Proceedings of the Royal Society of Edinburgh, **131A**, 1091–1111, *2001*

Solutions and multiple solutions for quasilinear hemivariational inequalities at resonance

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(MS received 7 February 2000; accepted 6 December 2000)

In this paper we consider quasilinear hemivariational inequalities at resonance. We obtain existence theorems using Landesman–Lazer-type conditions and multiplicity theorems for problems with strong resonance at infinity. Our method of proof is based on the non-smooth critical point theory for locally Lipschitz functions and on a generalized version of the Ekeland variational principle.

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with C^1-boundary Γ and let $2 \leqslant p < +\infty$. In this paper we study the following quasilinear hemivariational inequality at resonance:

$$\left.\begin{aligned} -\operatorname{div}(\|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}\nabla x(z)) - \lambda_1 |x(z)|^{p-2}x(z) \in \partial j(z, x(z)) \\ \text{almost everywhere on } Z, \\ x|_\Gamma = 0. \end{aligned}\right\} \quad \text{(HVI)}$$

By λ_1 we denote the first eigenvalue of the negative p-Laplacian

$$-\Delta_p x = -\operatorname{div}(\|\nabla x\|_{\mathbb{R}^N}^{p-2}\nabla x)$$

with the Dirichlet boundary condition (i.e. of $(-\Delta_p, W_0^{1,p}(Z))$). By $j : Z \times \mathbb{R} \mapsto \mathbb{R}$ we mean a functional, which is measurable in the first variable and locally Lipschitz in the second variable. By $\partial j(z, \zeta)$ we denote the subdifferential of $j(z, \cdot)$ in the sense of Clarke [7] (see § 2). Our work here continues in the direction of the two recent papers by the authors (see [8, 9]). It is also related to the recent work of Goeleven *et al.* [10], who examined semilinear (i.e. for $p = 2$) hemivariational inequalities at resonance. Hemivariational inequalities are a new type of variational inequalities, where the convex subdifferential is replaced by the subdifferential in the sense of Clarke of a locally Lipschitz function. Such inequalities arise in the problems of mechanics and engineering, when one wants to consider more realistic mechanical laws of non-monotone and multivalued nature. This leads to non-smooth and non-convex energy functionals. Concrete applications of hemivariational inequalities in

https://doi.org/10.1017/S0308210500001281 Published online by Cambridge University Press

mechanics and engineering problems can be found in the books of Naniewicz and Panagiotopoulos [18] and Panagiotopoulos [19].

First we prove two existence results, by employing some Landesman–Lazer-type conditions. These results extend theorem 5.2 of Goeleven *et al.* [10, p. 178], who deal with semilinear problems. In addition, they assume that the subdifferential of the locally Lipschitz function admits a continuous selection. Recalling that the Clarke subdifferential as a multifunction is only upper semicontinuous, we see that such a hypothesis is rather restrictive. Also, both existence results extend theorem 4 of [8], where the hypotheses on $j(z,\zeta)$ are more restrictive, namely it is assumed that the asymptotic values of the generalized potential $j(z,\zeta)$ exist as $\zeta \to \pm\infty$. In § 4, we prove two multiplicity results for strongly resonant elliptic case, i.e. we assume that for almost all $z \in Z$, function $j(z,\zeta)$ has finite limits as $\zeta \to \pm\infty$. In this section our work is similar to the semilinear (i.e. for $p = 2$), smooth (i.e. for $j(z,\cdot)$ being C^1-function) works of Bartolo *et al.* [3], Goncalves and Miyagaki [11, 12], Landesman *et al.* [15], Thews [22] and Ward [24]. Of the aforementioned works, only [11, 12] and [15] have multiplicity results. Moreover, in all these works, function j is of the form $j(z,\zeta) = \int_0^x f(r)\,\mathrm{d}r$, with f being a continuous function.

Our approach is variational and is based on the critical point theory for non-smooth locally Lipschitz functionals of Chang [6]. For the convenience of the reader, in the next section we recall some basic definitions and facts from that theory, which we will need in the sequel.

2. Preliminaries

Let X be a Banach space and X^* its topological dual. By $\|\cdot\|$ we will denote the norm in X, by $\|\cdot\|_*$ the norm in X^* and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X, X^*). A function $\phi : X \mapsto \mathbb{R}$ is said to be locally Lipschitz if, for every $x \in X$, there exists a neighbourhood U of x and a constant $k > 0$ depending on U such that $|\phi(z) - \phi(y)| \leqslant k\|z - y\|$ for all $z, y \in U$. From convex analysis, we know that a proper, convex and lower semicontinuous function $g : X \mapsto \bar{\mathbb{R}} \overset{\mathrm{df}}{=} \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom} g \overset{\mathrm{df}}{=} \{x \in X : g(x) < +\infty\}$. In analogy with the directional derivative of a convex function, we define the generalized directional derivative of a locally Lipschitz function ϕ at $x \in X$ in the direction $h \in X$ by

$$\phi^0(x;h) \overset{\mathrm{df}}{=} \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{\phi(x' + th) - \phi(x')}{t}.$$

The function $X \ni h \mapsto \phi^0(x;h) \in \mathbb{R}$ is sublinear, continuous and, by the Hahn–Banach theorem, it is the support function of a non-empty, convex and w^*-compact set

$$\partial\phi(x) \overset{\mathrm{df}}{=} \{x^* \in X^* : \langle x^*, h\rangle \leqslant \phi^0(x;h) \text{ for all } h \in X\}.$$

The set $\partial\phi(x)$ is called the ‘generalized’ or ‘Clarke’ subdifferential of ϕ at x. If $\phi, \psi : X \mapsto \mathbb{R}$ are locally Lipschitz functions, then $\partial(\phi + \psi)(x) \subseteq \partial\phi(x) + \partial\psi(x)$ and $\partial(t\phi)(x) = t\partial\phi(x)$ for all $t \in \mathbb{R}$ and all $x \in X$. Moreover, if $\phi : X \mapsto \mathbb{R}$ is also convex, then the subdifferential of ϕ in the sense of convex analysis coincides with the generalized subdifferential introduced above. If ϕ is strictly differentiable at x (in particular, if ϕ is continuously Gateaux differentiable at x), then $\partial\phi(x) = \{\phi'(x)\}$.

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Let $\phi : X \mapsto \mathbb{R}$ be a locally Lipschitz function on a Banach space X. A point $x \in X$ is said to be a 'critical point' of ϕ if $0 \in \partial\phi(x)$. If $x \in X$ is a critical point of ϕ, then the value $c \stackrel{\mathrm{df}}{=} \phi(x)$ is called a 'critical value' of ϕ. It is easy to see that if $x \in X$ is a local extremum of ϕ, then $0 \in \partial\phi(x)$. Moreover, the multifunction $X \ni x \mapsto \partial\phi(x) \in 2^{X^*}$ is upper semicontinuous, where the space X^* is equipped with the w^*-topology, i.e. for any w^*-open set $U \subseteq X^*$, the set $\{x \in X : \partial\phi(x) \subseteq U\}$ is open in X (see [13]). For more details on the generalized subdifferential, we refer to [7, ch. 2].

The critical point theory for smooth functions uses a compactness condition known as 'the Palais–Smale condition' (PS condition). In our present non-smooth setting, the condition takes the following form.

> A locally Lipschitz function $\phi : X \mapsto \mathbb{R}$ satisfies the 'non-smooth PS condition if any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\{\phi(x_n)\}_{n\geqslant 1}$ is bounded and $m(x_n) \stackrel{\mathrm{df}}{=} \min\{\|x^*\|_* : x^* \in \partial\phi(x_n)\} \to 0$ as $n \to +\infty$ has a strongly convergent subsequence.

If $\phi \in \mathcal{C}^1(X)$, then, since $\partial\phi(x_n) = \{\phi'(x_n)\}$, we see that the above definition of the PS condition coincides with the classical one (see [20]).

A weaker form of the PS condition was introduced in the context of the smooth theory by Cerami [5]. In our non-smooth setting, this condition takes the following form.

> A locally Lipschitz function $\phi : X \mapsto \mathbb{R}$ satisfies the 'non-smooth Cerami condition' (non-smooth C condition) if, for any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that the sequence of values $\{\phi(x_n)\}_{n\geqslant 1}$ is bounded and $(1 + \|x_n\|)m(x_n) \to 0$ as $n \to +\infty$, there exists a strongly convergent subsequence.

It was proved in the smooth case by Bartolo *et al.* [3, theorem 1.3, p. 985] that this weaker condition suffices to obtain a deformation lemma and from that derive minimax principles that generate the existence of critical points. The same can be done in the context of the non-smooth theory by modifying the arguments of [3], with the help of lemmata 3.1–3.4 of [6] or by using a recent generalization of the Ekeland variational principle due to Zhong [25] (for details, we refer to [14]). Evidently, the non-smooth PS condition implies the non-smooth C condition. We can have a 'local' version of these concepts.

> A locally Lipschitz function $\phi : X \mapsto \mathbb{R}$ satisfies the 'non-smooth C condition at level c' (respectively, the 'non-smooth PS condition at level c') if, for any sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $\phi(x_n) \to c$ and $(1 + \|x_n\|)m(x_n) \to 0$ (respectively, $m(x_n) \to 0$), there exists a strongly convergent subsequence.

If the above property is true for every level $c \in \mathbb{R}$, then we recover the previously introduced 'global' version of the non-smooth C condition or the non-smooth PS condition.

The first theorem gives the basic usage of the non-smooth PS condition (see, for example, [6, theorem 3.5, p. 118]).

THEOREM 2.1. *If X is a reflexive Banach space and $\phi : X \mapsto \mathbb{R}$ is a bounded below and locally Lipschitz functional that satisfies the non-smooth PS condition, then $c \stackrel{\mathrm{df}}{=} \inf\{\phi(x) : x \in X\}$ is a critical value of ϕ.*

The next theorem is the non-smooth extension of the well-known mountain-pass theorem, due to Ambrosetti and Rabinowitz [2] (see, for example, [6,14]).

THEOREM 2.2. *If*

(i) *X is a reflexive Banach space and $\phi : X \mapsto \mathbb{R}$ is a locally Lipschitz functional that satisfies the non-smooth C condition at level c,*

(ii) *there exist real number $r > 0$ and point $x_1 \in X$ such that $\|x_1\| > r$ and $\max\{\phi(0), \phi(x_1)\} < \inf\{\phi(x) : \|x\| = r\}$,*

(iii) *$c = \inf_{\gamma\in\Gamma_1} \max_{0\leqslant t\leqslant 1}\{\phi(\gamma(t))\}$, where*

$$\Gamma_1 \stackrel{\mathrm{df}}{=} \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = 0,\ \gamma(1) = x_1\},$$

then $c \geqslant \inf\{\phi(x) : \|x\| = r\}$ and there exists $x \in X$ such that $0 \in \partial\phi(x)$ and $\phi(x) = c$.

The third theorem is due to Zhong (see [25, theorem 1.1, p. 239]) and extends the Ekeland variational principle. Here we put the particular version of this theorem with $x_0 = 0$, $h(r) = r$ and $\lambda = 1$ (the notation is taken from [25]).

THEOREM 2.3. *If X is a reflexive Banach space and $\phi : X \mapsto \mathbb{R}$ is a lower semicontinuous functional that is bounded below, then, for any $\varepsilon > 0$ and any $\bar{x} \in X$ such that $\phi(\bar{x}) < \inf_{x\in X} \phi(x) + \varepsilon$, there exists $x \in X$ such that*

(i) *$\phi(x) \leqslant \phi(\bar{x})$,*

(ii) *$\phi(x) \leqslant \phi(u) + (\varepsilon\|x - u\|_X)/(1 + \|x\|_X)$ for all $u \in X$.*

In the formulation of (HVI), we encounter λ_1, which is the first eigenvalue of the negative p-Laplacian with the Dirichlet boundary condition. More precisely, let us consider the following nonlinear eigenvalue problem:

$$\left.\begin{aligned} -\operatorname{div}(\|\nabla x(z)\|^{p-2}_{\mathbb{R}^N}\nabla x(z)) = \lambda|x(z)|^{p-2}x(z) \quad &\text{almost everywhere on } Z,\\ x|_\Gamma = 0. \end{aligned}\right\} \tag{EP}$$

The least real number λ for which (EP) has a non-trivial solution is called the first eigenvalue λ_1 of $(-\Delta_p, W^{1,p}_0(Z))$. This first eigenvalue λ_1 is positive, isolated and simple (i.e. the associated eigenspace is one dimensional). Moreover, we have a variational characterization of λ_1 via the Rayleigh quotient, i.e.

$$\lambda_1 = \min\left\{\frac{\|\nabla x\|_p^p}{\|x\|_p^p} : x \in W^{1,p}_0(Z),\ x \neq 0\right\}. \tag{2.1}$$

The above minimum is realized at the normalized eigenfunction u_1. Note that if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$, and so we infer that the first eigenfunction u_1 does not change sign on Z. In fact, we can show that $u_1(z) \neq 0$

https://doi.org/10.1017/S0308210500001281 Published online by Cambridge University Press

almost everywhere on Z and so we can assume that $u_1 > 0$ almost everywhere on Z. Moreover, by the non-smooth elliptic regularity theory, we know that the solution of (EP) is continuous and one can even have that $u_1 \in C^{1,\beta}_{\mathrm{loc}}(Z)$ with $0 < \beta < 1$ (see [23, theorem 1, p. 127]). For details on the first eigenvalue, we refer to [17].

Let $Y_{u_1} \subseteq W^{1,p}_0(Z)$ be a topological complement to the one-dimensional eigenspace $\mathbb{R}u_1$ (i.e. $W^{1,p}_0(Z) = \mathbb{R}u_1 \oplus Y_{u_1}$). Since $\lambda_1 > 0$ is isolated, we have that

$$\lambda_{2,Y_{u_1}} \stackrel{\mathrm{df}}{=} \inf\left\{ \frac{\|\nabla y\|_p^p}{\|y\|_p^p} : y \in Y_{u_1},\ y \neq 0 \right\} > \lambda_1. \tag{2.2}$$

Let $\bar{\lambda}_2 \stackrel{\mathrm{df}}{=} \sup\{\lambda_{2,Y_{u_1}} : Y_{u_1} \subset W^{1,p}_0(Z)\}$, where the supremum is taken over all Y_{u_1}, topological complements of $\mathbb{R}u_1$. Recall that since $\mathbb{R}u_1$ is finite dimensional, a topological complement always exists (see [13, p. 502]). If $p = 2$, then $\bar{\lambda}_2 = \lambda_2$ is the second eigenvalue of the negative Laplacian with the Dirichlet boundary condition (i.e. of $(-\Delta, H^1_0(Z))$).

3. Landesman–Lazer-type condition

In this section we prove two existence theorems for (HVI) using Landesman–Lazer-type conditions.

In the sequel, we will assume that $p \geqslant 2$ and that p' is such that $1/p + 1/p' = 1$. By p^* we will denote the Sobolev critical exponent, defined by

$$p^* \stackrel{\mathrm{df}}{=} \begin{cases} Np/(N-p) & \text{if } p < N, \\ +\infty & \text{if } p \geqslant N, \end{cases}$$

and by $p^{*\prime}$ the number such that $1/p^* + 1/p^{*\prime} = 1$. Note that

$$1 \leqslant p^{*\prime} < p' \leqslant 2 \leqslant p < p^* \leqslant +\infty.$$

Our hypotheses on the generalized potential function $j(z,\zeta)$ are the following.

Hypotheses $H(j)_1$. $j : Z \times \mathbb{R} \mapsto \mathbb{R}$ is a function such that:

(i) for all $\zeta \in \mathbb{R}$, function $Z \ni z \mapsto j(z,\zeta) \in \mathbb{R}$ is measurable and $j(\cdot,0) \in L^{p^{*\prime}}(Z)$;

(ii) for almost all $z \in Z$, function $\mathbb{R} \ni \zeta \mapsto j(z,\zeta) \in \mathbb{R}$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $\eta \in \partial j(z,\zeta)$, we have $|\eta| \leqslant a(z)$ with some $a \in L^{q'}(Z)$, where $p^{*\prime} < q' \leqslant p'$;

(iv) there exist functions $v_+, v_- \in L^1(Z)$ such that, uniformly for almost all $z \in Z$, we have

$$v_+(z) = \sup_{\{v_n\}} \limsup_{n\to+\infty} v_n(z) \quad \text{and} \quad v_-(z) = \inf_{\{v_n\}} \liminf_{n\to+\infty} v_n(z),$$

where the supremum (respectively, the infimum) is taken over all sequences $\{v_n\}_{n\geqslant 1} \subseteq L^{p^{*\prime}}(Z)$ such that $v_n(z) \in \partial j(z,\zeta_n)$ with $\zeta_n \to +\infty$ (respectively, $\zeta_n \to -\infty$) and

$$\int_Z v_+(z)u_1(z)\,\mathrm{d}z < 0 < \int_Z v_-(z)u_1(z)\,\mathrm{d}z.$$

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Hypothesis $H(j)_1$ (iv) is of the so-called Landesman–Lazer type.

Let $\phi : W_0^{1,p}(Z) \mapsto \mathbb{R}$ be the energy functional defined by

$$\phi(x) \stackrel{\mathrm{df}}{=} \frac{1}{p}\|\nabla x\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z j(z, x(z))\,\mathrm{d}z.$$

Let $\psi : W_0^{1,p}(Z) \mapsto \mathbb{R}$ be defined by

$$\psi(x) \stackrel{\mathrm{df}}{=} \int_Z j(z, x(z))\,\mathrm{d}z.$$

By virtue of hypothesis $H(j)_1$ (iii) and theorem 2.7.5 of [7, p. 83], we see that ψ is locally Lipschitz. Furthermore, functionals $W_0^{1,p}(Z) \ni x \mapsto \|\nabla x\|_p^p \in \mathbb{R}$ and $W_0^{1,p}(Z) \ni x \mapsto \|x\|_p^p \in \mathbb{R}$ are convex, continuous, and hence locally Lipschitz on $W_0^{1,p}(Z)$. Therefore, ϕ is locally Lipschitz.

Proposition 3.1. *If hypotheses $H(j)_1$ hold, then ϕ satisfies the non-smooth PS condition.*

Proof. Let $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that $|\phi(x_n)| \leqslant M_1$ for all $n \geqslant 1$ and $m(x_n) \to 0$ as $n \to +\infty$. Let $x_n^* \in \partial\phi(x_n)$ be such that $m(x_n) = \|x_n^*\|_*$ for $n \geqslant 1$. For every $n \geqslant 1$, its existence is a consequence of the fact that $\partial\phi(x_n) \subseteq (W_0^{1,p}(Z))^* = W^{-1,p'}(Z)$ is weakly compact and the norm functional is weakly lower semicontinuous. Let $A : W_0^{1,p}(Z) \mapsto W^{-1,p'}(Z)$ be the nonlinear operator defined by

$$\langle Ax, v\rangle \stackrel{\mathrm{df}}{=} \int_Z \|\nabla x(z)\|_{\mathbb{R}^N}^{p-2}(\nabla x(z), \nabla v(z))_{\mathbb{R}^N}\,\mathrm{d}z \quad \forall x, v \in W_0^{1,p}(Z)$$

(by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,p'}(Z))$). It is straightforward to check that A is demicontinuous and strongly monotone, hence maximal monotone (see [13, corollary III.1.35, p. 309]). For every $n \geqslant 1$, we have

$$x_n^* = Ax_n - \lambda_1|x_n|^{p-2}x_n - u_n^*, \tag{3.1}$$

where $u_n^* \in \partial(\psi|_{W_0^{1,p}(Z)})(x_n)$, with $\psi : L^{p^{*\prime}}(Z) \mapsto \mathbb{R}$ defined by

$$\psi(x) \stackrel{\mathrm{df}}{=} \int_Z j(z, x(z))\,\mathrm{d}z.$$

From theorem 2.2 of [6, p. 110] and theorem 2.7.5 of [7, p. 83], we know that $u_n^* \in L^{p^{*\prime}}(Z)$ and $u_n^*(z) \in \partial j(z, x_n(z))$ almost everywhere on Z.

First we will show that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose that this is not true. Then, by passing to a subsequence if necessary, we may assume that $\|x_n\| \to +\infty$ as $n \to +\infty$. Let $y_n = x_n/\|x_n\|$ for $n \geqslant 1$. Then, by passing to another subsequence if necessary, we may assume that

$$y_n \to y \quad \text{weakly in } W_0^{1,p}(Z) \text{ as } n \to +\infty, \tag{3.2}$$

$$y_n \to y \quad \text{in } L^p(Z) \text{ as } n \to +\infty, \tag{3.3}$$

$$y_n \to y \quad \text{almost everywhere on } Z \text{ as } n \to +\infty,$$

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with some $y \in W_0^{1,p}(Z)$ and $|y_n(z)| \leqslant k(z)$ almost everywhere on Z, for all $n \geqslant 1$ and with some $k \in L^p(Z)$ (see [4, theorem IV.9, p. 58]). From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$, for all $n \geqslant 1$, we have

$$\frac{|\phi(x_n)|}{\|x_n\|^p} \leqslant \frac{M_1}{\|x_n\|^p},$$

so

$$\limsup_{n\to+\infty}\left(\frac{1}{p}\|\nabla y_n\|_p^p - \frac{\lambda_1}{p}\|y_n\|_p^p - \int_Z \frac{j(z,x_n(z))}{\|x_n\|^p}\,\mathrm{d}z\right) \leqslant 0. \tag{3.4}$$

By virtue of the Lebourg mean-value theorem (see [16] or [7, theorem 2.3.7, p. 41]), we know that for all $n \geqslant 1$ and almost all $z \in Z$, we can find $w_n(z) \in \partial j(z, t_n x_n(z))$ with $0 < t_n < 1$ such that

$$|j(z,x_n(z)) - j(z,0)| = |w_n(z)x_n(z)|.$$

From hypothesis $H(j)_1$ (iii), for almost all $z \in Z$, we have that

$$|j(z,x_n(z))| \leqslant |j(z,0)| + a(z)|x_n(z)|, \tag{3.5}$$

where $a \in L^{p^{*\prime}}(Z)$. So, from (3.5), hypothesis $H(j)_1(i)$ and the continuity of the embedding $W_0^{1,p}(Z) \subseteq L^{p^*}(Z)$, we have

$$\begin{aligned}\int_Z \frac{|j(z,x_n(z))|}{\|x_n\|^p}\,\mathrm{d}z &\leqslant \int_Z\left(\frac{|j(z,0)|}{\|x_n\|^p} + \frac{a(z)|x_n(z)|}{\|x_n\|^p}\right)\mathrm{d}z \\ &\leqslant \frac{\|j(\cdot,0)\|_1}{\|x_n\|^p} + \frac{\|a\|_{p^{*\prime}}\|x_n\|_{p^*}}{\|x_n\|^p} \\ &\leqslant \frac{c_1\|j(\cdot,0)\|_{p^{*\prime}}}{\|x_n\|^p} + \frac{c_2\|a\|_{p^{*\prime}}}{\|x_n\|^{p-1}},\end{aligned}$$

with $c_1 \overset{\mathrm{df}}{=} |Z|^{p^{*\prime}/(p^{*\prime}-1)}$ and some $c_2 > 0$, and thus

$$\int_Z \frac{j(z,x_n(z))}{\|x_n\|^p}\,\mathrm{d}z \to 0 \quad \text{as } n \to +\infty.$$

Also, from (3.3), we have

$$\frac{1}{p}\|y_n\|_p^p \to \frac{1}{p}\|y\|_p^p \quad \text{as } n \to +\infty,$$

so, from (3.4), we have

$$\frac{1}{p}\limsup_{n\to+\infty}\|\nabla y_n\|_p^p \leqslant \frac{\lambda_1}{p}\|y\|_p^p. \tag{3.6}$$

From (3.2), the weak lower semicontinuity of the norm functional and the Rayleigh quotient (see (2.1)), we have

$$\frac{\lambda_1}{p}\|y\|_p^p \leqslant \frac{1}{p}\|\nabla y\|_p^p \leqslant \frac{1}{p}\liminf_{n\to+\infty}\|\nabla y_n\|_p^p. \tag{3.7}$$

So from (3.6) and (3.7), it follows that

$$\|\nabla y\|_p^p = \lambda_1\|y\|_p^p \tag{3.8}$$

and

$$\|\nabla y_n\|_p^p \to \|\nabla y\|_p^p \quad \text{as } n \to +\infty.$$

Since we already know that $\nabla y_n \to \nabla y$ weakly in $L^p(Z;\mathbb{R}^N)$ as $n \to +\infty$ and space $L^p(Z;\mathbb{R}^N)$ is uniformly convex, from the Kadec–Klee property (see [13, definition I.1.72(d) and lemma I.1.74, p. 28]), we have that $\nabla y_n \to \nabla y$ in $L^p(Z;\mathbb{R})$ and so $y_n \to y$ in $W_0^{1,p}(Z)$ as $n \to +\infty$. Since $\|y_n\| = 1$, we have that $\|y\| = 1$, i.e. $y \neq 0$. Therefore, from (3.8), we infer that $y = \pm u_1$ (see the Rayleigh quotient (2.1)). Without any loss of generality, we can assume that $y = u_1$ (the case $y = -u_1$ is treated similarly). Since $u_1(z) > 0$ for all $z \in Z$, we have that $x_n(z) \to +\infty$ for all $z \in Z$. Because $\|x_n^*\|_* \to 0$, from (3.1), at least for a subsequence, we have that

$$\langle Ax_n, x_n\rangle - \lambda_1(|x_n|^{p-2}x_n, x_n)_{pp'} - \int_Z u_n^*(z)x_n(z)\,\mathrm{d}z \leqslant \frac{1}{n}\|x_n\|,$$

and so

$$\|\nabla x_n\|_p^p - \lambda_1\|x_n\|_p^p - \int_Z u_n^*(z)x_n(z)\,\mathrm{d}z \leqslant \frac{1}{n}\|x_n\|.$$

From the Rayleigh quotient (see (2.1)), we have

$$-\int_Z u_n^*(z)x_n(z)\,\mathrm{d}z \leqslant \frac{1}{n}\|x_n\|.$$

Dividing the last inequality by $\|x_n\|$, we obtain

$$-\int_Z u_n^*(z)y_n(z)\,\mathrm{d}z \leqslant \frac{1}{n}. \tag{3.9}$$

Recall that $\{u_n^*\}_{n\geqslant 1} \subseteq L^{p^{*\prime}}(Z)$ and, by virtue of hypothesis $H(j)_1$ (iii), this sequence is bounded. So, passing to a subsequence if necessary, we may assume that $u_n^* \to u^*$ weakly in $L^{p^*}(Z)$ as $n \to +\infty$. As $y_n \to u_1$ in $W_0^{1,p}(Z)$, so also $y_n \to u_1$ in $L^{p^{*\prime}}(Z)$ as $n \to +\infty$. Thus, passing to the limit in (3.9) as $n \to +\infty$, we obtain

$$-\int_Z u^*(z)u_1(z)\,\mathrm{d}z \leqslant 0.$$

Invoking proposition VII.3.9 of [13, p. 694], we have that $u^*(z) \leqslant v_+(z)$ almost everywhere on Z (see hypothesis $H(j)_1$ (iv)). As $u_1 > 0$, so we obtain

$$-\int_Z v_+(z)u_1(z)\,\mathrm{d}z \leqslant 0,$$

which contradicts hypothesis $H(j)_1$ (iv). This proves that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. Hence we may assume that $x_n \to x$ weakly in $W_0^{1,p}(Z)$ and, from the compactness of the embedding $W_0^{1,p}(Z) \subseteq L^q(Z)$ (where q is such that $1/q + 1/q' = 1$; note that $p \leqslant q < p^*$), we have that $x_n \to x$ in $L^q(Z)$. Because sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded and $\|x_n^*\|_* \to 0$ as $n \to +\infty$ so, at least for a subsequence, we have that

$$|\langle x_n^*, x_n - x\rangle| \leqslant \frac{1}{n}.$$

From (3.1), we obtain

$$\langle Ax_n, x_n - x\rangle - \lambda_1(|x_n|^{p-1}x_n, x_n - x)_{pp'} - (u_n^*, x_n - x)_{qq'} \leqslant \frac{1}{n} \tag{3.10}$$

(by $(\cdot,\cdot)_{qq'}$ we denote the duality brackets for the pair $(L^q(Z), L^{q'}(Z)))$. Because $x_n \to x$ in $L^q(Z)$, $x_n \to x$ in $L^p(Z)$ as $n \to +\infty$, and from the continuity of the operator $L^p(Z) \ni x \mapsto |x|^{p-2}x \in L^{p'}(Z)$, we have that $|x_n|^{p-2}x_n \to |x|^{p-2}x$ in $L^{p'}(Z)$ as $n \to +\infty$. From hypothesis $H(j)_1$ (iii), we know that the sequence $\{u_n^*\}_{n\geqslant 1} \subseteq L^{q'}(Z)$ is bounded and thus $(u_n^*, x_n - x)_{qq'} \to 0$ as $n \to +\infty$. So, passing to the limit in (3.10), we obtain

$$\limsup_{n\to+\infty}\langle Ax_n, x_n - x\rangle \leqslant 0.$$

Employing the maximal monotonicity of A, the Kadec–Klee property of uniformly convex spaces and arguing as before, we obtain that $x_n \to x$ in $W_0^{1,p}(Z)$. So ϕ satisfies the non-smooth PS condition. □

Proposition 3.2. *If hypotheses $H(j)_1$ hold, then ϕ is coercive* (*i.e. if $\|x\| \to +\infty$, then $\phi(x) \to +\infty$*).

Proof. Let us suppose that this is not true. Then we can find $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ and $M_2 > 0$ such that $\|x_n\| \to +\infty$ and $|\phi(x_n)| \leqslant M_2$ for all $n \geqslant 1$. Let $y_n \stackrel{\mathrm{df}}{=} x_n/\|x_n\|$ for all $n \geqslant 1$. Arguing as in the proof of proposition 3.1, we can check that $y_n \to \pm u_1$ weakly in $W_0^{1,p}(Z)$ as $n \to +\infty$ (at least for a subsequence). Assume that the last limit is u_1 (the case when it is $-u_1$ is treated similarly). Then we have $x_n(z) \to +\infty$ almost everywhere on Z. Let $Z_{0n} \stackrel{\mathrm{df}}{=} \{z \in Z : x_n(z) \neq 0\}$ and

$$g_n(z) \stackrel{\mathrm{df}}{=} \begin{cases} j(z, x_n(z))/x_n(z) & \text{if } z \in Z_{0n}, \\ 0 & \text{if } z \in Z \setminus Z_{0n}. \end{cases}$$

First we will show that for almost all $z \in Z$, we have

$$\limsup_{n\to+\infty} g_n(z) \leqslant v_+(z). \tag{3.11}$$

For this purpose, let $0 < \varepsilon < 1$. From the Lebourg mean-value theorem, for almost all $z \in Z$, we have

$$j(z, x_n(z)) = j(z, \varepsilon x_n(z)) + w_n(z)(1-\varepsilon)x_n(z), \tag{3.12}$$

with $w_n(z) \in \partial j(z, r_n(z))$, where $r_n(z) = (1-\xi_n)x_n(z) + \xi_n\varepsilon x_n(z)$ and $0 < \xi_n < 1$. Recall that for almost all $z \in Z$, we have $x_n(z) \to +\infty$ as $n \to +\infty$. Hence, for almost all $z \in Z$, we get

$$\begin{aligned} r_n(z) &= x_n(z) - \xi_n(1-\varepsilon)x_n(z) \\ &\geqslant x_n(z) - (1-\varepsilon)x_n(z) \\ &= \varepsilon x_n(z). \end{aligned}$$

So $r_n(z) \to +\infty$ for almost all $z \in Z$ as $n \to +\infty$. From (3.12), for $z \in Z_{0n}$, we have

$$\frac{j(z, x_n(z))}{x_n(z)} = \frac{j(z, \varepsilon x_n(z))}{x_n(z)} + w_n(z)(1-\varepsilon).$$

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As in the proof of proposition 3.1 (see (3.5)), we get that

$$|j(z,\varepsilon x_n(z))| \leqslant |j(z,0)| + a(z)\varepsilon|x_n(z)|$$

for almost all $z \in Z$. So, for $n \geqslant 1$ large enough and almost all $z \in Z_{0n}$, we have

$$\frac{j(z,x_n(z))}{x_n(z)} \leqslant \frac{|j(z,0)|}{x_n(z)} + a(z)\varepsilon + w_n(z)(1-\varepsilon).$$

From the definition of v_+, we see that $\limsup_{n\to+\infty} w_n(z) \leqslant v_+(z)$ for almost all $z \in Z$. Thus, finally, for almost all $z \in Z$, we can write that

$$\limsup_{n\to+\infty} \frac{j(z,x_n(z))}{x_n(z)} \leqslant a(z)\varepsilon + v_+(z)(1-\varepsilon)$$

(recall that $|Z_{0n}| \to |Z|$ as $n \to +\infty$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$). As the last inequality holds for any $0 < \varepsilon < 1$, it follows that

$$\limsup_{n\to+\infty} \frac{j(z,x_n(z))}{x_n(z)} \leqslant v_+(z)$$

for almost all $z \in Z$, which proves (3.11).

From the definition of g_n, for $n \geqslant 1$, we have

$$\int_Z \frac{j(z,x_n(z))}{\|x_n\|}\,\mathrm{d}z = \int_{Z_{0n}} g_n(z)y_n(z)\,\mathrm{d}z + \int_{Z\setminus Z_{0n}} \frac{j(z,0)}{\|x_n\|}\,\mathrm{d}z. \tag{3.13}$$

Note that since $j(\cdot,0) \in L^{p^{*\prime}}(Z)$ and $\|x_n\| \to +\infty$ as $n \to +\infty$, we have

$$\int_{Z\setminus Z_{0n}} \frac{j(z,0)}{\|x_n\|}\,\mathrm{d}z \to 0 \quad \text{as } n \to +\infty.$$

Note that $y_n\chi_{Z_{0n}} = y_n$ for all $n \geqslant 1$ (as $y_n|_{Z\setminus Z_{0n}} \equiv 0$). So, at least for a subsequence, we have

$$\chi_{Z_{0n}} y_n \to u_1 \quad \text{in } L^p(Z) \quad \text{as } n \to +\infty. \tag{3.14}$$

From (3.13), we obtain

$$\limsup_{n\to+\infty} \int_Z \frac{j(z,x_n(z))}{\|x_n\|}\,\mathrm{d}z = \limsup_{n\to+\infty} \int_Z g_n(z)(\chi_{Z_{0n}} y_n)(z)\,\mathrm{d}z. \tag{3.15}$$

Using (3.14), (3.11) and Fatou's lemma, from (3.15) it follows that

$$\limsup_{n\to+\infty} \int_Z \frac{j(z,x_n(z))}{\|x_n\|}\,\mathrm{d}z \leqslant \int_Z v_+(z)u_1(z)\,\mathrm{d}z. \tag{3.16}$$

From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$, we have

$$\phi(x_n) = \frac{1}{p}\|\nabla x_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p - \int_Z j(z,x_n(z))\,\mathrm{d}z \leqslant M_2,$$

so, from the Rayleigh quotient (see (2.1)), we get

$$-\int_Z j(z,x_n(z))\,\mathrm{d}z \leqslant M_2.$$

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Dividing both sides of the last inequality by $\|x_n\|$ and using (3.16), we get

$$\int_Z v_+(z)u_1(z)\,\mathrm{d}z \geqslant 0.$$

This contradicts the Landesman–Lazer-type condition in $H(j)_1$ (iv). Therefore, ϕ is coercive. □

Using propositions 3.1 and 3.2, we can prove our first existence theorem concerning problem (HVI).

THEOREM 3.3. *If hypotheses $H(j)_1$ hold, then (HVI) has a solution $x_0 \in W_0^{1,p}(Z)$.*

Proof. From proposition 3.2, we know that ϕ is bounded below. Also, by proposition 3.1, it satisfies the non-smooth PS condition. So we apply theorem 2.1 and obtain $x_0 \in W_0^{1,p}(Z)$, such that $\phi(x_0) = \inf\{\phi(x) : x \in W_0^{1,p}(Z)\}$. Then $0 \in \partial\phi(x_0)$ and so

$$Ax_0 - \lambda_1|x_0|^{p-2}x_0 = u^* \quad \text{in } W^{-1,q}(Z),$$

with $u^* \in \partial\psi(x_0)$, hence $u^* \in L^{p^{*\prime}}(Z)$ and $u^*(z) \in \partial j(z, x_0(z))$ almost everywhere on Z. We have

$$\langle Ax_0, \vartheta\rangle = \lambda_1(|x_0|^{p-2}x_0, \vartheta)_{pp'} + (u^*, \vartheta)_{p^*p^{*\prime}} \quad \forall \vartheta \in C_0^\infty(Z)$$

and, by Green's theorem,

$$\langle -\operatorname{div}(\|\nabla x_0\|_{\mathbb{R}^N}^{p-2}\nabla x_0), \vartheta\rangle = \lambda_1(|x_0|^{p-2}x_0, \vartheta)_{pp'} + (u^*, \vartheta)_{p^*p^{*\prime}} \quad \forall \vartheta \in C_0^\infty(Z).$$

Note that from the representation theorem for the elements in the dual space $W^{-1,p'}(Z) = (W_0^{1,p}(Z))^*$ (see [1, theorem 3.10, p. 50]), we have that

$$\operatorname{div}(\|\nabla x_0\|^{p-2}\nabla x_0) \in W^{-1,p'}(Z).$$

Since $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$, we deduce that

$$-\operatorname{div}(\|\nabla x_0(z)\|_{\mathbb{R}^N}^{p-2}\nabla x_0(z)) - \lambda_1|x_0(z)|^{p-2}x_0(z) = u^*(z) \in \partial j(z, x_0(z)) \quad \text{almost everywhere on } Z,$$

$$x_0|_\Gamma = 0,$$

and so x_0 is a solution of (HVI). □

We can have another existence result, with the reverse Landesman–Lazer-type condition, by adding an additional hypothesis, dictating a subresonant behaviour near the origin. More precisely, our hypotheses on $j(z, \zeta)$ are the following.

HYPOTHESES $H(j)_2$. $j : Z \times \mathbb{R} \mapsto \mathbb{R}$ is a function such that:

(i) for all $\zeta \in \mathbb{R}$, function $Z \ni z \mapsto j(z, \zeta) \in \mathbb{R}$ is measurable, $j(\cdot, 0) \in L^\infty(Z)$ and $\int_Z j(z, 0)\,\mathrm{d}z \geqslant 0$;

(ii) for almost all $z \in Z$, function $\mathbb{R} \ni \zeta \mapsto j(z, \zeta) \in \mathbb{R}$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $\eta \in \partial j(z, \zeta)$, we have $|\eta| \leqslant a(z)$ with some $a \in L^\infty(Z)$;

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(iv) there exist functions $\hat{v}_+, \hat{v}_- \in L^1(Z)$ such that, for almost all $z \in Z$, we have

$$\hat{v}_+(z) = \liminf_{n\to+\infty} v_n(z) \quad \text{and} \quad \hat{v}_-(z) = \limsup_{n\to+\infty} v_n(z)$$

where $\{v_n\}_{n\geqslant 1} \subseteq L^{p^{*\prime}}(Z)$ is such that $v_n(z) \in \partial j(z, \zeta_n)$ with $\zeta_n \to +\infty$ (respectively, $\zeta_n \to -\infty$) and

$$\int_Z \hat{v}_-(z) u_1(z)\,\mathrm{d}z < 0 < \int_Z \hat{v}_+(z) u_1(z)\,\mathrm{d}z;$$

(v) there exists $\mu > \lambda_1$ such that

$$\limsup_{\zeta\to 0} \frac{pj(z,\zeta)}{|\zeta|^p} \leqslant -\mu$$

uniformly for almost all $z \in Z$.

Proposition 3.4. *If hypotheses $H(j)_2$ hold, then there exist $\beta_1, \beta_2 > 0$ such that, for all $x \in W_0^{1,p}(Z)$, we have*

$$\phi(x) \geqslant \beta_1 \|x\|^p - \beta_2 \|x\|^\vartheta,$$

with $p < \vartheta \leqslant p^*$.

Proof. From hypothesis $H(j)_2$ (v), we can find $\delta > 0$ such that, for almost all $z \in Z$ and all $|\zeta| \leqslant \delta$, we have

$$j(z,\zeta) \leqslant -\frac{\lambda_1}{p}|\zeta|^p$$

(recall that $\mu > \lambda_1$). On the other hand, from the Lebourg mean-value theorem and hypotheses $H(j)_2$ (i) and (iii), one can show that for almost all $z \in Z$ and all $|\zeta| > \delta$, we have

$$|j(z,\zeta)| \leqslant c_3 + c_4|\zeta|,$$

with some $c_3, c_4 > 0$. Thus, for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have

$$j(z,\zeta) \leqslant -\frac{\lambda_1}{p}|\zeta|^p + c_5|\zeta|^\vartheta,$$

with $c_5 = (c_3 + c_4\delta)\delta^{-\vartheta} + (\lambda_1/p)\delta^{p-\vartheta}$ and $p < \vartheta \leqslant p^*$. Using this, we obtain that

$$\begin{aligned}
\phi(x) &= \frac{1}{p}\|\nabla x\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z j(z, x(z))\,\mathrm{d}z \\
&\geqslant \frac{1}{p}\|\nabla x\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p + \frac{\lambda_1}{p}\|x\|_p^p - c_5\|x\|_\vartheta^\vartheta \\
&= \frac{1}{p}\|\nabla x\|_p^p - c_5\|x\|_\vartheta^\vartheta.
\end{aligned}$$

Since $\vartheta \leqslant p^*$, from the Sobolev embedding theorem we have that $W_0^{1,p}(Z)$ is embedded continuously in $L^\vartheta(Z)$. So, using Poincaré's inequality, it follows that

$$\phi(x) \geqslant \beta_1\|x\|^p - \beta_2\|x\|^\vartheta$$

for some $\beta_1, \beta_2 > 0$ and all $x \in W_0^{1,p}(Z)$. □

THEOREM 3.5. *If hypotheses $H(j)_2$ hold, then problem (HVI) has a non-trivial solution $x_0 \in W_0^{1,p}(Z)$.*

Proof. From proposition 3.4, we know that there exist $\beta_1, \beta_2 > 0$ such that, for all $x \in W_0^{1,p}(Z)$, we have

$$\phi(x) \geqslant \beta_1 \|x\|^p - \beta_2 \|x\|^\vartheta,$$

with some $p < \vartheta \leqslant p^*$. Evidently, if we choose $r > 0$ small enough, we will have that $\phi(x) \geqslant c_6 > 0$ for all $x \in W_0^{1,p}(Z)$ such that $\|x\| = r$ and some $c_6 > 0$.

Next let $t > 0$ and let us consider the quantity $\phi(tu_1)$. Using the fact that $\|\nabla u_1\|_p^p = \lambda_1 \|u_1\|_p^p$, we have

$$\phi(tu_1) = \frac{t^p}{p} \|\nabla u_1\|_p^p - \frac{\lambda_1 t^p}{p} \|u_1\|_p^p - \int_Z j(z, tu_1(z))\,\mathrm{d}z = -\int_Z j(z, tu_1(z))\,\mathrm{d}z.$$

By a simple modification of the argumentation for (3.11) in the proof of proposition 3.2, we can verify that

$$\liminf_{t \to +\infty} \frac{j(z, tu_1(z))}{tu_1(z)} \geqslant \hat{v}_+(z) \quad \text{almost everywhere on } Z.$$

If $t_n \to +\infty$, using Fatou's lemma, we have that

$$\liminf_{n \to +\infty} \int_Z \frac{j(z, t_n u_1(z))}{t_n u_1(z)} u_1(z)\,\mathrm{d}z \geqslant \int_Z \hat{v}_+(z) u_1(z)\,\mathrm{d}z > 0.$$

Because

$$\int_Z j(z, t_n u_1(z))\,\mathrm{d}z = t_n \int_Z \frac{j(z, t_n u_1(z))}{t_n u_1(z)} u_1(z)\,\mathrm{d}z,$$

we have

$$\int_Z j(z, t_n u_1(z))\,\mathrm{d}z \to +\infty \quad \text{as } n \to +\infty.$$

Therefore, it follows that for $n \geqslant 1$ large enough, we will have $\phi(t_n u_1) \leqslant 0$. Also, $\phi(0) \leqslant 0$ (see hypothesis $H(j)_2$ (i)). Finally, by a simple modification of the proof of proposition 3.1, we can check that ϕ satisfies the non-smooth PS condition. So we can apply theorem 2.2 and obtain $x_0 \in W_0^{1,p}(Z)$, such that

$$\phi(x_0) \geqslant \inf\{\phi(x) : \|x\| = r\} \geqslant c_6 > 0 \geqslant \phi(0)$$

(hence $x_0 \neq 0$) and $0 \in \partial\phi(x_0)$. As in the proof of theorem 3.3, we can verify that x_0 is a solution of (HVI). □

4. Multiplicity results

In this section we prove the multiplicity result for problem (HVI) under the condition of strong resonance at infinity. The hypotheses on $j(z, \zeta)$ are the following.

HYPOTHESES $H(j)_3$. $j : Z \times \mathbb{R} \mapsto \mathbb{R}$ is a function such that:

(i) for all $\zeta \in \mathbb{R}$, function $Z \ni z \mapsto j(z, \zeta) \in \mathbb{R}$ is measurable, $j(\cdot, 0) \in L^1(Z)$ and $\int_Z j(z, 0)\,\mathrm{d}z \geqslant 0$;

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(ii) for almost all $z \in Z$, function $\mathbb{R} \ni \zeta \mapsto j(z,\zeta) \in \mathbb{R}$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $\eta \in \partial j(z,\zeta)$, we have $|\eta| \leqslant a(z)$ with some $a \in L^{\infty}(Z)$;

(iv) there exist functions $j_+, j_- \in L^1(Z)$ such that $j(z,\zeta) \to j_+(z)$ as $\zeta \to +\infty$ and $j(z,\zeta) \to j_-(z)$ as $\zeta \to -\infty$ uniformly for almost all $z \in Z$, $\int_Z j_{\pm}(z)\,\mathrm{d}z > 0$;

(v) there exists $\mu > \lambda_1$ such that

$$\limsup_{\zeta \to 0} \frac{pj(z,\zeta)}{|\zeta|^p} \leqslant -\mu$$

uniformly for almost all $z \in Z$.

(vi) there exist $t_- < 0 < t_+$ such that

$$\int_Z j_{\pm}(z)\,\mathrm{d}z < \int_Z j(z, t_{\pm}u_1(z))\,\mathrm{d}z;$$

(vii) for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have $pj(z,\zeta) \leqslant (\lambda_{2,Y_{u_1}} - \lambda_1)|\zeta|^p$ with some topological complement Y_{u_1} of $\mathbb{R}u_1$ (see § 2) and

$$\int_Z u_n^*(z)x_n(z)\,\mathrm{d}z \to 0 \quad \text{as } n \to +\infty$$

for any sequences $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ and $\{u_n^*\}_{n\geqslant 1} \subseteq L^{p^{*\prime}}(Z)$ such that $u_n^*(z) \in \partial j(z, x_n(z))$ and $|x_n(z)| \to +\infty$ almost everywhere on Z.

We can also modify hypothesis $H(j)_3$ (iv) and still have a multiplicity result. Namely we assume the following.

Hypotheses $H(j)_4$. $j : Z \times \mathbb{R} \mapsto \mathbb{R}$ is a function satisfying $H(j)_3$, with $H(j)_3$ (iv) and (vi) replaced by:

(iv) there exist functions $\hat{\jmath}_+, \hat{\jmath}_- \in L^1(Z)$ such that

$$\limsup_{|\zeta|\to+\infty} j(z,\zeta) = \hat{\jmath}_+(z) \quad \text{and} \quad \liminf_{|\zeta|\to+\infty} j(z,\zeta) = \hat{\jmath}_-(z)$$

uniformly for almost all $z \in Z$ and

$$\int_Z \hat{\jmath}_+(z)\,\mathrm{d}z > 0, \qquad \int_Z \hat{\jmath}_-(z)\,\mathrm{d}z \geqslant 0;$$

(vi) there exist $t_- < 0 < t_+$ such that

$$\int_Z \hat{\jmath}_{\pm}(z)\,\mathrm{d}z < \int_Z j(z, t_{\pm}u_1(z))\,\mathrm{d}z.$$

Remark 4.1. Hypotheses $H(j)_3$ (iv) and $H(j)_4$ (iv) are the strong resonance conditions, since they imply that, for almost all $z \in Z$, the limits $\lim_{\zeta\to\pm\infty} j(z,\zeta)$ are finite (the term 'strong resonance' was first used by Bartolo *et al.* [3]). Evidently,

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the growth condition in hypothesis $H(j)_{3,4}$ (vii) is automatically satisfied in a neighbourhood of the origin, by virtue of hypothesis $H(j)_{3,4}$ (v). Moreover, the growth condition in hypothesis $H(j)_{3,4}$ (vii) is analogous to hypothesis H_∞ of Goncalves and Miyagaki [12, theorem 1, p. 266]. Let

$$j(z,\zeta) = \int_0^\zeta f(z,r)\,\mathrm{d}r,$$

with $f : Z \times \mathbb{R} \mapsto \mathbb{R}$ a measurable function such that, for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have $|f(z,\zeta)| \leqslant a(z)$, with $a \in L^\infty(Z)$. Then $j(z,\zeta)$ satisfies hypotheses $H(j)_{3,4}$ (i)–(iii). If we set

$$f_1(z,\zeta) = \liminf_{\zeta' \to \zeta} f(z,\zeta'), \qquad f_2(z,\zeta) = \limsup_{\zeta' \to \zeta} f(z,\zeta'),$$

and if, for $i = 1, 2$, we assume that $f_i(z,\zeta)\zeta \to 0$ as $|\zeta| \to +\infty$, then the second part of hypothesis $H(j)_{3,4}$ (vii) is satisfied. This setting corresponds to problems with a discontinuous right-hand side (see [6, problem (0.1), p. 102 and § 5, pp. 122–128]). Hypothesis $H(j)_{3,4}$ (v) is needed in order to obtain the third non-trivial solution. Without it, we can not guarantee that the third solution is also non-trivial. When hypothesis $H(j)_{3,4}$ (v) is present, we will see in the sequel that the third solution is obtained via the mountain-pass theorem (see theorem 2.2). Without $H(j)_{3,4}$ (v), the third solution can be established using the saddle-point theorem (see [20]).

PROPOSITION 4.2. *If hypotheses $H(j)_3$ or $H(j)_4$ hold, then there exists function $b \in L^1(Z)$ such that $|j(z,\zeta)| \leqslant b(z)$ for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$.*

Proof. Let us assume that hypotheses $H(j)_3$ hold. By virtue of $H(j)_3$ (iv), we can find $M_3 > 0$ such that, for almost all $z \in Z$, we have

$$|j(z,\zeta) - j_+(z)| \leqslant 1 \quad \forall \zeta \geqslant M_3,$$
$$|j(z,\zeta) - j_-(z)| \leqslant 1 \quad \forall \zeta \leqslant -M_3.$$

So, for almost all $z \in Z$, we have

$$\left.\begin{aligned} |j(z,\zeta)| &\leqslant 1 + |j_+(z)| \quad \forall \zeta \geqslant M_3, \\ |j(z,\zeta)| &\leqslant 1 + |j_-(z)| \quad \forall \zeta \leqslant -M_3. \end{aligned}\right\} \tag{4.1}$$

On the other hand, using the Lebourg mean-value theorem (see [16] or [7, theorem 2.3.7, p. 41]) and hypothesis $H(j)_3$ (iii), for all $\zeta \in \mathbb{R}$ and almost all $z \in Z$, we have $|j(z,\zeta)| \leqslant |j(z,0)| + |a(z)||\zeta|$. Thus, for almost all $z \in Z$, we have

$$|j(z,\zeta)| \leqslant |j(z,0)| + M_3|a(z)| \quad \forall |\zeta| \leqslant M_3. \tag{4.2}$$

From (4.1) and (4.2), for all $\zeta \in \mathbb{R}$ and almost all $z \in Z$, we get $|j(z,\zeta)| \leqslant b(z)$, with $b \in L^1(Z)$, namely $b(z) \overset{\mathrm{df}}{=} \max\{1 + |j_+(z)|, 1 + |j_-(z)|, M_3|a(z)|\}$, which finishes the proof. The proof is similar when we assume that hypotheses $H(j)_4$ are in effect. □

As in § 3, let energy functional $\phi : W_0^{1,p}(Z) \mapsto \mathbb{R}$ be defined by

$$\phi(x) \overset{\mathrm{df}}{=} \frac{1}{p}\|\nabla x\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z j(z,x(z))\,\mathrm{d}z.$$

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PROPOSITION 4.3. *If hypotheses $H(j)_3$ hold, then ϕ satisfies the non-smooth C condition at all levels $c \neq -\int_Z j_\pm(z)\,\mathrm{d}z$.*

Proof. Let $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that $\phi(x_n) \to c$ as $n \to +\infty$, with $c \neq -\int_Z j_\pm(z)\,\mathrm{d}z$ and let $(1+\|x_n\|)m(x_n) \to 0$ as $n \to +\infty$.

We will show that $\{x_n\}_{n\geqslant 1}$ is bounded. Suppose that this is not true. Passing to a subsequence if necessary, we may assume that $\|x_n\| \to +\infty$. Let $y_n \overset{\mathrm{df}}{=} x_n/\|x_n\|$ for $n \geqslant 1$. Arguing as in the proof of the proposition 3.1, we can show that $y_n \to \pm u_1$ in $W_0^{1,p}(Z)$ as $n \to +\infty$. From this it follows that $x_n(z) \to \pm\infty$ almost everywhere on Z as $n \to +\infty$. Let us choose any $\varepsilon > 0$. Since $\phi(x_n) \to c$, we can find $n_0(\varepsilon) \geqslant 1$ such that, for all $n \geqslant n_0(\varepsilon)$, we have

$$c - \varepsilon \leqslant \phi(x_n) \leqslant c + \varepsilon$$

and so

$$c - \varepsilon \leqslant \frac{1}{p}\|\nabla x_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p - \int_Z j(z, x_n(z))\,\mathrm{d}z \leqslant c + \varepsilon. \tag{4.3}$$

Let $x_n^* \in \partial\phi(x_n)$ be such that $m(x_n) = \|x_n^*\|$ for $n \geqslant 1$. Since $(1+\|x_n\|)m(x_n) \to 0$, $\|x_n\| \cdot \|x_n^*\|_* \to 0$ as $n \to +\infty$ and, at least for a subsequence, we have

$$-\frac{1}{n} \leqslant \langle x_n^*, x_n\rangle \leqslant \frac{1}{n}$$

for all $n \geqslant 1$. However, recall that $x_n^* = Ax_n - \lambda_1|x_n|^{p-2}x_n - u_n^*$, with $A : W_0^{1,p}(Z) \mapsto W^{-1,p'}(Z)$ as in the proof of proposition 3.1 and $u_n^* \in L^{p^{*\prime}}(Z)$ such that $u_n(z) \in \partial j(z, x_n(z))$ almost everywhere on Z. So we have

$$-\frac{1}{n} \leqslant \|\nabla x_n\|_p^p - \lambda_1\|x_n\|_p^p - \int_Z u_n^*(z)x_n(z)\,\mathrm{d}z \leqslant \frac{1}{n}.$$

By virtue of $H(j)_3$ (vii), we have that

$$\int_Z u_n^*(z)x_n(z)\,\mathrm{d}z \to 0 \quad \text{as } n \to +\infty$$

and so we infer that $\|\nabla x_n\|_p^p - \lambda_n\|x_n\|_p^p \to 0$ as $n \to +\infty$. Using this fact in (4.3) and applying the Lebesgue dominated convergence theorem for the sequence $j(\cdot, x_n(\cdot))$ (note that, by virtue of proposition 4.2, its usage is allowed), we have

$$c - \varepsilon \leqslant -\int_Z j_\pm(z)\,\mathrm{d}z \leqslant c + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, so we conclude that $c = -\int_Z j_\pm(z)\,\mathrm{d}z$, thus we reach a contradiction. This proves the boundedness of $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$. Arguing as in the proof of proposition 3.1, via the Kadec–Klee property, we can show that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ has a strongly convergent subsequence. □

We can have a similar result, if hypotheses $H(j)_4$ are in effect.

PROPOSITION 4.4. *If hypotheses $H(j)_4$ hold, then ϕ satisfies the non-smooth C condition at all levels*

$$c \in \left(-\infty, -\int_Z \hat{j}_+(z)\,\mathrm{d}z\right) \cup (0, +\infty).$$

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Proof. Let $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that $\phi(x_n) \to c$ as $n \to +\infty$, with

$$c \in \left(-\infty, -\int_Z \hat{\jmath}_+(z)\,\mathrm{d}z\right) \cup (0, +\infty)$$

and let $(1 + \|x_n\|)m(x_n) \to 0$ as $n \to +\infty$.

We will show that $\{x_n\}_{n\geqslant 1}$ is bounded. As before, let us suppose that this is not true. Passing to a subsequence if necessary, we may assume that $\|x_n\| \to +\infty$. Let $y_n \overset{\mathrm{df}}{=} x_n/\|x_n\|$ for $n \geqslant 1$. Arguing as in the proof of proposition 3.1, we can show that $y_n \to \pm u_1$ in $W_0^{1,p}(Z)$ as $n \to +\infty$, hence $x_n(z) \to \pm\infty$ almost everywhere on Z as $n \to +\infty$. Using the Rayleigh quotient (see (2.1)), we obtain

$$-\int_Z j(z, x_n(z))\,\mathrm{d}z \leqslant \phi(x_n).$$

By proposition 4.2, for all $n \geqslant 1$ and almost all $z \in Z$, we have $|j(z, x_n(z))| \leqslant b(z)$, with $b \in L^1(Z)$, so we can use Fatou's lemma and obtain

$$\begin{aligned}
-\int_Z \hat{\jmath}_+(z)\,\mathrm{d}z &= -\int_Z \limsup_{n\to+\infty} j(z, x_n(z))\,\mathrm{d}z \\
&\leqslant -\limsup_{n\to+\infty} \int_Z j(z, x_n(z))\,\mathrm{d}z \\
&\leqslant \lim_{n\to+\infty} \phi(x_n),
\end{aligned}$$

so

$$-\int_Z \hat{\jmath}_+(z)\,\mathrm{d}z \leqslant c. \tag{4.4}$$

We have

$$\phi(x_n) = \frac{1}{p}\|\nabla x_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p - \int_Z j(z, x_n(z))\,\mathrm{d}z.$$

As in the proof of proposition 4.3, since $(1 + \|x_n\|)m(x_n) \to 0$ and using hypothesis $H(j)_4$ (vii), we have that

$$\frac{1}{p}\|\nabla x_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p \to 0 \quad \text{as } n \to +\infty.$$

So, by Fatou's lemma (note that proposition 4.2 allows its usage) and by hypothesis $H(j)_4$ (iv), we obtain

$$c = \lim_{n\to+\infty} \phi(x_n) \leqslant -\liminf_{n\to+\infty} \int_Z j(z, x_n(z))\,\mathrm{d}z \leqslant -\int_Z \hat{\jmath}_-(z)\,\mathrm{d}z \leqslant 0.$$

Also using (4.4), we have that

$$c \in \left[-\int_Z \hat{\jmath}_+(z)\,\mathrm{d}z, 0\right],$$

which contradicts our choice of c. So we have proved that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. Then, as in the proof of proposition 3.1, we produce a strongly convergent subsequence. □

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PROPOSITION 4.5. *If hypotheses $H(j)_3$ or $H(j)_4$ hold, then energy functional ϕ is bounded below.*

Proof. From the Rayleigh quotient (see (2.1)) and proposition 4.2, for all $x \in W_0^{1,p}(Z)$, we have

$$\begin{aligned}\phi(x) &= \frac{1}{p}\|\nabla x\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z j(z,x(z))\,\mathrm{d}z \\ &\geqslant -\int_Z |j(z,x(z))|\,\mathrm{d}z \geqslant -\int_Z |b(z)|\,\mathrm{d}z \\ &= -\|b\|_1,\end{aligned}$$

which shows that ϕ is indeed bounded below. □

Let Y_{u_1} be a topological complement of the one-dimensional eigenspace $\mathbb{R}u_1$ as in hypothesis $H(j)_{3,4}$ (vii), i.e. $W_0^{1,p}(Z) = \mathbb{R}u_1 \oplus Y_{u_1}$ (see [13, p. 502]).

PROPOSITION 4.6. *If hypotheses $H(j)_3$ or $H(j)_4$ hold, then $\phi|_{Y_{u_1}} \geqslant 0$.*

Proof. Let $y \in Y_{u_1}$. Using hypothesis $H(j)_{3,4}$ (vii) and the definition of $\lambda_{2,Y_{u_1}}$ (see (2.2)), for all $y \in Y_{u_1}$, we have

$$\begin{aligned}p\phi(y) &= \|\nabla y\|_p^p - \lambda_1\|y\|_p^p - p\int_Z j(z,y(z))\,\mathrm{d}z \\ &\geqslant \|\nabla y\|_p^p - \lambda_1\|y\|_p^p - \int_Z (\lambda_{2,Y_{u_1}} - \lambda_1)|y(z)|^p\,\mathrm{d}z \\ &= \|\nabla y\|_p^p - \lambda_{2,Y_{u_1}}\|y\|_p^p \geqslant 0,\end{aligned}$$

so, indeed, $\phi|_{Y_{u_1}} \geqslant 0$. □

One can see that proposition 3.4 is also valid under hypotheses $H(j)_{3,4}$ (v). Now we are ready to state and prove our multiplicity results.

THEOREM 4.7. *If hypotheses $H(j)_3$ hold, then problem (HVI) has at least three distinct non-trivial solutions in $W_0^{1,p}(Z)$.*

Proof. We introduce the open sets

$$U^{\pm} \overset{\mathrm{df}}{=} \{x \in W_0^{1,p}(Z) : x = \pm t u_1 + y,\ t > 0,\ y \in Y_{u_1}\}.$$

We will show that ϕ attains its infimum on both U^+ and U^-. To this end, let $\eta_+ \overset{\mathrm{df}}{=} \inf\{\phi(x) : x \in U^+\} = \inf\{\phi(x) : x \in \bar{U}^+\} < 0$ (since ϕ is locally Lipschitz and using hypothesis $H(j)_3$ (iv) and (vi), which says that $\phi(t_+u_1) < 0$ and $t_+u_1 \in U^+$). Let us set

$$\phi_0(x) \overset{\mathrm{df}}{=} \begin{cases} \phi(x) & \text{if } x \in \bar{U}^+, \\ +\infty & \text{if } x \in W_0^{1,p}(Z) \setminus \bar{U}^+. \end{cases}$$

Evidently, ϕ_0 is lower semicontinuous and bounded bellow (see proposition 4.5). So we can apply theorem 2.3 with $\varepsilon = 1/n$ for all $n \geqslant 1$ and generate a sequence $\{x_n\}_{n\geqslant 1} \subseteq U^+$ such that $\phi_0(x_n) = \phi(x_n) \searrow \eta_+ < 0$ and

$$\phi_0(x_n) \leqslant \phi_0(u) + \frac{(1/n)\|x_n - u\|}{1 + \|x_n\|} \quad \forall u \in W_0^{1,p}(Z),$$

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so

$$-\frac{(1/n)\|x_n-u\|}{1+\|x_n\|} \leqslant \phi_0(u)-\phi_0(x_n) \quad \forall u \in W_0^{1,p}(Z).$$

Let $u=x_n+tw$, with $t>0$ and $w \in W_0^{1,p}(Z)$. Because $x_n \in U^+$ and the latter is an open set, we can find $\delta>0$ such that $x_n+tw \in U^+$ for all $0 \leqslant t<\delta$. Thus we can write

$$-\frac{(1/n)\|w\|}{1+\|x_n\|} \leqslant \frac{\phi(x_n+tw)-\phi(x_n)}{t} \quad \forall 0<t<\delta \quad \forall w \in W_0^{1,p}(Z),$$

and so

$$-\frac{(1/n)\|w\|}{1+\|x_n\|} \leqslant \phi^0(x_n;w) \quad \forall w \in W_0^{1,p}(Z).$$

Let

$$\vartheta_n(w) \stackrel{\text{df}}{=} \frac{1+\|x_n\|}{1/n}\phi^0(x_n;w).$$

Then $\vartheta_n(\cdot)$ is a sublinear continuous function and $\vartheta_n(0)=0$. Moreover, $-\|w\| \leqslant \vartheta_n(w)$ for all $w \in W_0^{1,p}(Z)$. Thus we can apply lemma 1.3 of [21, p. 81] and obtain $y_n^* \in W^{-1,p'}(Z)$ with $\|y_n^*\|_* \leqslant 1$ and $\langle y_n^*, w\rangle \leqslant \vartheta_n(w)$ for all $w \in W_0^{1,p}(Z)$ and all $n \geqslant 1$. Set

$$x_n^* \stackrel{\text{df}}{=} \frac{(1/n)y_n^*}{1+\|x_n\|}.$$

We have $\langle x_n^*, w\rangle \leqslant \phi^0(x_n;w)$ for all $w \in W_0^{1,p}(Z)$ and so $x_n^* \in \partial\phi(x_n)$ for $n \geqslant 1$. Also,

$$(1+\|x_n\|)m(x_n) \leqslant (1+\|x_n\|)\|x_n^*\|_* \leqslant \frac{1}{n}\|y_n^*\|_* \leqslant \frac{1}{n} \to 0.$$

Note that, by virtue of hypothesis $H(j)_3$ (vi), we have that

$$\eta_+ < -\int_Z j_\pm(z)\,\mathrm{d}z,$$

and so we can apply proposition 4.3 and obtain that there exists $y_1 \in W_0^{1,p}(Z)$ such that, at least for a subsequence, we have $x_n \to y_1$ in $W_0^{1,p}(Z)$ as $n \to +\infty$. If $y_1 \in \partial U^+ = Y_{u_1}$, then $\eta_+ \geqslant 0$ (see proposition 4.6). But we know that $\eta_+ < 0$. So $y_1 \in U^+$ and y_1 is a local minimum of ϕ. Therefore, $0 \in \partial\phi(y_1)$. In a similar fashion, working with the set U^-, we obtain $y_2 \in U^-$ minimizing $\phi|_{U^-}$. Again, $0 \in \partial\phi(y_2)$ and clearly $y_1 \neq y_2$, $y_1 \neq 0$ and $y_2 \neq 0$.

By virtue of proposition 3.4, we can find $0<r<\min\{-t_-, t_+\}$ such that

$$\inf\{\phi(x) : \|x\|=r\} > 0 > \eta_\pm.$$

Since $\phi(0) \leqslant 0$, we can apply theorem 2.2 with $y=t_+u_1$ or $y=t_-u_1$ and obtain $y_3 \neq 0$ such that $\phi(y_3) \geqslant \inf\{\phi(x) : \|x\|=r\} > 0 > \eta_\pm$. Then $y_3 \neq y_1$ and $y_3 \neq y_2$.

Finally, since $0 \in \partial\phi(y_i)$, $i=1,2,3$, as before, we can check that y_1, y_2 and y_3 are three non-trivial solutions of (HVI). □

We can have the same multiplicity result if we assume hypotheses $H(j)_4$.

THEOREM 4.8. *If hypotheses $H(j)_4$ hold, then problem (HVI) has at least three distinct non-trivial solutions in $W_0^{1,p}(Z)$.*

Proof. The proof is identical to that of theorem 4.7, using this time proposition 4.4. Note that we have

$$\eta_{\pm} < -\int_Z \hat{\jmath}_+(z)\,\mathrm{d}z$$

(see hypothesis $H(j)_4$ (vi), and this permits the use of proposition 4.4). □

REMARK 4.9. These are the first multiplicity results for quasilinear hemivariational inequalities at resonance. In fact, to our knowledge, these are the first theorems that prove the existence of at least three non-trivial solutions for quasilinear equations involving the p-Laplacian and having strong resonance at infinity, even if the potential function is C^1. Moreover, if

$$j(z,\zeta) = \int_0^{\zeta} f(z,r)\,\mathrm{d}r,$$

with $f : Z \times \mathbb{R} \mapsto \mathbb{R}$ measurable, then our formulation incorporates problems with discontinuities, which were studied in the context of semilinear equations (i.e. for $p = 2$) by Chang [6].

Finally, a careful reading of § 4 reveals that the same multiplicity results are still valid if hypotheses $H(j)_{3,4}$ (iii) are replaced by:

(iii) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $\eta \in \partial j(z,\zeta)$, we have $|\eta| \leqslant a(z) + c|\zeta|^{\mu-1}$ with some $a \in L^{\infty}(Z)$, $c > 0$ and $0 < \mu < p$.

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(*Issued 19 October 2001*)