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RECOVERY OF ZEROTH ORDER COEFFICIENTS IN NON-LINEAR WAVE EQUATIONS

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Abstract This paper is concerned with the resolution of an inverse problem related to the recovery of a function V from the source to solution map of the semi-linear equation $(\Box_g + V)u + u^3 = 0$ on a globally hyperbolic Lorentzian manifold $(\mathcal{M}, g)$. We first study the simpler model problem, where $(\mathcal{M}, g)$ is the Minkowski space, and prove the unique recovery of V through the use of geometric optics and a three-fold wave interaction arising from the cubic non-linearity. Subsequently, the result is generalized to globally hyperbolic Lorentzian manifolds by using Gaussian beams.

Keywords: Inverse problems; non-linear wave equations; Gaussian beams; Lorentzian manifolds

2010 *Mathematics subject classification*: Primary 35R30

Contents

1. Introduction

Let $n \geqslant 2$ and let $(\mathcal{M}, g)$ be a $1+n$ dimensional Lorentzian manifold. We consider the non-linear hyperbolic equation

$$\Box_g u + Vu + u^3 = f, \tag{1}$$

where $\Box_g$ denotes the wave operator and $V \in \mathcal{C}^\infty(\mathcal{M}; \mathbb{C})$ is an a priori unknown complex-valued potential function. In this paper, we study the inverse problem of *remote sensing* of the unknown potential V through applying various small localized sources f in a fixed open subset $\mho$ of the space–time $\mathcal{M}$ and subsequently observing the corresponding solutions u in the same set $\mho$. In physical terms, our main result (Theorem 2) says that the potential V can be uniquely recovered from such measurements in the optimal causal diamond, that is, in the largest set $\mathbb{D}$ such that waves generated in $\mho$ can reach each $x \in \mathbb{D}$ and travel from x back to $\mho$. We emphasize that the presence of the non-linearity is the key to the generality of our result. Indeed, the corresponding problem for the linear operator $\Box_g + V$ is open to this date.

Inverse problems for non-linear wave equations in the geometric context have been under active study recently. Starting from [15], the recovery of leading order coefficients has been studied in a series of works that we will briefly review below. However, the only previous work considering the recovery of subleading terms is [5], where the first order coefficients were recovered. Lower order terms have a weaker effect on solutions. To illustrate the difference between the zeroth order case, studied in the present paper, and the second and first order cases in the previous literature, let us consider a parametrix Q for a linear wave operator P. The leading order coefficients of P affect the Lagrangian geometry of Q and the first order coefficients of P the principal symbol of Q, while the zeroth order coefficients of P affect Q only at the subprincipal level.

To access information on the subprincipal level, we modify the approach of [15] substantially by using wave packets instead of microlocal analysis based on conormal distributions. Wave packets are easy to construct in the Minkowski geometry using classical geometric optics, and as an introduction, we give first a proof in this case (Theorem 1). Then we proceed to show our main result (Theorem 2) on a globally hyperbolic Lorentzian manifold $(\mathcal{M}, g)$. In this case, the wave packets that we use are Gaussian beams.

Finally, let us mention that although a cubic leading order non-linearity is considered here, the approach also works, with minor modifications, for higher order non-linearities. The case of quadratic non-linearity presents additional technical challenges and is not covered by the analysis here.

1.1. Previous literature

As mentioned above, Kurylev, Lassas and Uhlmann introduced an approach to solve inverse coefficient determination problems for non-linear hyperbolic equations in [15]. The approach is based on considering multi-parameter families of solutions, and simultaneous linearizations with respect to each of the parameters. If only a one-parameter family of solutions is employed, the linearization yields simply a solution to the linearized version of the non-linear hyperbolic equation under consideration. However, simultaneous linearizations cause solutions to the linearized equation to interact in a non-linear manner, and this leads to richer dynamics in propagation of singularities (or wave packets) than in the case of linear hyperbolic equations.

In [15], the approach was applied to the determination of the conformal class of the Lorentzian metric tensor giving the leading order coefficients in a wave equation with quadratic non-linearity. The recovery of leading order coefficients has been considered also in the context of Einstein equations in [14] and subsequently in [18, 24]. We mention also [25] where the leading order coefficients were recovered in the presence of a quadratic derivative non-linearity, [17] where the approach of [15] was applied to the recovery of coefficients appearing in non-linear terms, and [9] where it was applied to a problem arising in seismic imaging.

Recently two approaches different from [15] were used by Nakamura and Vashisth to recover time-independent leading order coefficients, as well as coefficients in non-linear terms [19], and by Kian to recover a general function corresponding to the non-linearity and also including zeroth order coefficients [13]. The latter result is based on a reduction via linearization to the problem to recover the zeroth order coefficient in a linear wave equation. For this reason, contrary to our result, the geometric context in [13] is confined to the cases where results are available for linear wave equations.

2. The case of Minkowski geometry

We consider $\mathbb{R}^{1+n}$, with $n \geqslant 2$, and write $(x^0, x^1, \ldots, x^n) = (t, x') = x$ for the Cartesian coordinates. Let $r, T > 0$ and write

$$\mho = (0, T) \times B(0, r), \tag{2}$$

where $B(0, r)$ denotes the ball centered at the origin and radius r in $\mathbb{R}^n$. We will formulate an inverse coefficient determination problem with data given on $\mho$. Let $\kappa > 0$ be a fixed sufficiently large integer and define $\mathscr{C}$ as a small neighborhood of the origin in the $\mathcal{C}^\kappa_c(\mho)$ topology. Let $V \in \mathcal{C}^\infty(\mathbb{R}^{1+n})$, and for each $f \in \mathscr{C}$, consider the non-linear wave equation

$$\begin{cases} \Box u + Vu + u^3 = f, & \forall (t, x') \in (0, T) \times \mathbb{R}^n, \\ u(0, x') = 0,\ \partial_t u(0, x') = 0, & \forall x' \in \mathbb{R}^n, \end{cases} \tag{3}$$

where $\Box$ is the d'Alembert operator, that is,

$$\Box u = \partial_t^2 u - \sum_{j=1}^n \partial_{x^j}^2 u.$$

When κ is large enough and $\mathscr{C}$ is small enough, there exists a unique solution u to equation (3). We subsequently define the source to solution map for equation (3) as

$$L_V(f) = u|_{\mho}, \quad \forall f \in \mathscr{C}.$$

The *inverse coefficient determination problem* is to find V given the map L_V, up to the natural obstruction due to the finite speed of propagation.

In order to be able to determine $V(p)$ for a point $p \in \mathbb{R}^{1+n}$, there must be a signal, in the form of a non-vanishing solution to (3), from $\mho$ to p and from p to $\mho$. Due to the finite speed of propagation, a signal from a point $q = (t_0, x_0') \in \mathbb{R}^{1+n}$ can reach only the set

$$\mathscr{J}_+(q) = \{(t, x') \in \mathbb{R}^{1+n} \,|\, t \geqslant t_0,\ |x' - x_0'| \leqslant t - t_0\}, \tag{4}$$

called the future of q. We define also the past of q by

$$\mathscr{J}_-(q) = \{(t, x') \in \mathbb{R}^{1+n} \,|\, t \leqslant t_0,\ |x' - x_0'| \leqslant t_0 - t\}, \tag{5}$$

and write $\mathscr{J}_\pm(\mho) = \bigcup_{q\in\mho} \mathscr{J}_\pm(q)$. Then L_V contains no information on V outside the causal diamond

$$\mathbb{D} := \mathscr{J}_+(\mho) \cap \mathscr{J}_-(\mho) = \{(t, x') \in (0, T) \times \mathbb{R}^n \,|\, |x'| \leqslant r + t,\ |x'| \leqslant r + T - t\};$$

see Figure 1. On the other hand, we will show the following theorem saying that L_V determines V on $\mathbb{D}$.

Theorem 1. *Let L_{V_1}, L_{V_2} denote the source to solution map for equation (3) subject to functions $V_1, V_2 \in C^\infty(\mathbb{R}^{1+n})$ respectively. Then*

$$L_{V_1}(f) = L_{V_2}(f) \quad \forall f \in \mathscr{C} \implies V_1 = V_2 \text{ on } \mathbb{D}.$$

The non-trivial content of the theorem is the remote determination on $\mathbb{D} \setminus \mho$, as it is straightforward to see that L_V determines V on $\mho$. To see this, let $f \in C_c^\infty(\mho)$ and consider the one-parameter family of sources $f_\epsilon := \epsilon f$, $\epsilon \in \mathbb{R}$. For small enough ϵ, it holds that $f_\epsilon \in \mathscr{C}$, and we let u_ϵ denote the unique solution to (3) subject to this source term. Then $u := \partial_\epsilon u_\epsilon|_{\epsilon=0}$ solves the linear wave equation

$$\begin{cases} \Box u + V u = f, & \forall (t, x') \in (0, T) \times \mathbb{R}^n, \\ u(0, x') = 0,\ \partial_t u(0, x') = 0, & \forall x' \in \mathbb{R}^n, \end{cases} \tag{6}$$

and $u|_\mho = \partial_\epsilon L_V(f_\epsilon)|_{\epsilon=0}$. Observe that if $u(q) \neq 0$ for a point $q \in \mho$, then

$$V(q) = \frac{f(q) - \Box u(q)}{u(q)}.$$

It remains to show that for any $q \in \mho$, there is $f \in C_c^\infty(\mho)$ such that the solution u of (6) satisfies $u(q) \neq 0$. But this follows simply by taking any $u \in C_c^\infty(\mho)$ with this property, and setting $f = \Box u + V u$.

Let us also point out that it is an open question if $V|_\mathbb{D}$ is determined by the linearized source to solution map,

$$\mathcal{L}_V f = u|_\mho, \quad \forall f \in C_c^\kappa(\mho),$$

where u is the solution of (6). Only in the case that $V(t, x')$ is real-analytic in t, this is known to hold due to the variant of the boundary control method by Eskin [7]. The boundary control method fails to generalize to the case of smooth V since the method depends on the sharp unique continuation result by Tataru [23], which is known not to hold for smooth V due to a counter-example by Alinhac [1].

In addition, the approach to recover time-dependent coefficients in wave equations based on geometric optics, originating from [22], fails since no wave packet leaving $\mho$ returns there. For this approach to work, the set on which the data is given (i.e., $\mho$ in our case) needs to enclose the region where V is to be determined (i.e., $\mathbb{D}$ in our case). Like [22], the approach in the present section is based on geometric optics, but the difference is that the cubic non-linearity in (3) allows us to solve the inverse problem with the optimal relation between $\mho$ and $\mathbb{D}$.

Before entering into the proof of Theorem 1 in detail, let us briefly explain how the non-linearity is used. Let $f_1, f_2, f_3 \in C_c^\infty(\mho)$ and consider the three-parameter family of sources

$$f_\epsilon := \epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3, \quad \forall \epsilon := (\epsilon_1, \epsilon_2, \epsilon_3) \in \mathbb{R}^3. \tag{7}$$

For small enough ϵ_1, ϵ_2 and ϵ_3, it holds that $f_\epsilon \in \mathscr{C}$, and we let u_ϵ denote the unique solution to (3) subject to this source term. Then

$$u := \partial_{\epsilon_1} \partial_{\epsilon_2} \partial_{\epsilon_3} u_\epsilon|_{\epsilon=0} \tag{8}$$

solves the linear wave equation (6) with

$$f = -6u^{(1)} u^{(2)} u^{(3)}, \quad u^{(j)} = \partial_{\epsilon_j} u_\epsilon \Big|_{\epsilon=0}, \tag{9}$$

and $u^{(j)}$ satisfies the same equation with $f = f_j$. In the proof below, we will use sources f_j that generate geometric optics solutions $u^{(j)}$ with carefully chosen phases. This will allows us to employ u as a highly structured signal from a point $p \in \mathbb{D}$ back to $\mho$.

The remainder of this section is organized as follows. We start by briefly reviewing the geometric optic solutions for the linearized equation (6) in Section 2.1. In Section 2.2, we show that sources supported in $\mho$ can be explicitly chosen so that they generate the geometric optics solutions for (6) that pass through $\mho$, and in Section 2.3, we use these sources to construct the three-parameter family (7). Then, in Section 2.4, we consider the interaction of $u^{(1)}$, $u^{(2)}$ and $u^{(3)}$, as encoded by u in (8), and conclude the proof of Theorem 1.

The proof in the case of general, globally hyperbolic Lorentzian manifolds reflects the proof in Sections 2.2–2.4. Our main theorem is formulated in Section 3, and in Section 4, we present the Gaussian beam construction that replaces the classical geometric optics of Section 2.1 in the general case. Finally in Section 5, we perform the analogues of the steps in Sections 2.2–2.4 in the general case.

2.1. Geometric optics

In this section, we recall the classical construction of approximate geometric optics solutions to the wave equation

$$\Box u + V u = 0 \quad \text{in } \mathbb{R}^{1+n}. \tag{10}$$

The construction is based on the ansatz

$$u_\tau(x) = e^{i\tau\xi\cdot x} a_\tau(x) = e^{i\tau\xi\cdot x} \left(\sum_{k=0}^{N} \frac{a_k(x)}{\tau^k} \right), \tag{11}$$

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where $\tau > 0$ is a large parameter, and $\xi \in \mathbb{R}^{1+n}$ and a large integer $N > 0$ are fixed. Here the notation $\xi \cdot x = \sum_{j=0}^{n} \xi_j x^j$ is used. We will view $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \dots, \xi_n)$ as a covector, and it needs to be non-zero and light-like, that is to say,

$$|\xi_0|^2 = |\xi'|^2 := |\xi_1|^2 + \cdots + |\xi_n|^2.$$

We denote by $\xi^\sharp$ the vector version of ξ with respect to the Minkowski metric. In other words, $\xi^\sharp = (-\xi_0, \xi')$. Let $q = (t_0, x_0')$ be a point in $\mathbb{R}^{1+n}$. We will construct the amplitude functions a_k, $k = 0, 1, \dots, N$, so that u_τ satisfies (10) up to a remainder term that tends to zero as $\tau \to \infty$ and that u_τ is supported near the line

$$\gamma_{q,\xi}(s) := s\xi^\sharp + q = (-s\xi_0 + t_0,\ s\xi' + x_0'), \quad \forall s \in \mathbb{R}. \tag{12}$$

As ξ is light-like, it holds that

$$(\Box + V)(e^{i\tau\xi\cdot x} a_\tau) = e^{i\tau\xi\cdot x}(-2i\tau \mathcal{T}_\xi a_\tau + (\Box + V)a_\tau), \tag{13}$$

where $\mathcal{T}_\xi = -\xi_0 \partial_{x_0} + \sum_{j=1}^{n} \xi_j \partial_{x_j}$. The construction of the amplitudes a_k is driven by the requirement that expression (13) vanishes in powers of τ. In particular, this imposes the transport equation

$$\mathcal{T}_\xi a_0 = 0 \tag{14}$$

on a_0. Note that if $\omega \in \mathbb{R}^{1+n}$ satisfies $\xi^\sharp \cdot \omega = 0$, then for any $\chi \in \mathcal{C}^1(\mathbb{R})$ it holds that

$$\mathcal{T}_\xi(\chi(\omega \cdot (x - q))) = 0.$$

We choose $\omega_j' \in \mathbb{R}^n$ so that the covectors

$$\frac{\xi'}{|\xi'|}, \omega_1', \dots, \omega_{n-1}' \tag{15}$$

form an orthonormal basis for $\mathbb{R}^n$ with respect to the Euclidean metric, and write $\omega_j = (0, \omega_j')$. Observe that $\xi^\sharp \cdot \omega_j = 0$ and that

$$\{\gamma_{q,\xi}(s) \mid s \in \mathbb{R}\} = \{x \in \mathbb{R}^{1+n} \mid \xi \cdot (x - q) = \omega_1 \cdot (x - q) = \cdots = \omega_{n-1} \cdot (x - q) = 0\}.$$

Let $\delta > 0$ and let $\chi_\delta \in \mathcal{C}_c^\infty((-\delta, \delta))$. We choose

$$a_0(x) = \chi_\delta(|\xi_0|^{-1} \xi \cdot (x - q)) \prod_{j=1}^{n-1} \chi_\delta(\omega_j \cdot (x - q)). \tag{16}$$

Then (14) holds and

$$\operatorname{supp}(a_0(t, \cdot)) \subset H(t, \delta), \quad \forall t \in \mathbb{R}, \tag{17}$$

where $H(t, \delta)$ is the hypercube in $\mathbb{R}^n$ with side length 2δ, centered at the unique point $x' \in \mathbb{R}^n$ satisfying $(t, x') = \gamma_{q,\xi}(s)$ for some $s \in \mathbb{R}$, and with the edges pointing to directions (15).

The subsequent terms a_k with $k \geqslant 1$ are chosen iteratively through the transport equations

$$-2i\mathcal{T}_\xi a_k + (\Box + V)a_{k-1} = 0. \tag{18}$$

We impose vanishing initial conditions on the hyperplane

$$\Sigma_{q,\xi} = \{x \in \mathbb{R}^{1+n} \mid \xi^\sharp \cdot (x - q) = 0\},$$

and obtain

$$a_k(s\xi^\sharp + y) = \frac{1}{2i}\int_0^s ((\Box + V)a_{k-1})(\tilde{s}\xi^\sharp + y)\, d\tilde{s}, \tag{19}$$

where $s \in \mathbb{R}$ and $y \in \Sigma_{q,\xi}$. It follows from (17), via an induction, that also $\operatorname{supp}(a_k(t,\cdot)) \subset H(t,\delta)$, and therefore u_τ is supported near $\gamma_{q,\xi}$. Moreover, equations (14) and (18), together with (13), imply that

$$\|(\Box + V)u_\tau\|_{\mathcal{C}^k((0,T)\times\mathbb{R}^n)} \lesssim \tau^{-N+k}. \tag{20}$$

2.2. Source terms

As in the previous section, let $\xi \in \mathbb{R}^{1+n}$ be non-zero and light-like and let $q = (t_0, x_0') \in \mathbb{R}^{1+n}$. We will assume, furthermore, that $q \in \mho$, and proceed to construct a source $f \in \mathcal{C}_c^\infty(\mho)$ such that the solution to the linear wave equation (6) is close to the approximate geometric optics solution (11) in a sense that will be made precise below. For this construction to work, it is necessary to require that $\delta > 0$ in (16) is small enough so that

$$H(t_0, \delta) \subset B(0, r); \tag{21}$$

cf. (2) and (17).

It follows from (17) and (21) that there exists $\rho > 0$ such that

$$\operatorname{supp}(u_\tau(t,\cdot)) \subset B(0,r), \quad \forall t \in (t_0 - \rho, t_0 + \rho). \tag{22}$$

Next, we choose two non-negative functions $\zeta_\pm \in C^\infty(\mathbb{R}^{1+n})$ such that

$$\zeta_-(t,x') = \begin{cases} 0, & t < t_0 - \rho \\ 1, & t > t_0, \end{cases} \quad \text{and} \quad \zeta_+(t,x') = \begin{cases} 0, & t > t_0 + \rho \\ 1, & t < t_0. \end{cases} \tag{23}$$

We are now ready to define the source. Emphasizing the dependence on τ, q and ξ, we write

$$f_{\tau,q,\xi} = \zeta_+(\Box + V)(\zeta_- u_\tau). \tag{24}$$

Note that since $\zeta_-(t) = 0$ for $t < t_0 - \rho$ and $\zeta_+(t) = 0$ for $t > t_0 + \rho$, it follows together with (22) that

$$\operatorname{supp} f_{\tau,q,\xi} \subset (t_0 - \rho, t_0 + \rho) \times B(0,r) \subset \mho.$$

Recall that V is known on $\mho$, thanks to the knowledge of L_V and therefore the amplitude terms given by (19) are also known over the support of $f_{\tau,q,\xi}$. Hence $f_{\tau,q,\xi}$ is determined from the source to solution map.

We write $\mathcal{U}_\tau = u$, where u is the solution of the linear wave equation (6) with the source $f = f_{\tau,q,\xi}$. For the remainder of this section, we aim to show that the solution $\mathcal{U}_\tau$ will be approximately equal to the geometric optic ansatz u_τ, for times $t > t_0 + \rho$. To this end, note that as $\zeta_- = 1$ on the support of $1 - \zeta_+$, it holds that

$$(\Box + V)(\zeta_- u_\tau) - f_{\tau,q,\xi} = (1 - \zeta_+)(\Box + V)u_\tau,$$

and (20) implies the estimate

$$\|(\Box + V)(\zeta_- u_\tau) - f_{\tau,q,\xi}\|_{H^k((0,T)\times\mathbb{R}^n)} \lesssim \tau^{-N+k}.$$

Next, by combining the above estimate with the usual energy estimate for the wave equation and the Sobolev embedding of $\mathcal{C}((0,T)\times\mathbb{R}^n)$ in $H^{k+1}((0,T)\times\mathbb{R}^n)$ for $k > (n-1)/2$, we obtain

$$\|\zeta_- u_\tau - \mathcal{U}_\tau\|_{\mathcal{C}((0,T)\times\mathbb{R}^n)} \lesssim \tau^{-2} \tag{25}$$

when $N \geqslant k + 2$.

We will also need a test function whose construction differs from that of $f_{\tau,q,\xi}$ only to the extent that the roles of ζ_+ and ζ_- are reversed in (24). That is, we define

$$f^+_{\tau,q,\xi} = \zeta_-(\Box + V)(\zeta_+ u_\tau) \in \mathcal{C}^\infty_c(\mho). \tag{26}$$

Again, L_V determines $f^+_{\tau,q,\xi}$, and the analogue of (25) reads as

$$\|\zeta_+ u_\tau - \mathcal{U}_\tau\|_{\mathcal{C}((0,T)\times\mathbb{R}^n)} \lesssim \tau^{-2}, \tag{27}$$

where $\mathcal{U}_\tau$ is now the solution of the linear wave equation

$$\begin{cases} \Box u + V u = f, & \forall (t, x') \in (0, T) \times \mathbb{R}^n, \\ u(T, x') = 0,\ \partial_t u(T, x') = 0, & \forall x' \in \mathbb{R}^n, \end{cases} \tag{28}$$

with $f = f^+_{\tau,q,\xi}$.

2.3. Three-parameter family of sources

Let $p = (t_1, x'_1) \in \mathbb{D} \setminus \mho$. In this section, we will construct a three-parameter family of sources f_ϵ of form (7) so that cross derivative (8) will act as a structured signal from the point p back to $\mho$.

As $p \in \mathbb{D}$, there are $q^- = (t_0, x'_0) \in \mho$ and non-zero, light-like $\xi^- \in \mathbb{R}^{1+n}$ such that $t_1 > t_0$ and $p = \gamma_{q^-,\xi^-}(s_0)$ for some $s_0 \in \mathbb{R}$. Here we use notation (12). We also normalize the covector $\xi^- = (\xi^-_0, \dots, \xi^-_n)$ so that $\xi^-_0 = -1$. Then $s_0 = t_1 - t_0$. Using again the fact that $p \in \mathbb{D}$, we see that the point $q^+ := (t_0 + 2s_0, x'_0)$ is in $\mho$. Moreover, setting $\xi^+ = (-1, -\xi^-_1, \dots, -\xi^-_n)$, it holds that $p = \gamma_{q^+,\xi^+}(-s_0)$; see Figure 1.

We will next choose two light-like covectors, that are small perturbations of ξ^-, in such a way that ξ^+ can be written as a linear combination of ξ^- and the two perturbations. Noting that the time coordinate of p is the average of the time coordinates of $q^\pm$ and that the spatial coordinates of $q^\pm$ are identical, we may rotate the coordinate system in

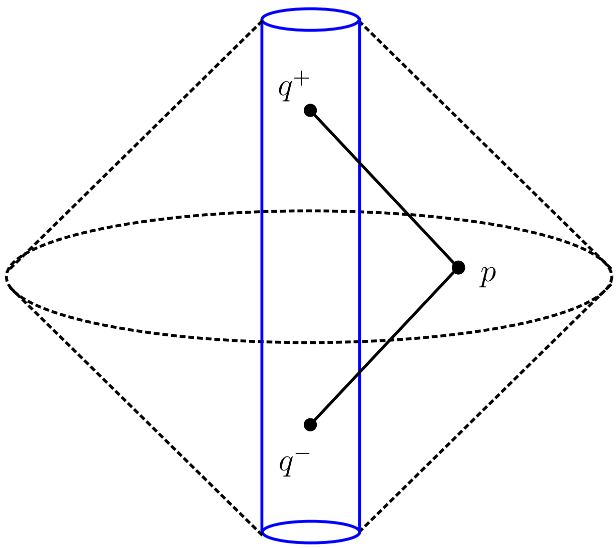

Figure 1. The causal diamond $\mathbb{D}$ is drawn with dashed curves, and $\mho$ is the blue cylinder. The black line segments are on $\gamma_{q^\pm,\xi^\pm}$, joining $q^\pm$ to p. We write also $q^+ = q^{(0)}$ and $q^- = q^{(1)}$, and denote by $q^{(2)}$ and $q^{(3)}$ two perturbations of $q^{(1)}$.

the spatial variables $(x^1, \dots, x^n) \in \mathbb{R}^n$ so that in the rotated coordinates, ξ^+ and ξ^- are represented by

$$\tilde{\xi}^{(0)} = (-1, -1, \underbrace{0, \dots, 0}_{n-1\,\text{times}}), \quad \tilde{\xi}^{(1)} = (-1, 1, \underbrace{0, \dots, 0}_{n-1\,\text{times}}), \tag{29}$$

respectively. We define for small $\sigma > 0$

$$\tilde{\xi}^{(2)} = (-1, \sqrt{1-\sigma^2}, \sigma, \underbrace{0, \dots, 0}_{n-2\,\text{times}}), \quad \tilde{\xi}^{(3)} = (-1, \sqrt{1-\sigma^2}, -\sigma, \underbrace{0, \dots, 0}_{n-2\,\text{times}}), \tag{30}$$

and have

$$\sigma^2 \tilde{\xi}^{(0)} + \kappa_1 \tilde{\xi}^{(1)} + \kappa_2 \tilde{\xi}^{(2)} + \kappa_3 \tilde{\xi}^{(3)} = 0, \tag{31}$$

where

$$\kappa_1 = 2(1 + \sqrt{1-\sigma^2}) - \sigma^2, \quad \kappa_2 = \kappa_3 = -1 - \sqrt{1-\sigma^2}. \tag{32}$$

Finally, we define the covector $\xi^{(0)}$ to be the representation of $\sigma^2 \tilde{\xi}^{(0)}$, after passing back to the original coordinate system, and $\xi^{(j)}$ for $j = 1, 2, 3$ to be the analogous representations of the covectors $\kappa_j \tilde{\xi}^{(j)}$. Then $\xi^{(0)} = \sigma^2 \xi^+$, $\xi^{(1)} = \kappa_1 \xi^-$ and both $\xi^{(2)}$ and $\xi^{(3)}$ are small perturbations of $-2\xi^-$. Note also that κ_1 is close to 4.

Define

$$q^{(0)} = \gamma_{p,\xi^{(0)}}(s_0/\sigma^2), \quad q^{(j)} = \gamma_{p,\xi^{(j)}}(-s_0/\kappa_j), \quad \text{for } j = 1, 2, 3. \tag{33}$$

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Then $q^{(0)} = q^+$, $q^{(1)} = q^-$ and $q^{(2)}, q^{(3)} \in \mho$ for small enough $\sigma > 0$. We are now ready to define the following three-parameter family of sources

$$f_{\epsilon,\tau} = \epsilon_1 f_{\tau,q^{(1)},\xi^{(1)}} + \epsilon_2 f_{\tau,q^{(2)},\xi^{(2)}} + \epsilon_3 f_{\tau,q^{(3)},\xi^{(3)}}, \tag{34}$$

where each $f_{\tau,q^{(j)},\xi^{(j)}}$ is defined by (24).

2.4. Recovery of V

Let $f_{\epsilon,\tau}$ be as in (34). Recall that p is an arbitrary point in $\mathbb{D} \setminus \mho$. In this section, we will prove Theorem 1 by showing that $V(p)$ is determined by L_V.

For a fixed $\tau > 0$ and small enough $\epsilon_j > 0$, it holds that $f_{\epsilon,\tau} \in \mathscr{C}$, and we let $u_{\epsilon,\tau}$ denote the unique solution to (3) subject to this source term. We write $\mathcal{U}_\tau^{(j)} = \partial_{\epsilon_j} u_{\epsilon,\tau}|_{\epsilon=0}$ for $j = 1, 2, 3$. Then the function $\mathcal{U}_\tau^{(j)}$ is close, in the sense of estimate (25), to the approximate geometric optics solution of form (11) supported near the line $\gamma_{q^{(j)},\xi^{(j)}}$. Moreover, it follows from (9) that the function

$$v_\tau = -\frac{1}{6} \frac{\partial^3 u_{\epsilon,\tau}}{\partial\epsilon_1 \partial\epsilon_2 \partial\epsilon_3}|_{\epsilon=0}$$

satisfies the equation

$$\begin{cases} \Box v_\tau + V v_\tau = \mathcal{U}_\tau^{(1)} \mathcal{U}_\tau^{(2)} \mathcal{U}_\tau^{(3)}, & \forall (t, x') \in (0, T) \times \mathbb{R}^n, \\ v_\tau(0, x') = 0, \ \partial_t v_\tau(0, x') = 0, & \forall x' \in \mathbb{R}^n. \end{cases} \tag{35}$$

Recall that $q^{(0)}$ is defined by (33) and, modulo a rotation and the rescaling by σ^2, $\xi^{(0)}$ is defined by (29). Consider the test function $f^+_{\tau,q^{(0)},\xi^{(0)}}$ defined by (26) and denote by $\mathcal{U}_\tau^{(0)}$ the solution of (28) with $f = f^+_{\tau,q^{(0)},\xi^{(0)}}$. As $f^+_{\tau,q^{(0)},\xi^{(0)}}$ is supported in $\mho$, there holds

$$-\frac{1}{6} \int_{(0,T)\times\mathbb{R}^n} \partial_{\epsilon_1} \partial_{\epsilon_2} \partial_{\epsilon_3} L_V(f_{\epsilon,\tau})|_{\epsilon=0} \, f^+_{\tau,q^{(0)},\xi^{(0)}} \, dx = \int_{(0,T)\times\mathbb{R}^n} v_\tau \, (\Box + V) \mathcal{U}_\tau^{(0)} \, dx.$$

After integrating by parts twice, we see that L_V determines the integral

$$\mathcal{I} = \int_{(0,T)\times\mathbb{R}^n} \mathcal{U}_\tau^{(0)} \mathcal{U}_\tau^{(1)} \mathcal{U}_\tau^{(2)} \mathcal{U}_\tau^{(3)} \, dx.$$

It follows from (25) and (27) that $\mathcal{U}_\tau^{(j)}$, with $j = 0, 1, 2, 3$, coincides with the corresponding approximate geometric optics solution up to an error of order τ^{-2}. We denote by $a_k^{(j)}$ the corresponding amplitude functions. We will expand $\mathcal{I}$ in the powers of τ,

$$\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_{-1} \tau^{-1} + \mathcal{O}(\tau^{-2}).$$

Observe that as $a_0^{(0)}$ is supported near $\gamma_{q^{(0)},\xi^{(0)}}$ and as $a_0^{(1)}$ is supported near $\gamma_{q^{(1)},\xi^{(1)}}$, their product is supported near the point p; cf. (33). For this reason, the cut-off functions ζ_- and ζ_+ in (25) and (27) do not appear in $\mathcal{I}_0$ and $\mathcal{I}_{-1}$. Moreover, (31) implies that the phases of the four approximate geometric optics solutions cancel each other under the product in $\mathcal{I}$. Therefore

$$\mathcal{I}_0 = \int_{(0,T)\times\mathbb{R}^n} a_0^{(0)} a_0^{(1)} a_0^{(2)} a_0^{(3)} dx,$$

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and analogously,

$$\mathcal{I}_{-1} = \sum_{|e|=1} \int_{(0,T)\times\mathbb{R}^n} a_{e_0}^{(0)} a_{e_1}^{(1)} a_{e_2}^{(2)} a_{e_3}^{(3)}\, dx,$$

where $e = (e_0, e_1, e_2, e_3) \in \{0, 1, \dots\}^4$ is a multi-index. In particular, L_V determines $\mathcal{I}_{-1}$.

Recall that $a_1^{(j)}$ is of the form $a_1^{(j)} = b_1^{(j)} + c_1^{(j)}$, where for $s \in \mathbb{R}$ and $y \in \Sigma_{q^{(j)},\xi^{(j)}}$,

$$b_1^{(j)}(s\xi^{(j)\sharp} + y) = \frac{1}{2i} \int_0^s (\Box a_0^{(j)})(\tilde{s}\xi^{(j)\sharp} + y)\, d\tilde{s}, \tag{36}$$

$$c_1^{(j)}(s\xi^{(j)\sharp} + y) = \frac{1}{2i} \int_0^s (V a_0^{(j)})(\tilde{s}\xi^{(j)\sharp} + y)\, d\tilde{s};$$

cf. (19).

Remark 1. We emphasize that the functions $a_0^{(j)}(x)$, $b_1^{(j)}(x)$ and $c_1^{(j)}(x)$ are defined for all $x \in \mathbb{R}^{1+n}$ but that for each fixed $j = 0, 1, 2, 3$, the value at each point is given by the (s, y) coordinate representation of the point x with respect to $\gamma_{q^{(j)},\xi^{(j)}}$, with $s \in \mathbb{R}$ and $y \in \Sigma_{q^{(j)},\xi^{(j)}}$. We also note that these functions are all supported in δ neighborhoods of the rays $\gamma_{q^{(j)},\xi^{(j)}}$. More precisely, the amplitudes $a_0^{(j)}$, $b_1^{(j)}$ and $c_1^{(j)}$ are supported in the set

$$P_\delta^{(j)} = \left\{ x \in \mathbb{R}^{1+n} \;\middle|\; \left| \frac{\xi^{(j)}}{\xi_0^{(j)}} \cdot (x - q^{(j)}) \right| \leqslant \delta,\ |\omega_k^{(j)} \cdot (x - q^{(j)})| \leqslant \delta \quad k = 1, \dots, n-1 \right\}.$$

We return to the expression $\mathcal{I}_{-1}$, which is determined by L_V. As $a_0^{(j)}$ is independent from V, so is $b_1^{(j)}$. Therefore L_V determines the quantity

$$\mathcal{J} = \int_{(0,T)\times\mathbb{R}^n} c_1^{(0)} a_0^{(1)} a_0^{(2)} a_0^{(3)}\, dx + \int_{(0,T)\times\mathbb{R}^n} a_0^{(0)} c_1^{(1)} a_0^{(2)} a_0^{(3)}\, dx$$
$$+ \int_{(0,T)\times\mathbb{R}^n} a_0^{(0)} a_0^{(1)} c_1^{(2)} a_0^{(3)}\, dx + \int_{(0,T)\times\mathbb{R}^n} a_0^{(0)} a_0^{(1)} a_0^{(2)} c_1^{(3)}\, dx.$$

For each $j = 0, 1, 2, 3$, we let the cut-off function χ_δ in the definition of the leading amplitude (16) converge to the indicator function of the interval $(-\delta, \delta)$. Then $\mathcal{J}$ converges to

$$\mathcal{J}_\delta = \sum_{j=0}^{3} \int_{P_\delta} c^{(j)} dx,$$

where

$$P_\delta = \bigcap_{j=0}^{3} P_\delta^{(j)}$$

is a small polygonal neighborhood of the point p and for each $j = 0, 1, 2, 3$, we have

$$c^{(j)}(s\xi^{(j)\sharp} + y) = \frac{1}{2i} \int_0^s V(\tilde{s}\xi^{(j)\sharp} + y)\, d\tilde{s} \quad \text{for } s \in \mathbb{R} \text{ and } y \in \Sigma_{q^{(j)},\xi^{(j)}}.$$

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Since $c^{(j)}$, $j = 0, 1, 2, 3$, are smooth functions, the Lebesgue differentiation theorem applies to obtain

$$\lim_{\delta \to 0} \frac{\mathcal{J}_\delta}{|P_\delta|} = \sum_{j=0}^{3} c^{(j)}(p).$$

Due to (33), we have

$$2ic^{(0)}(p) = \int_0^{-s_0/\sigma^2} V(\tilde{s}\xi^{(0)\sharp} + q^{(0)})\, d\tilde{s} = \sigma^{-2} \int_0^{s_0} V(-s\xi^{+\sharp} + q^{(0)})\, ds,$$

and

$$2ic^{(j)}(p) = \int_0^{s_0/\kappa_j} V(\tilde{s}\xi^{(j)\sharp} + q^{(j)})\, d\tilde{s}, \quad \forall j = 1, 2, 3.$$

Recalling the explicit dependence (32) of κ_j on σ, we see that

$$2i \lim_{\sigma \to 0} \sigma^2 \sum_{j=0}^{3} c^{(j)}(p) = \int_0^{s_0} V(-s\xi^{+\sharp} + q^{(0)})\, ds.$$

Repeating the same analysis for points on $\gamma^{(0)}$ in a small neighborhood of the point p, together with the fact that V is known on the set $\mho$, we conclude that the map L_V determines the truncated integrals

$$\int_0^{s} V(-\tilde{s}\xi^{+\sharp} + q^{(0)})\, d\tilde{s},$$

for all s in a small neighborhood of s_0. Keeping $q^{(0)}$ fixed, differentiating with respect to s and evaluating at $s = s_0$ shows that $V(p)$ is determined by L_V. This concludes the proof of Theorem 1.

3. The case of globally hyperbolic Lorentzian geometries

The rest of this paper is concerned with generalizing Theorem 1 to more general Lorentzian geometries. We begin by reviewing some key concepts from Lorentzian geometry, following the notations and definitions in [20].

Let us consider a smooth $1 + n$ dimensional Lorentzian manifold $(\mathcal{M}, g)$, with $n \geqslant 2$. The metric tensor g is taken to have the signature $(-, +, \ldots, +)$, and writing $\langle v, w \rangle_g = \sum_{i,j=0}^{n} g_{ij} v^i w^j$ for vectors $v, w \in T_p\mathcal{M}$, $p \in \mathcal{M}$, we recall that

> v is *time-like* (resp. *space-like*) if $\langle v, v \rangle_g < 0$ (resp. > 0),
>
> v is *light-like* (or *null*) if $\langle v, w \rangle_g = 0$.

The manifold $\mathcal{M}$ is assumed to be time-orientable in the sense that there exists a globally defined, smooth time-like vector field Z on M. A curve α on $\mathcal{M}$ is said to be *causal* if its tangent vector $\dot{\alpha}$ is time-like or light-like for all points on the curve, and a causal curve is *future-pointing* if $\langle \dot{\alpha}, Z \rangle_g < 0$. We write $p \preccurlyeq q$ if there exists a causal future-pointing curve from p to q (the case $p = q$ is allowed). The generalizations of (4) and (5) read as

$$\mathscr{J}_+(p) = \{q \in \mathcal{M} \mid p \preccurlyeq q\}, \quad \mathscr{J}_-(p) = \{q \in \mathcal{M} \mid q \preccurlyeq p\}, \quad \forall p \in \mathcal{M}.$$

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We will make the typical assumption that $(\mathcal{M}, g)$ is *globally hyperbolic*. This guarantees that the natural, linear wave equation is well-posed on $\mathcal{M}$. There are several equivalent characterizations of global hyperbolicity, and we recall the one in [4]: there are no closed causal paths on $\mathcal{M}$, and the intersection $\mathscr{J}_+(p) \cap \mathscr{J}_-(q)$ is compact for any pair of points $p, q \in \mathcal{M}$.

Global hyperbolicity implies that there is a global splitting in "time" and "space" in the sense that $(\mathcal{M}, g)$ is isometric to $\mathbb{R} \times M$ with the metric

$$g = -\beta(t, x')\, dt \otimes dt + g_0(t, x'), \quad \forall t \in \mathbb{R}, \ x' \in M, \tag{37}$$

where β is a smooth positive function and g_0 is a Riemannian metric on the n dimensional manifold M smoothly depending on the parameter t. Moreover, each set $\{t\} \times M$ is a Cauchy hypersurface in $\mathcal{M}$, that is to say, any causal curve intersects it at most once.

To simplify the notation, we fix a global splitting of form (37) and use it throughout the rest of the paper. Analogously to (2), we set

$$\mho = (0, T) \times \mathcal{O},$$

where $T > 0$ and $\mathcal{O}$ is an open, bounded set in M. Analogously with the Minkowski case, we write again $\mathscr{J}_\pm(\mho) = \bigcup_{q \in \mho} \mathscr{J}_\pm(q)$ and define

$$\mathbb{D} = \mathscr{J}_+(\mho) \cap \mathscr{J}_-(\mho).$$

We will denote by ∇^g the Levi-Civita connection on $\mathcal{M}$ and let div_g denote the divergence operator on $\mathcal{M}$. The wave (or Laplace–Beltrami) operator $\Box_g$ acting on smooth functions $\mathcal{C}^\infty(\mathcal{M})$ is subsequently defined through $\Box_g u = -\mathrm{div}_g \nabla^g u$. In local coordinates $x = (t := x^0, x^1, \dots, x^n) = (t, x')$, we have

$$\Box_g u = -\sum_{i,j=0}^{n} |g|^{-\frac{1}{2}} \frac{\partial}{\partial x_i} \left(|g|^{\frac{1}{2}} g^{ij} \frac{\partial u}{\partial x_j} \right).$$

We consider the following Cauchy problem:

$$\begin{cases} \Box_g u + V u + u^3 = f, & \forall x \in (0, T) \times M, \\ u(0, x') = 0, \ \partial_t u(0, x') = 0, & \forall x' \in M, \end{cases} \tag{38}$$

where $V \in \mathcal{C}^\infty(\mathcal{M})$. This equation is well-posed for $f \in \mathscr{C}$, where $\mathscr{C}$ denotes a neighborhood of origin in the $\mathcal{C}_c^\kappa(\mho)$ topology with κ a sufficiently large but explicit constant. In other words, for each $f \in \mathscr{C}$, there exists a unique small solution $u \in H^1((0, T) \times M)$ to (38).

We define the source to solution map L_V associated with the Cauchy problem (38) through

$$L_V f := u|_{\mho}, \quad \forall f \in \mathscr{C}. \tag{39}$$

As in the Minkowski case, we are interested in the problem of determining the unknown potential function V on the causal diamond $\mathbb{D}$, given the map L_V. We have the following result.

Theorem 2. *The source to solution map L_V determines V on $\mathbb{D}$ in the sense that*

$$L_{V_1} f = L_{V_2} f \quad \forall f \in \mathscr{C} \qquad \Longrightarrow V_1 = V_2 \text{ on } \mathbb{D}.$$

The rest of this paper is concerned with the proof of this theorem. This is organized as follows. In Section 4, we recall the construction of Gaussian beams, which generalizes the Minkowski geometric optic construction to the globally hyperbolic manifold $(\mathcal{M}, \bar{g})$. Section 5 begins with the construction of the appropriate source terms that produce these Gaussian beams. Next, in Section 5.2, we consider the geometry of null geodesics and study the intersections of such curves. Finally, in Section 5.3, we study the interaction of waves corresponding to a three-fold linearization of the semi-linear equation and derive uniqueness of V.

4. Gaussian beams

This section is concerned with the review of (formal) Gaussian beams for the wave equation. Gaussian beams are a classical construction that was introduced in [2, 21]. The construction in [21] works in general Lorentzian manifolds. We also refer the reader to [12], where an analogous construction is carried out in static Lorentzian manifolds that are products of a time interval with a Riemannian manifold. Gaussian beams were first used in the context of inverse problems in [3, 11]. For the convenience of the reader and to make the notation self-contained, we will present the construction here.

4.1. Fermi coordinates

In this section, we recall Fermi coordinates (or geodesic coordinates) near a null geodesic γ, that is, a geodesic with a light-like tangent vector $\dot{\gamma}$. For similar constructions in the context of stationary Lorentzian geometries or Riemannian geometries with a product structure, we refer the reader to [8] and [6], respectively.

Lemma 1 (Fermi coordinates). *Let $\delta > 0$, $a < b$ and let $\gamma : (a-\delta, b+\delta) \to \mathcal{M}$ be a null geodesic on $\mathcal{M}$. There exists a coordinate neighborhood (U, Φ) of $\gamma([a, b])$, with the coordinates denoted by $(z^0 := s, z^1, \ldots, z^n)$, such that:*

- (i) $\Phi(U) = (a-\delta', b+\delta') \times B(0, \delta')$, *where $B(0, \delta')$ denotes a ball in $\mathbb{R}^n$ with a small radius $\delta' > 0$.*
- (ii) $\Phi(\gamma(s)) = (s, \underbrace{0, \ldots, 0}_{n \text{ times}})$.

Moreover, the metric tensor g satisfies in this coordinate system

$$g|_\gamma = 2ds \otimes dz^1 + \sum_{\alpha=2}^{n} dz^\alpha \otimes dz^\alpha, \tag{40}$$

and $\partial_i g_{jk}|_\gamma = 0$ for $i, j, k = 0, \ldots, n$. Here, $|_\gamma$ denotes the restriction on the curve γ.

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Proof. Write $q = \gamma(a-\delta)$ and $e_0 = \dot\gamma(a-\delta)$. Note that $g(e_0, e_0) = 0$. There are non-zero $c_0 \in \mathbb{R}$ and $e_0' \in T_q M$ such that $e_0 = c_0\partial_t + e_0'$. We set

$$e_1 = \frac{1}{\beta c_0^2}(-c_0\partial_t + e_0').$$

Then $g(e_1, e_1) = 0$ and $g(e_0, e_1) = 2$. Finally, we choose vectors $e_2, \ldots e_n \in T_q\mathcal{M}$ such that $g(e_k, e_k) = 1$ for all $k = 2, \ldots, n$, and $g(e_i, e_j) = 0$ for all $i = 0, 1, \ldots, n$ and $j = 2, \ldots, n$ with $i \neq j$. Then $e_0, \ldots, e_n$ is a pseudo-orthonormal basis on $T_q\mathcal{M}$. For each $k = 0, \ldots, n$, let $E_k(s) \in T_{\gamma(s)}\mathcal{M}$ denote the parallel transport of e_k along γ to the point $\gamma(s)$. Observe that $E_0 = \dot\gamma$. Then $E_0(s), \ldots, E_n(s)$ is a pseudo-orthonormal basis on $T_{\gamma(s)}\mathcal{M}$.

We now define the coordinate system $(z^0 := s, \ldots, z^n)$ through the map

$$\mathcal{F}(s, z^1, \ldots, z^n) = \exp_{\gamma(s)}\left(\sum_{k=1}^{n} z^k E_k(s)\right),$$

where $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ denotes the exponential map on $\mathcal{M}$ at a point p. Clearly,

$$\mathcal{F}(s, \underbrace{0, \ldots, 0}_{n \text{ times}}) = \gamma(s), \quad \forall s \in (a-\delta, b+\delta)$$

is injective as γ is not self-intersecting due to global hyperbolicity. Furthermore,

$$\frac{\partial}{\partial z^k}\mathcal{F}(s, \underbrace{0, \ldots, 0}_{n \text{ times}}) = E_k(s), \quad \text{for } k = 0, \ldots, n.$$

The inverse function theorem applies, and we conclude that $\mathcal{F}$ is a smooth diffeomorphism in a neighborhood of $(a-\delta, b+\delta) \times \{0\}$. We define $\Phi = \mathcal{F}^{-1}$ and note that (i) and (ii) are satisfied.

Since $E_0(s), \ldots, E_n(s)$ is a pseudo-orthonormal basis, (40) holds. Let us now study the derivatives of g on γ. Let $(s, a^1, \ldots a^n) \in \Phi(U)$ be fixed and consider the path $h(t) = \exp_{\gamma(s)}(t\sum_{i=1}^n a^i E_i(s))$. As h is a geodesic, it satisfies

$$\ddot h^k + \Gamma^k_{\alpha\beta}\dot h^\alpha \dot h^\beta = 0,$$

where $\Gamma^k_{\alpha\beta}$ are the Christoffel symbols of the second kind for g. In the Fermi coordinates, $h^0 = s$ and $h^i = ta^i$ for $i = 1, \ldots, n$, and therefore $\ddot h^k = 0$ for all $k = 0, \ldots, n$. By varying $(a^1, \ldots, a^n)$, we see that $\Gamma^k_{\alpha\beta} = 0$ for $k = 0, \ldots, n$ and $\alpha, \beta = 1, \ldots, n$.

As E_α is defined as a parallel transport, there holds

$$\nabla^g_{\partial_0}\partial_j = \nabla^g_{\dot\gamma(s)} E_j(s) = 0,$$

and therefore, using the symmetry of the Levi-Civita connection, $\Gamma^k_{0j} = \Gamma^k_{j0} = 0$ for $k, j = 0, \ldots, n$. Thus all the Christoffel symbols Γ^k_{ij}, $i, j, k = 0, \ldots, n$, vanish on γ. Hence, there holds on γ,

$$\partial_k g_{ij} = \langle \nabla^g_{\partial_k}\partial_i, \partial_j\rangle_g + \langle \partial_i, \nabla^g_{\partial_k}\partial_j\rangle_g = \Gamma_{kij} + \Gamma_{ikj} = 0,$$

where $\Gamma_{\alpha ij} = g_{\alpha\beta}\Gamma^\beta_{ij}$ are the Christoffel symbols of the first kind. □

4.2. WKB approximation

We use the shorthand notation

$$\mathcal{P}_V u = (\Box_g + V)u.$$

Analogously to the approximate geometric optics solutions in Section 2.1, we will construct approximate solutions to $\mathcal{P}_V u = 0$, which concentrate on a given null geodesic $\gamma : (a-\delta, b+\delta) \to \mathcal{M}$. We write $I = [a-\delta', b+\delta']$ with $\delta' > 0$ as in Lemma 1, and define the tubular set

$$\mathcal{V} = \{x \in \mathcal{M} \,|\, s \in I,\ \ |z'| := \sqrt{|z^1|^2 + \cdots + |z^n|^2} < \delta'\}.$$

We consider the WKB ansatz

$$u_\tau(s, z') = e^{i\tau\phi(s,z')} a_\tau(s, z')$$

in the Fermi coordinates $z = (s, z')$ near γ. The complex-valued phase $\phi \in C^\infty(\mathcal{V})$ and amplitude $a_\tau \in C_c^\infty(\mathcal{V})$ will be constructed below.

We have

$$\mathcal{P}_V(e^{i\tau\phi}a_\tau) = e^{i\tau\phi}\left(\tau^2(\mathcal{H}\phi)a_\tau - i\tau\mathcal{T}a_\tau + \mathcal{P}_V a_\tau\right), \tag{41}$$

where the operators $\mathcal{H}, \mathcal{T} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ are defined through

$$\mathcal{H}\phi := \langle d\phi, d\phi\rangle_g, \quad \mathcal{T}a := 2\langle d\phi, da\rangle_g - (\Box_g\phi)a. \tag{42}$$

We make the following ansatz for ϕ and a, respectively:

$$\phi = \sum_{j=0}^{N} \phi_j(s, z') \quad \text{and} \quad a_\tau(s, z') = \chi\left(\frac{|z'|}{\delta'}\right)\sum_{k=0}^{N} \tau^{-k} v_k(s, z'),$$
$$v_k(s, z') = \sum_{j=0}^{N} v_{k,j}(s, z'), \tag{43}$$

where for each $j, k = 0, \ldots, N$, ϕ_j and $v_{k,j}$ are complex-valued homogeneous polynomials of degree j with respect to the variables z^i with $i = 1, \ldots, n$, and $\chi(t)$ is a non-negative smooth function of compact support such that $\chi(t) = 1$ for $|t| \leqslant \frac{1}{4}$ and $\chi = 0$ for $|t| \geqslant \frac{1}{2}$.

The equation $\mathcal{H}\phi = 0$ is often called the eikonal equation, and we require that it is satisfied in the following sense on γ,

$$\frac{\partial^\alpha}{\partial z^\alpha}(\mathcal{H}\phi)(s, 0, \ldots, 0) = 0, \quad \forall s \in I, \tag{44}$$

for all multi-indices $\alpha = \{0, 1, \ldots\}^{1+n}$ with $|\alpha| \leqslant N$. Here I is the interval in Lemma 1. We also require that the following transport type equations are satisfied on γ by the leading v_0 and subsequent v_k, $k = 1, \ldots, N$, amplitudes,

$$\frac{\partial^\alpha}{\partial z^\alpha}(\mathcal{T}v_0)(s, 0, \ldots, 0) = 0, \quad \forall s \in I, \tag{45}$$

$$\frac{\partial^\alpha}{\partial z^\alpha}(-i\mathcal{T}v_k + \mathcal{P}_V v_{k-1})(s, 0, \ldots, 0) = 0, \quad \forall s \in I, \tag{46}$$

for all $\alpha = \{0, 1, \ldots\}^{1+n}$ with $|\alpha| \leqslant N$. With these notations, we will define the following.

Definition 1. An approximate Gaussian beam of order N along γ is a function $u_\tau = e^{i\tau\phi}a_\tau$ with ϕ, a_τ defined as in (43) such that the following properties hold:

(i) Equations (44)–(46) hold.

(ii) $\Im(\phi)|_\gamma = 0$, that is, the imaginary part of ϕ vanishes on γ.

(iii) $\Im(\phi)(z) \geqslant C|z'|^2$ for all points $z \in \mathcal{V}$, where $C > 0$.

It follows from (i) and (iii) that u_τ is an approximate solution to equation $\mathcal{P}_V u = 0$ in the sense of the following lemma.

Lemma 2. *Let u_τ be an approximate Gaussian beam of order N along γ in the sense of Definition 1. Suppose that the end points of γ are outside $[0, T] \times M$ in the sense that $\gamma(a), \gamma(b) \notin [0, T] \times M$. Then for all $\tau > 0$,*

$$\|\mathcal{P}_V u_\tau\|_{H^k((0,T)\times M)} \lesssim \tau^{-K}, \quad \|u_\tau\|_{\mathcal{C}((0,T)\times M)} \lesssim 1,$$

where $K = \frac{N+1}{2} + \frac{n}{4} - k - 2$.

Proof. The second estimate follows trivially from (43) and (iii). Note that equations (44), (45) and (46) imply that

$$|\partial_z^\alpha \mathcal{P}_V u_\tau| \lesssim \tau^{|\alpha|}|e^{i\tau\phi}|(C_0\tau^2|z'|^{N+1} + C_1\tau|z'|^{N+1} + C_2\tau^{-N}).$$

Moreover, $|e^{i\tau\phi}| \leqslant e^{-C\tau|z'|^2}$ by (iii). Writing $r = |z'|$, we obtain the first estimate by using the estimates

$$\left|\int_{B(0,\delta')} e^{-C\tau|z'|^2}|z'|^{2j}\, dz'\right| \leqslant \left|\int_{\mathbb{R}} e^{-C\tau r^2} r^{2j} r^{n-1}\, dr\right| \leqslant C_j \tau^{-j-\frac{n}{2}} \quad \forall\, j \in \mathbb{N} \cup \{0\},$$

for all δ' small, where $C_j = \int_{\mathbb{R}} e^{-C\rho^2}\rho^{2j+n-1} d\rho$. □

Observe also that if $u_\tau = e^{i\tau\phi}a_\tau$ is an approximate Gaussian beam, then also

$$\tilde{u}_\tau = e^{-i\tau\bar{\phi}}\bar{a}_\tau \tag{47}$$

satisfies the estimates in Lemma 2. Here, the notation $\bar{\cdot}$ means complex conjugation.

4.2.1. The phase function. Let us now construct ϕ and a_τ satisfying (44)–(46). We begin by constructing the expansion of the phase function ϕ in such a way that equation (44) holds. For $|\alpha| = 0$, we obtain the equation on γ

$$\sum_{k,l=0}^{n} g^{kl} \frac{\partial\phi}{\partial z^k}\frac{\partial\phi}{\partial z^l} = 0.$$

Using (40), this reduces to

$$2\partial_0\phi\,\partial_1\phi + \sum_{k=2}^{n}(\partial_k\phi)^2 = 0. \tag{48}$$

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Recalling that for all $i, j, k = 0, \ldots, n$, we have $\partial_i g^{jk} = 0$ on γ, we obtain similarly for $|\alpha| = 1$,

$$\sum_{k,l=0}^{n} g^{kl} \partial^2_{ik}\phi \, \partial_l \phi = 0 \tag{49}$$

for all $i = 1, \ldots, n$. Equations (48) and (49) are satisfied setting

$$\phi_0 = 0 \quad \text{and} \quad \phi_1 = z^1. \tag{50}$$

Indeed, (48) holds since $\partial_0 \phi = \partial_k \phi = 0$ for $k = 2, \ldots, n$, and (49) holds since $g^{kl}\partial_l \phi \neq 0$ on γ only if $k = 0$ and $l = 1$, and since $\partial^2_{i0}\phi = 0$ on γ for all $i = 1, \ldots, n$.

Next, we write

$$\phi_2(s, z') := \sum_{1 \leqslant i, j \leqslant n} H_{ij}(s) z^i z^j,$$

where $H_{ij} = H_{ji}$ is a complex-valued matrix. By Definition 1, we require that the imaginary part of H is positive definite, that is,

$$\Im H(s) > 0, \quad \forall s \in I. \tag{51}$$

Equation (44) with $|\alpha| = 2$ is equivalent to

$$\sum_{k,l=0}^{n} (2g^{kl}\partial^3_{kij}\phi \, \partial_l \phi + 2g^{kl}\partial^2_{ki}\phi \, \partial^2_{lj}\phi + \partial^2_{ij} g^{kl} \partial_k \phi \, \partial_l \phi + 4\partial_i g^{kl} \partial^2_{jk}\phi \, \partial_l \phi) = 0$$

for all $i, j = 1, \ldots, n$. Using (40) and (50), $\partial^2_{i0}\phi = 0$, $i = 1, \ldots, n$ and $\partial_i g^{kl} = 0$, this reduces to

$$2g^{10}\partial^3_{0ij}\phi + 2\sum_{k=2}^{n} \partial^2_{ki}\phi \, \partial^2_{kj}\phi + \partial^2_{ij} g^{11} = 0.$$

Noting that $\partial^2_{ij}\phi = 2H_{ij}$, we obtain the following Riccati equation for $H(s)$:

$$\frac{d}{ds}H + HCH + D = 0, \quad \forall s \in I, \tag{52}$$

where C and D are the matrices defined through

$$\begin{cases} C_{11} = 0 & \\ C_{ii} = 2 & i = 2, \ldots, n, \\ C_{ij} = 0 & \text{otherwise}, \end{cases} \quad D_{ij} = \frac{1}{4}\partial^2_{ij} g^{11}. \tag{53}$$

We recall the following result from [12, Section 8] regarding solvability of the Riccati equation.

Lemma 3. *Let $\hat{s}_0 \in I$ and let H_0 be a symmetric matrix with $\Im H_0 > 0$. The Riccati equation (52), together with the initial condition $H(\hat{s}_0) = H_0$, has a unique solution $H(s)$*

for all $s \in I$. We have $\Im H > 0$ and $H(s) = Z(s)Y^{-1}(s)$, where the matrix-valued functions $Z(s), Y(s)$ solve the first order linear system

$$\frac{d}{ds}Y = CZ \quad \textit{and} \quad \frac{d}{ds}Z = -DY, \quad \textit{subject to } Y(\hat{s}_0) = I, Z(\hat{s}_0) = H_0.$$

Moreover, the matrix $Y(s)$ is non-degenerate on I, and there holds

$$\det(\Im H(s)) \cdot |\det(Y(s))|^2 = \det(\Im(H_0)).$$

We refer the reader to [8, Section 3.5] for a geometrically invariant interpretation of the function $Y(s)$ above. With the help of Lemma 3, we have so far succeeded in determining the coefficients of ϕ up to the third term in (43). The remaining terms can be solved through linear first order ordinary differential equations (ODEs).

We will describe only the case $j = 3$ in detail, the cases $j > 3$ being analogous. We see that equation (44) with $\partial_z^\alpha = \partial_p \partial_q \partial_r$ is equivalent to

$$2\sum_{k,l=0}^{n} (g^{kl}\partial_k \partial_z^\alpha \phi\, \partial_l \phi + g^{kl}\partial^3_{kpq}\phi\, \partial^2_{lr}\phi + g^{kl}\partial^3_{kpr}\phi\, \partial^2_{lq}\phi + g^{kl}\partial^3_{kqr}\phi\, \partial^2_{lp}\phi) + \mathcal{F}_\alpha = 0,$$

where $\mathcal{F}_\alpha$ depends only on ϕ_j with $j \leqslant 2$. It holds on γ that

$$\sum_{k,l=0}^{n} g^{kl}\partial_k \partial_z^\alpha \phi\, \partial_l \phi = \partial_s \partial_z^\alpha \phi,$$

and we see that the coefficients $\partial_z^\alpha \phi$ with $|\alpha| = 3$ satisfy a system of linear ODEs with the right-hand side depending on ϕ_j and $\partial_s \phi_j$ with $j \leqslant 2$. Solving this system with any fixed initial condition gives $\partial_z^\alpha \phi$ with $|\alpha| = 3$, and the polynomials ϕ_j of higher degree are constructed analogously.

4.2.2. The amplitude function. We study next the leading amplitude function v_0 by determining the terms $\{v_{0,k}\}_{k\geqslant 0}$ in such a way that equation (45) holds for all $m = 0, \ldots, N$. For $|\alpha| = 0$, using the definition of $\mathcal{T}$, we obtain on γ

$$2\sum_{k,l=0}^{n} g^{kl}\frac{\partial \phi}{\partial z^k}\frac{\partial v_0}{\partial z^l} - (\Box_g \phi)v_0 = 0, \quad \forall s \in I.$$

Recalling Lemma 1, we have on γ

$$-\Box_g \phi = \sum_{i,j=0}^{n} g^{ij}\partial^2_{ij}\phi = \sum_{i=2}^{n}\partial^2_{ii}\phi = \mathrm{Tr}(CH),$$

and therefore

$$2\frac{d}{ds}v_{0,0} + \mathrm{Tr}(CH)v_{0,0} = 0, \quad \forall s \in I.$$

Lemma 3 yields

$$\mathrm{Tr}(CH) = \mathrm{Tr}(\dot{Y}Y^{-1}) = \mathrm{Tr}\,\frac{d}{ds}\log Y = \frac{d}{ds}\log\det Y,$$

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which implies that we can set

$$v_{0,0}(s) = (\det Y(s))^{-\frac{1}{2}}, \quad \forall s \in I. \tag{54}$$

The subsequent terms $v_{0,k}$ with $k = 1, \ldots, N$ can be constructed by solving linear first order ODEs. Indeed, taking $m = k$ in equation (45) and recalling the definition of $\mathcal{T}$, we obtain the following equation for the homogeneous polynomial $v_{0,k}(s, z')$:

$$2\frac{\partial}{\partial s} v_{0,k} + \mathrm{Tr}(CH) v_{0,k} + \mathcal{E}_k = 0 \quad \forall k \geqslant 1,\ s \in I \quad \text{and} \quad v_{0,k}(\hat{s}_0) = 0, \tag{55}$$

where $\mathcal{E}_k$ is a homogeneous polynomial of degree k in the z' coordinates with the coefficients only depending on $\{v_{0,l}\}_{l=0}^{k-1}$ and $\{\phi_l\}_{l=0}^{k+2}$. These first order differential equations can be solved uniquely by prescribing zero initial data at $\hat{s}_0 \in I$.

To determine the subsequent terms v_j, we need to solve equation (46), but this can be accomplished analogously to the above argument and is therefore omitted for the sake of brevity. However, let us establish in detail the analogue of (36) that will be needed later. Equation (46) with $k = 1$ and $|\alpha| = 0$ reads on γ as

$$2\frac{d}{ds} v_{1,0} + \mathrm{Tr}(CH) v_{1,0} = \Box_g v_0 + V v_{0,0},$$

and therefore we may take

$$\begin{aligned} v_{1,0}(s) &= b_{1,0}(s) + c_{1,0}(s), \\ b_{1,0}(s) &= -\frac{i}{2}(\det Y(s))^{-\frac{1}{2}} \int_{\hat{s}_0}^{s} (\Box_g v_0)(\tilde{s}, 0)(\det Y(\tilde{s}))^{\frac{1}{2}}\, d\tilde{s}, \\ c_{1,0}(s) &= -\frac{i}{2}(\det Y(s))^{-\frac{1}{2}} \int_{\hat{s}_0}^{s} V(\tilde{s}, 0)\, d\tilde{s}. \end{aligned} \tag{56}$$

This completes the construction of solutions ϕ and a_τ to equations (44)–(46). The function $u_\tau = e^{i\tau\phi} a_\tau$ is then a formal Gaussian of order N when $\delta' > 0$ in (43) is sufficiently small. Indeed, conditions (i) and (ii) in Definition 1 follow from (44)–(46) and (50), respectively, and (iii) follows from (51) for small $\delta' > 0$.

5. Proof of Theorem 2

5.1. Source terms

Let γ be a null geodesic and suppose that the end points of γ are outside $[0, T] \times M$ in the sense of Lemma 2. Suppose also that γ intersects $\mho$ and write

$$\gamma(0) = q \in \mho, \quad \dot{\gamma}(0) = Z \in T_q\mathcal{M}. \tag{57}$$

Let u_τ be a formal Gaussian beam along γ (with $\hat{s}_0 = 0$). Analogously to (24), we can choose cut-off functions $\zeta_\pm$ such that the solution $\mathcal{U}_\tau$ of

$$\begin{cases} \mathcal{P}_V \mathcal{U}_\tau = f_{\tau,q,Z}, & \forall (t, x') \in (0, T) \times M, \\ \mathcal{U}_\tau(0, x') = 0,\ \partial_t \mathcal{U}_\tau(0, x') = 0, & \forall x' \in M \end{cases} \tag{58}$$

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with the source

$$f_{\tau,q,Z} = \zeta_+(\Box_g + V)(\zeta_- u_\tau) \in \mathcal{C}_c^\infty(\mho)$$

satisfies the estimate

$$\|\mathcal{U}_\tau - \zeta_- u_\tau\|_{H^{k+1}((0,T)\times M)} \lesssim \tau^{-K}. \tag{59}$$

Here K is as in Lemma 2 and $\zeta_- = 1$ in $\mathscr{J}_+(q)$.

It is important to choose the higher order amplitudes $v_{k,j}$, $k \geqslant 1$, $j \geqslant 0$, of the formal Gaussian beam u_τ so that their initial conditions are set to be zero at the point q. A concrete example is given by $v_{1,0}$ in (56). We take $\hat{s}_0 = 0$ in (56), and then the term $c_{1,0}(s)$ that depends on the potential V is known for small s. Indeed, the map L_V determines $V|_\mho$ uniquely, and therefore V is known near $\gamma(0) = q$; cf. (57). As in the Minkowski case, for suitable cut-off functions $\zeta_\pm$, the source $f_{\tau,q,Z}$ can be then constructed given L_V. Finally, we can also construct a test function $f^+_{\tau,q,Z}$ analogously to (26).

5.2. Three-parameter family of sources

We start by recalling a key lemma; see [16, Lemma 2.3]. We will reproduce the proof for completeness, and toward that end, we recall some concepts from Lorentzian geometry, namely, the notions of lengths of causal curves $\alpha : I \to \mathbb{R}$ and time separation between points $p, q \in \mathcal{M}$. We can define the length L of a causal curve $\alpha : I \to \mathbb{R}$ as follows:

$$L(\alpha) = \int_I \sqrt{-g(\dot{\alpha}(s), \dot{\alpha}(s))}\, ds.$$

We also define the time-separation function $\tau(p, q) \in [0, \infty)$ for $p \preccurlyeq q$ through

$$\tau(p, q) = \sup\{L(\alpha) \,|\, \alpha \text{ is a future-pointing causal curve from } p \text{ to } q\}. \tag{60}$$

We set τ to be zero if $p \preccurlyeq q$ does not hold. Under the global hyperbolicity assumption, $\tau : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is continuous. Heuristically, time separation in globally hyperbolic Lorentzian geometries plays the role of Riemannian distance in Riemannian geometries. Indeed, we have the well-known proposition that given $p \preccurlyeq q$, there exists a causal geodesic from p to q of length $\tau(p, q)$. Furthermore given a curve α, we say that a path α on $\mathcal{M}$ is a pre-geodesic if it is $\mathcal{C}^1$ smooth and admits a parametrization $\alpha : I \to \mathcal{M}$ such that $\dot{\alpha}(t) \neq 0$ for all $t \in I$ and $t \mapsto \alpha(t)$ is a geodesic.

Lemma 4. *For any point $p \in \mathbb{D} \setminus \mho$, there are a point $q^+ = (t^+, x'^+) \in \mho$ and a null geodesic γ^+ such that $q^+ = \gamma^+(0)$ and $p = \gamma^+(s_0)$ for some $s_0 < 0$ and that the following statement holds. Given any $\tilde{s}_0 \geqslant s_0$ in a small neighborhood of s_0, there are a point $q^- = (t^-, x'^-) \in \mho$ and a null geodesic γ^- going through the points q^- and $\gamma^+(\tilde{s}_0)$ such that $\gamma^+(\tilde{s}_0)$ is the only point in $[t^-, t^+] \times M$ where the null geodesics $\gamma^\pm$ intersect.*

Proof. By the definition of $\mathbb{D}$, there is a point $\hat{q}^+ = (\hat{t}^+, \hat{x}'^+)$ in $\mho$ such that $p \preccurlyeq \hat{q}^+$. Analogous to [16], we define the earliest observation time,

$$\tilde{t}^+ = \inf\{t \in [0, T] \,|\, \tau(p, (t, \hat{x}'^+)) > 0\},$$

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and set $\tilde{q}^+ = (\tilde{t}^+, \hat{x}'^+)$. As $p \notin \mho$, the points $\tilde{q}^+$ and p are distinct. Writing $p = (t_p, x'_p)$, this implies that $t_p < \tilde{t}^+$. Since $\mathcal{M}$ is globally hyperbolic, τ is continuous and therefore

$$\tau(p, \tilde{q}^+) = 0.$$

But then [20, Proposition 10.46] implies that the causal curve from p to $\tilde{q}^+$ is a null pre-geodesic. In particular, there is a null geodesic γ^+ going through p and $\tilde{q}^+$. Finally, we choose $q^+ = (t^+, x'^+) \in \mho$ on γ^+ such that it lies strictly between p and $\tilde{q}^+$. We also parametrize γ^+ so that $\gamma^+(0) = q^+$ and $p = \gamma^+(s_0)$ for some $s_0 < 0$.

Let us now consider $\tilde{s}_0 \geqslant s_0$ in a small neighborhood of s_0 so that $\gamma^+(\tilde{s}_0) \in \mathbb{D} \setminus \mho$. Again, by the definition of $\mathbb{D}$, there is a point $\hat{q}^- = (\hat{t}^-, \hat{x}'^-) \in \mho$ such that $\hat{q}^- \preccurlyeq \gamma^+(\tilde{s}_0)$. We define the earliest observation time,

$$\tilde{t}^- = \sup\{t \in [0, T] \mid \tau((t, \hat{x}'^-), \gamma^+(\tilde{s}_0)) > 0\},$$

and set $\tilde{q}^- = (\tilde{t}^-, \hat{x}'^-)$. Analogously as above, we conclude that there exists a null geodesic γ^- that goes through the points $\tilde{q}^-$ and $\gamma^+(\tilde{s}_0)$. Finally, we choose $q^- = (t^-, x'^-) \in \mho$ on γ^- such that it lies strictly between p and $\tilde{q}^-$ and p.

We remark that as it stands, the two null geodesics γ^+ and γ^- could be different parametrizations of the same geodesic. Next, we show how to remove this possibility. Indeed, we define $\hat{q}^-_\epsilon = (\tilde{t}^- - \epsilon, \hat{x}'^-)$, where ϵ is sufficiently small so that this point lies in the set $\mho$. Then, the path from $\hat{q}^-_\epsilon$ to $\gamma^+(\tilde{s}_0)$, consisting of a time-like path from $\hat{q}^-_\epsilon$ to $\tilde{q}^-$ together with γ^- from $\tilde{q}^-$ to $\gamma^+(\tilde{s}_0)$, is not a null pre-geodesic. Therefore [20, Proposition 10.46] implies that $\tau(\hat{q}^-_\epsilon, \gamma^+(\tilde{s}_0)) > 0$. The same is true for all points in a small neighborhood of $\hat{q}^-_\epsilon$, and therefore we could replace $\hat{q}^-$ with such a point in the above construction and proceed as before to determine γ^- and q^-. Hence, we may assume without loss of generality that γ^+ and γ^- are not segments of the same null geodesic.

To get a contradiction, let us assume that the two null geodesics γ^+ and γ^- intersect at a point $\tilde{q} = (t, x') \in [t^-, t^+] \times M$ and $\tilde{q} \neq \gamma^+(\tilde{s}_0) = (t_1, x'_1)$. Suppose for the moment that $t_1 < t \leqslant t^+$. Recall that $t^+ < \tilde{t}^+$. Following γ^- from $\gamma^+(\tilde{s}_0)$ to $\tilde{q}$ and then γ^+ from $\tilde{q}$ to $\tilde{q}^+$ gives a causal path from $\gamma^+(\tilde{s}_0)$ to $\tilde{q}^+$. As γ^+ and γ^- are not segments of the same geodesic, this path is not a pre-geodesic. Therefore it follows from [20, Proposition 10.46] that $\tau(\gamma^+(\tilde{s}_0), \tilde{q}^+) > 0$, a contradiction with $\tau(p, \tilde{q}^+) = 0$. The other scenario $t^- \leqslant t < t_1$ can be treated analogously. □

We now proceed as in Section 2.3 with some minor modifications. Consider a point $p = (t_p, x'_p) \in \mathbb{D} \setminus \mho$. Let $\gamma^{(0)} := \gamma^+$ and $q^+ = \gamma^{(0)}(0)$ be as in Lemma 4. We write $p = \gamma^{(0)}(s_0)$ for some $s_0 < 0$ and let $\tilde{s}_0 \geqslant s_0$ be in a small neighborhood of s_0 so that $\gamma^{(0)}(\tilde{s}_0) \in \mathbb{D} \setminus \mho$. We define

$$\tilde{p} = \gamma^{(0)}(\tilde{s}_0) \tag{61}$$

and let the null geodesic $\gamma^{(1)} := \gamma^-$ and the point $q^- \in \mho$ be as given by Lemma 4. We emphasize here that the two null geodesics $\gamma^{(0)}$ and $\gamma^{(1)}$ only intersect at $\tilde{p}$. Next, denote by $\xi^{(0)\sharp}, \xi^{(1)\sharp} \in T_{\tilde{p}}\mathcal{M}$ the tangent vector to $\gamma^{(0)}$ and $\gamma^{(1)}$, respectively, at the point $\tilde{p}$. We may reparametrize $\gamma^{(j)}$ and choose local coordinates near $\tilde{p}$ so that g coincides with the Minkowski metric at $\tilde{p}$, and that

$$\xi^{(0)\sharp} = (1, \cos\theta, \sin\theta, \underbrace{0, \ldots, 0}_{n-2\,\text{times}}), \quad \xi^{(1)\sharp} = (1, 1, 0, \underbrace{0, \ldots, 0}_{n-2\,\text{times}})$$

with $\theta \in [0, 2\pi]$. Here, we emphasize the contrast with the Minkowski setting, where without loss of generality, θ above could be taken to be π. In the Minkowski case, the geodesic from $\tilde{p}$ with the direction $(1, -1, 0, \ldots, 0)$ returns to $\mho$ since this direction is obtained by reversing the spatial component of $\xi^{(1)\sharp}$. Due to lack of symmetries in the general case under consideration, the geodesic to the spatially reversed direction may not intersect $\mho$ (and it may intersect $\gamma^{(1)}$ several times).

We now define

$$\xi^{(2)\sharp} = (1, \sqrt{1-\sigma^2}, \sigma, \underbrace{0, \ldots, 0}_{n-2\,\text{times}}), \quad \xi^{(3)\sharp} = (1, \sqrt{1-\sigma^2}, -\sigma, \underbrace{0, \ldots, 0}_{n-2\,\text{times}}),$$

where $\sigma \in (0, 1)$. By [5, Lemma 1], we have

$$\sigma^2 \xi^{(0)\sharp} + \underbrace{(2b(\theta) + \mathcal{O}(\sigma))}_{\kappa_1} \xi^{(1)\sharp} + \underbrace{(-b(\theta) + \mathcal{O}(\sigma))}_{\kappa_2} \xi^{(2)\sharp} + \underbrace{(-b(\theta) + \mathcal{O}(\sigma))}_{\kappa_3} \xi^{(3)\sharp} = 0, \tag{62}$$

where $b(\theta) = 1 - \cos\theta$. As $\xi^{(0)\sharp} \neq \xi^{(1)\sharp}$ by Lemma 4, it holds that $b(\theta) \neq 0$. In particular, $\lim_{\sigma\to 0} \kappa_j(\sigma)$ is finite and non-zero for $j = 1, 2, 3$. We will write also $\kappa_0(\sigma) = \sigma^2$.

Let $\gamma^{(j)}$, $j = 2, 3$, be null geodesics with the tangent vectors $\xi^{(j)\sharp}$ at the point $\tilde{p}$. We choose σ sufficiently small so that the geodesics $\gamma^{(j)}$ intersect the set $\mho$ at some points $q^{(j)}$ near $q^{(1)} := q^-$. We write also $q^{(0)} := q^+ = \gamma^{(0)}(0)$. We may choose the parametrizations of $\gamma^{(j)}$, $j = 1, 2, 3$, so that $q^{(j)} = \gamma^{(j)}(0)$. Then $\gamma^{(j)}(\tilde{s}_j) = \tilde{p}$ for some $\tilde{s}_j > 0$, $j = 1, 2, 3$. We write also $\dot{\gamma}^{(j)}(0) = Z^{(j)}$ for $j = 0, 1, 2, 3$. To summarize, we have

$$\gamma^{(j)}(0) = q^{(j)}, \quad \dot{\gamma}^{(j)}(0) = Z^{(j)}, \quad j = 0, 1, 2, 3 \tag{63}$$

and

$$\gamma^{(j)}(\tilde{s}_j) = \tilde{p}, \quad \dot{\gamma}^{(j)}(\tilde{s}_j) = \xi^{(j)\sharp}, \quad j = 0, 1, 2, 3.$$

To simplify the discussion, we assume that κ_j in (62) satisfy

$$\kappa_1 > 0 \quad \text{and} \quad \kappa_2, \kappa_3 < 0. \tag{64}$$

Other cases can be treated in a similar manner and are omitted for the sake of brevity. We then construct formal Gaussian beams $u_\tau^{(j)}$, in the Fermi coordinates associated with $\gamma^{(j)}$, of order

$$N \geqslant \frac{3n}{2} + 10, \tag{65}$$

and the form

$$\begin{aligned} u_\tau^{(0)} &= e^{i\kappa_0\tau\phi^{(0)}} a^{(0)}_{\kappa_0\tau}, \quad u_\tau^{(1)} = e^{i\kappa_1\tau\phi^{(1)}} a^{(1)}_{\kappa_1\tau}, \\ u_\tau^{(2)} &= e^{i\kappa_2\tau\bar{\phi}^{(2)}} \bar{a}^{(2)}_{\kappa_2\tau}, \quad u_\tau^{(3)} = e^{i\kappa_3\tau\bar{\phi}^{(3)}} \bar{a}^{(3)}_{\kappa_3\tau}, \end{aligned} \tag{66}$$

where the functions $\phi^{(j)}, a^{(j)}$ are exactly as in Section 4.2 with the initial conditions for all ODEs assigned at the points $q^{(j)} = \gamma^{(j)}(0)$ in the sense that $\hat{s}_0^{(j)} = 0$; see (55) and (56).

Remark 2. Recall that we are considering the case that (64) holds. Because of this, we need the Gaussian beams corresponding to $\gamma^{(2)}$ and $\gamma^{(3)}$ to involve the complex

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conjugation of the phase and amplitude terms. The reason for this specific choice will become clear in the proof of Lemma 5. If alternatively, for example, $\kappa_1 > 0$, $\kappa_2 > 0$ and $\kappa_3 < 0$, the complex conjugation will only appear for $u_\tau^{(3)}$.

We consider the source terms $f_{\tau,q^{(j)},Z^{(j)}}$ for $j = 1, 2, 3$ and the test function $f^+_{\tau,q^{(0)},Z^{(0)}}$ as discussed in Section 5.1. Moreover, we define again the three-parameter family of source $f_{\epsilon,\tau}$ with $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ by (34).

5.3. Recovery of V

We start by considering the formal Gaussian beams $u_\tau^{(j)}$ given by (66). Let $f_{\epsilon,\tau}$ be the three-parameter family constructed in the previous section. In this section, we will complete the proof of Theorem 2 by showing that $V(p)$ is determined from L_V for all $p \in \mathbb{D} \setminus \mho$.

Repeating the argument in the beginning of Section 2.4 shows that L_V determines the integral

$$\mathcal{I} = \int_{\mathcal{M}} \mathcal{U}_\tau^{(0)} \mathcal{U}_\tau^{(1)} \mathcal{U}_\tau^{(2)} \mathcal{U}_\tau^{(3)} \, dV_g,$$

where $dV_g = \sqrt{|g|}\, dt\, dx^1 \dots dx^n$ denotes the volume form on $(\mathcal{M}, g)$. Using the Sobolev embedding, we obtain from (59),

$$\|\mathcal{U}_\tau^{(j)} - \zeta_-^{(j)} u_\tau^{(j)}\|_{\mathcal{C}((0,T)\times M)} \lesssim \tau^{-\frac{n+1}{2}-2}, \quad \forall j = 1, 2, 3, \tag{67}$$

where $\zeta_-^{(j)} = 1$ in $\mathscr{I}_+(q^{(j)})$. Indeed, choice (65) guarantees that $K \geqslant (n+1)/2 + 2$, where K is as in Lemma 2 with $k = n/2 + 1$. Analogously, $\mathcal{U}_\tau^{(0)}$ satisfies the estimate

$$\|\mathcal{U}_\tau^{(0)} - \zeta_+^{(0)} u_\tau^{(0)}\|_{\mathcal{C}((0,T)\times M)} \lesssim \tau^{-\frac{n+1}{2}-2}, \tag{68}$$

with $\zeta_+^{(0)} = 1$ in $\mathscr{I}_-(q^{(j)})$.

We proceed to asymptotically analyze $\mathcal{I}$. Applying estimates (67)–(68) together with the boundedness of formal Gaussian beams (see Lemma 2), we have

$$\tau^{\frac{n+1}{2}} \mathcal{I} = \tau^{\frac{n+1}{2}} \int_{\mathcal{M}} u_\tau^{(0)} u_\tau^{(1)} u_\tau^{(2)} u_\tau^{(3)} \, dV_g + \mathcal{O}(\tau^{-2}). \tag{69}$$

We will use the method of stationary phase to analyze the product of the four formal Gaussian beams in (69), and need the following lemma. In the lemma, we choose d to be an auxiliary distance function on $\mathcal{M}$.

Lemma 5. *Consider the formal Gaussian beams along the geodesics $\gamma^{(k)}$, $k = 0, 1, 2, 3$, in (66), and recall that these four geodesics intersect at $\tilde{p} = \gamma^{(0)}(\tilde{s}_0)$. Recall also that $\kappa_0, \kappa_1 > 0$ while $\kappa_2, \kappa_3 < 0$. Then the function*

$$S := \kappa_0 \phi^{(0)} + \kappa_1 \phi^{(1)} + \kappa_2 \bar{\phi}^{(2)} + \kappa_3 \bar{\phi}^{(3)}$$

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is well defined in a small neighborhood of the point $\tilde{p}$ and there holds the following:

(i) $S(\tilde{p}) = 0$;

(ii) $\nabla^g S(\tilde{p}) = 0$;

(iii) $\Im S(q) \geqslant m\, d(q, \tilde{p})^2$ *for q in a neighborhood of $\tilde{p}$. Here $m > 0$ is a constant.*

Proof. Note that the first claim is trivial as each of the four phases $\phi^{(k)}$ vanishes along $\gamma^{(k)}$ and therefore the sum must vanish at the point of intersection $\tilde{p}$. For the second claim, we note that equation (50) applies to show that along each null geodesic $\gamma^{(k)}$, we have $\nabla^g \phi^{(k)}|_{\gamma^{(k)}} = \dot{\gamma}^{(k)}$. Together with (62), the second claim follows.

Let us now consider the last claim. Note that it suffices to show that

$$D^2 \Im S(X, X) > 0 \quad \forall X \in T_{\tilde{p}}\mathcal{M} \setminus 0.$$

First, note that $\Im S = \sum_{k=0}^{3} |\kappa_k| \Im \phi^{(k)}$, implying that $D^2 \Im S(X, X) \geqslant 0$. Indeed, using the Fermi coordinates, we see that for each $k = 0, 1, 2, 3$,

$$D^2 \Im \phi^{(k)}(X, X) \geqslant 0 \quad \forall X \in T_{\tilde{p}}\mathcal{M},$$
$$D^2 \Im \phi^{(k)}(X, X) > 0 \quad \forall X \in T_{\tilde{p}}\mathcal{M} \setminus \operatorname{span} \xi^{(k)\sharp}$$

due to (50) and (51) and the fact that the Christoffel symbols vanish on $\gamma^{(k)}$ in these coordinates (see Lemma 1). Since $\xi^{(0)\sharp}$ and $\xi^{(1)\sharp}$ are linearly independent, the claim follows. □

We know from Lemma 4 that p is the only point of intersection of the four null geodesics $\gamma^{(j)}$, $j = 0, 1, 2, 3$. Thus the product

$$u_\tau^{(0)} u_\tau^{(1)} u_\tau^{(2)} u_\tau^{(3)} = e^{i\tau S} a_{\kappa_0 \tau}^{(0)} a_{\kappa_1 \tau}^{(1)} \bar{a}_{\kappa_2 \tau}^{(2)} \bar{a}_{\kappa_3 \tau}^{(3)}$$

is supported in a small neighborhood U of $\tilde{p}$. Let us record the following bounds that follow from Lemma 5:

$$\tau^{n+1} \int_U |e^{i\tau S}|^2 \, dV_g + \tau\, \tau^{n+1} \int_U |e^{i\tau S}|^2 \, d(\cdot, p)^2 \, dV_g \leqslant C, \tag{70}$$

where $C > 0$ is independent of τ.

We return to (69) and expand the amplitudes $a_{\kappa_j \tau}^{(j)}$ in terms of the functions $v_k^{(j)}$ as in (43). Applying (70), we obtain

$$\tau^{\frac{n+1}{2}} \mathcal{I} - \mathcal{J} = \tau^{-1} \tau^{\frac{n+1}{2}} \int_U e^{i\tau S} F \, dV_g + \mathcal{O}(\tau^{-\frac{3}{2}}), \tag{71}$$

where

$$\mathcal{J} = \tau^{\frac{n+1}{2}} \int_U e^{i\tau S} v_0^{(0)} v_0^{(1)} \bar{v}_0^{(2)} \bar{v}_0^{(3)} \, dV_g$$

is known (since there is no dependence on V here) and

$$F = \frac{1}{\kappa_0} v_{10}^{(0)} v_{00}^{(1)} \bar{v}_{00}^{(2)} \bar{v}_{00}^{(3)} + \frac{1}{\kappa_1} v_{00}^{(0)} v_{10}^{(1)} \bar{v}_{00}^{(2)} \bar{v}_{00}^{(3)} + \frac{1}{\kappa_2} v_{00}^{(0)} v_{00}^{(1)} \bar{v}_{10}^{(2)} \bar{v}_{00}^{(3)} + \frac{1}{\kappa_3} v_{00}^{(0)} v_{00}^{(1)} \bar{v}_{00}^{(2)} \bar{v}_{10}^{(3)}.$$

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We now consider the right-hand side of (71) and apply the method of stationary phase; see e.g., Theorem 7.7.5 in [10]. Observe that the assumptions of the theorem are satisfied in view of Lemma 5 and the fact that the integrand in (71) is supported in a small neighborhood of $\tilde{p}$. The method of stationary phase gives

$$\tau^{-1}\left(\frac{c_0}{\kappa_0}v_{10}^{(0)}(\tilde{p})+\frac{c_1}{\kappa_1}v_{10}^{(1)}(\tilde{p})+\frac{c_2}{\kappa_2}\bar{v}_{10}^{(2)}(\tilde{p})+\frac{c_3}{\kappa_3}\bar{v}_{10}^{(3)}(\tilde{p})\right)c_4+\mathcal{O}(\tau^{-\frac{3}{2}}),$$

where c_j, $j=0,1,2,3$, are non-zero constants resulting from the leading amplitudes $v_{00}^{(j)}$ and c_4 contains the determinant factor from the stationary phase. In particular, the constants c_j, $j=0,\dots,4$, do not depend on the potential V, and for $j=0,1,2,3$, they do not depend on σ in (62).

Using the splitting $v_{1,0}^{(j)}=b_{1,0}^{(j)}+c_{1,0}^{(j)}$ as in (56) (recall here that $\hat{s}_0^{(j)}=0$) together with the fact that $b_{1,0}^{(j)}$ does not depend on V, we see that the map L_V uniquely determines the expressions

$$\sum_{j=0}^{1}\frac{c_j}{\kappa_j}\int_0^{\tilde{s}_j}V(\gamma^{(j)}(s))\,ds+\sum_{j=2}^{3}\frac{c_j}{\kappa_j}\int_0^{\tilde{s}_j}\bar{V}(\gamma^{(j)}(s))\,ds, \tag{72}$$

where $\gamma^{(j)}(\tilde{s}_j)=\tilde{p}$ for each $j=0,1,2,3$ and in accordance with Lemma 1 and the initial condition (63), the integration along the null geodesics $\gamma^{(j)}$ is with respect to the parametrization that was fixed in (63). Finally, noting that $\lim_{\sigma\to 0}\kappa_j\neq 0$ for $j=1,2,3$ and $\lim_{\sigma\to 0}\kappa_0=0$, and that $\gamma^{(0)}=\gamma^{+}$, we deduce that the knowledge of the source to solution map L_V uniquely determines the integral

$$\int_0^{\tilde{s}_0}V(\gamma^{+}(s))\,ds, \tag{73}$$

where we recall by (61) that $\gamma^{+}(\tilde{s}_0)=\tilde{p}$. Again, we emphasize that the choice of the parametrization along $\gamma^{+}=\gamma^{(0)}$ is fixed here subject to (63). Recall also that $p=\gamma^{+}(s_0)$ and that $\tilde{s}_0\geqslant s_0$ can vary in a neighborhood of s_0 by Lemma 4. Hence, we can differentiate the preceding expression with respect to the parameter $\tilde{s}_0$ and evaluate it at $\tilde{s}_0=s_0$ to conclude that the source to solution map L_V uniquely determines the value $V(p)$ for all $p\in\mathbb{D}\setminus\mho$. This completes the proof of Theorem 2 since V had also been determined on $\mho$.

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