{\rho_1})$, $(I^0_{\rho_2})$, $(I^1_{\rho_3})$ *and* $(I^0_{\rho_4})$ *hold.*

## 3. Some non-existence results

We now consider the auxiliary Hammerstein integral equation,

$$u(t) = \int_0^1 k_S(t,s) g(s) f(s, u(s)) \, \mathrm{d}s := Su(t), \tag{3.1}$$

where the kernel $k_S$ is given by the formula

$$k_S(t,s) = \frac{\gamma(t)}{D}[(1-\beta[\delta])\mathcal{K}_A(s) + \alpha[\delta]\mathcal{K}_B(s)] \\ + \frac{\delta(t)}{D}[\beta[\gamma]\mathcal{K}_A(s) + (1-\alpha[\gamma])\mathcal{K}_B(s)] + k(t,s).$$

The operator $S$ shares a number of useful properties with $T$ (primarily the cone invariance and compactness). The proof follows directly from (C1)–(C8) and is omitted.

Lemma 3.1. *The operator (3.1) maps* $K$ *into* $K$ *and is compact.*

A key property that is also useful is the one given by the following theorem; the proof is similar to that in [53, lemma 2.8 and theorem 2.9] and is omitted.

Lemma 3.2. *The operators* $S$ *and* $T$ *have the same fixed points in* $K$*. Furthermore, if* $u \neq Tu$ *for* $u \in \partial D_K$*, then* $i_K(T, D_K) = i_K(S, D_K)$*.*

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We define the constants

$$\frac{1}{m_S} := \sup_{t\in[0,1]} \left\{ \max \left\{ \int_0^1 k_S^+(t,s)g(s)\,\mathrm{d}s, \int_0^1 k_S^-(t,s)g(s)\,\mathrm{d}s \right\} \right\}$$

and

$$\frac{1}{M_S(a,b)} = \frac{1}{M_S} := \inf_{t\in[a,b]} \int_a^b k_S(t,s)g(s)\,\mathrm{d}s,$$

and we prove the following non-existence results.

THEOREM 3.3. *Assume that one of the following conditions holds:*

(1) $f(t,u) < m_S|u|$ *for every* $t \in [0,1]$ *and* $u \in \mathbb{R} \setminus \{0\}$*;*

(2) $f(t,u) > M_S u$ *for every* $t \in [a,b]$ *and* $u \in \mathbb{R}^+$.

*Then (2.1) and (3.1) have no non-trivial solution in* $K$.

*Proof.* In view of lemma 3.2 we prove the theorem using the operator $S$.

(1) Assume, on the contrary, that there exists $u \in K$, $u \not\equiv 0$, such that $u = Su$ and let $t_0 \in [0,1]$ such that $\|u\| = |u(t_0)|$. Then we have

$$\begin{aligned}
\|u\| &= |u(t_0)| \\
&= \left| \int_0^1 k_S(t_0,s)g(s)f(s,u(s))\,\mathrm{d}s \right| \\
&\leqslant \max \left\{ \int_0^1 k_S^+(t_0,s)g(s)f(s,u(s))\,\mathrm{d}s, \int_0^1 k_S^-(t_0,s)g(s)f(s,u(s))\,\mathrm{d}s \right\} \\
&< \max \left\{ \int_0^1 k_S^+(t_0,s)g(s)m_S|u(s)|\,\mathrm{d}s, \int_0^1 k_S^-(t_0,s)g(s)m_S|u(s)|\,\mathrm{d}s \right\} \\
&\leqslant \max \left\{ \int_0^1 k_S^+(t_0,s)g(s)\,\mathrm{d}s, \int_0^1 k_S^-(t_0,s)g(s)\,\mathrm{d}s \right\} m_S\|u\| \\
&\leqslant \|u\|,
\end{aligned}$$

a contradiction.

(2) Assume, on the contrary, that there exists $u \in K$, $u \not\equiv 0$, such that $u = Su$ and let $\eta \in [a,b]$ be such that $u(\eta) = \min_{t\in[a,b]} u(t)$. For $t \in [a,b]$ we have

$$\begin{aligned}
u(t) &= \int_0^1 k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s \\
&\geqslant \int_a^b k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s \\
&> M_S \int_a^b k_S(t,s)g(s)u(s)\,\mathrm{d}s.
\end{aligned}$$

Taking the infimum for $t \in [a,b]$, we have

$$\min_{t\in[a,b]} u(t) > M_S \inf_{t\in[a,b]} \int_a^b k_S(t,s)g(s)u(s)\,\mathrm{d}s.$$

Thus, we obtain

$$u(\eta) > M_S u(\eta) \inf_{t\in[a,b]} \int_a^b k_S(t,s)g(s)\,\mathrm{d}s = u(\eta),$$

a contradiction. □

## 4. Eigenvalue criteria for the existence of non-trivial solutions

In this section we assume the additional hypothesis that the functionals $\alpha$ and $\beta$ are given by *positive measures*.

In order to state our eigenvalue comparison results, we consider the following operators on $C[0,1]$:

$$Lu(t) := \int_0^1 |k_S(t,s)|g(s)u(s)\,\mathrm{d}s, \qquad \tilde{L}u(t) := \int_a^b k_S^+(t,s)g(s)u(s)\,\mathrm{d}s.$$

By proofs similar to those of [53, lemma 2.6 and theorem 2.7], we study the properties of these operators.

Theorem 4.1. *The operators $L$ and $\tilde{L}$ are compact and map $P$ into $P\cap K$.*

*Proof.* Note that the operators $L$ and $\tilde{L}$ map $P$ into $P$ (because they have a positive integral kernel) and are compact. We now show that they map $P$ into $P\cap K$. We do this for the operator $L$; the proof of $\tilde{L}$ is similar.

Firstly, we observe that

$$\begin{aligned}|k_S(t,s)| &\leqslant \frac{|\gamma(t)|}{D}((1-\beta[\delta])\mathcal{K}_A(s)+\alpha[\delta]\mathcal{K}_B(s)) \\ &\quad + \frac{|\delta(t)|}{D}(\beta[\gamma]\mathcal{K}_A(s)+(1-\alpha[\gamma])\mathcal{K}_B(s)) + |k(t,s)| \\ &\leqslant \frac{\|\gamma\|}{D}((1-\beta[\delta])\mathcal{K}_A(s)+\alpha[\delta]\mathcal{K}_B(s)) \\ &\quad + \frac{\|\delta\|}{D}(\beta[\gamma]\mathcal{K}_A(s)+(1-\alpha[\gamma])\mathcal{K}_B(s)) + |k(t,s)| \\ &\leqslant \Upsilon(s)+\Phi(s) \\ &=: \Psi(s),\end{aligned}$$

where

$$\Upsilon(s) = \frac{\|\gamma\|}{D}((1-\beta[\delta])\mathcal{K}_A(s)+\alpha[\delta]\mathcal{K}_B(s)) + \frac{\|\delta\|}{D}(\beta[\gamma]\mathcal{K}_A(s)+(1-\alpha[\gamma])\mathcal{K}_B(s)).$$

Moreover, we have, for $t\in[a,b]$,

$$\begin{aligned}|k_S(t,s)| &= k_S(t,s) \\ &\geqslant \frac{c_2\|\gamma\|}{D}[(1-\beta[\delta])\mathcal{K}_A(s)+\alpha[\delta]\mathcal{K}_B(s)] \\ &\quad + \frac{c_3\|\delta\|}{D}[\beta[\gamma]\mathcal{K}_A(s)+(1-\alpha[\gamma])\mathcal{K}_B(s)] + c_1\Phi(t) \\ &\geqslant c\Psi(s),\end{aligned}$$

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and thus

$$\min_{t\in[a,b]} k_S(t,s) \geqslant c\Psi(s). \tag{4.1}$$

Also we have $g\Psi \in L^1[0,1]$ and we obtain that, for $u \in P$ and $t \in [0,1]$,

$$Lu(t) \leqslant \int_0^1 \Psi(s)g(s)u(s)\,\mathrm{d}s$$

such that, taking the supremum on $t \in [0,1]$, we get

$$\|Lu\| \leqslant \int_0^1 \Psi(s)g(s)u(s)\,\mathrm{d}s.$$

On the other hand,

$$\min_{t\in[a,b]} Lu(t) \geqslant c\int_0^1 \Psi(s)g(s)u(s)\,\mathrm{d}s \geqslant c\|Lu\|.$$

Furthermore, since $\alpha$ and $\beta$ are given by positive measures,

$$\alpha[Lu] = \int_0^1\int_0^1 |k_S(t,s)|g(s)u(s)\,\mathrm{d}s\,\mathrm{d}A(t) \geqslant 0$$

and

$$\beta[Lu] = \int_0^1\int_0^1 |k_S(t,s)|g(s)u(s)\,\mathrm{d}s\,\mathrm{d}B(t) \geqslant 0.$$

Hence, we have $Lu \in K$. □

We recall that $\lambda$ is an *eigenvalue* of a linear operator $\Gamma$ with corresponding eigenfunction $\varphi$ if $\varphi \neq 0$ and $\lambda\varphi = \Gamma\varphi$. The reciprocals of non-zero eigenvalues are called *characteristic values* of $\Gamma$. We shall denote the *spectral radius* of $\Gamma$ by $r(\Gamma) := \lim_{n\to\infty} \|\Gamma^n\|^{1/n}$ and its *principal characteristic value* (the reciprocal of the spectral radius) by $\mu(\Gamma) := 1/r(\Gamma)$.

The following theorem is analogous to those in [53, 55] and is proven by using the facts that the operators considered leave $P$ invariant and $P$ is reproducing, combined with the well-known Krein–Rutman theorem. Condition (C3) is used to show that $r(L) > 0$.

Theorem 4.2. *The spectral radius of $L$ is non-zero and is an eigenvalue of $L$ with an eigenfunction in $P$. A similar result holds for $\tilde{L}$.*

Remark 4.3. As a consequence of the two previous theorems, the above-mentioned eigenfunction is in $P \cap K$.

We use the operator on $C[a,b]$ defined, for $t \in [a,b]$, by

$$\bar{L}u(t) := \int_a^b k_S^+(t,s)g(s)u(s)\,\mathrm{d}s$$

and the cone $P_{[a,b]}$ of positive functions in $C[a,b]$.

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In the recent papers [50, 52], Webb developed an elegant theory valid for $u_0$-positive linear operators relative to two cones. It turns out that our operator $\bar{L}$ fits within this setting and, in particular, satisfies the assumptions of [52, theorem 3.4]. We state here a special case of [52, theorem 3.4] that can be used for $\bar{L}$.

THEOREM 4.4. *Suppose that there exist $u \in P_{[a,b]} \setminus \{0\}$ and $\lambda > 0$ such that*

$$\lambda u(t) \geqslant \bar{L}u(t) \quad \text{for } t \in [a,b].$$

*Then we have $r(\bar{L}) \leqslant \lambda$.*

We define the following extended real numbers:

$$\left.\begin{aligned} f^0 &= \overline{\lim_{u\to 0}} \frac{\text{ess sup}_{t\in[0,1]} f(t,u)}{|u|}, & f_0 &= \underline{\lim_{u\to 0^+}} \frac{\text{ess inf}_{t\in[a,b]} f(t,u)}{u}, \\ f^\infty &= \overline{\lim_{|u|\to +\infty}} \frac{\text{ess sup}_{t\in[0,1]} f(t,u)}{|u|}, & f_\infty &= \underline{\lim_{u\to +\infty}} \frac{\text{ess inf}_{t\in[a,b]} f(t,u)}{u}. \end{aligned}\right\} \tag{4.2}$$

In order to prove the following theorem, we adapt some of the proofs of [55, theorems 3.2–3.5] to this new context.

THEOREM 4.5. *We have the following.*

(1) *If $0 \leqslant f^0 < \mu(L)$, then there exists $\rho_0 > 0$ such that $i_K(T, K_\rho) = 1$ for each $\rho \in (0, \rho_0]$.*

(2) *If $0 \leqslant f^\infty < \mu(L)$, then there exists $R_0 > 0$ such that $i_K(T, K_R) = 1$ for each $R > R_0$.*

(3) *If $\mu(\tilde{L}) < f_0 \leqslant \infty$, then there exists $\rho_0 > 0$ such that $i_K(T, K_\rho) = 0$ for each $\rho \in (0, \rho_0]$.*

(4) *If $\mu(\tilde{L}) < f_\infty \leqslant \infty$, then there exists $R_1 > 0$ such that $i_K(T, K_R) = 0$ for each $R \geqslant R_1$.*

*Proof.* We show the statements for the operator $S$ instead of $T$, in view of lemma 3.2.

(1) Let $\tau$ be such that $f^0 \leqslant \mu(L) - \tau$. Then there exists $\rho_0 \in (0, 1)$ such that for all $u \in [-\rho_0, \rho_0]$ and almost every $t \in [0, 1]$ we have

$$f(t, u) \leqslant (\mu(L) - \tau)|u|.$$

Let $\rho \in (0, \rho_0]$. We prove that $Su \neq \lambda u$ for $u \in \partial K_\rho$ and $\lambda \geqslant 1$, which implies $i_K(S, K_\rho) = 1$. In fact, if we assume otherwise, then there exist $u \in \partial K_\rho$ and $\lambda \geqslant 1$

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such that $\lambda u = Su$. Therefore,

$$\begin{aligned}|u(t)| \leqslant \lambda |u(t)| &= |Su(t)| \\ &= \left| \int_0^1 k_S(t,s) g(s) f(s,u(s))\,\mathrm{d}s \right| \\ &\leqslant \int_0^1 |k_S(t,s)| g(s) f(s,u(s))\,\mathrm{d}s \\ &\leqslant (\mu(L)-\tau) \int_0^1 |k_S(t,s)| g(s) |u(s)|\,\mathrm{d}s \\ &= (\mu(L)-\tau) L|u|(t).\end{aligned}$$

Thus, we have, for $t \in [0,1]$,

$$\begin{aligned}|u(t)| &\leqslant (\mu(L)-\tau)L[(\mu(L)-\tau)L|u|(t)] \\ &= (\mu(L)-\tau)^2 L^2 |u|(t) \\ &\leqslant \cdots \leqslant (\mu(L)-\tau)^n L^n |u|(t).\end{aligned}$$

Thus, taking the norms, $1 \leqslant (\mu(L)-\tau)^n \|L^n\|$, and then

$$1 \leqslant (\mu(L)-\tau) \lim_{n\to\infty} \|L^n\|^{1/n} = \frac{\mu(L)-\tau}{\mu(L)} < 1,$$

a contradiction.

(2) Let $\tau \in \mathbb{R}^+$ such that $f^\infty < \mu(L)-\tau$. Then there exists $R_1 > 0$ such that, for every $|u| \geqslant R_1$ and almost every $t \in [0,1]$,

$$f(t,u) \leqslant (\mu(L)-\tau)|u|.$$

Also, by (C4) there exists $\phi_{R_1} \in L^\infty[0,1]$ such that $f(t,u) \leqslant \phi_{R_1}(t)$ for all $u \in [-R_1, R_1]$ and almost every $t \in [0,1]$. Hence,

$$f(t,u) \leqslant (\mu(L)-\tau)|u| + \phi_{R_1}(t) \quad \text{for all } u \in \mathbb{R} \text{ and almost every } t \in [0,1]. \tag{4.3}$$

Denote by Id the identity operator and observe that $\mathrm{Id} - (\mu(L)-\tau)L$ is invertible since $(\mu(L)-\tau)L$ has a spectral radius less than 1. Furthermore, by the Neumann series expression,

$$[\mathrm{Id} - (\mu(L)-\tau)L]^{-1} = \sum_{k=0}^{\infty} [(\mu(L)-\tau)L]^k.$$

Therefore, $[\mathrm{Id} - (\mu(L)-\tau)L]^{-1}$ maps $P$ into $P$, since $L$ does.

Let

$$C := \int_a^b \Phi(s) g(s) \phi_{R_1}(s)\,\mathrm{d}s \quad \text{and} \quad R_0 := \|[\mathrm{Id} - (\mu(L)-\tau)L]^{-1} C\|.$$

Now we prove that, for each $R > R_0$, $Su \neq \lambda u$ for all $u \in \partial K_R$ and $\lambda \geqslant 1$, which implies $i_K(S, K_R) = 1$. Assume otherwise: there exist $u \in \partial K_R$ and $\lambda \geqslant 1$ such

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that $\lambda u = Su$. Taking into account the inequality (4.3), for $t \in [0,1]$ we have

$$\begin{aligned}|u(t)| \leqslant \lambda |u(t)| &= |Su(t)| \\ &= \left| \int_0^1 k_S(t,s) g(s) f(s,u(s))\,\mathrm{d}s \right| \\ &\leqslant \int_0^1 |k_S(t,s)| g(s) f(s,u(s))\,\mathrm{d}s \\ &\leqslant (\mu(L)-\tau) \int_0^1 |k_S(t,s)| g(s) |u(s)|\,\mathrm{d}s + C \\ &= (\mu(L)-\tau) L|u|(t) + C,\end{aligned}$$

which implies

$$[\mathrm{Id} - (\mu(L)-\tau)L]|u|(t) \leqslant C.$$

Since $(\mathrm{Id} - (\mu(L)-\tau)L)^{-1}$ is non-negative, we have

$$|u|(t) \leqslant [\mathrm{Id} - (\mu(L)-\tau)L]^{-1} C \leqslant R_0.$$

Therefore, we have $\|u\| \leqslant R_0 < R$, a contradiction.

(3) There exists $\rho_0 > 0$ such that for all $u \in [0, \rho_0]$ and all $t \in [a,b]$ we have

$$f(t,u) \geqslant \mu(\tilde{L}) u.$$

Let $\rho \in (0, \rho_0]$. Let us prove that $u \neq Su + \lambda \varphi_1$ for all $u$ in $\partial K_\rho$ and $\lambda \geqslant 0$, where $\varphi_1 \in K \cap P$ is the eigenfunction of $\tilde{L}$ with $\|\varphi_1\| = 1$ corresponding to the eigenvalue $1/\mu(\tilde{L})$. This implies that $i_K(S, K_\rho) = 0$.

Assume, on the contrary, that there exist $u \in \partial K_\rho$ and $\lambda \geqslant 0$ such that $u = Su + \lambda \varphi_1$.

We distinguish two cases. Firstly, we discuss the case $\lambda > 0$. We have, for $t \in [a,b]$,

$$\begin{aligned}u(t) &= \int_0^1 k_S(t,s) g(s) f(s,u(s))\,\mathrm{d}s + \lambda \varphi_1(t) \\ &\geqslant \int_a^b k_S^+(t,s) g(s) f(s,u(s))\,\mathrm{d}s + \lambda \varphi_1(t) \\ &\geqslant \mu(\tilde{L}) \int_a^b k_S^+(t,s) g(s) u(s)\,\mathrm{d}s + \lambda \varphi_1(t) \\ &= \mu(\tilde{L}) \tilde{L} u(t) + \lambda \varphi_1(t).\end{aligned}$$

Moreover, we have $u(t) \geqslant \lambda \varphi_1(t)$ and then $\tilde{L} u(t) \geqslant \lambda \tilde{L} \varphi_1(t) \geqslant (\lambda/\mu(\tilde{L})) \varphi_1(t)$ and therefore we obtain

$$u(t) \geqslant \mu(\tilde{L}) \tilde{L} u(t) + \lambda \varphi_1(t) \geqslant 2\lambda \varphi_1(t) \quad \text{for } t \in [a,b].$$

By iteration, we deduce that, for $t \in [a,b]$, we get

$$u(t) \geqslant n \lambda \varphi_1(t) \quad \text{for every } n \in \mathbb{N},$$

a contradiction because $\|u\| = \rho$.

Now we consider the case $\lambda = 0$. Let $\varepsilon > 0$ be such that for all $u \in [0, \rho_0]$ and almost every $t \in [a, b]$ we have

$$f(t, u) \geqslant (\mu(\tilde{L}) + \varepsilon)u.$$

We have, for $t \in [a, b]$,

$$\begin{aligned} u(t) &= \int_0^1 k_S(t, s) g(s) f(s, u(s))\, \mathrm{d}s \\ &\geqslant \int_a^b k_S^+(t, s) g(s) f(s, u(s))\, \mathrm{d}s \\ &\geqslant (\mu(\tilde{L}) + \varepsilon)\tilde{L}u(t). \end{aligned}$$

Since $\tilde{L}\varphi_1(t) = r(\tilde{L})\varphi_1(t)$ for $t \in [0, 1]$, we have, for $t \in [a, b]$,

$$\bar{L}\varphi_1(t) = \tilde{L}\varphi_1(t) = r(\tilde{L})\varphi_1(t),$$

and we obtain $r(\bar{L}) \geqslant r(\tilde{L})$. On the other hand, we have, for $t \in [a, b]$,

$$u(t) \geqslant (\mu(\tilde{L}) + \varepsilon)\tilde{L}u(t) = (\mu(\tilde{L}) + \varepsilon)\bar{L}u(t),$$

where $u(t) > 0$. Thus, using theorem 4.4, we have $r(\bar{L}) \leqslant 1/(\mu(\tilde{L}) + \varepsilon)$ and therefore $r(\tilde{L}) \leqslant 1/(\mu(\tilde{L}) + \varepsilon)$. This gives $\mu(\tilde{L}) + \varepsilon \leqslant \mu(\tilde{L})$, a contradiction.

(4) Let $R_1 > 0$ such that

$$f(t, u) > \mu(\tilde{L})u$$

for all $u \geqslant cR_1$, $c$ as in (4.1) and all $t \in [a, b]$.

Let $R \geqslant R_1$. We prove that $u \neq Su + \lambda\varphi_1$ for all $u$ in $\partial K_R$ and $\lambda \geqslant 0$, which implies $i_K(S, K_R) = 0$.

Assume now, on the contrary, that there exist $u \in \partial K_R$ and $\lambda \geqslant 0$ such that $u = Su + \lambda\varphi_1$. Observe that for $u \in \partial K_R$ we have $u(t) \geqslant c\|u\| = cR \geqslant cR_1$ for $t \in [a, b]$. Hence, we have $f(t, u(t)) > \mu(\tilde{L})u(t)$ for $t \in [a, b]$. Proceeding as in the proof of (3), this implies that, for the case $\lambda > 0$,

$$u(t) \geqslant \mu(\tilde{L})\tilde{L}u(t) + \lambda\varphi_1(t) \geqslant 2\lambda\varphi_1(t) \quad \text{for } t \in [a, b].$$

Then, for $t \in [a, b]$, we have $u(t) \geqslant n\lambda\varphi_1(t)$ for every $n \in \mathbb{N}$, a contradiction because $\|u\| = R$. The proof in the case $\lambda = 0$ is treated as in the proof of (3). □

The following theorem, along the lines of [53, 56], applies the index results of lemmas 2.7 and 2.10 and theorem 4.5 in order to obtain some results on the existence of multiple non-trivial solutions for (2.1).

THEOREM 4.6. *Assume that conditions* (C1)–(C8) *hold with* $\alpha$ *and* $\beta$ *given by positive measures.*

*The integral equation (2.1) has at least one non-trivial solution in* $K$ *if one of the following conditions holds:*

(H1) $0 \leqslant f^0 < \mu(L)$ *and* $\mu(\tilde{L}) < f_\infty \leqslant \infty$,

(H2) $0 \leqslant f^\infty < \mu(L)$ *and* $\mu(\tilde{L}) < f_0 \leqslant \infty$.

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*The integral equation (2.1) has at least two non-trivial solutions in $K$ if one of the following conditions holds:*

(Z1) $0 \leqslant f^0 < \mu(L)$, $f_{\rho,\rho/c} > M_S(a,b)$ *for some* $\rho > 0$ *and* $0 \leqslant f^\infty < \mu(L)$;

(Z2) $\mu(\tilde{L}) < f_0 \leqslant \infty$, $f^{-\rho,\rho} < m_S$ *for some* $\rho > 0$ *and* $\mu(\tilde{L}) < f_\infty \leqslant \infty$.

*The integral equation (2.1) has at least three non-trivial solutions in $K$ if one of the following conditions holds:*

(T1) *there exist* $0 < \rho_1 < \rho_2 < \infty$, *such that*

$$\mu(\tilde{L}) < f_0 \leqslant \infty, \quad f^{-\rho_1,\rho_1} < m_S, \quad f_{\rho_2,\rho_2/c} > M_S(a,b), \quad 0 \leqslant f^\infty < \mu(L);$$

(T2) *there exist* $0 < \rho_1 < c\rho_2 < \infty$, *such that*

$$0 \leqslant f^0 < \mu(L), \quad f_{\rho_1,\rho_1/c} > M_S(a,b), \quad f^{-\rho_2,\rho_2} < m_S, \quad \mu(\tilde{L})_1 < f_\infty \leqslant \infty.$$

It is possible to give criteria for the existence of an arbitrary number of non-trivial solutions by extending the list of conditions. We omit the routine statement of such results.

The following lemma sheds some light on the relation between some of these constants.

Lemma 4.7. *The following relations hold:*

$$M_S(a,b) \geqslant \mu(\tilde{L}) \geqslant \mu(L) \geqslant m_S.$$

*Proof.* The fact that $\mu(L) \geqslant m_S$ essentially follows from [55, theorem 2.8]. The comment that follows [55, theorem 3.4] also applies in our case, giving $\mu(\tilde{L}) \geqslant \mu(L)$.

We now prove $M_S(a,b) \geqslant \mu(\tilde{L})$. Let $\varphi \in P \cap K$ be a corresponding eigenfunction of norm 1 of $1/\mu(\tilde{L})$ for the operator $\tilde{L}$, i.e. $\varphi = \mu(\tilde{L})\tilde{L}(\varphi)$ and $\|\varphi\| = 1$. Then, for $t \in [a,b]$ we have

$$\varphi(t) = \mu(\tilde{L}) \int_a^b k_S(t,s)g(s)\varphi(s)\,\mathrm{d}s \geqslant \mu(\tilde{L}) \min_{t\in[a,b]} \varphi(t) \int_a^b k_S(t,s)g(s)\,\mathrm{d}s.$$

Taking the infimum over $[a,b]$, we obtain

$$\min_{t\in[a,b]} \varphi(t) \geqslant \mu(\tilde{L}) \min_{t\in[a,b]} \varphi(t)/M_S(a,b),$$

i.e. $M_S(a,b) \geqslant \mu(\tilde{L})$. □

In order to present an index zero result of a different nature, we introduce the operator

$$L_+u(t) := \int_0^1 k_S^+(t,s)g(s)u(s)\,\mathrm{d}s,$$

for which a result similar to theorems 4.1 and 4.2 holds.

In the next theorem we use the following notation, with $c$ as in (4.1):

$$\tilde{f}_0 = \varliminf_{u\to 0} \frac{\operatorname{ess\,inf}_{t\in[0,1]} f(t,u)}{|u|}, \qquad \tilde{c} := \frac{1}{c} \sup_{t\in[0,1]} \frac{\int_0^1 k_S^-(t,s)g(s)\,\mathrm{d}s}{\int_a^b k_S^+(t,s)g(s)\,\mathrm{d}s}. \tag{4.4}$$

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THEOREM 4.8. *If $\mu(L_+) < \tilde{f}_0 - \tilde{c}f^0$, then there exists $\rho_0 > 0$ such that, for each $\rho \in (0, \rho_0]$, if $u \neq Tu$ for $u \in \partial K_\rho$, $i_K(T, K_\rho) = 0$ is satisfied.*

*Proof.* Firstly, since $u \in K$ we have, for $t \in [0, 1]$,

$$\int_0^1 k_S^-(t,s)g(s)|u(s)|\,\mathrm{d}s \leqslant \int_0^1 k_S^-(t,s)g(s)\|u\|\,\mathrm{d}s \leqslant \tilde{c}\int_a^b k_S^+(t,s)g(s)c\|u\|\,\mathrm{d}s$$
$$\leqslant \tilde{c}\int_a^b k_S^+(t,s)g(s)|u(s)|\,\mathrm{d}s \leqslant \tilde{c}L_+|u|(t).$$

Observe that the hypothesis $\mu(L_+) < \tilde{f}_0 - \tilde{c}f^0$ implies $\tilde{f}_0, f^0 < \infty$. Let $\rho_0 > 0$ such that

$$f(t,u) \geqslant (\mu(L_+) + \tilde{c}f^0)|u| \quad \text{and} \quad f(t,u) \leqslant (f^0 + \tfrac{1}{2}\mu(L_+))|u|$$

for all $u \in [-\rho_0, \rho_0]$ and almost all $t \in [0, 1]$.

Let $\rho \leqslant \rho_0$. We shall prove that $u \neq Su + \lambda\varphi_+$ for all $u$ in $\partial K_\rho$ and $\lambda > 0$, where $\varphi_+ \in K$ is an eigenfunction of $L_+$ related to the eigenvalue $1/\mu(L_+)$ such that $\|\varphi_+\| = 1$.

Assume now, on the contrary, that there exist $u \in \partial K_\rho$ and $\lambda > 0$ such that $u(t) = Su(t) + \lambda\varphi_+(t)$ for all $t \in [0, 1]$. Hence, we have

$$u(t) = -\int_0^1 k_S^-(t,s)g(s)f(s,u(s))\,\mathrm{d}s + \int_0^1 k_S^+(t,s)g(s)f(s,u(s))\,\mathrm{d}s + \lambda\varphi_+(t).$$

On the one hand we have

$$u(t) + \int_0^1 k_S^-(t,s)g(s)f(s,u(s))\,\mathrm{d}s$$
$$\leqslant |u(t)| + [f^0 + \tfrac{1}{2}\mu(L_+)]\int_0^1 k_S^-(t,s)g(s)|u(s)|\,\mathrm{d}s$$
$$\leqslant |u(t)| + \tilde{c}[f^0 + \tfrac{1}{2}\mu(L_+)]L_+|u|(t).$$

On the other we have

$$\int_0^1 k_S^+(t,s)g(s)f(s,u(s))\,\mathrm{d}s + \lambda\varphi_+(t) \geqslant (\mu(L_+) + \tilde{c}f^0)L_+|u|(t) + \lambda\varphi_+(t).$$

Therefore, we obtain

$$(\mu(L_+) + \tilde{c}f^0)L_+|u|(t) + \lambda\varphi_+ \leqslant |u(t)| + \tilde{c}[f^0 + \tfrac{1}{2}\mu(L_+)]L_+|u|(t)$$

or, equivalently,

$$\tfrac{1}{2}\mu(L_+)L_+|u|(t) + \lambda\varphi_+(t) \leqslant |u(t)|.$$

Hence, we get

$$\lambda\varphi_+(t) \leqslant |u(t)|.$$

Reasoning as in the proof of theorem 4.5(3), we obtain

$$|u(t)| \geqslant \lambda\tfrac{1}{2}\mu(L_+)L_+\varphi_+(t) + \lambda\varphi_+(t) = \tfrac{3}{2}\lambda\varphi_+(t).$$

By induction we deduce that $|u(t)| \geqslant (\frac{1}{2}n + 1)\lambda\varphi_+(t)$ for every $n \in \mathbb{N}$, a contradiction since $\|u\| = \rho$. □

As in theorem 4.6, results on the existence of multiple non-trivial solutions can be established. We omit such results here.

REMARK 4.9. The hypothesis of theorem 4.8 implies that $\tilde{c} \in (0,1)$. Also, if $\tilde{f}_0 = f^0 = f_0$, then the hypothesis is equivalent to $\mu(L^+)/(1-\tilde{c}) < \tilde{f}_0 < \infty$. Furthermore, if $[a,b] = [0,1]$, then $L = L^+ = \tilde{L}$ and the growth condition becomes $\mu(L) < \tilde{f}_0 < \infty$, which is theorem 4.5(3) for $f_0 < \infty$.

## 5. Study of the Green functions of the BVPs (1.4), (1.5)

In this section we study the properties of the Green function of the BVP

$$\epsilon u''(t) + \omega^2 u(t) = y(t), \quad u'(0) = u'(1) = 0,$$

where $y \in L^1[0,1]$, $\epsilon = \pm 1$ and $\omega \in \mathbb{R}^+$. We discuss two separate cases.

CASE 1 ($\epsilon = -1$). The Green function $k$ of the BVP

$$-u''(t) + \omega^2 u(t) = y(t), \quad u'(0) = u'(1) = 0,$$

is given by (see, for example, [46, 59]),

$$\omega \sinh \omega k(t,s) := \begin{cases} \cosh \omega(1-t) \cosh \omega s, & 0 \leqslant s \leqslant t \leqslant 1, \\ \cosh \omega(1-s) \cosh \omega t, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Note that $k$ is continuous, positive and satisfies some symmetry properties such as

$$k(t,s) = k(s,t) = k(1-t, 1-s).$$

Observe that

$$\frac{\partial k}{\partial t}(t,s) < 0 \quad \text{for } s < t$$

and

$$\frac{\partial k}{\partial t}(t,s) > 0 \quad \text{for } s > t.$$

Therefore, we choose

$$\Phi(s) := \sup_{t \in [0,1]} k(t,s) = k(s,s).$$

For a fixed $[a,b] \subset [0,1]$ we have

$$c(a,b) := \min_{t \in [a,b]} \min_{s \in [0,1]} \frac{k(t,s)}{\Phi(s)} = \frac{\min\{\cosh \omega a, \cosh \omega(1-b)\}}{\cosh \omega}.$$

The choice of $g \equiv 1$ gives

$$\frac{1}{m} = \sup_{t \in [0,1]} \int_0^1 k(t,s) \, \mathrm{d}s,$$

and, by direct calculation, we obtain that $m = \omega^2$. The constant $M$ can be computed as follows:

$$\frac{1}{M(a,b)} := \inf_{t\in[a,b]} \int_a^b k(t,s)\,\mathrm{d}s$$
$$= \frac{1}{\omega^2} - \sup_{t\in[a,b]} \frac{\sinh\omega a\cosh\omega(1-t) + \sinh\omega(1-b)\cosh\omega t}{\omega^2\sinh\omega}.$$

Let $\xi_1(t) := \sinh\omega a\cosh\omega(1-t) + \sinh\omega(1-b)\cosh\omega t$. Then we have $\xi_1''(t) = \omega^2\xi(t) \geqslant 0$. Therefore, the supremum of $\xi_1$ must be attained in one of the endpoints of the interval $[a,b]$. Thus, we have

$$\frac{1}{M(a,b)} = \frac{1}{\omega^2} - \frac{\max\{\xi_1(a),\xi_1(b)\}}{\omega^2\sinh\omega}.$$

Note that

$$\xi_1(b) - \xi_1(a) = -2\sinh^2(\tfrac{1}{2}(b-a)\omega)\sinh\omega(a+b-1),$$

and therefore $\xi_1(b) \geqslant \xi_1(a)$ if and only if $a+b \leqslant 1$. Hence, we obtain

$$\frac{1}{M(a,b)} = \frac{1}{\omega^2} - \frac{1}{\omega^2\sinh\omega} \times \begin{cases} \sinh\omega a\cosh\omega(1-b) + \sinh\omega(1-b)\cosh\omega b, & a+b \leqslant 1, \\ \sinh\omega a\cosh\omega(1-a) + \sinh\omega(1-b)\cosh\omega a, & a+b > 1. \end{cases}$$

Case 2 ($\epsilon = 1$). The Green function $k$ of the BVP

$$u''(t) + \omega^2 u(t) = y(t), \quad u'(0) = u'(1) = 0$$

is given by

$$\omega\sin\omega k(t,s) := \begin{cases} \cos\omega(1-t)\cos\omega s, & 0 \leqslant s \leqslant t \leqslant 1, \\ \cos\omega(1-s)\cos\omega t, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

In the following lemma we describe the sign properties of this Green function with respect to the parameter $\omega$. Similar studies have been done, for different BVPs, in [5, theorem 4.3] and [6, lemma 5.2]. The proofs are straightforward and omitted.

Lemma 5.1. *We have the following:*

(1) *$k$ is positive for $\omega \in (0, \frac{1}{2}\pi)$;*

(2) *$k$ is positive for $\omega = \frac{1}{2}\pi$, except at the points $(0,0)$ and $(1,1)$, where it is zero;*

(3) *$k$ is positive on the strip $(1-\pi/(2\omega), \pi/(2\omega)) \times [0,1]$ if $\omega \in (\frac{1}{2}\pi, \pi)$;*

(4) *if $\omega > \pi$, there is no strip of the form $(a,b) \times [0,1]$, where $k$ is positive.*

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Consider $\omega \in (0, \pi)$. Fix $s \in [0, 1]$ and note that $(\partial k/\partial t)(t, s)$ never changes sign for $t \in [0, s)$ or for $t \in (s, 1]$. Thus, we can take

$$\begin{aligned}\Phi(s) &:= \sup_{t\in[0,1]} |k(t,s)| \\ &= \max\{|k(0,s)|, |k(1,s)|, |k(s,s)|\} \\ &= \frac{\max\{|\cos\omega(1-s)|, |\cos\omega s|, |\cos\omega s\cos\omega(1-s)|\}}{\omega\sin\omega} \\ &= \frac{\max\{\cos\omega(1-s), \cos\omega s\}}{\omega\sin\omega}.\end{aligned}$$

The last equality holds because $\cos(\omega s) \geqslant -\cos\omega(1-s) \geqslant 0$ for $s \leqslant 1 - \pi/(2\omega)$ and $\cos(1-\omega s) \geqslant -\cos\omega s \geqslant 0$ for $s \geqslant \pi/(2\omega)$.

On the other hand, for $[a, b] \subset (\max\{0, 1 - \pi/(2\omega)\}, \min\{1, \pi/(2\omega)\})$, we have

$$\inf_{t\in[a,b]} k(t,s) = \begin{cases}\min\{k(a,s), k(b,s)\}, & s \in [0,1]\setminus[a,b], \\ \min\{k(a,s), k(s,s), k(b,s)\}, & s \in [a,b].\end{cases}$$

Now, we study the three intervals $[0, a)$, $[a, b]$ and $(b, 1]$ separately.

If $s \in [0, a)$, we have

$$\begin{aligned}&\inf_{s\in[0,a)} \frac{\min\{k(a,s), k(b,s)\}}{\Phi(s)} \\ &= \inf_{s\in[0,a)} \frac{\min\{\cos\omega(1-a)\cos\omega s, \cos\omega(1-b)\cos\omega s\}}{\max\{\cos\omega(1-s), \cos\omega s\}} \\ &= \inf_{s\in[0,a)} \min\left\{\cos\omega(1-a), \cos\omega(1-b), \frac{\cos\omega(1-a)\cos\omega s}{\cos\omega(1-s)}, \frac{\cos\omega(1-b)\cos\omega s}{\cos\omega(1-s)}\right\} \\ &= \min\left\{\cos\omega(1-a), \cos\omega(1-b), \cos\omega a, \cos\omega(1-b)\frac{\cos\omega a}{\cos\omega(1-a)}\right\} \\ &= \min\{\cos\omega(1-a), \cos\omega(1-b), \cos\omega a\},\end{aligned}$$

where these equalities hold because $(\cos\omega s)/\cos\omega(1-s)$ is a decreasing function for $s \in [\max\{0, 1 - \pi/(2\omega)\}, 1]$ and the function cosine is decreasing in $[0, \pi]$.

If $s \in [a, b]$, we have

$$\begin{aligned}&\inf_{s\in[a,b]} \frac{\min\{k(a,s), k(s,s), k(b,s)\}}{\Phi(s)} \\ &\quad= \inf_{s\in[a,b]} \frac{\min\{\cos\omega a\cos\omega(1-s), \cos\omega s\cos\omega(1-s), \cos\omega(1-b)\cos\omega s\}}{\max\{\cos\omega(1-s), \cos\omega s\}} \\ &\quad= \inf_{s\in[a,b]} \min\left\{\cos\omega a, \cos\omega(1-b), \cos\omega s, \cos\omega(1-s), \cos\omega a\frac{\cos\omega(1-s)}{\cos\omega s},\right. \\ &\qquad\qquad\left.\cos\omega(1-b)\frac{\cos\omega s}{\cos\omega(1-s)}\right\} \\ &\quad= \min\{\cos\omega a, \cos\omega(1-b), \cos\omega b, \cos\omega(1-a)\}.\end{aligned}$$

If $s \in (b, 1]$, we have

$$\begin{aligned}
&\inf_{s\in(b,1]} \frac{\min\{k(a,s),k(b,s)\}}{\Phi(s)} \\
&\quad = \inf_{s\in(b,1]} \frac{\min\{\cos\omega a\cos\omega(1-s),\cos\omega b\cos\omega(1-s)\}}{\max\{\cos\omega(1-s),\cos\omega s\}} \\
&\quad = \inf_{s\in(b,1]} \min\left\{\cos\omega a,\cos\omega b,\cos\omega a\frac{\cos\omega(1-s)}{\cos\omega s},\cos\omega b\frac{\cos\omega(1-s)}{\cos\omega s}\right\} \\
&\quad = \min\left\{\cos\omega a,\cos\omega b,\cos\omega a\frac{\cos\omega(1-b)}{\cos\omega b},\cos\omega(1-b)\right\} \\
&\quad = \min\{\cos\omega a,\cos\omega b,\cos\omega(1-b)\}.
\end{aligned}$$

Therefore, taking into account these three infima, we obtain that

$$c(a,b) := \inf_{s\in[0,1]} \frac{\inf_{t\in[a,b]} k(t,s)}{\Phi(s)} = \min\{\cos\omega a,\cos\omega(1-a),\cos\omega b,\cos\omega(1-b)\}.$$

In order to compute the constant $m$ we use lemma 5.1 and the fact that $k(t,s) = k(s,t)$ for all $t, s \in [0,1]$.

If $\omega \in (0, \frac{1}{2}\pi)$, the function $k$ is positive and therefore

$$m = \omega^2.$$

If $\omega \in [\frac{1}{2}\pi, \pi)$, we have

$$\begin{aligned}
\zeta(t) &:= \int_0^1 k^+(t,s)\,\mathrm{d}s \\
&= \begin{cases}
\displaystyle\int_{1-\pi/2\omega}^1 k(t,s)\,\mathrm{d}s = \frac{1}{\omega^2}\frac{\cos\omega t}{\sin\omega}, & t\in\left[0,1-\dfrac{\pi}{2\omega}\right), \\
\dfrac{1}{\omega^2}, & t\in\left[1-\dfrac{\pi}{2\omega},\dfrac{\pi}{2\omega}\right], \\
\displaystyle\int_0^{\pi/2\omega} k(t,s)\,\mathrm{d}s = \frac{1}{\omega^2}\frac{\cos\omega(1-t)}{\sin\omega}, & t\in\left(\dfrac{\pi}{2\omega},1\right].
\end{cases}
\end{aligned}$$

Since

$$0 < \frac{1}{\omega^2} = \int_0^1 k(t,s)\,\mathrm{d}s = \int_0^1 k^+(t,s)\,\mathrm{d}s - \int_0^1 k^-(t,s)\,\mathrm{d}s,$$

we obtain that

$$\int_0^1 k^+(t,s)\,\mathrm{d}s > \int_0^1 k^-(t,s)\,\mathrm{d}s$$

and therefore we obtain

$$m = \frac{1}{\max_{t\in[0,1]}\zeta(t)} = \omega^2\sin\omega.$$

Also we have

$$\frac{1}{M(a,b)} = \frac{1}{\omega^2} - \sup_{t\in[a,b]} \frac{\cos\omega(1-t)\sin\omega a + \cos\omega t\sin\omega(1-b)}{\omega^2\sin\omega}.$$

Define

$$\xi_3(t) := \cos\omega(1-t)\sin\omega a + \cos\omega t\sin\omega(1-b),$$

and observe that

$$\xi_3(t) = \omega^2\sin\omega\biggl(\int_0^1 k(t,s)\,\mathrm{d}s - \int_a^b k(t,s)\,\mathrm{d}s\biggr),$$

and therefore we have $\xi_3(t) \geqslant 0$ for $t \in [a,b]$. Then, we have

$$\xi_3'(a)\xi_3'(b) = -4\omega^2\cos[\tfrac{1}{2}\omega(2-a+b)]\cos[\tfrac{1}{2}\omega(a+b)]\sin^2[\tfrac{1}{2}\omega(a-b)]\sin\omega(1-b)\sin\omega a.$$

Now, $\xi_3'(a)\xi_3'(b) < 0$ if and only if $2-\pi/\omega < a+b < \pi/\omega$, which is always satisfied for $[a,b] \subset (1-\pi/(2\omega), \pi/(2\omega))$. In such a case, $\xi_3$ has a maximum in $[a,b]$, precisely at the unique point $t_0$ satisfying

$$\sin\omega t_0 = \frac{\sin\omega\sin\omega a}{\cos\omega\sin\omega\alpha + \sin\omega(1-b)}\cos\omega t_0.$$

Thus, we obtain

$$\begin{aligned}\xi_3(t_0) &= \cos\omega\cos\omega b\cos\omega t_0 + \cos\omega\sin\omega a\cos\omega t_0 - \cos\omega\sin\omega b\cos\omega t_0\\ &\qquad + \sin\omega\sin\omega a\sin\omega t_0\\ &= \biggl(\cos\omega\cos\omega b + \cos\omega\sin\omega a - \cos\omega\sin\omega b\\ &\qquad\qquad + \frac{(\sin\omega\sin\omega a)^2}{\cos\omega\sin\omega\alpha + \sin\omega(1-b)}\biggr)\cos\omega t_0\\ &= \biggl|\cos\omega\cos\omega b + \cos\omega\sin\omega a - \cos\omega\sin\omega b + \frac{(\sin\omega\sin\omega a)^2}{\cos\omega\sin\omega\alpha + \sin\omega(1-b)}\biggr|\\ &\qquad \times\biggl(\sqrt{\biggl(\frac{\sin\omega\sin\omega a}{\cos\omega\sin\omega\alpha + \sin\omega(1-b)}\biggr)^2 + 1}\biggr)^{-1}.\end{aligned}$$

Remark 5.2. In the particular case $a+b=1$, we have

$$\xi_3(t) = \sin\omega a[\cos\omega(1-t) + \cos\omega t].$$

In this case, observe that $\xi_3(t) = \xi_3(1-t)$ and recall that $\xi_3''(t) = -\omega^2\xi_3(t) \geqslant 0$ ($\xi_3$ is not constantly zero in any open subinterval). Therefore, the maximum is reached at the only point where $t = 1-t$, i.e. $t = \frac{1}{2}$. Hence, we obtain

$$\frac{1}{M(a,b)} = \frac{1 - 2\cos\frac{1}{2}\omega\sin\omega a}{\omega^2\sin\omega}.$$

Remark 5.3. The constants $m$, $M(a,b)$, $c(a,b)$ and the function $\Phi$ improve on and complement some of those used in [41–43, 45, 46, 58, 59].

## 6. Examples

In this section we present some examples illustrating some of the constants that occur in our theory and the applicability of our theoretical results. Note that the constants that occur are rounded to the third decimal place unless they are exact.

In the first example we study the existence of multiple non-trivial solutions of a (local) Neumann BVP.

Example 6.1. Consider the BVP

$$u''(t) + \left(\frac{7\pi}{12}\right)^2 u(t) = \frac{\tau_1 u^2(t)}{1+t^2}\mathrm{e}^{-\tau_2|u(t)|}, \quad t \in [0,1], \ u'(0) = u'(1) = 0, \tag{6.1}$$

where $\tau_1, \tau_2 > 0$.

In this case $\omega = \frac{7}{12}\pi$ and, by lemma 5.1, the Green function is positive on the strip $(\frac{1}{7}, \frac{6}{7}) \times [0,1]$. We illustrate remark 5.2 by choosing $[\frac{1}{4}, \frac{3}{4}] \subset (\frac{1}{7}, \frac{6}{7})$ and we prove, by using theorem 4.6, the existence of two non-trivial solutions of the BVP (6.1) that are (strictly) positive on the interval $[\frac{1}{4}, \frac{3}{4}]$.

In order to do this, note that in our case we have

$$f(t,u) = \frac{\tau_1 u^2}{1+t^2}\mathrm{e}^{-\tau_2|u|} \quad \text{and} \quad f^0 = f^\infty = 0.$$

Furthermore, using the results in the previous section, we have

$$c(\tfrac{1}{4}, \tfrac{3}{4}) = \cos(\tfrac{7}{16}\pi) = \tfrac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{2}}} = 0.195 \tag{6.2}$$

and

$$M = M(\tfrac{1}{4}, \tfrac{3}{4}) = 7.029.$$

Henceforth, we work in the cone

$$K = \Big\{u \in C[0,1] \colon \min_{t \in [1/4, 3/4]} u(t) \geqslant c\|u\|\Big\},$$

with $c$ given by (6.2).

We set

$$\hat{f}_0 := 2\frac{c-1}{\ln c}c^{c/(c-1)}M = 10.289.$$

We now prove that if $\tau_1/\tau_2 > \hat{f}_0$, then condition (Z1) is satisfied. Let

$$\hat{f}(u) := \inf_{t \in [0,1]} \frac{\tau_1 u^2}{1+t^2}\mathrm{e}^{-\tau_2 u} = \tfrac{1}{2}\tau_1 u^2 \mathrm{e}^{-\tau_2 u}, \quad u \in [0, +\infty).$$

Note that $\hat{f}'$ only vanishes at 0 and $2/\tau_2$; $\hat{f}$ is strictly increasing in the interval $(0, 2/\tau_2)$ and is strictly decreasing in the interval $(2/\tau_2, +\infty)$. Thus, $\hat{f}$ assumes the maximum at the unique point $2/\tau_2$ and, since $\hat{f}(0) = 0$ and $\lim_{x \to +\infty} \hat{f}(x) = 0$, the inverse image by $\hat{f}$ of any strictly positive real number different from $\hat{f}(2/\tau_2)$ has either two points or no points. For $x \in [0, +\infty)$, let

$$l(x) := \hat{f}(x) - \hat{f}\left(\frac{x}{c}\right).$$

Take $\varepsilon \in (0, 2c/\tau_2)$ and note that $l(\varepsilon) < 0$ in view of the strict monotonicity of $\hat{f}$. Moreover, if $\eta > 2/\tau_2$, then $l(\eta) > 0$. Since the function $l$ is continuous, there exists a point $\bar{x} \in (\varepsilon, \eta)$ such that $l(\bar{x}) = 0$, i.e. $\hat{f}(\bar{x}) = \hat{f}(\bar{x}/c) = p$. From the type of monotonicity of $f$, for $x \in [\bar{x}, \bar{x}/c]$ we have $p \leqslant \hat{f}(x)$. Hence, we have

$$\hat{f}(\bar{x}) = \hat{f}\left(\frac{\bar{x}}{c}\right) \implies \bar{x} = \exp\left(\tau_2\left(\frac{\bar{x}}{c} - \bar{x}\right)\right)\frac{\bar{x}}{c} \implies \bar{x} = \frac{2c\ln c}{\tau_2(c-1)}, \quad \frac{\bar{x}}{c} = \frac{2\ln c}{\tau_2(c-1)}.$$

Thus, if we impose $p > M\bar{x}$, we obtain

$$M\frac{2c\ln c}{\tau_2(c-1)} = M\bar{x} < \hat{f}(\bar{x}) = \hat{f}\left(\frac{\bar{x}}{c}\right) = \tau_1\left(\frac{2\ln c}{\tau_2(c-1)}\right)^2 c^{-2/(c-1)},$$

i.e. $\tau_1/\tau_2 > \hat{f}_0$.

We now present an example of a BVP subject to two non-local BCs.

Example 6.2. Consider the BVP

$$\left.\begin{aligned} u''(t) + \omega^2 u(t) = \mathrm{e}^{-|u(t)|}, \quad t \in [0,1], \\ u'(0) = u(0) + u(1), \qquad u'(1) = \int_0^1 u(t)\sin \pi t \,\mathrm{d}t, \end{aligned}\right\} \tag{6.3}$$

where $\omega \in (\frac{1}{2}\pi, \pi)$. We rewrite the BVP (6.3) in the integral form

$$Tu(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t,s)f(s,u(s))\,\mathrm{d}s,$$

where

$$\gamma(t) = \frac{\cos\omega(1-t)}{\omega\sin\omega}, \qquad \delta(t) = \frac{\cos(\omega t)}{\omega\sin\omega},$$
$$\alpha[u] = u(0) + u(1), \qquad \beta[u] = \int_0^1 u(t)\sin\pi t\,\mathrm{d}t.$$

In order to verify (S1) of theorem 2.12, we take $[a,b] \subset (1 - \pi/(2\omega), \pi/(2\omega))$ and let $f(u) = \mathrm{e}^{-|u|}$.

Note that the condition $f^\infty = 0$ implies that the condition $(I_\rho^1)$ is satisfied for $\rho$ sufficiently large (hence, $i_K(T, K_R) = 1$ for $R$ big enough).

Now it is left to prove that $i_K(T, V_\rho) = 0$ for $\rho$ small enough (condition $(I_\rho^0)$). We have

$$\alpha[\gamma] = \alpha[\delta] = \sqrt{2}\frac{\sin(\frac{1}{4}\pi + \omega)}{\omega\sin\omega}, \qquad \beta[\gamma] = \beta[\delta] = \frac{\pi\cot(\frac{1}{2}\omega)}{\pi^2\omega - \omega^3},$$
$$D := D(\omega) = \frac{(\pi^2\omega - \omega^3)\sin(\frac{1}{2}\omega) - (\pi + \pi^2 - \omega^2)\cos(\frac{1}{2}\omega)}{(\pi^2\omega - \omega^3)\sin(\frac{1}{2}\omega)},$$

$$\mathcal{K}_A(s) = \frac{\cos\omega s + \cos(\omega[1-s])}{\omega\sin\omega}, \quad \mathcal{K}_B(s) = \frac{\pi\cos\omega s\cot(\frac{1}{2}\omega) - \omega\sin\pi s + \pi\sin\omega s}{\pi^2\omega - \omega^3}.$$

Observe that $\alpha[\gamma], \alpha[\delta], \beta[\gamma], \beta[\delta], \mathcal{K}_A(s), \mathcal{K}_B(s) \geqslant 0$ and $\alpha[\gamma], \beta[\delta] < 1$ for $\omega \in (\frac{1}{2}\pi, \pi)$.

Also, we have $D(\omega) > 0$ for $\omega \in (\frac{1}{2}\pi, \pi)$. In fact, $D(\omega)$ is a strictly increasing function (since $D'(\omega) > 0$ for $\omega \in (0, \pi)$), $\lim_{\omega\to 0^+} D(\omega) = -\infty$ and $D(\pi) = 1 -$

$1/4\pi > 0$, so there is a unique zero $\omega_0$ of $D$ in $(0,\pi)$ but $\omega_0 = 1.507 < \frac{1}{2}\pi$. Now, $\gamma$ is increasing and $\delta$ is decreasing. Therefore, $c_2 = \gamma(a)/\gamma(1) = \cos(\omega[1-a])$, $c_3 = \delta(b)/\delta(0) = \cos\omega b$. On the other hand, we have

$$\begin{aligned} f_{\rho,\rho/c} &= \frac{f(\rho/c)}{(\rho/c)} = \frac{\mathrm{e}^{-\rho/c}c}{\rho}, \\ c(a,b) &= \min\{\cos\omega a, \cos\omega(1-a), \cos\omega b, \cos\omega(1-b)\}, \\ \int_a^b \mathcal{K}_A(s)\,\mathrm{d}s &= \frac{\sin\omega b - \sin\omega a + \sin\omega(1-a) - \sin\omega(1-b)}{\omega^2\sin\omega}, \\ \int_a^b \mathcal{K}_B(s)\,\mathrm{d}s &= \frac{\pi^2(\cot(\frac{1}{2}\omega)(\sin(b\omega)-\sin(a\omega)) + \cos(a\omega) - \cos(b\omega))}{\omega^2(\pi^3-\pi\omega^2)} \\ &\quad - \frac{\omega^2(\cos(\pi a)+\cos(\pi b))}{\omega^2(\pi^3-\pi\omega^2)}. \end{aligned}$$

Taking $a+b=1$, we obtain

$$\begin{aligned} \int_a^b \mathcal{K}_A(s)\,\mathrm{d}s &= \frac{2\csc(\frac{1}{2}\omega)\sin(\frac{1}{2}(\omega-2a\omega))}{\omega^2}, \\ \int_a^b \mathcal{K}_B(s)\,\mathrm{d}s &= -\frac{2(\omega^2\cos(\pi a) - \pi^2\cos(a\omega) + \pi^2\cot(\frac{1}{2}\omega)\sin(a\omega))}{\omega^2(\pi^3-\pi\omega^2)}, \\ c &= \cos\omega a. \end{aligned}$$

Condition $(I_\rho^0)$ is equivalent to

$$f_{\rho,\rho/c}\cdot \inf_{t\in[a,b]}\left\{q(t,\omega,a) + \int_a^b k(t,s)\,\mathrm{d}s\right\} > 1,$$

where

$$\begin{aligned} &q(t,\omega,a) \\ &\quad = \frac{2\csc(\omega)(\pi\csc(\frac{1}{2}\omega)\sin(\frac{1}{2}(\omega-2a\omega))(\pi\cos(t\omega) + (\pi-\omega)(\omega+\pi)\cos(\omega-t\omega)))}{\pi\omega^2((\pi-\omega)\omega(\omega+\pi) - (-\omega^2+\pi^2+\pi)\cot(\frac{1}{2}\omega))} \\ &\qquad - \frac{2\omega\csc(\omega)\cos(\pi a)(\sin(t\omega) - \sin(\omega-t\omega) + \omega\cos(t\omega))}{\pi\omega^2((\pi-\omega)\omega(\omega+\pi) - (-\omega^2+\pi^2+\pi)\cot(\frac{1}{2}\omega))}. \end{aligned}$$

Using remark 2.11, it is enough to check

$$f_{\rho,\rho/c}\cdot\left(\inf_{t\in[a,b]} q(t,\omega,a) + \frac{1}{M(a,b)}\right) > 1.$$

It can be checked numerically that $\inf_{t\in[a,b]} q(t,\omega,a) = q(a,\omega,a)$. Hence, we need

$$\frac{\exp(-\rho/\cos\omega a)\cos\omega a}{\rho}\left(q(a,\omega,a) + \frac{1-2\cos\frac{1}{2}\omega\sin\omega a}{\omega^2\sin\omega}\right) > 1.$$

Since

$$\lim_{\rho\to 0}\frac{\exp(-\rho/\cos\omega a)}{\rho} = +\infty,$$

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the inequality is satisfied for $\rho$ small enough, and hence we have proved that the BVP (6.3) has at least a non-trivial solution in the cone $K$.

We now study an example that occurs in an earlier article by Bonanno and Pizzimenti [3].

Example 6.3. Consider the BVP

$$-u''(t)+u(t)=\lambda t\mathrm{e}^{u(t)},\quad t\in[0,1],\ u'(0)=u'(1)=0. \tag{6.4}$$

In [3] Bonanno and Pizzimenti establish the existence of at least one positive solution such that $\|u\|<2$ for $\lambda\in(0,2\mathrm{e}^{-2})$.

The BVP (6.4) is equivalent to the following integral problem:

$$u(t)=\int_0^1 k(t,s)g(s)f(u(s))\,\mathrm{d}s,$$

where

$$g(s)=s,\qquad f(u)=\lambda\mathrm{e}^u$$

and

$$k(t,s):=\frac{1}{\sinh(1)}\begin{cases}\cosh(1-t)\cosh s, & 0\leqslant s\leqslant t\leqslant 1,\\ \cosh(1-s)\cosh t, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

By the results in § 5, the kernel $k$ is positive and it satisfies (C1)–(C8) with $[a,b]=[0,1]$. Thus, we work in the cone

$$K=\Big\{u\in C[0,1]\colon \min_{t\in[0,1]}u(t)\geqslant c\|u\|\Big\},$$

where

$$c=c(0,1)=\frac{1}{\cosh 1}=0.648.$$

We can compute the following constants:

$$m=\frac{\mathrm{e}+1}{2}=1.859,$$
$$M(0,1)=\frac{\mathrm{e}+1}{\mathrm{e}-1}=2.163,$$
$$f^{0,\rho}=f_{\rho,\rho/c}=\lambda\frac{\mathrm{e}^\rho}{\rho}.$$

Taking $\rho_2=2$, $(I^1_{\rho_2})$ is satisfied for $\lambda<(\mathrm{e}+1)\mathrm{e}^{-2}$, and taking $0<\rho_1<\frac{1}{2}c$ we have $(I^0_{\rho_1})$ for $\lambda>[(\mathrm{e}+1)/(\mathrm{e}-1)]\rho_1\mathrm{e}^{-\rho_1}$.

Hence, (S1) of theorem 2.12 is satisfied whenever

$$\lambda\in\left(0,\frac{\mathrm{e}+1}{\mathrm{e}^2}\right)\supset(0,2\mathrm{e}^{-2}).$$

Furthermore, reasoning as in [26], when $\lambda=\frac{1}{4}$ the choice of $\rho_2=0.16$ and $\rho_1=0.1$ gives the following localization for the solution:

$$0.064\leqslant u(t)\leqslant 0.16,\quad t\in[0,1].$$

An application of theorem 3.3 yields that for

$$\lambda > \frac{\mathrm{e}+1}{\mathrm{e}(\mathrm{e}-1)} = 0.797$$

there are no solutions in $K$ (the trivial solution does not satisfy the differential equation). Furthermore, note that $T\colon P \to K$; this shows that there are no positive solutions for the BVP (6.4) when $\lambda > (\mathrm{e}+1)/\mathrm{e}(\mathrm{e}-1)$.

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