

Resonance oscillations of a drop (bubble) in a vibrating fluid

D. V. Lyubimov¹, T. P. Lyubimova^{1,2,†} and A. A. Cherepanov¹

¹Perm State University, 15, Bukireva St, Perm 614990, Russia

²Institute of Continuous Media Mechanics, Ural Branch of RAS, 1, Koroleva St, Perm 614013, Russia

(Received 15 May 2020; revised 2 October 2020; accepted 21 October 2020)

The paper deals with the resonance oscillations of a drop (bubble) surrounded by a fluid of different density in a container subjected to small amplitude vibrations in zero gravity conditions. The drop size is considered to be large in comparison with both the vibration amplitude and the thickness of viscous Stokes layers. The calculations for parametrically excited oscillations of the drop are carried out in the linear approximation, for inviscid and low viscous media, neglecting compressibility effects. The resonant oscillation is a doublet of neighbouring modes of eigen-oscillations of the drop, for which the sum of frequencies coincides with the frequency of the forced vibrations. This means that the basic state becomes unstable against quasi-periodic oscillations. The finite viscosity implies a finite threshold for the excitation of resonance. On the other hand, the viscosity plays a destabilizing role; at non-zero (even infinitesimal) viscosity the width of the instability frequency range turns out to be greater than in the case of inviscid fluids.

Key words: drops, parametric instability, capillary waves

1. Introduction

It is known that vibrations change considerably the behaviour of mechanical systems. Even in a case as simple as a mechanical pendulum, the action of vibration can lead to opposite effects: at some frequencies vertical oscillations of the suspension point can lead to the parametric excitation of the oscillations while, on the other hand, high frequency vibrations can result in the stabilization of equilibrium states, which are unstable in the absence of vibrations (Stephenson 1908; Kapitsa 1951; Landau & Lifshitz 1976).

In many situations, a hydrodynamic system in the absence of vibrations is capable of performing periodic motions and has a spectrum of eigenfrequencies. Examples of this kind are capillary–gravitational waves on the surface of a liquid or interface of liquids, free oscillations of a bubble suspended in a liquid matrix, etc. In the absence of external forces, due to viscous dissipation, free oscillations, as a rule, damp. The pumping of energy into the system due to vibrations can lead to resonant excitation of oscillations. Although the pioneering work of Faraday (1831), where the parametric resonance was first described, is devoted specifically to the vibrational excitation of capillary–gravitational waves and came up almost two centuries ago, the question of the vibrational excitation of resonant

† Email address for correspondence: lyubimova@psu.ru

oscillations in hydrodynamic systems still cannot be considered fully investigated. The present paper deals with the oscillations of an isolated drop or bubble in a vibrating fluid of different density. The study of drop (bubble) oscillations is very important not only from a theoretical point of view, but also for applications. The measurements of the frequencies and damping coefficients of oscillations of drops levitating in an acoustic or electromagnetic field and of pendant drops are used for the determination of the material properties of the media (Keene *et al.* 1986; Asaki, Thiessen & Marston 1995; Przyborowski *et al.* 1995; Tian, Holt & Apfel 1997; Lyubimov *et al.* 2011; Abi Chebel *et al.* 2019; Shao *et al.* 2020).

A drop of liquid (or a gas bubble) suspended in a liquid of a different density is an example of a hydrodynamic system with a spectrum of free oscillations. Rayleigh (1879) was the first to calculate the frequencies of free oscillations of non-viscous spherical liquid drop. Later on, Lamb (1881) generalized the formulas obtained by Rayleigh, taking into account the viscosity. The damping of the viscous drop oscillations was considered by Chandrasekhar (1959) and Reid (1960).

There are large number of works where the drop (bubble) oscillations are studied for acoustic fields, where compressibility plays an important role. Under the action of a modulated pressure field far from the bubble, the compressible gas in the bubble is able to change its volume, including with the preservation of a spherical shape, which is described by the well-known Rayleigh–Plesset equation (Rayleigh 1917; Plesset 1949). Marston & Apfel (1980) experimentally observed quadrupole oscillations of an air bubble and liquid drop in a liquid matrix, induced by two acoustic waves with close high frequencies. The frequency of bubble oscillations was found to be equal to the difference of frequencies of the imposed fields. In a theoretical paper (Marston 1980) it was shown that these oscillations are not parametric since there is no threshold for their excitation. The excitation of oscillations of a bubble in a liquid subjected to a monochromatic acoustic field was studied theoretically in Mei & Zhou (1991). It has been shown that, due to the interaction of the acoustic field with non-symmetric modes of the bubble eigen-oscillations, radially symmetric oscillations become unstable when the wave power exceeds some critical value.

In Leal (1992), the stability of a non-stationary state of a gaseous bubble surrounded by a liquid is considered. It is assumed that in the basic state the bubble centroid is quiescent and its radius is a function of time. For such symmetry of a system, in the linear stability problem the spherical harmonics do not interact, and for each of them we obtain an independent problem. For periodic time dependence of the bubble radius this problem is reduced to the Mathieu equation. A quite different situation should take place if in the basic state the bubble undergoes translational motions. In this case, the spherical harmonics, which we can expand the basic state perturbations in, should all be coupled to one another, even in the linear approximation. The study of this situation is one of the goals of the present work.

In another paper (Feng & Leal 1995) it is shown that the interaction of two adjacent modes of shape oscillations of a bubble can lead to a translational instability. The mechanism of this instability is essentially related to the bubble volume variations. The subject of the present study, in some sense, is opposite to one discussed in Feng & Leal (1995); we are interested in the energy transfer from forced translational oscillations to the connected adjacent modes of eigen-oscillations. As it will be shown below, this mechanism does not require variations of the bubble volume and can be observed in an incompressible fluid.

In experiments (Shen, Xie & Wei 2010), a parametric instability arising from modulation of the sound pressure was discovered that breaks the axial symmetry of a liquid drop suspended in air by the acoustic levitation method.

Large amplitude shape oscillations of drops and bubbles immersed in an immiscible liquid have been investigated in Trinh, Thiessen & Holt (1998) using the ultrasonic radiation pressure technique. The interaction between axisymmetric and non-axisymmetric $l = 3$ and $l = 2$ modes (l is the degree of the Legendre polynomial) has been documented; if large amplitude drop shape oscillations corresponding to $l = 3$ are excited then the $l = 2$ mode is also excited in a subharmonic way, moreover, the frequency of oscillations for the $l = 2$ mode is twice lower than that of the $l = 3$ mode. A similar behaviour has been found for the other resonant pairs too.

In most of the cited papers, a considerable role is played by the compressibility effects, since for an incompressible medium inside the bubble radially symmetric oscillations are impossible.

In the present paper we consider oscillations of a drop or bubble suspended in a fluid of differing density, in a container subjected to vibrations under zero gravity conditions. The study is performed neglecting the compressibility of media. Since a bubble (or a drop) surrounded by a fluid of differing density is an oscillatory system with a discrete spectrum of eigen-oscillations, one should expect the resonance excitation of oscillations when certain relations between the vibration frequency and eigenfrequencies of oscillations of the bubble (drop) are fulfilled.

2. Equations and boundary conditions

It is known (Faraday 1831) that vibrations of a container filled with a fluid or a system of fluids can lead to parametrically excited waves (Faraday ripple) at a free surface of the fluid or at the fluid interface. Similar phenomena can take place for a drop (bubble) suspended in a medium of differing density when such a system is subjected to vibrations. Since this system possesses eigenfrequencies, at certain ratios between them and the vibration frequency one should expect resonant phenomena. To study the conditions for parametric excitation of the drop oscillations, we consider the following problem.

Let a fluid of density ρ_1 fill a container, where a drop of different fluid (of density ρ_2) immiscible with the surrounding medium, is suspended. In the absence of gravity and other external fields, the drop, affected by the surface tension, assumes a spherical shape of radius R_0 . We suppose that the container size and the distance of the drop from its walls by far exceeds the drop size. Let the container undergo translational sinusoidal vibrations with frequency ω and amplitude a . In this case, the inertia forces arise in the reference frame of the container. If the densities of the drop and the surrounding medium are different, these forces are non-uniform, which sets both fluids in motion.

Let the vibration frequency be such that the corresponding sound wavelength is larger than the container size, and the quantity $a\omega$, which determines the order of magnitude of the velocities of the fluids, is small compared with the sound velocity. In this case, the fluids may be considered as incompressible. Additionally, at the first stage of calculations we will neglect the media viscosities, assuming the flow of the fluids to be potential.

In this case, the velocity potentials in both media φ_j satisfy the Laplace equations

$$\Delta\varphi_j = 0, \quad (2.1)$$

($j = 1, 2$; the subscript 1 refers to the outer fluid, and 2 to the inner one).

On the rigid walls of the container the impermeability condition is satisfied

$$\nabla\varphi_1 \cdot \mathbf{n} = 0, \tag{2.2}$$

where \mathbf{n} is the unit vector normal to the surface.

Since the drop size is assumed to be small as compared with the distance from the container walls, condition (2.2) may be replaced by the condition of vanishing fluid velocity far from the drop

$$\nabla\varphi_1 \rightarrow 0 \quad \text{at } r \rightarrow \infty, \tag{2.3}$$

where r is the length of the radius vector drawn from the inertial centre of the drop.

The conditions at the drop surface described by equation $G(\mathbf{r}, t) = 0$, which has to be determined itself, are the following:

- (i) The continuity condition for the normal components of fluid velocities

$$[\nabla\varphi] \cdot \mathbf{n} = 0. \tag{2.4}$$

- (ii) The kinematic condition connecting the displacement of the drop surface with the fluids velocities

$$\frac{\partial G}{\partial t} + \nabla\varphi \cdot \nabla G = 0. \tag{2.5}$$

- (iii) The condition of the normal stress balance accounting for the inertial forces and surface tension

$$\left[\rho \left(\frac{\partial\varphi}{\partial t} + \frac{1}{2} (\nabla\varphi)^2 + a\omega^2 z \cos \omega t \right) \right] = \alpha \operatorname{div} \mathbf{n} + \text{const}. \tag{2.6}$$

Here, the brackets denote the jump of the value across the interface, α is the surface tension coefficient and the z -axis is directed along the vibration axis. Equation (2.6) results from combining Bernoulli’s equation with the normal stress balance at the interface. The constant in this equation, generally speaking, can be a function of time, but not coordinates since it results from the integration of the Euler equations describing the flow in the absence of viscosity.

Let us introduce dimensionless variables. We choose the equilibrium radius of a drop as the length scale and the inverse frequency of vibrations ω^{-1} and ωR_0^2 as the scales for time and the velocity potential, respectively. For the densities of the fluids we take the scale $(\rho_1 + \rho_2)$.

Dimensionless equations (2.1) and the boundary conditions (2.3)–(2.5) have the same form as earlier and condition (2.6) takes the form

$$\left[\tilde{\rho} \left(\frac{\partial\varphi}{\partial t} + \frac{1}{2} (\nabla\varphi)^2 \right) \right] + [\tilde{\rho}] \tilde{a} z \cos t = \frac{1}{We} \operatorname{div} \mathbf{n} + \text{const}. \tag{2.7}$$

Here

$$\tilde{\rho}_j = \frac{\rho_j}{\rho_1 + \rho_2}, \quad \tilde{a} = \frac{a}{R_0} \tag{2.8a,b}$$

are the dimensionless densities of two media and the vibration amplitude and

$$We = \frac{(\rho_1 + \rho_2) \omega^2 R_0^3}{\alpha} \tag{2.9}$$

is the Weber number. Note that, for the scales chosen, the dimensionless densities satisfy the relation

$$\tilde{\rho}_1 + \tilde{\rho}_2 = 1. \tag{2.10}$$

Below we omit the tilde for the dimensionless density and vibration amplitude.

Let the amplitude of the imposed vibrations be so small that the inequality

$$a \ll 1 \tag{2.11}$$

holds.

In a real situation in the absence of vibrations the drop oscillations will be damped due to the dissipative effects. Periodic forcing (vibrations, as in our case), in principle, can lead to the resonance excitation of undamped oscillations. Such oscillations can also occur in the case of a small deviation of the vibration frequency from the exact resonance value. In our paper, we consider only small amplitudes of vibrations for which condition (2.11) is satisfied. In this case, the resonance oscillations will take place only at small dissipation of energy. That is why we will restrict ourselves to media with zero or very small viscosities.

Thus, in the problem under consideration arise two different, generally independent, small quantities: the vibration amplitude and the deviation of the frequency from the resonance value. In this situation it is suitable to introduce a formal small dimensionless parameter ε and to search for the solution of the problem in the form of a power series with respect to this parameter. Let us set

$$a = \varepsilon a_1, \tag{2.12}$$

$$We = We_0 + \varepsilon We_1. \tag{2.13}$$

In (2.13) the Weber number, which can be interpreted as the squared vibration frequency, is expanded into a power series with respect to ε instead of frequency itself. By We_0 we denote the resonance value of We which is to be determined from the solution of the problem).

Let us consider small deviations of a drop shape from equilibrium

$$r = 1 + \varepsilon f(\theta, t). \tag{2.14}$$

In (2.14), r and θ are the spherical coordinates, the polar angle θ is measured from the vibrations axis. Due to the symmetry of the problem, axisymmetrical flows are considered, and all the variables are assumed to be independent of the azimuthal angle.

As ε is small (and, consequently, so are the φ_j), the problem may be solved by expanding the variables into power series with respect to ε . Assuming that the φ_j are of the same order of magnitude as ε , we write down the potential in the form $\varphi_j = \varepsilon \tilde{\varphi}_j$, where $\tilde{\varphi}_j$ is a finite quantity.

The boundary conditions (2.4), (2.5), (2.7) are to be imposed at the drop surface, i.e. at $r = 1 + \varepsilon f$. Expanding these conditions in Taylor series with respect to the deviation of the drop shape from the spherical one, one obtains the relations to be satisfied at $r = 1$. For example, condition (2.4), written in a detailed form, is

$$r = 1 + \varepsilon f : \left[\varepsilon \tilde{\varphi}_r - \frac{\varepsilon^2}{r^2} f_{\vartheta} \tilde{\varphi}_{\vartheta} \right] = 0, \tag{2.15}$$

since for the axisymmetric case and small deviations of the drop shape from the spherical one we have

$$\nabla \tilde{\varphi} = \tilde{\varphi}_r \mathbf{e}_r + \frac{1}{r} \tilde{\varphi}_{\vartheta} \mathbf{e}_{\vartheta}, \quad \mathbf{n} = \mathbf{e}_r - \frac{1}{r} f_{\vartheta} \mathbf{e}_{\vartheta}. \tag{2.16a,b}$$

By changing

$$\tilde{\varphi}_r|_{r=1+\varepsilon f} = \tilde{\varphi}_r|_{r=1} + \varepsilon f \tilde{\varphi}_{rr}|_{r=1} + O(\varepsilon^2), \tag{2.17}$$

we obtain

$$r = 1 : \quad \varepsilon [\tilde{\varphi}_r] + \varepsilon^2 [f \tilde{\varphi}_{rr} - f_\theta \tilde{\varphi}_\theta] + O(\varepsilon^3) = 0, \tag{2.18}$$

or, dividing by ε , and omitting the tilde for the potential

$$\varphi_{1r} + \varepsilon(\varphi_{1rr}f - \varphi_{1\theta}f_\theta) = \varphi_{2r} + \varepsilon(\varphi_{2rr}f - \varphi_{2\theta}f_\theta). \tag{2.19}$$

Applying the same operations to the boundary conditions (2.5), (2.7) we obtain

$$f_t = \varphi_{2r} + \varepsilon(\varphi_{2rr}f - \varphi_{2\theta}f_\theta), \tag{2.20}$$

$$\begin{aligned} & -\rho_2 \left[\varepsilon \varphi_{2t} + \varepsilon^2 \left(\varphi_{2rr}f + \frac{1}{2} \varphi_{2r}^2 + \frac{1}{2} \varphi_{2\theta}^2 \right) \right] \\ & + \rho_1 \left[\varepsilon \varphi_{1t} + \varepsilon^2 \left(\varphi_{1rr}f + \frac{1}{2} \varphi_{1r}^2 + \frac{1}{2} \varphi_{1\theta}^2 \right) \right] \\ & = \frac{1}{We} \left[2 - \varepsilon(2f + f_{\theta\theta} + f_\theta \cot \theta) + 2\varepsilon^2 f(f + f_{\theta\theta} + f_\theta \cot \theta) \right] \\ & + (\rho_2 - \rho_1)\varepsilon a_1 \cos \theta \cos t + (\rho_2 - \rho_1)\varepsilon^2 a_1 f \cos \theta \cos t + \text{const.} \end{aligned} \tag{2.21}$$

Letter subscripts in (2.20)–(2.21) stand for differentiation with respect to the corresponding variables; the curvature of the surface $\text{div } \mathbf{n}$ is calculated with an accuracy of up to ε^2 .

The summary of the assumptions used in our theory in the inviscid case is the following:

$$\omega \ll 2\pi c/R_0, \quad a \ll R_0, \quad We - We_0 \ll 1. \tag{2.22a-c}$$

3. Forced oscillations of the drop

At zeroth order, the constant in (2.21) is evaluated, and to the next order we obtain the following problem for the determination of the fluid velocities and the drop shape (the corresponding variables of the linear problem are capitalized):

$$\Delta \Phi_j = 0, \tag{3.1}$$

at $r \rightarrow \infty$

$$\nabla \Phi_1 = 0, \tag{3.2}$$

and at $r = 1$

$$F_t = \Phi_{2r}, \tag{3.3}$$

$$\Phi_{1r} = \Phi_{2r}, \tag{3.4}$$

$$-\rho_2 \Phi_{2t} + \rho_1 \Phi_{1t} - (\rho_2 - \rho_1)a_1 \cos \theta \cos t = -\frac{1}{We} (2F + F_{\theta\theta} + F_\theta \cot \theta). \tag{3.5}$$

Here, now Φ is the leading part of the velocity potential.

The solution of the linear problem (3.1)–(3.5) is

$$F = \frac{2}{3}\mu \cos \theta \cos t, \tag{3.6}$$

$$\Phi_1 = \frac{1}{3r^2}\mu \cos \theta \sin t, \tag{3.7}$$

$$\Phi_2 = -\frac{2}{3}\mu r \cos \theta \sin t, \tag{3.8}$$

where the following notation is introduced

$$\mu = \frac{3(\rho_1 - \rho_2)}{2\rho_2 + \rho_1} a_1. \tag{3.9}$$

Thus, to the first order with respect to ε we obtain the solution which corresponds to the translational oscillation of the drop with the amplitude $2\mu\varepsilon/3$ and the frequency equal to that of the container oscillations. Moreover, the shape of the drop does not change. The amplitude of these oscillations becomes zero at $\rho_1 = \rho_2$, since in this case the inertial forces are uniform. The phase of oscillations of the drop is determined by the density difference: the oscillations of a denser drop occur in the counterphase relative to the container oscillations, the phase of oscillations of a lighter drop is the same as that of the container.

4. Stability analysis

To analyse the stability of the obtained solution, it is convenient to transit to the reference framework moving with the drop. In this framework, the basic solution (3.6)–(3.8) has the form

$$F = 0, \tag{4.1}$$

$$\Phi_2 = 0, \tag{4.2}$$

$$\Phi_1 = \frac{2}{3}\mu \left(r + \frac{1}{2r^2} \right) \cos \theta \sin t. \tag{4.3}$$

The problem for the perturbations of the basic solution (4.1)–(4.3) is obtained from (2.1)–(2.3) and (2.20)–(2.21) in the linear approximation, taking into account the modification of the inertial forces with the transition to the new reference frame. The perturbations of the velocity potentials satisfy the Laplace equations, they vanish far from the drop (for the outer medium) and satisfy the following conditions at the drop surface, $r = 1$:

$$f_t = \varphi_{2r}, \tag{4.4}$$

$$\varphi_{2r} - \varphi_{1r} = 2\mu \sin t \left(f \cos \theta + \frac{1}{2}f_\theta \sin \theta \right), \tag{4.5}$$

$$\rho_1\varphi_{1t} - \rho_2\varphi_{2t} + \frac{1}{We} (2f + f_{\theta\theta} + f_\theta \cot \theta) = \mu\rho_1(\varphi_{1\theta} \sin \theta \sin t + f \cos \theta \cos t). \tag{4.6}$$

For the perturbations we use the same notations as for the functions themselves.

The solution to the Laplace equation that vanishes at infinity and does not have any singularities at the drop centre, has the form

$$\varphi_1 = \sum \frac{A_k(t)}{r^{k+1}} P_k(\cos \theta), \tag{4.7}$$

$$\varphi_2 = \sum B_k(t) r^k P_k(\cos \theta). \tag{4.8}$$

At the same time, the following equality holds

$$f = \sum f_k P_k(\cos \theta), \tag{4.9}$$

where P_k are Legendre polynomials of k th order.

Substituting (4.7)–(4.9) into (4.4)–(4.6), at the zeroth order with respect to μ we arrive at the known problem (Landau & Lifshitz 1987) of the eigen-oscillations of the drop. In this case, the spectrum of eigenfrequencies is

$$\Omega_k^2 = \frac{1}{We} \frac{k(k^2 - 1)(k + 2)}{\rho_1 k + \rho_2(k + 1)}, \tag{4.10}$$

where $k \geq 2$ since the case $k = 0$ is impossible because of incompressibility of the fluids, and $k = 1$ corresponds to a uniform displacement of the drop, and not to oscillations.

Analysis of (4.4)–(4.6) at the next order shows that, in principle, the resonant excitation of harmonics is possible under the following synchronism condition:

$$\Omega_{k+1} + \Omega_k = 1. \tag{4.11}$$

Let the value of the Weber number be in such a range that the synchronism condition (4.11) is satisfied for some pair of modes. Then, all the harmonics of the solution (4.7)–(4.9), except for the k th and $(k + 1)$ th, are non-resonant and, consequently, to find the resonance conditions it is sufficient to restrict ourselves to just this pair of the harmonics of (4.7)–(4.9), assuming that

$$f = f_k P_k + f_{k+1} P_{k+1}, \tag{4.12}$$

$$\varphi_1 = \frac{A_k}{r^{k+1}} P_k + \frac{A_{k+1}}{r^{k+2}} P_{k+1}, \tag{4.13}$$

$$\varphi_2 = B_k r^k P_k + B_{k+1} r^{k+1} P_{k+1}. \tag{4.14}$$

Here, the argument in Legendre polynomials is $\cos \theta$; f_k , A_k and B_k are the functions of time.

Substitution of (4.12)–(4.14) into (4.4)–(4.6) after omitting non-resonant harmonics yields the set of ordinary differential equations

$$\dot{f}_k = k B_k, \tag{4.15}$$

$$\dot{f}_{k+1} = (k + 1) B_{k+1}, \tag{4.16}$$

$$kB_k + (k + 1)A_k = -\mu \frac{k(k + 1)}{2k + 3} f_{k+1} \sin t, \tag{4.17}$$

$$(k + 1)B_{k+1} + (k + 2)A_{k+1} = \mu \frac{(k + 1)(k + 2)}{2k + 1} f_k \sin t, \tag{4.18}$$

$$\rho_1 \dot{A}_k - \rho_2 \dot{B}_k - \frac{\beta_k}{We} f_k = \rho_1 \mu \frac{k + 1}{2k + 3} [-(k + 2)A_{k+1} \sin t + f_{k+1} \cos t], \tag{4.19}$$

$$\rho_1 \dot{A}_{k+1} - \rho_2 \dot{B}_{k+1} - \frac{\beta_{k+1}}{We} f_{k+1} = \rho_1 \mu \frac{k + 1}{2k + 1} [kA_k \sin t + f_k \cos t]. \tag{4.20}$$

Here, overdots stand for differentiation of functions with respect to time and we denote

$$\beta_k = (k - 1)(k + 2). \tag{4.21}$$

The set (4.15)–(4.20) can be reduced to two equations for f_k and f_{k+1}

$$-\alpha_k \ddot{f}_k - \frac{\beta_k}{We} f_k = \rho_1 \mu \frac{2k + 1}{2k + 3} (\dot{f}_{k+1} \sin t + f_{k+1} \cos t), \tag{4.22}$$

$$-\alpha_{k+1} \ddot{f}_{k+1} - \frac{\beta_{k+1}}{We} f_{k+1} = -\rho_1 \mu \dot{f}_k \sin t, \tag{4.23}$$

where

$$\alpha_k = \frac{\rho_2}{k} + \frac{\rho_1}{k + 1}. \tag{4.24}$$

To study (4.22)–(4.23), it is convenient to use the multiple scale method (Nayfeh 1981). In accordance with it, we introduce a hierarchy of time scales, so that the time derivative is represented as the series

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \dots \tag{4.25}$$

The functions f_k and f_{k+1} are also presented in series form

$$f_k = f_k^{(0)} + \varepsilon f_k^{(1)} + \dots, \tag{4.26}$$

$$f_{k+1} = f_{k+1}^{(0)} + \varepsilon f_{k+1}^{(1)} + \dots \tag{4.27}$$

Substitution of series (4.25)–(4.27) into the set of equations (4.22)–(4.23) yields, to the leading order,

$$\ddot{f}_k^{(0)} + \Omega_k^2 f_k^{(0)} = 0, \tag{4.28}$$

$$\ddot{f}_{k+1}^{(0)} + \Omega_{k+1}^2 f_{k+1}^{(0)} = 0, \tag{4.29}$$

i.e. the solutions are independent oscillations with the frequencies Ω_k and Ω_{k+1} , defined by (4.10). Thus,

$$f_k^{(0)} = C \exp(i\Omega_k t_0) + C^* \exp(-i\Omega_k t_0), \tag{4.30}$$

$$f_{k+1}^{(0)} = D \exp(i\Omega_{k+1} t_0) + D^* \exp(-i\Omega_{k+1} t_0), \tag{4.31}$$

where ‘*’ stands for complex conjugation, the amplitudes C and D do not depend on the ‘fast’ time t_0 , but, generally speaking, are functions of ‘slow’ time t_1 .

To the next order, (4.22)–(4.23) with allowance for (4.25)–(4.27) yield

$$\begin{aligned}
 & -\alpha_k \frac{\partial^2 f_k^{(1)}}{\partial t_0^2} - 2\alpha_k \frac{\partial^2 f_k^{(0)}}{\partial t_0 \partial t_1} - \frac{\beta_k}{We_0} f_k^{(1)} + \frac{\beta_k}{We_0^2} We_1 f_k^{(0)} \\
 & = \rho_1 \mu \frac{2k+1}{2k+3} \left(\frac{\partial f_{k+1}^{(0)}}{\partial t_0} \sin t_0 + f_{k+1}^{(0)} \cos t_0 \right), \tag{4.32}
 \end{aligned}$$

$$-\alpha_{k+1} \frac{\partial^2 f_{k+1}^{(1)}}{\partial t_0^2} - 2\alpha_{k+1} \frac{\partial^2 f_{k+1}^{(0)}}{\partial t_0 \partial t_1} - \frac{\beta_{k+1}}{We_0} f_{k+1}^{(1)} + \frac{\beta_{k+1}}{We_0^2} We_1 f_{k+1}^{(0)} = -\rho_1 \mu \frac{\partial f_k^{(0)}}{\partial t_0} \sin t_0. \tag{4.33}$$

Here, We_0 is the value of the Weber number at which the synchronism condition (4.11) is satisfied and εWe_1 is a small deviation from this value.

If condition (4.11) is satisfied, the non-uniformities of equations (4.32)–(4.33), which determine f_k and f_{k+1} , contain the resonance terms. The requirement of the absence of secular solutions to this set determines the evolution of C and D as

$$-2i\alpha_k \Omega_k \frac{\partial C}{\partial t_1} + \frac{\beta_k}{We_0^2} We_1 C = \frac{1}{2} \rho_1 \mu \frac{2k+1}{2k+3} (1 - \Omega_{k+1}) D^*, \tag{4.34}$$

$$2i\alpha_{k+1} \Omega_{k+1} \frac{\partial D^*}{\partial t_1} + \frac{\beta_{k+1}}{We_0^2} We_1 D^* = \frac{1}{2} \rho_1 \mu \Omega_k C. \tag{4.35}$$

Since the set (4.34)–(4.35) is linear, it admits a solution proportional to $\exp(\kappa t_1)$. Substituting this solution into (4.34)–(4.35), one gets the set of linear algebraic equations for the corresponding amplitudes. These solutions are non-zero if the determinant equals zero. This condition yields the equation for the increment κ

$$\begin{aligned}
 & 4\alpha_k \alpha_{k+1} \Omega_k \Omega_{k+1} \kappa^2 + 2i \frac{We_1}{We_0^2} \kappa (\alpha_{k+1} \Omega_{k+1} \beta_k - \alpha_k \Omega_k \beta_{k+1}) \\
 & + \beta_k \beta_{k+1} \frac{We_1^2}{We_0^4} - \frac{\mu^2}{4} \frac{2k+1}{2k+3} \Omega_k^2 \rho_1^2 = 0. \tag{4.36}
 \end{aligned}$$

In the parameter range where the increment κ is positive (instability area) the solutions of (4.34)–(4.35) are exponentially growing and expansions (4.26), (4.27) become invalid at large time. From algebraic equation (4.36) it follows that the solutions of equations (4.34)–(4.35) are growing if the discriminant of (4.36) is positive, i.e. if

$$\rho_1^2 \mu^2 \frac{2k+1}{2k+3} \Omega_k^2 \alpha_k \alpha_{k+1} \Omega_k \Omega_{k+1} > \frac{We_1^2}{We_0^4} (\alpha_k \Omega_k \beta_{k+1} + \alpha_{k+1} \Omega_{k+1} \beta_k)^2. \tag{4.37}$$

5. Discussion of the results for the inviscid case

Thus, the instability range in the coordinates a_1 , We_1 is limited by the straight lines

$$a_1 = \pm q_k We_1, \quad q_k = \frac{2\rho_2 + \rho_1}{3\rho_1 |\rho_2 - \rho_1|} \frac{1}{\Omega_k We_0^2} \sqrt{\frac{2k+3}{2k+1} (\alpha_{k+1} \Omega_{k+1} \beta_k + \alpha_k \Omega_k \beta_{k+1})}. \tag{5.1}$$

Each pair of neighbouring modes is related to its own instability region, since the values We_0 , which satisfy (4.11), and the coefficient q_k depend on the number of interacting pairs.

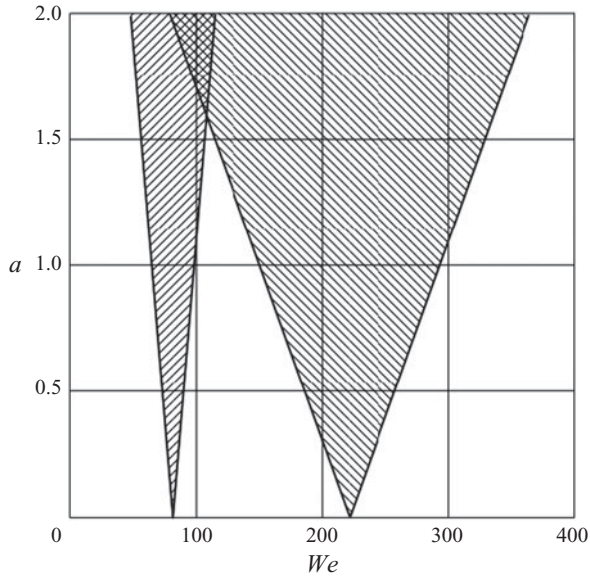


FIGURE 1. First two resonance zones ($k = 2$ and $k = 3$) for the case of inviscid media with $\rho_1 = 0.55$, $\rho_2 = 0.45$.

In figure 1 the two first instability regions are shown for $\rho_1 = 0.55$ and $\rho_2 = 0.45$, which correspond to a kerosene drop in water. The resonance $\Omega_2 + \Omega_3 = 1$ is related to $We_0 = 81.5$ and the next resonance to $We_0 = 221.8$.

Although in the considered problem one cannot derive directly the Mathieu-type equation for the perturbations, the stability map, typical for the parametric resonance, shows that the specific parametric resonance takes place. Namely, the frequency of the forcing is decomposed into two eigenfrequencies. However, these frequencies are not identical, as for the Mathieu equation. Here, they are different and correspond to the neighbouring modes of eigen-oscillations. This situation is encountered when parametric oscillations of coupled systems are studied (see Schmidt 1975).

The slope coefficient q_k of the straight lines bounding the resonance region is a function of the densities of the drop and the surrounding medium. In figure 2 the dependence of q_k on ρ_1 is presented for the first resonant doublet ($k = 2, k + 1 = 3$). Recall that, in the adopted notations, once determined, ρ_1 quite simply determines $\rho_2 = 1 - \rho_1$. As one can see, the resonance is impossible for $\rho_1 = 0.5$, since in this case of fluids of equal densities the inertial forces are uniform, and the drop does not displace relative to the surrounding medium under vibrations of the container. The resonance is also impossible for $\rho_1 = 0$, that is, when the density of the surrounding medium is negligible as compared with the drop density. At $\rho_1 < 0.5$ (a heavy drop in a light matrix) the dependence $q_k(\rho_1)$ is non-monotonic. For the resonance under consideration, there exists a minimum close to $\rho_1 \approx 0.29$, where the width of the resonance region is maximal, since the straight lines, bounding the instability region, have the smallest slope with respect to the axis We_1 . At $\rho_1 > 0.5$ the value of q_k monotonically decreases as ρ_1 grows, and the resonance region is wider the denser the surrounding medium as compared with the drop density.

If one considers non-axisymmetric perturbations with azimuthal number m , then the results do not change qualitatively. As known (Landau & Lifshitz 1987), the spectrum of eigenfrequencies of the drop oscillations (4.10) is degenerate and does not depend

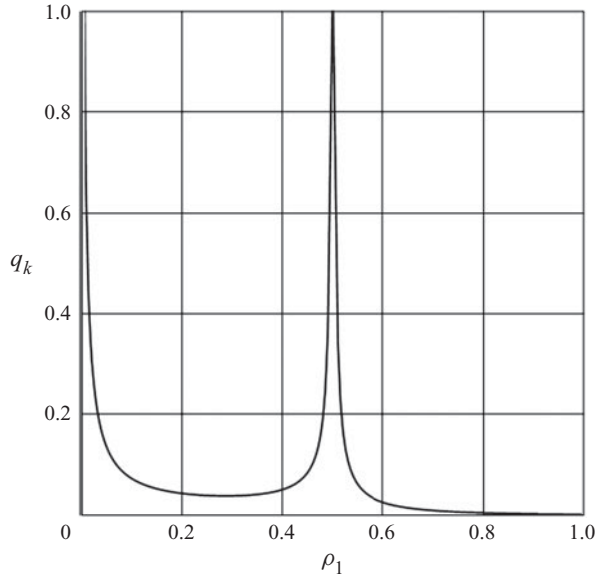


FIGURE 2. Value of q_k versus ρ_1 for the first resonance zone.

on the value of m . Calculations show that, when all the other conditions are the same, the resonance region for perturbations with $m \neq 0$ is always narrower than that for the axisymmetric ones and it is always located inside the domain bounded by the straight lines (5.1). This means that axisymmetric oscillations are always the most dangerous ones.

6. Accounting for small viscosity

The threshold of excitation of parametric oscillations in figure 1 equals zero at $We_1 = 0$, i.e. when the resonance condition (4.11) is satisfied identically. The absence of a finite threshold for parametric excitation is due to the inviscid media model.

For fluids of low viscosity ν_j , when the condition

$$\frac{\nu_j}{\omega R_0^2} \ll 1 \tag{6.1}$$

holds, one can account for the viscous dissipation with the aid of the following scheme. Since we assume that inequality (6.1) is satisfied, then the viscous corrections to the eigenfrequency for the spherical k th mode (for comparable densities of media) to the leading order have the form (Chandrasekhar 1959)

$$\omega_{1k} = \varepsilon(i - 1)\gamma_k, \tag{6.2}$$

where

$$\gamma_k = \frac{1}{2} \sqrt{\frac{\Omega_k}{2}} (2k + 1)^2 \frac{W e^{-1/4}}{\alpha_k} \frac{\rho_1 \rho_2 \sqrt{\nu_1 \nu_2}}{\rho_1 \sqrt{\nu_1} + \rho_2 \sqrt{\nu_2}}. \tag{6.3}$$

Here, ν_j denote dimensionless analogues of the viscosity coefficients scaled with $\sqrt{\alpha R_0 / (\rho_1 + \rho_2)}$.

We remark that the order of magnitude for the viscosity coefficients is explicitly predetermined. It follows from (6.2) that, in the approximation applied, one should take the viscosity as the small quantity of the second order.

As seen from (6.3), the correction depends on the mode number, and its presence, since ω_{1k} is complex, leads not only to the damping of eigen-oscillations but also to the viscous shift of their frequencies.

Since we consider low viscosity fluids, then γ_k is a small quantity and to account for the viscous dissipation it suffices to introduce this quantity in (4.34)–(4.35) just replacing

$$\frac{\partial}{\partial t_1} \rightarrow \frac{\partial}{\partial t_1} - i(i - 1)\gamma_k. \tag{6.4}$$

Such a way of accounting for the weak dissipation is certainly approximate and has to be looked at as a phenomenological one. It does not allow for nonlinear dissipative effects and requires an additional assumption regarding the order of the damping coefficient smallness. However, a quite similar approach is traditionally used in the studies of waves at the free surface or interface, particularly, including studies of parametrically excited oscillations (see, for example, Miles & Henderson 1990; Edwards & Fauve 1994; Kumar & Tuckerman 1994; Christiansen, Alstrom & Levinsen 1995; Lyubimova *et al.* 2019) and often allows one to obtain qualitatively good results. In Kumar & Tuckerman (1994) the instability threshold and critical wavelength obtained using a phenomenological approach and by numerical simulation based on the full Navier–Stokes equations (FHS) are compared. It is found that the wavelengths predicted by the two approaches do not differ significantly for low viscosity. The authors also compare the results of the phenomenological and FHS approaches with the experimental results obtained for a viscous glycerine–water mixture in contact with air in Edwards & Fauve (1993). It is found that both the phenomenological and FHS approaches agree reasonably well with the experimentally measured wavelengths. In Lyubimova *et al.* (2019), the Faraday waves on a band pattern formed in two-phase systems near the critical point under zero gravity conditions were studied. Comparison of the instability threshold values obtained analytically using a phenomenological approach, numerically by direct numerical simulation based on the Navier–Stokes equations and experimentally has shown good agreement.

Further calculations for the region of parametric resonance are carried out according to the above-described scheme. The neutral curve obtained in this way has the following form in the coordinates a_1, We_1 :

$$\begin{aligned} \gamma_k \gamma_{k+1} & \left[\frac{(\alpha_k \Omega_k \beta_{k+1} + \alpha_{k+1} \Omega_{k+1} \beta_k)^2 We_1^2}{\alpha_k \alpha_{k+1} \Omega_k \Omega_{k+1} (\gamma_k + \gamma_{k+1})^2 We_0^4} + \frac{4(\alpha_k \Omega_k \beta_{k+1} + \alpha_{k+1} \Omega_{k+1} \beta_k) We_1}{\gamma_k + \gamma_{k+1} We_0^2} \right] \\ & + 8\gamma_k \gamma_{k+1} \alpha_k \alpha_{k+1} \Omega_k \Omega_{k+1} - \frac{9\rho_1^2 (\rho_2 - \rho_1)^2}{4(2\rho_2 + \rho_1)^2} a_1^2 \frac{2k + 1}{2k + 3} \Omega_k^2 = 0. \end{aligned} \tag{6.5}$$

Viscous dissipation, if taken into account, leads to two different effects. First, the parametric excitation for the resonance becomes that of a finite amplitude type so that the oscillations arise (in the minimum of the neutral curve) when a_1 exceeds some critical value

$$a^* = \frac{4(2\rho_2 + \rho_1)}{3\rho_1 \Omega_k |\rho_2 - \rho_1|} \sqrt{\frac{2k + 3}{2k + 1} \gamma_k \gamma_{k+1} \alpha_k \alpha_{k+1} \Omega_k \Omega_{k+1}}. \tag{6.6}$$

Second, a viscous shift of the frequency from the value determined by (4.11) takes place. The minimum of the neutral curve occurs not at $We_1 = 0$ but when it assumes the value

$$We_{1*} = -\frac{2We_0^2(\gamma_k + \gamma_{k+1})\alpha_k\alpha_{k+1}\Omega_k\Omega_{k+1}}{\alpha_k\Omega_k\beta_{k+1} + \alpha_{k+1}\Omega_{k+1}\beta_k}. \tag{6.7}$$

The threshold value of the vibration amplitude (6.6) decreases as the number k grows. However, the threshold value of the amplitude of the vibration velocity, which is equal the product $a\omega$ and experimentally characterizes the power loss at the vibrating plate, increases with k . This means that the most easily excited resonance is the one where the second and third modes of eigen-oscillations interact.

Formula (6.5) has one important feature. When the viscosity tends to zero, it does not turn into (5.1), yielding instead

$$a_1 = \pm \Gamma_k q_k We_1, \tag{6.8}$$

where

$$\Gamma_k^2 = \frac{4\alpha_k\alpha_{k+1}\sqrt{\Omega_k\Omega_{k+1}}(2k+1)^2(2k+3)^2}{[\alpha_{k+1}\sqrt{\Omega_k}(2k+1)^2 + \alpha_k\sqrt{\Omega_{k+1}}(2k+3)^2]^2}. \tag{6.9}$$

The value Γ_k is always less than unity, therefore the straight lines (6.8) are inclined to the We_1 -axis more than those given by (5.1). Thus, the presence of an infinitesimal dissipation considerably changes the resonance properties as compared with the case $\nu_1 = \nu_2 = 0$. Cases of such paradoxical behaviour are known in the mechanics of elastic discrete systems (Schmidt 1975). However, the situation is different in the case of Faraday ripple at a flat interface and for the Mathieu equation, which makes this result paradoxical. In the next subsection we show that, in reality, the increase of the width of the instability zone including a small viscosity is a general situation whereas Faraday ripple is a particular case where the destabilizing effect is reduced due to the symmetry properties of the system.

In figure 3 the stability map is presented for the second and third resonant modes for the dimensionless densities $\rho_1 = 0.555$, $\rho_2 = 0.445$. Dotted lines correspond to $\nu_1 = \nu_2 = 0$ (that is the same resonance as in figure 1), dashed lines correspond to $\nu_1, \nu_2 \rightarrow 0$. Solid line is plotted for a kerosene drop of the radius 1 cm in water. As one can see, at small deviations of We from the resonance value (that is, at small εWe_1), viscosity stabilizes the equilibrium by increasing the excitation threshold as compared with the case of inviscid media. However, at further deviation of We from the resonance value, the solid line crosses the dashed and dotted lines. This means that, in these regions, the viscosity plays a destabilizing role.

If the density of the drop is very small as compared with the density of the surrounding medium (a gaseous bubble in a liquid matrix), one should take a different value for the damping coefficient. In this case, instead of (6.2)–(6.3) we take

$$\omega_{1k} = i\gamma_k, \quad \gamma_k = \nu_1(2k+1)(k+2). \tag{6.10}$$

Then we obtain

$$a^{*2} = \frac{16\Omega_{k+1}\nu_1^2(k+3)(2k+1)(2k+3)(3k+5)}{9\Omega_k(k+1)(3k+2)}. \tag{6.11}$$

As one can see, unlike the case of fluids with comparable densities, the resonance excitation threshold is now proportional to the viscosity and not to its square root as in (6.6). That is, in the low viscosity approximation, oscillations of a gas bubble are induced

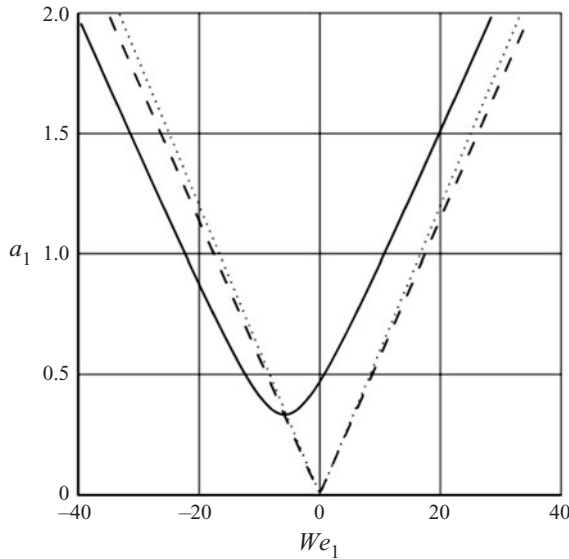


FIGURE 3. First resonance zone ($k = 2$) for different dimensionless viscosities: dotted lines - $\nu_1 = 0$, $\nu_2 = 0$; dashed lines - $\nu_1 = 10^{-6}$, $\nu_2 = 10^{-6}$; solid lines - $\nu_1 = 0.00190$, $\nu_2 = 0.00133$.

by the container oscillations with smaller amplitude than the oscillations of a heavy drop. There is no viscous shift of frequency in this case.

Thus, the vibrations of a container lead to excitation of a specific parametric resonance owing to the coupling of the neighbouring modes of the eigen-oscillations of the drop suspended in a fluid of different density. The threshold value of the vibration velocity amplitude for this resonance is determined by the viscous dissipation and grows with the increase of the number of resonating modes and, consequently, with the increase of the frequency of the container oscillations.

7. On the paradoxical influence of the viscosity on parametric instability

Unlike a usual resonance of forced oscillations, for the parametric resonance the transfer of energy from an external source does not occur directly but is due to the control of the interaction of different modes of eigen-oscillations. In this case, for weak interaction, the synchronism conditions should be satisfied. If there are two modes of eigen-oscillation with frequencies ω_1 and ω_2 interacting with the external source with frequency Ω , then

$$\Omega = \omega_1 + \omega_2. \quad (7.1)$$

A deviation from exact synchronism does not prevent the resonance, but requires a finite amplitude of forcing, which could provide synchronization of oscillations. For two modes with frequencies only slightly different from those satisfying condition (7.1), in the absence of forcing and dissipation, we can derive equations for a slow evolution of amplitudes

$$\dot{X} + i\delta_1 X = 0, \quad (7.2)$$

$$\dot{Y} + i\delta_2 Y = 0, \quad (7.3)$$

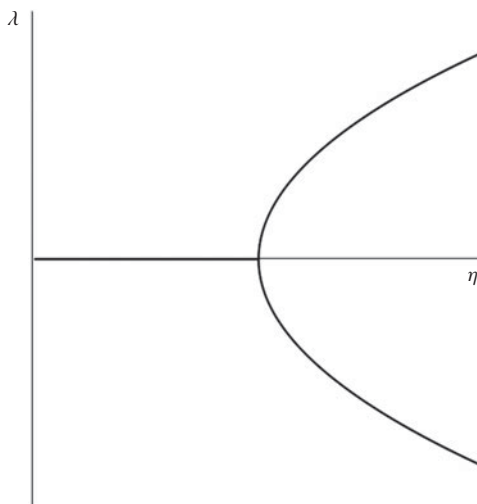


FIGURE 4. Dependence of the real part of growth rate on the parameter of forcing $\eta = \eta_1 \eta_2$.

where δ_1 and δ_2 are frequency mismatch parameters. In the case of interaction with an external energy source, the equations are modified

$$\dot{X} + i\delta_1 X + \eta_1 Y = 0, \tag{7.4}$$

$$\dot{Y} + i\delta_2 Y + \eta_2 X = 0. \tag{7.5}$$

Here, η_1 and η_2 are parameters proportional to the amplitude of forcing. Solutions to equation (7.4) are proportional to $\exp(\lambda t)$ where the growth rate λ is determined by the characteristic equation

$$\lambda^2 + i(\delta_1 + \delta_2)\lambda - \delta_1 \delta_2 - \eta_1 \eta_2 = 0. \tag{7.6}$$

Dependence of the real part of growth rate on the parameter of forcing $\eta = \eta_1 \eta_2$ is plotted in figure 4.

Critical (threshold) amplitude for excitation η_* differs from zero for different mismatches of two interacting modes

$$\eta_* = \frac{1}{4}(\delta_1 - \delta_2)^2. \tag{7.7}$$

If $\eta < \eta_*$, the growth rate is purely imaginary, which corresponds to the absence of synchronization.

To account for dissipation, one needs to add dissipative terms into (7.4)

$$\dot{X} + i\delta_1 X + \sigma_1 X + \eta_1 Y = 0, \tag{7.8}$$

$$\dot{Y} + i\delta_2 Y + \sigma_2 Y + \eta_2 X = 0. \tag{7.9}$$

Growing solutions of (7.8)–(7.9) appear when the parameter η reaches the threshold value η_d

$$\eta_d = \sigma_1 \sigma_2 \left[1 + \left(\frac{\delta_1 - \delta_2}{\sigma_1 + \sigma_2} \right)^2 \right]. \tag{7.10}$$

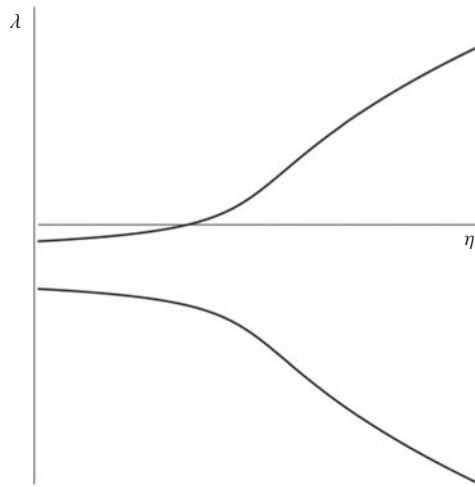


FIGURE 5. Typical dependence of the real part of the growth rate on η .

For $\sigma_1 = \sigma_2 = \sigma$ we obtain from (7.10) $\eta = \eta_e$, where

$$\eta_e = \frac{1}{4} (\delta_1 - \delta_2)^2 + \sigma^2 = \eta_* + \sigma^2, \tag{7.11}$$

i.e. $\eta_e > \eta_*$; at equal damping coefficients the dissipation increases the instability threshold. However, at $\sigma_1 \neq \sigma_2$, as one can see from (7.10), η_d can assume any small values. Moreover, for small enough but different damping coefficients, η_d is necessarily less than η_* . Indeed, by introducing the notation $\sigma_2/\sigma_1 = q$ we can rewrite (7.10) in the form

$$\eta_d = \frac{4q}{(1+q)^2} \eta_* + q\sigma_1^2, \tag{7.12}$$

from which we have

$$\eta_d - \eta_* = - \left(\frac{1-q}{1+q} \right)^2 \eta_* + \sigma_1\sigma_2. \tag{7.13}$$

The right-hand side of (7.13) is negative at any $k \neq 1$ and with small enough product $\sigma_1\sigma_2$. Typical dependence of the real part of growth rate on η is presented in figure 5.

Thus, the extension of the parameter range for excitation of parametric resonance when weak dissipation is accounted for is a general result. The situation is different only in some particular cases when $\sigma_1 = \sigma_2$. One of these particular cases is the excitation of Faraday ripple at a flat surface of a fluid. In that case, two interacting modes are two waves propagating in opposite directions. Due to the symmetry of the problem, for these waves $\delta_2 = -\delta_1$ and $\sigma_2 = \sigma_1$ and accounting for the dissipation results in a decrease of the instability range. In the situation under consideration, two interacting modes are different modes of eigen-oscillations of a drop with different spatial structures and consequently different damping coefficients.

The paradoxical extension of the instability domain in the parameter space under the influence of infinitesimal dissipative forces is also known for linear, autonomous, non-conservative mechanical systems (see, for example, Kirillov 2007).

8. Conclusions

Vibrations significantly affect the behaviour of an incompressible drop or bubble suspended in a fluid of different density.

Since the drop is an oscillatory system, under certain conditions, vibrations lead to a parametric resonance.

One interesting feature of this resonance is that it is twofold: vibrations induce coupled oscillations of two neighbouring modes.

Another peculiarity is the paradoxical effect of the viscosity on the conditions of the resonance excitation. As in the other cases, viscosity leads to a non-zero threshold of resonance excitation and a viscous shift of frequency. However, what is more, viscosity widens the resonance zone to such an extent that parametric resonance becomes possible at the vibration frequencies and amplitudes at which it is impossible for an inviscid fluid.

Declaration of interests

The authors report no conflict of interest.

REFERENCES

- ABI CHEBEL, N., PIEDFERT, A., LALANNE, B., DALMAZZONE, C., NOÏK, C., MASBERNAT, O. & RISSO, F. 2019 Interfacial dynamics and rheology of a crude-oil droplet oscillating in water at a high frequency. *Langmuir* **35** (29), 9441–9455.
- ASAKI, T. J., THIESSEN, D. B. & MARSTON, P. L. 1995 Effect of an insoluble surfactant on capillary oscillations of bubbles in water: observation of a maximum in the damping. *Phys. Rev. Lett.* **75**, 2686–2689.
- CHANDRASEKHAR, S. 1959 The oscillations of viscous liquid glob. *Proc. Lond. Math. Sci.* **9**, 141–149.
- CHRISTIANSEN, B., ALSTROM, P. & LEVINSSEN, M. T. 1995 Dissipation and ordering in capillary waves at high aspect ratios. *J. Fluid Mech.* **291**, 323–341.
- EDWARDS, W. S. & FAUVE, S. 1993 Parametrically excited quasicrystalline surface waves. *Phys. Rev. E* **47**, 788–791.
- EDWARDS, W. S. & FAUVE, S. 1994 Patterns and quasi-patterns in the Faraday experiment. *J. Fluid Mech.* **278**, 123–148.
- FARADAY, M. 1831 On a peculiar class acoustical figures and on certain forms assumed by a group of particles upon vibrating elastic surface. *Phil. Trans. R. Soc. Lond.* **121**, 299–340.
- FENG, Z. C. & LEAL, L. G. 1995 Translational instability of a bubble undergoing shape oscillations. *Phys. Fluids* **7**, 1325–1336.
- KAPITSA, P. L. 1951 Dynamical stability of pendulum at oscillating point of suspension. *J. Expl Theor. Phys.* **21**, 588–597.
- KEENE, B. J., MILLS, K. C., KASAMA, A., MCLEAN, A. & MILLER, W. A. 1986 Comparison of surface tension measurements using the levitated droplet method. *Metall. Trans. B* **17** (1), 159–162.
- KIRILLOV, O. N. 2007 On the stability of nonconservative systems with small dissipation. *J. Math. Sci.* **145** (5), 323–341.
- KUMAR, K. & TUCKERMAN, L. S. 1994 Parametric instability of the interface between two fluids. *J. Fluid Mech.* **279**, 49–68.
- LAMB, H. 1881 On the oscillations of a viscous spheroid. *Proc. Lond. Math. Sci.* **13**, 51–66.
- LANDAU, L. D. & LIFSHITZ, E. M. 1976 *Mechanics*, 3rd edn. Pergamon Press.
- LANDAU, L. D. & LIFSHITZ, E. M. 1987 *Fluid Mechanics*, 2nd edn. Pergamon Press.
- LEAL, G. 1992 *Laminar Flow and Convective Transport Processes. Scaling Principles and Asymptotic Analysis*. Butterworth-Heinemann.
- LYUBIMOV, D. V., KONOVALOV, V. V., LYUBIMOVA, T. P. & EGRY, I. 2011 Small amplitude shape oscillations of a spherical liquid drop with surface viscosity. *J. Fluid Mech.* **677**, 204–217.

- LYUBIMOVA, T., IVANTSOV, A., GARRABOS, Y., LECOUTRE, C. & BEYSENS, D. 2019 Faraday waves on band pattern under zero gravity conditions. *Phys. Rev. Fluids* **4**, 064001.
- MARSTON, P. L. 1980 Shape oscillation and static deformation of drops and bubbles driven by modulated radiation stress - theory. *J. Acoust. Soc. Am.* **67**, 15–26.
- MARSTON, P. L. & APFEL, R. E. 1980 Quadrupole resonance of drops driven by modulated acoustic radiation pressure. Experimental properties. *J. Acoust. Soc. Am.* **67**, 27–37.
- MEI, C. C. & ZHOU, X. 1991 Parametric resonance of a spherical bubble. *J. Fluid Mech.* **229**, 29–50.
- MILES, J. & HENDERSON, D. 1990 Parametrically forced surface waves. *Annu. Rev. Fluid Mech.* **22**, 143–165.
- NAYFEH, A. H. 1981 *Introduction to Perturbation Techniques*. John Wiley and Sons.
- PLESSET, M. W. 1949 The dynamics of cavitation bubbles. *Trans. ASME: J. Appl. Mech.* **16**, 277–282.
- PRZYBOROWSKI, M., HIBIYA, T., EGUCHI, M. & EGRY, I. 1995 Surface tension measurement of molten silicon by the oscillating drop method using electromagnetic levitation. *J. Cryst. Growth* **151** (1–2), 60–65.
- RAYLEIGH, LORD. 1879 The capillary phenomena of jets. *Proc. R. Soc. Lond.* **29**, 71–97.
- RAYLEIGH, LORD. 1917 On the pressure developed in a liquid during the collapse of a spherical cavity. *Phil. Mag.* **34** (200), 94–98.
- REID, W. 1960 The oscillations of a viscous liquid drop. *Q. Appl. Maths* **18**, 86–89.
- SCHMIDT, G. 1975 *Parameterregte Schwingungen*. Veb Deutscher Verlag der Wissenschaften.
- SHAO, X., FREDERICKS, S. A., SAYLOR, J. R. & BOSTWICK, J. B. 2020 A method for determining surface tension, viscosity, and elasticity of gels via ultrasonic levitation of gel drops. *J. Acoust. Soc. Am.* **147** (4), 2488–2498.
- SHEN, C. L., XIE, W. J. & WEI, B. 2010 Parametric resonance in acoustically levitated water drops. *Phys. Lett. A* **374**, 2301–2304.
- STEPHENSON, A. 1908 New type of dynamical stability. *Mem. Proc. Manch. Lit. Philos. Soc.* **52** (8), 1–10.
- TIAN, Y., HOLT, R. G. & APFEL, R. E. 1997 Investigation of liquid surface rheology of surfactant solutions by droplet shape oscillations: experiments. *J. Colloid Interface Sci.* **187** (1), 1–10.
- TRINH, E. H., THIESSEN, D. B. & HOLT, R. G. 1998 Driven and freely decaying nonlinear shape oscillations of drops and bubbles immersed in a liquid: experimental results. *J. Fluid Mech.* **364**, 253–272.