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A NOTE ON EXTENSIONS OF MULTILINEAR MAPS DEFINED ON MULTILINEAR VARIETIES

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Abstract Let $G_1, \dots, G_k$ be finite-dimensional vector spaces over a prime field $\mathbb{F}_p$. A multilinear variety of codimension at most d is a subset of $G_1 \times \cdots \times G_k$ defined as the zero set of d forms, each of which is multilinear on some subset of the coordinates. A map ϕ defined on a multilinear variety B is multilinear if for each coordinate c and all choices of $x_i \in G_i$, $i \neq c$, the restriction map $y \mapsto \phi(x_1, \dots, x_{c-1}, y, x_{c+1}, \dots, x_k)$ is linear where defined. In this note, we show that a multilinear map defined on a multilinear variety of codimension at most d coincides on a multilinear variety of codimension $O_k(d^{O_k(1)})$ with a multilinear map defined on the whole of $G_1 \times \cdots \times G_k$. Additionally, in the case of general finite fields, we deduce similar (but slightly weaker) results.

Keywords: multilinear maps; multilinear varieties; extensions of maps

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1. Introduction

In [4], the authors proved a quantitative version of the inverse theorem for the Gowers U^4 norm over finite fields. The proof depended on a series of results about maps that have bilinear behaviour on subsets of $\mathbb{F}_p^n$, which included the following theorems. In the statements, G_1, G_2, H are finite-dimensional vector spaces over $\mathbb{F}_p$ and $\omega = e^{2\pi i/p}$.

Theorem 1.1 (Gowers and Milićević [4, Theorem 7.7]). *Suppose that $r \geq 20d$ and that $\beta : G_1 \times G_2 \to \mathbb{F}_p^d$ is a bilinear map that satisfies $\mathbb{E}_{x\in G_1, y \in G_2}\, \omega^{\lambda\cdot\beta(x,y)} \leq p^{-r}$ for all $\lambda \in \mathbb{F}_p^d \setminus \{0\}$. Let $D = \{(x, y) \in G_1 \times G_2 \colon \beta(x, y) = 0\}$. Let $\phi\colon D \to H$ be a bilinear map, in the sense that for each $x \in G_1$, the map $\phi_x\colon \{y \in G_2 : (x, y) \in D\} \to H$ given by $y \mapsto \phi(x, y)$ is linear, and the analogous statement holds for the second coordinate. Then there is a bilinear map $\Phi : G_1 \times G_2 \to H$ such that $\Phi(x, y) = \phi(x, y)$ for all $(x, y) \in D$.*

The condition on β in the above theorem is equivalent to the statement that the bilinear form $\lambda \cdot \beta$ has rank at least r for every non-zero $\lambda \in \mathbb{F}_p^d$. Without it, the conclusion is not necessarily true; we shall provide a counterexample later in this section. The next theorem tells us that if the condition does not hold, then we can pass to small-codimensional subspaces where it does.

Theorem 1.2 (Gowers and Milićević [4, Theorem 5.2]). *Let $\beta : G_1 \times G_2 \to \mathbb{F}_p^d$ be a bilinear map and let r be a positive integer. Then, there are subspaces $V_1 \le G_1, V_2 \le G_2$ of codimension at most rd such that $\mathbb{E}_{x \in V_1, y \in V_2}\, \omega^{\lambda \cdot \beta(x,y)} \le p^{-r}$ for all $\lambda \in \mathbb{F}_p^d \setminus \{0\}$.*

Let us say that a set of the form $\{(x, y) \in G_1 \times G_2 \colon \alpha(x) = 0, \beta(y) = 0, \gamma(x, y) = 0\}$ for linear maps $\alpha : G_1 \to \mathbb{F}_p^{t_1}, \beta : G_2 \to \mathbb{F}_p^{t_2}$ and bilinear map $\gamma : G_1 \times G_2 \to \mathbb{F}_p^{t_3}$ is a *bilinear variety of codimension at most* $t = t_1 + t_2 + t_3$. We may combine the two theorems above into a single result.

Corollary 1.3. *Let $\beta : G_1 \times G_2 \to \mathbb{F}_p^d$ be a bilinear map, let $D = \{(x, y) \in G_1 \times G_2 : \beta(x, y) = 0\}$, and let $\phi : D \to H$ be a bilinear map in the sense of Theorem 1.1. Then there is a bilinear variety $B \subset D$ of codimension $O(d^2)$ and a bilinear map $\Phi \colon G_1 \times G_2 \to H$ such that ϕ agrees with Φ on B.*

As we have mentioned, bilinear maps defined on a bilinear variety cannot, in general, be extended to global bilinear maps, so Corollary 1.3 is the best we can hope for in a qualitative sense. For a simple example of a non-extendable map, take the variety $B = \{(x_1, x_2; y_1, y_2) \in \mathbb{F}_p^2 \times \mathbb{F}_p^2 : x_1 y_1 - x_2 y_2 = 0\}$. We may partition B into sets Z and B_λ, where $\lambda \in \mathbb{F}_p \setminus \{0\}$, defined by

$$
\begin{aligned}
Z = {} & \{(0, 0; y_1, y_2) : y_1, y_2 \in \mathbb{F}_p\} \cup \{(x_1, x_2; 0, 0) : x_1, x_2 \in \mathbb{F}_p\} \\
& \cup \{(0, x_2; y_1, 0) : x_2, y_1 \in \mathbb{F}_p\} \cup \{(x_1, 0; 0, y_2) : x_1, y_2 \in \mathbb{F}_p\}
\end{aligned}
$$

and

$$B_\lambda = \{(\lambda x, x; y, \lambda y) : x, y \in \mathbb{F}_p \setminus \{0\}\}.$$

Let $f : \mathbb{F}_p \setminus \{0\} \to \mathbb{F}_p$ be any map. Define a map $\phi : B \to \mathbb{F}_p$ by $\phi(x_1, x_2; y_1, y_2) = 0$, when $(x_1, x_2; y_1, y_2) \in Z$, and $\phi(x_1, x_2; y_1, y_2) = f(\lambda) x_2 y_1$, when $(x_1, x_2; y_1, y_2) \in B_\lambda$, $\lambda \ne 0$. It is easy to check that ϕ is a bilinear map on B for any choice of f.

To see that ϕ cannot be extended to a global bilinear map, it suffices to show that for some f the restriction $\psi \colon \{(x, x) : x \in \mathbb{F}_p\} \to \mathbb{F}_p$ defined by $\psi(x, x) = \phi(x, 1; 1, x)$, cannot be extended to a biaffine map on $\mathbb{F}_p \times \mathbb{F}_p$. Observe that $\psi(x, x) = f(x)$ when $x \ne 0$, and $\psi(0, 0) = 0$, so there are p^{p-1} different functions ψ we may create in this way, while there are only p^4 biaffine maps on $\mathbb{F}_p \times \mathbb{F}_p$.

The construction above works when $p > 5$. When $p \le 5$, we expect that there are similar examples of non-extendable maps, but we shall not pursue that here. Instead, let us mention a related phenomenon that happens when one studies subsets $A \subset G \times G$ that are subspaces in principal directions (that is, for each $x \in G$ we have $\{y \in G \colon (x, y) \in A\} \le G$ and we have an analogous property in the second direction). Again, it turns out that such a property does not guarantee that A is a bilinear variety, as shown by Bienvenu et al. [3] for any prime p. The case when $p = 2$ seems to be somewhat harder

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because it requires $\dim G \geq 2$. On the other hand, such a set A contains a large bilinear variety (while possibly not being equal to it); see [2, 5, 7].

The aim of this note is to generalize Corollary 1.3 to the multivariate case. Let $\mathbb{F}$ be a finite field, which we shall regard as fixed, and now let $G_1, \dots, G_k$ be finite-dimensional vector spaces over $\mathbb{F}$. (All vector spaces in this paper are finite-dimensional and we think of their dimension as large.) We define a *multilinear variety of codimension at most* d in $G_1 \times \cdots \times G_k$ to be a set of the form $V = \{(x_1, \dots, x_k) \in G_1 \times \cdots \times G_k : (\forall i \in [d]) \beta_i(x_{I_i}) = 0\}$, where the maps $\beta_i : \prod_{j \in I_i} G_j \to \mathbb{F}$ are multilinear (i.e. linear in each variable separately) forms for $i \in [d]$ and $\beta_i(x_{I_i})$ means that we use x_j for $j \in I_i$ as arguments of β_i. Observe that V has the property that if we fix values $x_1 \in G_1, \dots, x_{i-1} \in G_{i-1}, x_{i+1} \in G_{i+1}, \dots, x_k \in G_k$ for all coordinates but coordinate i, then the set

$$V_{x_1,\dots x_{i-1},x_{i+1},\dots,x_k} = \{y_i \in G_i \colon (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_k) \in V\}$$

is actually a non-empty subspace of G_i. Hence, if H is another vector space over $\mathbb{F}$, we define a map $\phi \colon V \to H$ to be *multilinear* if the restriction

$$\phi' \colon V_{x_1,\dots x_{i-1},x_{i+1},\dots,x_k} \to H, \quad \phi'(y_i) = \phi(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_k)$$

is linear for every choice of coordinate i and elements $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k$.

Our main theorem is the following. Note that it is stated for prime fields $\mathbb{F}_p$ instead of general finite fields.

Theorem 1.4. *For each positive integer k there are constants $C = C_k, D = D_k$ such that the following statement holds. Let $G_1, \dots, G_k$ be vector spaces over a prime field $\mathbb{F}_p$. Let B be a multilinear variety of codimension d in $G_1 \times \cdots \times G_k$ and let $\phi \colon B \to H$ be a multilinear map to a vector space H over $\mathbb{F}_p$. Then there is a global multilinear map $\Phi : G_1 \times \cdots \times G_k \to H$ such that the set $\{(x_1, \dots, x_k) \in B : \Phi(x_1, \dots, x_k) = \phi(x_1, \dots, x_k)\}$ contains a multilinear variety of codimension at most Cd^D.*

Note that the constants C and D do not depend on the prime p.* Also, a variety of codimension at most r is easily seen to have density at least p^{-kr} in the ambient space (see Lemma 2.3). This means that the set $\{(x_1, \dots, x_k) \in B : \Phi(x_1, \dots, x_k) = \phi(x_1, \dots, x_k)\}$ is necessarily large.

In the case of general finite fields, we prove the following result.

Theorem 1.5. *For each positive integer k and finite field $\mathbb{F}$, there are constants $C = C_{k,\mathbb{F}}, D = D_k$ such that the following statement holds. Let $G_1, \dots, G_k$ be vector spaces over $\mathbb{F}$. Let p be the characteristic of the field $\mathbb{F}$ and view $\mathbb{F}_p$ as a subfield of $\mathbb{F}$. Let B be a multilinear variety of codimension at most d in $G_1 \times \cdots \times G_k$ and let $\phi \colon B \to H$ be a multilinear map to a vector space H over $\mathbb{F}$. (Note that $G_1, \dots, G_k$ may be seen as vector spaces over $\mathbb{F}_p$ as well.) Then there are a global multilinear map $\Phi : G_1 \times \cdots \times G_k \to H$ and an $\mathbb{F}_p$-multilinear variety B' of codimension at most Cd^D such that $\Phi(x_1, \dots, x_k) = \phi(x_1, \dots, x_k)$ holds for all elements $(x_1, \dots, x_k) \in B'$.*

* The proof gives bounds $C \leq 2^{2^{2^{O(k^2)}}}$ and $D \leq 2^{2^{O(k^2)}}$.

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We remark that it is likely that the constant C in Theorem 1.5 does not depend on $\mathbb{F}$. However, we shall first prove Theorem 1.4 and then use that special case to deduce the second result, which will incur the cost of the additional dependence of C on $\mathbb{F}$. Note also that in Theorem 1.5, we only find an $\mathbb{F}_p$-multilinear variety of bounded codimension inside the set X of points where the given map coincides with a global multilinear map. However, it is likely that X contains an $\mathbb{F}$-multilinear variety of bounded codimension, which does not follow from the arguments given in this paper.

Theorem 1.4 relies crucially on power-type bounds for partition rank in terms of analytic rank, which were independently proved by Janzer [8] and the second author [12]. (The relevant definitions and a precise statement of the result will be given at the end of § 2.) Let us also note that Kazhdan and Ziegler generalized Theorem 1.1 in [9], but their result, like Theorem 1.1, has the crucial assumption that the domain of the given map is a variety of high rank. However, in higher dimensions, finding a high rank subvariety inside the given variety leads to significantly worse bounds than those in Theorem 1.4.

Organization of the paper. We begin our work with a short preliminary section that contains useful auxiliary results. Then, in §3 and §4, we prove Theorem 1.4 in the case of prime fields. Finally, we deduce the full result in § 5.

2. Preliminaries

Let $\mathbb{F}$ be a finite field, and write $\mathbf{f}$ for its cardinality $|\mathbb{F}|$. In this preliminary section, we do not assume that $\mathbb{F}$ is a prime field, and if we treat the case of prime fields we use the notation $\mathbb{F}_p$. Further, fix k and vector spaces $G_1, \dots, G_k$ over $\mathbb{F}$. We recall the following notational conventions, definitions and proposition from [12].

Notation. In the rest of the paper, we use the following abbreviations in situations where we have many indices appearing in predictable patterns. Given a sequence $x_1, \dots, x_m$, we shall denote it by $x_{[m]}$, and more generally if $I \subset [m]$ then we shall write x_I for the subsequence with indices that run through I. We shall do the same for products of the spaces G_i as well: $G_{[k]}$ will stand for $\prod_{i \in [k]} G_i$ and G_I for $\prod_{i \in I} G_i$. For example, instead of writing $\alpha\colon \prod_{i \in I} G_i \to \mathbb{F}$ and $\alpha(x_i \colon i \in I)$, we write $\alpha\colon G_I \to \mathbb{F}$ and $\alpha(x_I)$.

Recall that a map $\phi\colon U \to V$ between vector spaces is *affine* if $u \mapsto \phi(u) - \phi(0)$ is linear. A map $\alpha\colon G_{[k]} \to H$, where H is a vector space over $\mathbb{F}$, is *multiaffine* if it is affine in each variable separately. We refer to the zero set of a multiaffine map $\alpha\colon G_{[k]} \to H$ as a *variety*, and we say that such a variety has codimension at most $\dim H$. (Note that multilinear varieties are varieties in the sense of this definition.) Equivalently, we define codimension of a variety V as the least value of $\dim H$, where we range over all multiaffine maps $\alpha\colon G_{[k]} \to H$ that have V as their zero set.

Another convention we adopt is that we write $\mathbb{E}_x$, without specifying the set from which x is taken, when this causes no confusion. Frequently, we shall consider 'slices' of sets $S \subset G_{[k]}$, by which we mean sets $S_{x_I} = \{y_{[k]\setminus I} \in G_{[k]\setminus I} \colon (x_I, y_{[k]\setminus I}) \in S\}$, for $I \subset [k], x_I \in G_I$. (Here $(x_I, y_{[k]\setminus I})$ denotes not the concatenation of the two sequences but the sequence $w_{[k]}$, where $w_i = x_i$ when $i \in I$ and $w_i = y_i$ when $i \in [k] \setminus I$.) Occasionally, we might have a single element $z \in G_i$ instead of x_I, and in this case, we write $S_{i\colon z}$ for the resulting slice, since the direction i is not clear from the notation z, unlike in the case of x_I. In other words, $S_{i:z}$ is the set $\{y_{[k]\setminus\{i\}} : (z, y_{[k]\setminus\{i\}}) \in S\}$ (with a similar

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interpretation of $(z, y_{[k]\setminus\{i\}})$). Finally, for each vector space G_i, fix a dot product. We need this for the characterization of linear forms on G_i – each linear form $\phi\colon G_i \to \mathbb{F}$ takes the form $\phi(x) = x \cdot u$ for some element $u \in G_i$.

Define a graph $\mathcal{G}$ with vertex set $G_{[k]}$ by putting edges between points that differ in a single coordinate. We say that a set $S \subset G_{[k]}$ is *connected* if the induced graph $\mathcal{G}[S]$ is connected. The *diameter* of S is the largest distance between two vertices in the graph $\mathcal{G}[S]$. In the rest of the paper, we fix a non-trivial additive character $\chi\colon \mathbb{F} \to \mathbb{C}$.

Proposition 2.1 (one-sided regularity lemma [12, Proposition 10]). *Write $c_k = 4(k+1)$. Let $\rho\colon G_{[k]} \to \mathbb{F}$ and $\beta_i\colon G_{I_i} \to \mathbb{F}$ $(i = 1, 2, \dots, r)$ be multilinear forms, where $I_i \subset [k]$. Let $\mathcal{I} = \{i \in [r]\colon I_i = [k]\}$. Suppose that*

$$\mathop{\mathbb{E}}_{x_1,\dots,x_k} \chi\Big(\rho(x_{[k]}) - \sum_{i\in\mathcal{I}} \lambda_i \beta_i(x_{[k]})\Big) \le \eta = \mathbf{f}^{-c_k(r+1)},$$

for any choice of $\lambda \in \mathbb{F}^{\mathcal{I}}$. Then the set of $x_{[k]} \in G_{[k]}$ for which $\rho(x_{[k]}) \neq 0$ and $\beta_i(x_{I_i}) = 0$ for $i = 1, 2, \dots, r$ is connected and has diameter at most $(2k+1)(2^k - 1)$.

Note that the expression on the left-hand side of the displayed inequality is a non-negative real, so there is no need for absolute values.

Corollary 2.2. *Let $\rho, \beta_1, \dots, \beta_r$ be as in Proposition 2.1. Let $x_{[k]}, y_{[k]} \in G_{[k]}$ be such that $\rho(x_{[k]}), \rho(y_{[k]}) \neq 0$ and $\beta_i(x_{I_i}) = \beta_i(y_{I_i}) = 0$ for all $i \in [r]$. Then, there are points $q^0_{[k]}, q^1_{[k]}, \dots, q^s_{[k]} \in G_{[k]}$ with the following properties.*

(1) Any two consecutive points differ in exactly one coordinate.

(2) The first point $q^0_{[k]}$ is equal to $x_{[k]}$, and the last point $q^s_{[k]}$ is equal to $(\lambda_1 y_1, \dots, \lambda_k y_k)$, for some non-zero $\lambda_1, \dots, \lambda_k \in \mathbb{F}$.

(3) The number s is at most $(2k+1)(2^k - 1)$.

(4) We have $\rho(q^0_{[k]}) = \rho(q^1_{[k]}) = \cdots = \rho(q^s_{[k]})$ and $\beta_j(q^i_{I_j}) = 0$ for all $i \in [0, s], j \in [r]$.

Proof. By Proposition 2.1, the set $\{x_{[k]} \in G_{[k]}\colon (\forall i \in [r])\beta_i(x_{I_i}) = 0, \rho(x_{[k]}) \neq 0\}$ is connected and of diameter at most $(2k+1)(2^k - 1)$. Hence, there is a sequence $q^0_{[k]}, q^1_{[k]}, \dots, q^s_{[k]} \in G_{[k]}$ that satisfies the first three of the listed properties, $\rho(q^0_{[k]}), \dots, \rho(q^s_{[k]}) \neq 0$ and $\beta_j(q^i_{I_j}) = 0$ for all $i \in [0, s], j \in [r]$. By induction on $t \in [0, s]$, we show that there is a sequence $p^0_{[k]}, p^1_{[k]}, \dots, p^s_{[k]} \in G_{[k]}$ that satisfies the first three of the listed properties, where we relax the first property to allow consecutive points to be equal and that also satisfies a modified version of the last property, namely that $\rho(p^0_{[k]}) = \rho(p^1_{[k]}) = \cdots = \rho(p^t_{[k]}) \neq 0$, $\rho(p^{t+1}_{[k]}), \dots, \rho(p^s_{[k]}) \neq 0$, and $\beta_j(p^i_{I_j}) = 0$ for all $i \in [0, s], j \in [r]$. For $t = 0$, we may take $p^i_{[k]} = q^i_{[k]}$. Assume now that the claim holds for some $t < s$, and let $p^0_{[k]}, \dots, p^s_{[k]}$ be the sequence so far. Then, points $p^t_{[k]}$ and $p^{t+1}_{[k]}$ differ in a single coordinate, say $c \in [k]$. Let $\lambda \in \mathbb{F} \setminus \{0\}$ be such that $\rho(p^t_{[k]}) = \lambda\rho(p^{t+1}_{[k]})$. Modify all points $p^{t+1}_{[k]}, \dots, p^s_{[k]}$ by multiplying their c-coordinate by λ. It is easy to check that the modified sequence satisfies all the properties.

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Once we have a sequence for $t = s$, remove points that are equal to their predecessor to finish the proof. □

We shall also need to know that the set considered in the results above is necessarily non-empty. To prove this, we need two simple lemmas.

Lemma 2.3 (Milićević [12, Lemma 11]). *Let $B \subset G_{[k]}$ be a non-empty variety of codimension at most d. Then $|B| \geq \mathbf{f}^{-kd}|G_{[k]}|$.*

When $\alpha\colon G_{[k]} \to H$ is a multiaffine map, it is a simple linear-algebraic fact that α can be uniquely written as $\alpha(x_{[k]}) = \sum_{I \subset [k]} \alpha_I(x_I)$, for some multilinear maps $\alpha_I\colon G_I \to H$ for $I \subset [k]$ (where $\alpha_\emptyset$ is interpreted as a constant). We refer to $\alpha_{[k]}$ as the *multilinear part* of α. We shall write α^{lin} for $\alpha_{[k]}$, unless stated otherwise.

Lemma 2.4 (Lovett [11, Lemma 2.1]). *Suppose that $\alpha\colon G_{[k]} \to \mathbb{F}$ is a multiaffine form with multilinear part α^{lin}. Then*

$$\left|\mathop{\mathbb{E}}_{x_{[k]}} \chi(\alpha(x_{[k]}))\right| \leq \mathop{\mathbb{E}}_{x_{[k]}} \chi(\alpha^{lin}(x_{[k]})).$$

To save space, given multilinear forms $\beta_1, \ldots, \beta_r$ and $\lambda \in \mathbb{F}^r$, we shall write $\lambda \cdot \beta$ for the multilinear form $\sum_{i \in [r]} \lambda_i \beta_i$.

Lemma 2.5. *Let $\rho, \beta_1, \ldots, \beta_r\colon G_{[k]} \to \mathbb{F}$ be multilinear forms and let $m \in \mathbb{N}$ be such that for all choices of $\lambda \in \mathbb{F}^r$,*

$$\mathop{\mathbb{E}}_{x_{[k]}} \chi\left(\rho(x_{[k]}) + (\lambda \cdot \beta)(x_{[k]})\right) < \mathbf{f}^{-k(r+m)}.$$

Then for any multilinear forms $\gamma_i\colon G_{I_i} \to \mathbb{F}$, $\emptyset \neq I_i \subsetneq [k]$, $i = 1, 2, \ldots, m$, we may find $x_{[k]} \in G_{[k]}$ such that

(1) $\rho(x_{[k]}) = 1$,

(2) $(\forall i \in [r])\ \beta_i(x_{[k]}) = 0$, *and*

(3) $(\forall i \in [m])\ \gamma_i(x_{I_i}) = 0$.

Proof. Suppose that, on the contrary, whenever a point $x_{[k]}$ satisfies $\beta_i(x_{[k]}) = 0$ for all $i \in [r]$ and $\gamma_i(x_{I_i}) = 0$ for all $i \in [m]$, then $\rho(x_{[k]}) = 0$. The set of such points is a multilinear variety of codimension at most $r + m$, so by Lemma 2.3,

$$\begin{aligned}\mathbf{f}^{-k(r+m)} &\leq \mathop{\mathbb{E}}_{x_{[k]}} \mathbf{1}\left((\forall i \in [r])\beta_i(x_{[k]}) = 0 \wedge (\forall i \in [m])\gamma_i(x_{I_i}) = 0\right)\\ &= \mathop{\mathbb{E}}_{x_{[k]}} \chi(\rho(x_{[k]}))\mathbf{1}\left((\forall i \in [r])\beta_i(x_{[k]}) = 0 \ \wedge\ (\forall i \in [m])\ \gamma_i(x_{I_i}) = 0\right)\end{aligned}$$

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$$= \mathop{\mathbb{E}}_{x_{[k]}} \mathop{\mathbb{E}}_{\lambda\in\mathbb{F}^r,\mu\in\mathbb{F}^m} \chi\Big(\rho(x_{[k]}) + (\lambda\cdot\beta)(x_{[k]}) + \sum_{i\in[m]} \mu_i\gamma_i(x_{I_i})\Big)$$
$$\leq \mathop{\mathbb{E}}_{\lambda\in\mathbb{F}^r,\mu\in\mathbb{F}^m} \Big|\mathop{\mathbb{E}}_{x_{[k]}} \chi\Big(\rho(x_{[k]}) + (\lambda\cdot\beta)(x_{[k]}) + \sum_{i\in[m]} \mu_i\gamma_i(x_{I_i})\Big)\Big|.$$

By Lemma 2.4, this is at most $\mathbb{E}_{\lambda\in\mathbb{F}^r} |\mathbb{E}_{x_{[k]}} \chi(\rho(x_{[k]}) + (\lambda\cdot\beta)(x_{[k]}))|$, which by hypothesis is less than $\mathbf{f}^{-k(r+m)}$. This is a contradiction, so the lemma is proved. □

The purpose of the next lemma is to enable us to deduce the value that ϕ takes at certain points in a situation where, because ϕ is not defined everywhere, one cannot straightforwardly expand and use bilinearity.

Lemma 2.6. *Let $U \leq G_1$ and $V \leq G_2$ be subspaces over prime field $\mathbb{F}_p$ and let $\beta\colon G_1\times G_2 \to \mathbb{F}_p^r$ and $\rho\colon G_1\times G_2\to\mathbb{F}_p$ be bilinear. Let $B = \{(x, y) \in U\times V\colon \beta(x, y) = 0\}$ and let $B^0 = \{(x, y)\in B\colon \rho(x, y) = 0\}$. Let $(x, y), (z, w), (u, v) \in B$ be points such that $\rho(x, y) = \rho(z, w) = \rho(u, v) = 1$ and $\rho = 0$ for all other points in $\{x, z, u\}\times\{y, w, v\}$. Let $\phi\colon B^0 \to H$ be a bilinear map. Then, for all $\ell\in\mathbb{F}_p$, we have*

$$\begin{aligned}\phi(x-\ell z, \ell y + w) &= \phi(x - z, y + w) + (\ell-1)\phi(x-u, y+v) - (\ell-1)\phi(z-u, w+v)\\ &\quad - (\ell-1)\phi(x, v) - (\ell^2-1)\phi(z, y)\\ &\quad + (\ell-1)\phi(u, y) + (\ell-1)\phi(z, v) - (\ell-1)\phi(u, w).\end{aligned}$$

Also,

$$\begin{aligned}\phi(x-\ell z, \ell y + w) &= \ell\phi(x-u, y+v) - \ell\phi(z-u, w+v) + \phi(x, w) - \ell\phi(x, v)\\ &\quad - \ell^2\phi(z, y) + \ell\phi(u, y) + \ell\phi(z, v) - \ell\phi(u, w).\end{aligned}\tag{2.1}$$

Remark. *The proof of this lemma works only for prime fields.*

Notational remark. Here and in the rest of the paper, whenever ϕ is a map with domain D and we write an expression of the form $\phi(q)$, we are tacitly stating that the point q lies in D.

Proof. Note first that our hypotheses imply that all the points where we evaluate ϕ do indeed belong to B^0. We prove the claim by induction on ℓ. For $\ell = 1$, the claim is easy to check. Assume now that it holds for some $\ell - 1$. Then

$$\begin{aligned}\phi(x-\ell z, \ell y+w) &= \phi(x-\ell z, \ell y + w + v) - \phi(x, v) + \ell\phi(z, v)\\ &= \phi(x-(\ell-1)z - u, \ell y + w + v)\\ &\quad - \phi(z-u, \ell y + w + v) - \phi(x, v) + \ell\phi(z, v)\\ &= \phi(x-(\ell-1)z-u, \ell y+w+v)\\ &\quad - \phi(z-u, w+v) - \ell\phi(z, y) + \ell\phi(u, y) - \phi(x, v) + \ell\phi(z, v)\end{aligned}$$

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$$
\begin{aligned}
&= \phi(x-(\ell-1)z-u,(\ell-1)y+w)\\
&\quad + \phi(x-(\ell-1)z-u,y+v) - \phi(z-u,w+v)\\
&\quad - \ell\phi(z,y) + \ell\phi(u,y) - \phi(x,v) + \ell\phi(z,v)\\
&= \phi(x-(\ell-1)z-u,(\ell-1)y+w)\\
&\quad + \phi(x-u,y+v) - (\ell-1)\phi(z,y) - (\ell-1)\phi(z,v)\\
&\quad - \phi(z-u,w+v) - \ell\phi(z,y)\\
&\quad + \ell\phi(u,y) - \phi(x,v) + \ell\phi(z,v)\\
&= \phi(x-(\ell-1)z,(\ell-1)y+w)\\
&\quad - (\ell-1)\phi(u,y) - \phi(u,w) + \phi(x-u,y+v)\\
&\quad - (\ell-1)\phi(z,y) - (\ell-1)\phi(z,v) - \phi(z-u,w+v)\\
&\quad - \ell\phi(z,y) + \ell\phi(u,y) - \phi(x,v) + \ell\phi(z,v)\\
&= \phi(x-(\ell-1)z,(\ell-1)y+w)\\
&\quad + \phi(x-u,y+v) - \phi(z-u,w+v)\\
&\quad - \phi(x,v) - (2\ell-1)\phi(z,y)\\
&\quad + \phi(u,y) + \phi(z,v) - \phi(u,w)\\
&= \phi(x-z,y+w) + (\ell-1)\phi(x-u,y+v)\\
&\quad - (\ell-1)\phi(z-u,w+v)\\
&\quad - (\ell-1)\phi(x,v) - (\ell^2-1)\phi(z,y)\\
&\quad + (\ell-1)\phi(u,y) + (\ell-1)\phi(z,v) - (\ell-1)\phi(u,w),
\end{aligned}
$$

where we applied the induction hypothesis in the last line.

To deduce the second equality in the statement, use the first equality with $\ell = 0$ to write $\phi(x-z,y+w)$ in terms of other summands. □

Finally, we shall also need polynomial bounds for partition rank in terms of analytic rank, whose definitions we now recall. Let $\alpha\colon G_{[k]} \to \mathbb{F}$ be a multilinear form.

The *partition rank* of α, introduced by Naslund in [13], is the smallest r such that α can be written in the form $\alpha(x_{[k]}) = \sum_{i\in[r]} \beta_i(x_{I_i})\gamma_i(x_{[k]\setminus I_i})$, for further multilinear forms $\beta_i\colon G_{I_i} \to \mathbb{F}$ and $\gamma_i\colon G_{[k]\setminus I_i} \to \mathbb{F}$, where $\emptyset \neq I_i \neq [k]$. We write $\operatorname{prank}_{\mathbb{F}}(\alpha)$ for this quantity. The *analytic rank* of α, introduced by Gowers and Wolf in [6], is defined to be the quantity $-\log_{\mathbf{f}} \mathbb{E}_{x_{[k]}} \chi(\alpha(x_{[k]}))$.

When $k = 2$, it is straightforward to check that both the partition rank and the analytic rank are equal to the rank of α in the usual linear-algebraic sense. However, when $k \geq 3$, the situation is more complicated, partly because there are many competing algebraic definitions of rank. The fact that partition rank can be bounded in terms of analytic rank was proved by Bhowmick and Lovett in [1], where they obtained Ackermannian bounds. As was very recently proved, one may in fact take polynomial bounds.

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Theorem 2.7 (Janzer [8], Milićević [12]). *For every positive integer $k \geq 2$, there are constants $C = C_k^{\text{ranks}}, D = D_k^{\text{ranks}} > 0$ with the following property. Suppose that $\alpha\colon G_{[k]} \to \mathbb{F}$ is a multilinear form of analytic rank r. Then the partition rank of α is at most $C(r^D + 1)$.*

Note that the proof in [12] yields constants C_k and D_k that do not depend on the cardinality of the field $\mathbb{F}$. In the special case of polynomials on a single vector space, this was conjectured by Kazhdan and Ziegler [9, 10].

3. Extending multilinear maps using one-sided regularity

Important notational remark. In this section and the following one, we shall use the notation $\mathbb{F}$ to denote a *prime* field and $\mathbf{f}$ to denote its cardinality (so $\mathbf{f}$ is a prime). While one would normally write $\mathbb{F}_p$ in this situation, we wish to use the letter p to stand for points in our arguments. However, the fact that $\mathbb{F}$ is a prime field will play a role only at a single step (which is the application of Lemma 2.6), so the notation $\mathbb{F}$ should not be too misleading.

When two points $x_{[k]}, y_{[k]} \in G_{[k]}$ differ in a single coordinate, say d, we write $(x \ominus y)_{[k]}$ for the point with coordinates $(x \ominus y)_i = x_i = y_i$, when $i \neq d$, and $(x \ominus y)_d = x_d - y_d$. Notice that if B is a multilinear variety, then whenever $x_{[k]}, y_{[k]} \in B$ differ in a single coordinate, the point $x \ominus y$ belongs to B as well. Recall that a map $\phi\colon B \to H$, where H is another $\mathbb{F}$-vector space, is multilinear if the restriction

$$\phi'\colon B_{x_{[k]\setminus\{i\}}} \to H, \quad \phi'(y_i) = \phi(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_k)$$

is linear for every choice of coordinate i and elements $x_{[k]\setminus\{i\}}$. Notice that whenever $x_{[k]}, y_{[k]} \in B$ differ in a single coordinate then $\phi(x \ominus y) = \phi(x) - \phi(y)$.

Theorem 3.1. *Let $\rho\colon G_{[k]} \to \mathbb{F}$ and $\beta_i\colon G_{I_i} \to \mathbb{F}$, $i \in [m]$ be multilinear forms. Write $\mathcal{I} = \{i \in [m]\colon I_i = [k]\}$. Let $B = \{x_{[k]} \in G_{[k]}\colon (\forall i \in [m])\, \beta_i(x_{I_i}) = 0\}$ and let $B^0 = \{x_{[k]} \in B\colon \rho(x_{[k]}) = 0\}$. Let H be another $\mathbb{F}$-vector space and let $\phi\colon B^0 \to H$ be a multilinear map. Suppose that for each $\lambda \in \mathbb{F}^{\mathcal{I}}$*

$$\mathop{\mathbb{E}}_{x_{[k]}} \chi\left(\rho(x_{[k]}) + \sum_{i\in\mathcal{I}} \lambda_i \beta_i(x_{[k]})\right) < \frac{1}{2k^2}\mathbf{f}^{-(2k^2+k+1)(m+1)2^{2k+3}}. \tag{3.1}$$

Then, for each $z_{[k]} \in B \setminus B^0$ and $h_0 \in H$, there is a unique multilinear map $\phi^{\text{ext}}\colon B \to H$ such that $\phi^{\text{ext}}\big|_{B^0} = \phi$ and $\phi^{\text{ext}}(z_{[k]}) = h_0$.

Remark. The theorem says that if ρ is sufficiently quasirandom with respect to the other forms β_i, then we may uniquely extend ϕ to the larger variety B that we obtain by removing ρ from the definition of the domain of ϕ. This observation is crucial and it allows us to avoid strong assumptions such as the domain variety having high rank (as in the result of Kazhdan and Ziegler).

The proof splits up into several stages. We begin by explaining how the map ϕ^{ext} is defined. To simplify the writing slightly, we assume that $\rho(z_{[k]}) = 1$, which we may do

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without loss of generality. Let $x_{[k]} \in B \setminus B^0$ be given. By Corollary 2.2 (these $z_{[k]}$ and $x_{[k]}$ play the roles of $x_{[k]}$ and $y_{[k]}$ of the corollary, respectively) there is a sequence $z_{[k]} = q^0_{[k]}, q^1_{[k]}, \ldots, q^s_{[k]} = (\lambda_1 x_1, \ldots, \lambda_k x_k) \in G_{[k]}$ with the properties stated in the conclusion of that corollary, the fourth of which gives us that $\rho(q^s_{[k]}) = 1$ and therefore that $\rho(x_{[k]}) = \prod_{i\in[k]} \lambda_i^{-1}$.

Motivated by this, for an integer $\mathbf{s}$, we say that a sequence of points $t^0_{[k]}, t^1_{[k]}, \ldots, t^s_{[k]} \in G_{[k]}$ is $\mathbf{s}$-*good* if:

(1) we have $s \le \mathbf{s}$,

(2) any two consecutive points $t^i_{[k]}$ and $t^{i+1}_{[k]}$ differ in exactly one coordinate, and

(3) we have $\rho(t^0_{[k]}) = \rho(t^1_{[k]}) = \cdots = \rho(t^s_{[k]})$ and $\beta_j(t^i_{I_j}) = 0$ for all $i \in [0, s], j \in [r]$.

We call $t^0_{[k]}$ and $t^s_{[k]}$ the endpoints of the sequence. In particular, Corollary 2.2 says that for any $x_{[k]} \in B \setminus B^0$ (recall that $z_{[k]}$ was fixed) there is always a $(2k+1)(2^k-1)$-good sequence with endpoints $z_{[k]}$ and $(\lambda_1 x_1, \ldots, \lambda_k x_k)$ for some scalars λ_i.

Assume for a moment that $\phi^{\text{ext}} \colon B \to H$ is a multilinear map that extends ϕ. Then, since each $(q^{i+1} \ominus q^i)_{[k]} \in B^0$, we must have

$$\begin{aligned}
\phi^{\text{ext}}(x_{[k]}) &= \Big(\prod_{i\in[k]} \lambda_i^{-1}\Big) \phi^{\text{ext}}(q^s_{[k]}) \\
&= \rho(x_{[k]}) \phi^{\text{ext}}(q^s_{[k]}) \\
&= \rho(x_{[k]}) \left(\phi^{\text{ext}}(q^s_{[k]} \ominus q^{s-1}_{[k]}) + \phi^{\text{ext}}(q^{s-1}_{[k]})\right) \\
&= \rho(x_{[k]}) \left(\phi^{\text{ext}}(q^s_{[k]} \ominus q^{s-1}_{[k]}) + \cdots + \phi^{\text{ext}}(q^1_{[k]} \ominus q^0_{[k]}) + \phi^{\text{ext}}(q^0_{[k]})\right) \\
&= \rho(x_{[k]}) \left(\phi(q^s_{[k]} \ominus q^{s-1}_{[k]}) + \cdots + \phi(q^1_{[k]} \ominus q^0_{[k]}) + h_0\right).
\end{aligned}$$

From this, we see that if ϕ^{ext} exists, it has to be unique.

We use this observation to define the map ϕ^{ext}. For each $x_{[k]} \in B \setminus B^0$, we apply Corollary 2.2 to points $z_{[k]}$ and $x_{[k]}$ (playing the roles of $x_{[k]}$ and $y_{[k]}$ of the corollary respectively) and thus choose a sequence $q^0_{[k]} = z_{[k]}, q^1_{[k]}, q^2_{[k]}, \ldots, q^s_{[k]} = (\lambda_1 x_1, \ldots, \lambda_k x_k)$ in $B \setminus B^0$ such that ρ is equal at all points, any two consecutive points differ in exactly one coordinate, and $\lambda_1, \ldots, \lambda_k$ are non-zero elements of $\mathbb{F}$ and $s \le \mathbf{s} = (2k+1)(2^k-1)+1$. (The addition of 1 to the bound in Corollary 2.2 is intentional here: it will simplify the proof that the map ϕ^{ext} we are defining is multilinear.) We then take $\phi^{\text{ext}}(x_{[k]})$ to be

$$\rho(x_{[k]}) \left(\phi(q^s_{[k]} \ominus q^{s-1}_{[k]}) + \cdots + \phi(q^1_{[k]} \ominus q^0_{[k]}) + h_0\right). \tag{3.2}$$

If $x_{[k]} \in B^0$, then we simply set $\phi^{\text{ext}}(x_{[k]}) = \phi(x_{[k]})$.

It remains to show that ϕ^{ext} is well defined and multilinear.

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3.1. The extension map is well defined

Fix now a point $x_{[k]} \in B \setminus B^0$. Let $q^0_{[k]} = z_{[k]}, q^1_{[k]}, \dots, q^s_{[k]} = (\lambda_1 x_1, \dots, \lambda_k x_k)$ and $p^0_{[k]} = z_{[k]}, p^1_{[k]}, \dots, p^t_{[k]} = (\mu_1 x_1, \dots, \mu_k x_k)$ be two **s**-good sequences. In particular, $\prod_{i\in[k]} \lambda_i = \prod_{i\in[k]} \mu_i \neq 0$. We need to show that

$$\phi(q^s_{[k]} \ominus q^{s-1}_{[k]}) + \dots + \phi(q^1_{[k]} \ominus q^0_{[k]}) + \phi(p^0_{[k]} \ominus p^1_{[k]}) + \dots + \phi(p^{t-1}_{[k]} \ominus p^t_{[k]}) = 0.$$

As a slight digression, we note that if ϕ were a global multilinear map, then this would be trivial to prove, since $\phi(q^s_{[k]} \ominus q^{s-1}_{[k]})$ could be split as $\phi(q^s_{[k]}) - \phi(q^{s-1}_{[k]})$, and so on, and $\phi(q^s_{[k]}) = \phi(p^t_{[k]})$. We mimic this proof, by using Lemma 2.5 to find a point 'orthogonal' to the sequence $q^i_{[k]}$. First, we prove the following claim that exploits the properties of such a point (and explains the meaning of 'orthogonality' we have in mind).

In the proof below, and in subsequent arguments, when we write an expression of the form $\big((a_i)_{i\in F}, (b_i)_{i\in E\setminus F}\big)$, it should be understood as the sequence $(c_i)_{i\in E}$ such that $c_i = a_i$ when $i \in F$ and $c_i = b_i$ when $i \in E \setminus F$.

Proposition 3.2. *Let $q^0_{[k]} = z_{[k]}, q^1_{[k]}, \dots, q^s_{[k]} = (\lambda_1 x_1, \dots, \lambda_k x_k)$ be an* **s***-good sequence and let $\nu_1, \dots, \nu_k \in \mathbb{F}$ be non-zero scalars such that $\prod_{i\in[k]} \nu_i \cdot \prod_{i\in[k]} \lambda_i = 1$. Let $e_{[k]} \in G_{[k]}$ be a point that satisfies the conditions*

(1) $\rho(e_{[k]}) = -1$,

(2) $(\forall \emptyset \neq I \subsetneq [k])(\forall i \in [0, s])\ \rho(e_I, q^i_{[k]\setminus I}) = 0$,

(3) $(\forall i \in [0, s])(\forall j \in [m])(\forall \emptyset \neq J \subset I_j)\ \beta_j(e_J, q^i_{I_j\setminus J}) = 0$.

Then

$$\begin{aligned}\phi(q^s_{[k]} \ominus q^{s-1}_{[k]}) + \dots + \phi(q^1_{[k]} \ominus q^0_{[k]}) &= \Big(\prod_{i\in[k]} \lambda_i\Big)\phi(x_1 + \nu_1 e_1, \dots, x_k + \nu_k e_k)\\ &- \phi(z_1 + \nu_1\lambda_1 e_1, \dots, z_k + \nu_k \lambda_k e_k)\\ &- \sum_{\emptyset\neq I\subsetneq[k]} \Big(\prod_{i\in[k]} \lambda_i\Big)\phi\big((\nu_i e_i)_{i\in I}, (x_i)_{i\in[k]\setminus I}\big)\\ &+ \sum_{\emptyset\neq I\subsetneq[k]} \phi\big((\lambda_i\nu_i e_i)_{i\in I}, (z_i)_{i\in[k]\setminus I}\big).\end{aligned}$$

Proof. Suppose that $q^{i+1}_{[k]}$ and $q^i_{[k]}$ differ in coordinate d. To simplify the notation in the proof, we shall temporarily write $u_{[k]} = q^{i+1}_{[k]}$, $w_{[k]} = q^i_{[k]}$ and $\tilde{e}_i = \lambda_i \nu_i e_i$. Then

$$\begin{aligned}\phi((q^{i+1} \ominus q^i)_{[k]}) &= \phi((u \ominus w)_{[k]}) = \phi(w_{[d-1]}, u_d - w_d, w_{[d+1,k]})\\ &= \phi(w_1 + \tilde{e}_1, w_{[2,d-1]}, u_d - w_d, w_{[d+1,k]}) - \phi(\tilde{e}_1, w_{[2,d-1]}, u_d - w_d, w_{[d+1,k]})\end{aligned}$$

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$$= \phi(w_1 + \tilde{e}_1, w_{[2,d-1]}, u_d - w_d, w_{[d+1,k]}) - \phi(\tilde{e}_1, u_{[2,k]}) + \phi(\tilde{e}_1, w_{[2,k]})$$
$$= \phi(w_1 + \tilde{e}_1, w_2 + \tilde{e}_2, w_{[3,d-1]}, u_d - w_d, w_{[d+1,k]}) - \phi(w_1 + \tilde{e}_1, \tilde{e}_2, u_{[3,k]})$$
$$+ \phi(w_1 + \tilde{e}_1, \tilde{e}_2, w_{[3,k]}) - \phi(\tilde{e}_1, u_{[2,k]}) + \phi(\tilde{e}_1, w_{[2,k]}).$$

Repeating this argument once for each coordinate apart from the dth and using the fact that $w_j = u_j$ whenever $j \neq d$, we arrive at the expression

$$\begin{aligned}
&\phi\left(w_1 + \tilde{e}_1, \dots, w_{d-1} + \tilde{e}_{d-1}, u_d - w_d, w_{d+1} + \tilde{e}_{d+1}, \dots, w_k + \tilde{e}_k\right) \\
&\quad - \sum_{j \in [d-1]} \phi(u_1 + \tilde{e}_1, \dots, u_{j-1} + \tilde{e}_{j-1}, \tilde{e}_j, u_{[j+1,d-1]}, u_d, u_{[d+1,k]}) \\
&\quad + \sum_{j \in [d-1]} \phi(w_1 + \tilde{e}_1, \dots, w_{j-1} + \tilde{e}_{j-1}, \tilde{e}_j, w_{[j+1,d-1]}, w_d, w_{[d+1,k]}) \\
&\quad - \sum_{j \in [d+1,k]} \phi(u_1 + \tilde{e}_1, \dots, u_{d-1} + \tilde{e}_{d-1}, u_d, u_{d+1} + \tilde{e}_{d+1}, \dots, \\
&\quad\quad u_{j-1} + \tilde{e}_{j-1}, \tilde{e}_j, u_{[j+1,k]}) \\
&\quad + \sum_{j \in [d+1,k]} \phi(w_1 + \tilde{e}_1, \dots, w_{d-1} + \tilde{e}_{d-1}, w_d, w_{d+1} + \tilde{e}_{d+1}, \dots, \\
&\quad\quad w_{j-1} + \tilde{e}_{j-1}, \tilde{e}_j, w_{[j+1,k]}).
\end{aligned} \tag{3.3}$$

Expanding this out gives

$$\begin{aligned}
&\phi(u_1 + \tilde{e}_1, \dots, u_{d-1} + \tilde{e}_{d-1}, u_d + \tilde{e}_d, u_{d+1} + \tilde{e}_{d+1}, \dots, u_k + \tilde{e}_k) \\
&\quad - \phi(w_1 + \tilde{e}_1, \dots, w_{d-1} + \tilde{e}_{d-1}, w_d + \tilde{e}_d, w_{d+1} + \tilde{e}_{d+1}, \dots, w_k + \tilde{e}_k) \\
&\quad - \sum_{\emptyset \neq I \subset [k] \setminus \{d\}} \phi((\tilde{e}_j)_{j \in I}, (u_j)_{j \in [k] \setminus I}) \\
&\quad + \sum_{\emptyset \neq I \subset [k] \setminus \{d\}} \phi((\tilde{e}_j)_{j \in I}, (w_j)_{j \in [k] \setminus I}).
\end{aligned} \tag{3.4}$$

To see why, note that the first term in (3.3) expands to the first two terms in (3.4). And after that, each set I arises from the expansion of the jth summand in one of the sums in (3.3) only when $j = \max I$.

We now return to writing q^i and q^{i+1} instead of w and u. Writing $d_{i+1} \in [k]$ for the direction where $q^{i+1}_{[k]}$ and $q^i_{[k]}$ differ for $i \in [0, s-1]$, the work above yields the equality

$$\begin{aligned}
&\phi(q^{i+1}_{[k]} \ominus q^i_{[k]}) \\
&= \phi(q^{i+1}_1 + \tilde{e}_1, \dots, q^{i+1}_{d_{i+1}-1} + \tilde{e}_{d_{i+1}-1}, q^{i+1}_{d_{i+1}} + \tilde{e}_{d_{i+1}}, q^{i+1}_{d_{i+1}+1} + \tilde{e}_{d_{i+1}+1}, \dots, q^{i+1}_k + \tilde{e}_k) \\
&\quad - \phi(q^i_1 + \tilde{e}_1, \dots, q^i_{d_{i+1}-1} + \tilde{e}_{d_{i+1}-1}, q^i_{d_{i+1}} + \tilde{e}_{d_{i+1}}, q^i_{d_{i+1}+1} + \tilde{e}_{d_{i+1}+1}, \dots, q^i_k + \tilde{e}_k) \\
&\quad - \sum_{\emptyset \neq I \subset [k] \setminus \{d_{i+1}\}} \phi((\tilde{e}_j)_{j \in I}, (q^{i+1}_j)_{j \in [k] \setminus I}) + \sum_{\emptyset \neq I \subset [k] \setminus \{d_{i+1}\}} \phi((\tilde{e}_j)_{j \in I}, (q^i_j)_{j \in [k] \setminus I}).
\end{aligned}$$

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Using this equality, we obtain a telescoping sum from the first two terms, and therefore find that

$$\begin{aligned}\phi(q^s_{[k]} \ominus q^{s-1}_{[k]}) + \cdots + \phi(q^1_{[k]} \ominus q^0_{[k]}) &= \phi(q^s_1 + \tilde{e}_1, \ldots, q^s_k + \tilde{e}_k) - \phi(q^0_1 + \tilde{e}_1, \ldots, q^0_k + \tilde{e}_k)\\ &\quad - \sum_{\emptyset \neq I \subsetneq [k]} \sum_{\substack{i \in [1,s]\\ d_i \notin I}} \phi((\tilde{e}_j)_{j \in I}, (q^i_j)_{j \in [k]\setminus I})\\ &\quad + \sum_{\emptyset \neq I \subsetneq [k]} \sum_{\substack{i \in [0,s-1]\\ d_{i+1} \notin I}} \phi((\tilde{e}_j)_{j \in I}, (q^i_j)_{j \in [k]\setminus I}). \end{aligned} \tag{3.5}$$

Fix $\emptyset \neq I \subsetneq [k]$ temporarily. Let $1 \leq i'_1 \leq i''_1 < i'_2 \leq i''_2 < \cdots < i'_n \leq i''_n \leq s$ be indices such that

$$\{j \in [s] \colon d_j \notin I\} = [i'_1, i''_1] \cup [i'_2, i''_2] \cup \cdots \cup [i'_n, i''_n]$$

and $i''_j \leq i'_j - 2$. (We simply partition the set of indices j such that $d_j \notin I$ into contiguous parts.) The contribution to (3.5) coming from the set I after simple cancellation becomes

$$\sum_{\ell \in [n]} \phi((\tilde{e}_j)_{j \in I}, (q^{i'_\ell - 1}_j)_{j \in [k]\setminus I}) - \sum_{\ell \in [n]} \phi((\tilde{e}_j)_{j \in I}, (q^{i''_\ell}_j)_{j \in [k]\setminus I}). \tag{3.6}$$

Crucially, observe that points $((\tilde{e}_j)_{j \in I}, (q^{i''_\ell}_j)_{j \in [k]\setminus I})$ and $((\tilde{e}_j)_{j \in I}, (q^{i'_{\ell+1}-1}_j)_{j \in [k]\setminus I})$ are the same for each $\ell \in [n-1]$. To see this, we just need to check that $q^{i''_\ell}_j = q^{i'_{\ell+1}-1}_j$ for $j \in [k] \setminus I$. However, by definition of indices $d_1, \ldots, d_s$, the points $q^{i''_\ell}_{[k]}$ and $q^{i'_{\ell+1}-1}_{[k]}$ may differ only at coordinates $d_{i''_\ell+1}, d_{i''_\ell+2}, \ldots, d_{i'_{\ell+1}-1}$. However, these coordinates belong to I, so $q^{i''_\ell}_{[k]}$ and $q^{i'_{\ell+1}}_{[k]}$ agree on $[k] \setminus I$, as required. Similarly, $((\tilde{e}_j)_{j \in I}, (q^{i'_1-1}_j)_{j \in [k]\setminus I})$ equals $((\tilde{e}_j)_{j \in I}, (q^0_j)_{j \in [k]\setminus I})$ and $((\tilde{e}_j)_{j \in I}, (q^{i''_n}_j)_{j \in [k]\setminus I})$ equals $((\tilde{e}_j)_{j \in I}, (q^s_j)_{j \in [k]\setminus I})$. Hence, the contribution (3.6) coming from I is just

$$\phi((\tilde{e}_i)_{i \in I}, (q^0_i)_{i \in [k]\setminus I}) - \phi((\tilde{e}_i)_{i \in I}, (q^s_i)_{i \in [k]\setminus I}).$$

Hence, (3.5) is equal to

$$\begin{aligned}&\phi(q^s_1 + \tilde{e}_1, \ldots, q^s_k + \tilde{e}_k) - \phi(q^0_1 + \tilde{e}_1, \ldots, q^0_k + \tilde{e}_k) - \sum_{\emptyset \neq I \subsetneq [k]} \phi((\tilde{e}_i)_{i \in I}, (q^s_i)_{i \in [k]\setminus I})\\ &\quad + \sum_{\emptyset \neq I \subsetneq [k]} \phi((\tilde{e}_i)_{i \in I}, (q^0_i)_{i \in [k]\setminus I}).\end{aligned}$$

The claim follows after recalling that $q^0_{[k]} = z_{[k]}$ and $q^s_{[k]} = (\lambda_1 x_1, \ldots, \lambda_k x_k)$. □

To complete the proof that ϕ^{ext} is well defined, we shall need a point $e_{[k]}$ with slightly stronger properties than the ones used in Proposition 3.2. The first property is the same, the second and third are the same but now for two **s**-good sequences rather than just one, and the fourth is new.

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Proposition 3.3. *Given two* $\mathbf{s}$*-good sequences* $q^0_{[k]} = z_{[k]}, q^1_{[k]}, \dots, q^s_{[k]}$ *and* $p^0_{[k]} = z_{[k]}, p^1_{[k]}, \dots, p^t_{[k]}$*, there is a point* $e_{[k]}$ *that satisfies the following conditions.*

(i) $\rho(e_{[k]}) = -1$.

(ii) $(\forall \emptyset \neq I \subsetneq [k])(\forall i \in [0,s])\ \rho(e_I; q^i_{[k]\setminus I}) = 0$ *and* $(\forall \emptyset \neq I \subsetneq [k])(\forall i \in [0,t])$ $\rho(e_I; p^i_{[k]\setminus I}) = 0$.

(iii) $(\forall i \in [0,s])(\forall j \in [m])(\forall \emptyset \neq J \subset I_j)\ \beta_j(e_J, q^i_{I_j \setminus J}) = 0$ *and* $(\forall i \in [0,t])$ $(\forall j \in [m])(\forall \emptyset \neq J \subset I_j)\ \beta_j(e_J, p^i_{I_j \setminus J}) = 0$.

(iv) For all pairs of distinct coordinates $c_1, c_2 \in [k]$ *and all* $\lambda_{[k]\setminus\{c_1,c_2\}} \in (\mathbb{F} \setminus \{0\})^{[k]\setminus\{c_1,c_2\}}, \mu \in \mathbb{F}^{\mathcal{I}_{c_1,c_2}}$,

$$\mathop{\mathbb{E}}_{y_{c_1}, y_{c_2}} \chi\Big(\rho(y_{c_1}, y_{c_2}, (z_j - \lambda_j e_j)_{j \in [k]\setminus\{c_1,c_2\}}) - \sum_{i \in \mathcal{I}_{c_1,c_2}} \mu_i \beta_i(y_{c_1}, y_{c_2}, (z_j - \lambda_j e_j)_{j \in I_i \setminus \{c_1,c_2\}})\Big)$$

is at most $\mathbf{f}^{-(m+1)2^{k+2}}$*, where* $\mathcal{I}_{c_1,c_2} = \{i \in [m] \colon c_1, c_2 \in I_i\}$.

Proof. We begin the proof by using Lemma 2.5 to find at least one point that satisfies properties (i), (ii) and (iii). To achieve this, we consider the following multilinear forms.

(1) For each proper non-empty subset $I \subsetneq [k]$ and each $i \in [0,s]$ we take the form on G_I that maps x_I to $\rho(x_I, q^i_{[k]\setminus I})$.

(2) For each proper non-empty subset $I \subsetneq [k]$ and each $i \in [0,t]$ we take the form on G_I that maps x_I to $\rho(x_I, p^i_{[k]\setminus I})$.

(3) For each $i \in [0,s]$, each $j \in [m]$, and each non-empty proper subset $J \subset I_j$, we take the form on G_J that maps x_J to $\beta_j(x_J, q^i_{I_j \setminus J})$.

(4) For each $i \in [0,t]$, each $j \in [m]$, and each non-empty proper subset $J \subset I_j$, we take the form on G_J that maps x_J to $\beta_j(x_J, p^i_{I_j \setminus J})$.

(5) For each $i \in [m]$ such that $I_i = [k]$ we take the form β_i.

Recall that $\mathbf{s} = (2k+1)(2^k - 1) + 1$. We listed

$$(2^k - 2)(s+1) + (2^k - 2)(t+1) + \Big(\sum_{j \in [m]} (s+1)\Big)(2^{|I_j|} - 2)\Big) + \Big(\sum_{j \in [m]} (t+1)\Big)(2^{|I_j|} - 2)\Big) + m \le (m+1)\mathbf{s}\, 2^{k+1}$$

multilinear forms.

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Assumption (3.1) of Theorem 3.1 implies that for all $\lambda \in \mathbb{F}^{\mathcal{I}}$,

$$\mathop{\mathbb{E}}_{x_{[k]}} \chi\Big(\rho(x_{[k]}) - \sum_{i\in\mathcal{I}} \lambda_i \beta_i(x_{[k]})\Big) < \mathbf{f}^{-k(m+1)\mathbf{s}2^{k+1}},$$

where $\mathcal{I} = \{i \in [m] \colon I_i = [k]\}$. Therefore, by Lemma 2.5, we have at least one point $x_{[k]}$ which evaluates to zero under all the forms listed above (after suitable projections) and $\rho(x_{[k]}) = -1$. (Note that m here does not have the same meaning as the parameter m in Lemma 2.5, the $r + m$ term of that lemma is the total number of multilinear forms we are using, which we bounded by $(m+1)\mathbf{s}\, 2^{k+1}$ in the current context.) But the set of such points is a non-empty variety of codimension at most $(m+1)\mathbf{s}2^{k+1} + 1$, so by Lemma 2.3, there are at least $\mathbf{f}^{-k(m+1)\mathbf{s}2^{k+1}-k}|G_{[k]}|$ of them.

On the other hand, for each $c_1, c_2 \in [k]$, $\mu \in \mathbb{F}^{\mathcal{I}_{c_1,c_2}}$, we have

$$\begin{aligned}&\mathop{\mathbb{E}}_{x_{[k]\setminus\{c_1,c_2\}}} \left(\mathop{\mathbb{E}}_{y_{c_1},y_{c_2}} \chi\left(\rho(y_{c_1}, y_{c_2}, x_{[k]\setminus\{c_1,c_2\}}) - \sum_{i\in\mathcal{I}_{c_1,c_2}} \mu_i\beta_i(y_{c_1}, y_{c_2}, x_{I_i\setminus\{c_1,c_2\}})\right)\right)\\ &= \left|\mathop{\mathbb{E}}_{x_{[k]\setminus\{c_1,c_2\}},y_{c_1},y_{c_2}} \chi\Big(\rho(y_{c_1}, y_{c_2}, x_{[k]\setminus\{c_1,c_2\}}) - \sum_{i\in\mathcal{I}_{c_1,c_2}} \mu_i\beta_i(y_{c_1}, y_{c_2}, x_{I_i\setminus\{c_1,c_2\}})\Big)\right|,\end{aligned}$$

since the inner expectation on the left-hand side is always a non-negative real. By Lemma 2.4, the right-hand side is at most

$$\mathop{\mathbb{E}}_{x_{[k]}} \chi\left(\rho(x_{[k]}) - \sum_{i\in\mathcal{I}} \mu_i\beta_i(x_{[k]})\right),$$

which, using assumption (3.1) of Theorem 3.1 again, is at most

$$\frac{1}{2k^2}\mathbf{f}^{-k(m+1)\mathbf{s}2^{k+1}-(m+1)2^{k+2}-m-k}.$$

From this, we deduce that the set $X_{c_1,c_2} \subset G_{[k]\setminus\{c_1,c_2\}}$ of points $x_{[k]\setminus\{c_1,c_2\}}$ such that for some $\mu \in \mathbb{F}^{\mathcal{I}_{c_1,c_2}}$

$$\mathop{\mathbb{E}}_{y_{c_1},y_{c_2}} \chi\left(\rho(y_{c_1}, y_{c_2}, x_{[k]\setminus\{c_1,c_2\}}) - \sum_{i\in\mathcal{I}_{c_1,c_2}} \mu_i\beta_i(y_{c_1}, y_{c_2}, x_{I_i\setminus\{c_1,c_2\}})\right) > \mathbf{f}^{-(m+1)2^{k+2}}$$

has size $|X_{c_1,c_2}| \le \frac{1}{2k^2}\mathbf{f}^{-k(m+1)\mathbf{s}2^{k+1}-k}|G_{[k]}|$. Thus, there is a choice of $e_{[k]}$ such that the properties (i), (ii) and (iii) hold and for each distinct $c_1, c_2 \in [k]$ and each $\lambda \in (\mathbb{F} \setminus \{0\})^{[k]\setminus\{c_1,c_2\}}$, the sequence $(z_i - \lambda_i e_i \colon i \in [k] \setminus \{c_1, c_2\})$ does not belong to X_{c_1,c_2}, which completes the proof. □

Next, we exploit the property (iv) to understand how the values of $\phi(z_1 + \lambda_1 e_1, \ldots, z_k + \lambda_k e_k)$ are related for different values of $\lambda_{[k]} \in (\mathbb{F} \setminus \{0\})^{[k]}$.

Proposition 3.4. *Suppose that $e_{[k]}$ has the properties listed in Proposition 3.3. Then, for any $\tau, \sigma \in \mathbb{F}^k$ such that $\prod_{i\in[k]} \tau_i = \prod_{i\in[k]} \sigma_i = 1$, we have*

$$\phi\left(z_1 + \tau_1 e_1, \dots, z_k + \tau_k e_k\right) - \sum_{\emptyset \neq I \subsetneq [k]} \phi\left((\tau_i e_i)_{i\in I}, (z_i)_{i\in[k]\setminus I}\right)$$
$$= \phi\left(z_1 + \sigma_1 e_1, \dots, z_k + \sigma_k e_k\right) - \sum_{\emptyset \neq I \subsetneq [k]} \phi\left((\sigma_i e_i)_{i\in I}, (z_i)_{i\in[k]\setminus I}\right).$$

Proof. Let $\mathcal{S}$ be the set of all sequences $\tau \in \mathbb{F}^k$ such that $\prod_{i\in[k]} \tau_i = 1$. For $\tau \in \mathcal{S}$, write $\Phi(\tau)$ for the value

$$\phi\left(z_1 + \tau_1 e_1, \dots, z_k + \tau_k e_k\right) - \sum_{\emptyset \neq I \subsetneq [k]} \phi\left((\tau_i e_i)_{i\in I}, (z_i)_{i\in[k]\setminus I}\right).$$

The claim can be rephrased as $\Phi(\tau) = \Phi(\sigma)$ for all $\tau, \sigma \in \mathcal{S}$. We say that two sequences $\sigma, \tau \in \mathcal{S}$ are *neighbouring* if they differ in exactly two coordinates (note that they cannot differ in only a single coordinate). Notice that for any two $\sigma, \tau \in \mathcal{S}$ we may find further sequences $\sigma^{(0)}, \dots, \sigma^{(t)} \in \mathcal{S}$ for some $t \geq 1$ such that $\sigma^{(0)} = \sigma, \sigma^{(t)} = \tau$ and for each $i \in [t]$, the sequences $\sigma^{(i-1)}$ and $\sigma^{(i)}$ are neighbouring or equal. Indeed, we simply set

$$\sigma^{(i)} = \left(\tau_1, \tau_2, \dots, \tau_i, \sigma_{i+1} \prod_{j\in[i]} \sigma_j \tau_j^{-1}, \sigma_{i+2}, \dots, \sigma_k\right)$$

which also belongs to $\mathcal{S}$. Suppose for a moment that we have proved the claim for neighbouring pairs of sequences. Then we would have $\Phi(\sigma) = \Phi(\sigma^{(0)}) = \dots = \Phi(\sigma^{(s)} = \Phi(\tau)$ and the general case would follow.

Therefore, it suffices to prove the claim for the case when $\sigma_i = \tau_i$ for $i \in [k] \setminus \{c_1, c_2\}$, for some pair of coordinates c_1, c_2, and $\sigma_{c_1} = \delta \tau_{c_1}, \sigma_{c_2} = \eta \tau_{c_2}$, where $\delta\eta = 1$. Furthermore, notice that we may without loss of generality assume that $\tau_i = 1$. This follows from the fact that the point $(\tau_1 e_1, \dots, \tau_k e_k)$ satisfies the same properties as $e_{[k]}$ and we may then use the special case for the sequence $(1, 1, \dots, 1) \in \mathcal{S}$ with the point $(\tau_1 e_1, \dots, \tau_k e_k)$ playing the role of $e_{[k]}$ in the proposition. Also, by symmetry, we may assume without loss of generality that $c_1 = 1$ and $c_2 = 2$. Write $\theta\colon G_{[2]} \to H$ for the map $\theta(x, y) = \phi(x, y, z_3 + e_3, \dots, z_k + e_k)$. The claim now reduces to showing that

$$\theta(z_1 + e_1, z_2 + e_2) - \theta(e_1, z_2) - \theta(z_1, e_2)$$
$$= \theta(z_1 + \delta e_1, z_2 + \eta e_2) - \delta\theta(e_1, z_2) - \eta\theta(z_1, e_2).$$

By property **(iv)** of Proposition 3.3 and by Lemma 2.5 applied to maps that map $(u, v) \in G_1 \times G_2$ to:

(1) $\rho(u, v, z_3 + e_3, \dots, z_k + e_k)$, $\rho(u, y, z_3 + e_3, \dots, z_k + e_k)$ where $y \in \{z_2, e_2\}$, $\rho(x, v, z_3 + e_3, \dots, z_k + e_k)$, where $x \in \{z_1, e_1\}$,

(2) $\beta_i(u, v, (z_j + e_j)_{j\in I_i\setminus\{1,2\}})$, $\beta_i(u, y, (z_j + e_j)_{j\in I_i\setminus\{1,2\}})$, where $y \in \{z_2, e_2\}$, $\beta_i(x, v, (z_j + e_j)_{j\in I_i\setminus\{1,2\}})$, where $x \in \{z_1, e_1\}$, when $1, 2 \in I_i$,

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(3) $\beta_i(u, (z_j + e_j)_{j \in I_i \setminus \{1\}})$ when $1 \in I_i$, $2 \notin I_i$, and

(4) $\beta_i(v, (z_j + e_j)_{j \in I_i \setminus \{2\}})$ when $2 \in I_i$, $1 \notin I_i$,

we conclude that there are $u \in G_1, v \in G_2$ such that $\rho(u, v, z_3 + e_3, \dots, z_k + e_k) = 1$, and all other values of the maps $\rho, \beta_{[m]}$ at points among $\{z_1, e_1, u\} \times \{z_2, e_2, v\} \times \{(z_3 + e_3, \dots, z_k + e_k)\}$, involving u or v, are zero. Therefore, by Lemma 2.6 (using statement (2.1) of the lemma for the second and fourth equalities)

$$\begin{aligned}
&\theta(z_1 + \delta e_1, z_2 + \eta e_2) - \delta\theta(e_1, z_2) - \eta\theta(z_1, e_2) \\
&\quad = \eta\theta(z_1 - \delta(-e_1), \delta z_2 + e_2) - \delta\theta(e_1, z_2) - \eta\theta(z_1, e_2) \\
&\quad = \eta(\delta\theta(z_1 - u, z_2 + v) - \delta\theta(-e_1 - u, e_2 + v) \\
&\qquad + \theta(z_1, e_2) - \delta\theta(z_1, v) - \delta^2\theta(-e_1, z_2) + \delta\theta(u, z_2) + \delta\theta(-e_1, v) - \delta\theta(u, e_2)) \\
&\qquad - \delta\theta(e_1, z_2) - \eta\theta(z_1, e_2) \\
&\quad = \theta(z_1 - u, z_2 + v) - \theta(-e_1 - u, e_2 + v) - \theta(z_1, v) + \theta(u, z_2) + \theta(-e_1, v) - \theta(u, e_2) \\
&\quad = \theta(z_1 + e_1, z_2 + e_2) - \theta(e_1, z_2) - \theta(z_1, e_2),
\end{aligned}$$

as desired. □

We now return to the proof that ϕ^{ext} is well defined. Recall that $q^0_{[k]} = z_{[k]}, \dots, q^s = (\lambda_1 x_1, \dots, \lambda_k x_k)$ and $p^0_{[k]} = z_{[k]}, \dots, p^t = (\mu_1 x_1, \dots, \mu_k x_k)$ are two $\mathbf{s}$-good sequences. Apply Proposition 3.3 to find a point $e_{[k]} \in G_{[k]}$ that has the properties described in that proposition. The assumptions of Proposition 3.2 are satisfied. Applying the proposition twice with $\nu_i = \lambda_i^{-1}$, we obtain

$$\begin{aligned}
\phi(q^s_{[k]} \ominus q^{s-1}_{[k]}) + \dots + \phi(q^1_{[k]} \ominus q^0_{[k]}) &= \left(\prod_{i \in [k]} \lambda_i\right) \phi\left(x_1 + \nu_1 e_1, \dots, x_k + \nu_k e_k\right) \\
&- \phi\left(z_1 + \nu_1\lambda_1 e_1, \dots, z_k + \nu_k\lambda_k e_k\right) \\
&- \sum_{\emptyset \neq I \subsetneq [k]} \left(\prod_{i \in [k]} \lambda_i\right) \phi\left((\nu_i e_i)_{i \in I}, (x_i)_{i \in [k] \setminus I}\right) \\
&+ \sum_{\emptyset \neq I \subsetneq [k]} \phi\left((\nu_i\lambda_i e_i)_{i \in I}, (z_i)_{i \in [k] \setminus I}\right),
\end{aligned}$$

and

$$\begin{aligned}
\phi(p^t_{[k]} \ominus p^{t-1}_{[k]}) + \dots + \phi(p^1_{[k]} \ominus p^0_{[k]}) &= \left(\prod_{i \in [k]} \mu_i\right) \phi\left(x_1 + \nu_1 e_1, \dots, x_k + \nu_k e_k\right) \\
&- \phi\left(z_1 + \nu_1\mu_1 e_1, \dots, z_k + \nu_k\mu_k e_k\right) \\
&- \sum_{\emptyset \neq I \subsetneq [k]} \left(\prod_{i \in [k]} \mu_i\right) \phi\left((\nu_i e_i)_{i \in I}, (x_i)_{i \in [k] \setminus I}\right) + \sum_{\emptyset \neq I \subsetneq [k]} \phi\left((\nu_i\mu_i e_i)_{i \in I}, (z_i)_{i \in [k] \setminus I}\right).
\end{aligned}$$

Our task is to prove that these two expressions are equal. Hence, it suffices to prove that

$$\phi\Big(z_1+\tau_1 e_1,\ldots,z_k+\tau_k e_k\Big)-\sum_{\emptyset\neq I\subsetneq[k]}\phi((\tau_i e_i)_{i\in I},(z_i)_{i\in[k]\setminus I})$$
$$=\phi\Big(z_1+e_1,\ldots,z_k+e_k\Big)-\sum_{\emptyset\neq I\subsetneq[k]}\phi((e_i)_{i\in I},(z_i)_{i\in[k]\setminus I}),$$

where $\tau_i=\mu_i\lambda_i^{-1}$. Since $\prod_{i\in[k]}\tau_i=1$, this follows from Proposition 3.4.

3.2. The extension map is multilinear

Let $x_{[k]},y_{[k]}\in B$ be arbitrary points that differ in a single coordinate. We need to show that $\phi^{\text{ext}}(x_{[k]})-\phi^{\text{ext}}(y_{[k]})=\phi^{\text{ext}}((x\ominus y)_{[k]})$. To begin, we show that ϕ^{ext} respects scalar multiplication in a single coordinate.

Claim 3.5. *Let $x_{[k]}\in B$ and let $\lambda\in\mathbb{F}$. Then*

$$\phi^{\text{ext}}(x_1,\ldots,x_{i-1},\lambda x_i,x_{i+1},\ldots,x_k)=\lambda\phi^{\text{ext}}(x_{[k]}).$$

Proof. If $x_{[k]}\in B^0$ or $\lambda=0$, we are done, so assume the contrary. By Corollary 2.2, there is a $(2k+1)(2^k-1)$-good sequence $q^0_{[k]}=z_{[k]},q^1_{[k]},\ldots,q^s_{[k]}=(\lambda_1 x_1,\ldots,\lambda_k x_k)$. Recall from (3.2) that ϕ^{ext} is defined by the formula

$$\phi^{\text{ext}}(x_{[k]})=\rho(x_{[k]})\left(\phi(q^s_{[k]}\ominus q^{s-1}_{[k]})+\cdots+\phi(q^1_{[k]}\ominus q^0_{[k]})+h_0\right).$$

Noting that the same **s**-good sequence can be used for $(x_1,\ldots,x_{i-1},\lambda x_i,x_{i+1},\ldots,x_k)$, we find that

$$\phi^{\text{ext}}(x_1,\ldots,x_{i-1},\lambda x_i,x_{i+1},\ldots,x_k)$$
$$=\lambda\rho(x_{[k]})\left(\phi(q^s_{[k]}\ominus q^{s-1}_{[k]})+\cdots+\phi(q^1_{[k]}\ominus q^0_{[k]})+h_0\right),$$

so the claim follows.

To finish the proof that ϕ^{ext} is multilinear, we distinguish two cases.

Case 1: at least one of the points $x_{[k]},y_{[k]},(x\ominus y)_{[k]}$ is in B^0.

Observe that $(x\ominus(x\ominus y))_{[k]}=y_{[k]}$, and also that $(y\ominus x)_{[k]}$ is equal to $(x\ominus y)_{[k]}$ except in the coordinate where x and y differ, which changes sign. Combining these observations and using the claim above, we may assume without loss of generality that $(x\ominus y)_{[k]}\in B^0$, which is equivalent to the statement that $\rho(x_{[k]})=\rho(y_{[k]})$. If $\rho(x_{[k]})=\rho(y_{[k]})=0$, then the map ϕ^{ext} at all three points equals ϕ, which we know to be multilinear. Hence, we may assume that $\rho(x_{[k]})=\rho(y_{[k]})\neq 0$. By Corollary 2.2, there is a $(2k+1)(2^k-1)$-good sequence $q^0_{[k]}=z_{[k]},\ldots,q^s_{[k]}=(\lambda_1 y_1,\ldots,\lambda_k y_k)$. But if we add the point

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$q^{s+1}_{[k]} = (\lambda_1 x_1, \ldots, \lambda_k x_k)$, then we get an $\mathbf{s}$-good sequence for $x_{[k]}$ as well, so

$$\begin{aligned}\phi^{\text{ext}}(x_{[k]}) &= \rho(x_{[k]}) \left(\phi(q^{s+1}_{[k]} \ominus q^{s}_{[k]}) + \phi(q^{s}_{[k]} \ominus q^{s-1}_{[k]}) + \cdots + \phi(q^{1}_{[k]} \ominus q^{0}_{[k]}) + h_0\right)\\ &= \rho(x_{[k]})\left(\prod_{i\in[k]} \lambda_i\right)\phi(x_{[k]} \ominus y_{[k]})\\ &\qquad + \rho(y_{[k]}) \left(\phi(q^{s}_{[k]} \ominus q^{s-1}_{[k]}) + \cdots + \phi(q^{1}_{[k]} \ominus q^{0}_{[k]}) + h_0\right)\\ &= \phi^{\text{ext}}(x_{[k]} \ominus y_{[k]}) + \phi^{\text{ext}}(y_{[k]}).\end{aligned}$$

Case 2: no point belongs to B^0.

In this case, we have that $\rho(x_{[k]}), \rho(y_{[k]}), \rho((x \ominus y)_{[k]}) \neq 0$. Let d be the coordinate in which $x_{[k]}$ and $y_{[k]}$ differ. By Corollary 2.2, there is a $(2k+1)(2^k-1)$-good sequence $q^0_{[k]} = z_{[k]}, q^1_{[k]}, \ldots, q^s_{[k]} = (\lambda_1 x_1, \ldots, \lambda_k x_k)$. Define points

$$p^1_{[k]} = (\lambda_1 x_1, \ldots, \lambda_{d-1} x_{d-1}, \mu y_d, \lambda_{d+1} x_{d+1}, \ldots, \lambda_k x_k)$$

and

$$p^2_{[k]} = (\lambda_1 x_1, \ldots, \lambda_{d-1} x_{d-1}, \nu(x_d - y_d), \lambda_{d+1} x_{d+1}, \ldots, \lambda_k x_k),$$

where ν, μ are such that $\rho(y_{[k]}) = \mu^{-1} \prod_{i\in[k]\setminus\{d\}} \lambda_i^{-1}$ and $\rho((x \ominus y)_{[k]}) = \nu^{-1} \prod_{i\in[k]\setminus\{d\}} \lambda_i^{-1}$. The sequences $q^0_{[k]}, \ldots, q^s_{[k]}, p^1_{[k]}$ and $q^0_{[k]}, \ldots, q^s_{[k]}, p^2_{[k]}$ are also $\mathbf{s}$-good, so

$$\phi^{\text{ext}}(x_{[k]}) = \rho(x_{[k]}) \left(\phi(q^{s}_{[k]} \ominus q^{s-1}_{[k]}) + \cdots + \phi(q^{1}_{[k]} \ominus q^{0}_{[k]}) + h_0\right),$$

$$\phi^{\text{ext}}(y_{[k]}) = \rho(y_{[k]}) \left(\phi(p^{1}_{[k]} \ominus q^{s}_{[k]}) + \phi(q^{s}_{[k]} \ominus q^{s-1}_{[k]}) + \cdots + \phi(q^{1}_{[k]} \ominus q^{0}_{[k]}) + h_0\right), \text{ and}$$

$$\begin{aligned}\phi^{\text{ext}}((x \ominus y)_{[k]}) = \rho((x \ominus y)_{[k]}) &\left(\phi(p^{2}_{[k]} \ominus q^{s}_{[k]})\right.\\ &\left.+\phi(q^{s}_{[k]} \ominus q^{s-1}_{[k]}) + \cdots + \phi(q^{1}_{[k]} \ominus q^{0}_{[k]}) + h_0\right).\end{aligned}$$

Hence, writing $\Lambda = \prod_{i\in[k]\setminus\{d\}} \lambda_i^{-1}$ and recalling that $\lambda_d \rho(x_{[k]}) = \mu\rho(y_{[k]}) = \nu\rho((x - y)_{[k]}) = \Lambda$, we have

$$\begin{aligned}&\phi^{\text{ext}}(y_{[k]}) + \phi^{\text{ext}}((x \ominus y)_{[k]}) - \phi^{\text{ext}}(x_{[k]})\\ &\quad= \rho(y_{[k]})\phi(p^1_{[k]} \ominus q^s_{[k]}) + \rho((x \ominus y)_{[k]})\phi(p^2_{[k]} \ominus q^s_{[k]})\\ &\quad= \rho(y_{[k]})\phi(\lambda_1 x_1, \ldots, \lambda_{d-1} x_{d-1}, \mu y_d - \lambda_d x_d, \lambda_{d+1} x_{d+1}, \ldots, \lambda_k x_k)\\ &\qquad + \rho((x \ominus y)_{[k]})\phi(\lambda_1 x_1, \ldots, \lambda_{d-1} x_{d-1}, \nu(x_d - y_d) - \lambda_d x_d, \lambda_{d+1} x_{d+1}, \ldots, \lambda_k x_k)\\ &\quad= \phi(\lambda_1 x_1, \ldots, \lambda_{d-1} x_{d-1}, \rho(y_{[k]})(\mu y_d - \lambda_d x_d)\\ &\qquad + \rho((x \ominus y)_{[k]})(\nu(x_d - y_d) - \lambda_d x_d), \lambda_{d+1} x_{d+1}, \ldots, \lambda_k x_k)\end{aligned}$$

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$$= \phi(\lambda_1 x_1, \ldots, \lambda_{d-1} x_{d-1}, (\rho(y_{[k]})\mu) y_d + (\rho((x \ominus y)_{[k]})\nu)(x_d - y_d) - (\rho(x_{[k]})\lambda_d) x_d, \lambda_{d+1} x_{d+1}, \ldots, \lambda_k x_k)$$
$$= \phi(\lambda_1 x_1, \ldots, \lambda_{d-1} x_{d-1}, \Lambda(y_d + (x_d - y_d) - x_d), \lambda_{d+1} x_{d+1}, \ldots, \lambda_k x_k) = 0,$$

completing the proof. □

4. From multilinear maps on general varieties to global multilinear maps

We are now ready to prove the main result, which will follow from the following proposition.

Proposition 4.1. *Let $\emptyset \in \mathcal{F} \subset \mathcal{P}[k]$ be a down-set*† *with a maximal set S. There are constants $C = C_{\mathcal{F}}, D = D_{\mathcal{F}}$ such that the following is true.*

Let $\beta_i \colon G_{I_i} \to \mathbb{F}$ be multilinear maps for $i \in [m]$, with $I_i \in \mathcal{F}$. Let $B = \{x_{[k]} \in G_{[k]} \colon (\forall i \in [m])\ \beta_i(x_{I_i}) = 0\}$ and let $\phi \colon B \to H$ be a multilinear map to a $\mathbb{F}$-vector space H. Then there exist $r \le Cm^D$, multilinear forms $\gamma_i \colon G_{J_i} \to \mathbb{F}$, $J_i \in \mathcal{F} \setminus \{S\}$, $i \in [r]$, and a multilinear map $\Phi \colon \{x_{[k]} \in G_{[k]} \colon (\forall i \in [r])\ \gamma_i(x_{J_i}) = 0\} \to H$ such that $\phi = \Phi$ on $\operatorname{dom} \phi \cap \operatorname{dom} \Phi$, *where* dom *stands for the domain of a given function.*

Proposition 4.1 implies Theorem 1.4. Let $C = \max\{\max_{\mathcal{F}} C_{\mathcal{F}}, 1\}$ and $D = \max\{\max_{\mathcal{F}} D_{\mathcal{F}}, 1\}$, where the maximums are taken over all non-empty down-sets $\mathcal{F} \subset \mathcal{P}[k]$ and $C_{\mathcal{F}}, D_{\mathcal{F}}$ are as in Proposition 4.1. Let $\mathcal{F}_1 = \mathcal{P}[k] \supsetneq \mathcal{F}_2 \supsetneq \cdots \supsetneq \mathcal{F}_{2^k} = \{\emptyset\}$ be a sequence of down-sets in $\mathcal{P}[k]$, where we remove a maximal set S_i from each down-set $\mathcal{F}_i$ to obtain the next one. Let ϕ be a multilinear map from a multilinear variety of codimension at most m to a vector space H. Apply Proposition 4.1 to $\mathcal{F}_1, S_1$ and ϕ to get a new multilinear map ϕ^1 such that $\phi = \phi^1$ on $\operatorname{dom}\phi \cap \operatorname{dom}\phi^1$ which is a multilinear variety of codimension at most $2Cm^D$. Then, apply Proposition 4.1 to $\mathcal{F}_2, S_2$ and ϕ^1 to get another multilinear map ϕ^2 such that $\phi^1 = \phi^2$ on $\operatorname{dom}\phi^1 \cap \operatorname{dom}\phi^2$ which is a multilinear variety of codimension at most $2C(2Cm^D)^D$ and proceed like this. The final map we get $\Phi = \phi^{2^k}$ is then a global multilinear map, and $\phi = \phi^{2^k}$ holds on $\operatorname{dom}\phi \cap \operatorname{dom}\phi^1 \cap \cdots \cap \operatorname{dom}\phi^{2^k}$, which is a multilinear variety of the codimension at most $(2C)^{D^{2^k-1} + D^{2^k-2} + \cdots + 1} m^{D^{2^k}}$, as claimed in Theorem 1.4. □

Proof of Proposition 4.1. Reordering the maps if necessary, we may assume that $I_1 = \cdots = I_s = S$ and $I_{s+1}, \ldots, I_m \neq S$. Let $\lambda^1, \ldots, \lambda^n \in \mathbb{F}^s$ be a maximal linearly independent sequence such that for each $i \in [n]$

$$\mathop{\mathbb{E}}_{x_S} \chi\left(\sum_{j \in [s]} \lambda^i_j \beta_j(x_S)\right) \ge \frac{1}{2k^2} \mathbf{f}^{-(2k^2+k+1)(m+1)2^{2k+3}}.$$

(We allow $n = 0$ if there are no linear combinations $\lambda \in \mathbb{F}^s$ with the displayed property.) Extend $\lambda^1, \ldots, \lambda^n$ with $\mu^1, \ldots, \mu^{s-n}$ to a basis of $\mathbb{F}^s$. Write $\rho_i = \sum_{j \in [s]} \mu^i_j \beta_j$ for $i \in [s-n]$

† A collection of sets closed under taking subsets.

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and $\alpha_i = \sum_{j\in[s]} \lambda^i_j \beta_j$ for $i \in [n]$. Then ϕ is defined on

$$\{x_{[k]} \in G_{[k]} \colon (\forall i \in [s-n])\ \rho_i(x_S) = 0\} \cap \{x_{[k]} \in G_{[k]} \colon (\forall i \in [n])\ \alpha_i(x_S) = 0\}$$
$$\cap\, \{x_{[k]} \in G_{[k]} \colon (\forall i \in [s+1,m])\ \beta_i(x_{I_i}) = 0\}.$$

Let $\mathcal{I} = \{i \in [s+1,m] \colon I_i \subset [k] \setminus S\}$. Then by Lemma 2.4, the maps satisfy

$$\begin{aligned}\Bigg|\mathop{\mathbb{E}}_{x_S} \chi\Bigg(\sum_{i\in[s-n]} \nu_i\rho_i(x_S) + \sum_{i\in[n]} \tau_i\alpha_i(x_S) + \sum_{i\in[s+1,m]\setminus\mathcal{I}} \sigma_i\beta_i(x_{S\cap I_i})\Bigg)\Bigg| \\ \leq \Bigg|\mathop{\mathbb{E}}_{x_S} \chi\Bigg(\sum_{i\in[s-n]} \nu_i\rho_i(x_S) + \sum_{i\in[n]} \tau_i\alpha_i(x_S)\Bigg)\Bigg| \\ < \frac{1}{2k^2}\mathbf{f}^{-(2k^2+k+1)(m+1)2^{2k+3}} \end{aligned} \tag{4.1}$$

when $\nu \in \mathbb{F}^{[s-n]} \setminus \{0\}, \tau \in \mathbb{F}^n, \sigma \in \mathbb{F}^{[s+1,m]\setminus\mathcal{I}}$ and

$$(\forall i \in [n])\ \mathop{\mathbb{E}}_{x_S} \chi(\alpha_i(x_S)) \geq \frac{1}{2k^2}\mathbf{f}^{-(2k^2+k+1)(m+1)2^{2k+3}}. \tag{4.2}$$

□

Claim 4.2. *For $i \in [0, s-n]$, there is a multilinear variety $B^i \subset G_{[k]\setminus S}$ of codimension at most im defined by maps whose coordinate sets belong to $\mathcal{F} \setminus \{S\}$, and a multilinear map $\psi^i \colon \operatorname{dom}\psi^i \to H$, where*

$$\begin{aligned}\operatorname{dom}\psi^i = (B^i \times G_S) \cap \{x_{[k]} \in G_{[k]} \colon (\forall j \in [i+1, s-n])\rho_j(x_S) = 0\} \\ \cap\, \{x_{[k]} \in G_{[k]} \colon (\forall j \in [n])\ \alpha_j(x_S) = 0\} \\ \cap\, \{x_{[k]} \in G_{[k]} \colon (\forall j \in [s+1,m])\ \beta_j(x_{I_j}) = 0\}\end{aligned} \tag{4.3}$$

such that $\psi^i = \phi$ on $\operatorname{dom}\phi \cap \operatorname{dom}\psi^i$.

Proof of Claim 4.2. We argue by induction on i. The base case is $i = 0$, where we may take $\psi^0 = \phi$. Assume now that the claim holds for some $i - 1 < s - n$, and let B^{i-1} and ψ^{i-1} be the corresponding variety and map. Take an arbitrary $z_S = z^{(i)}_S \in G_S$ such that $\rho_i(z_S) = 1$, $\rho_j(z_S) = 0$ for $j > i$, $\alpha_j(z_S) = 0$ for $j \in [n]$, and $\beta_j(z_{I_j}) = 0$ for $I_j \subsetneq S$. Such a point exists by Lemma 2.5.

We define $B^i = \{x_{[k]\setminus S} \colon (\forall j \in [s+1,m] \colon I_j \not\subset S)\ \beta_j(x_{I_j\setminus S}, z_{I_j\cap S}) = 0\} \cap B^{i-1}$. Notice that the coordinate sets of the maps defining B^i lie in $\mathcal{F} \setminus \{S\}$. Next, define $\psi^i \colon \operatorname{dom}\psi^i \to H$, where we first define $\operatorname{dom}\psi^i$ exactly as in (4.3) (with the current B^i and other relevant items), and we define values of ψ^i by extending the map $\tau_{x_{[k]\setminus S}} \colon (\operatorname{dom}\psi^{i-1})_{x_{[k]\setminus S}} \to H$ defined by $y_S \mapsto \psi^{i-1}(x_{[k]\setminus S}, y_S)$ by mapping z_S to 0, for each $x_{[k]\setminus S} \in B^i$. By Theorem 3.1, for each $x_{[k]\setminus S} \in B^i$ there is a unique multilinear extension $\theta_{x_{[k]\setminus S}}$ of $\tau_{x_{[k]\setminus S}}$ from the domain $(\operatorname{dom}\psi^{i-1})_{x_{[k]\setminus S}}$ to the domain $(\operatorname{dom}\psi^i)_{x_{[k]\setminus S}}$ that sends z_S to 0. The assumption (3.1) of Theorem 3.1 follows from (4.1). In particular, we use the fact that ρ_i is very quasirandom with respect to other forms. Note that $\operatorname{dom}\psi^i$ has

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the property that $\operatorname{dom}\psi^i = \bigcup_{x_{[k]\setminus S}\in B^i}\{x_{[k]\setminus S}\}\times(\operatorname{dom}\psi^i)_{x_{[k]\setminus S}}$ and for each $x_{[k]\setminus S}\in B^i$, $z_S\in(\operatorname{dom}\psi^i)_{x_{[k]\setminus S}}\setminus(\operatorname{dom}\psi^{i-1})_{x_{[k]\setminus S}}$. We thus define ψ^i for $(x_{[k]\setminus S}, y_S)\in\operatorname{dom}\psi^i$, by setting $\psi^i(x_{[k]\setminus S}, y_S)=\theta_{x_{[k]\setminus S}}(y_S)$.

It suffices to show that ψ^i is multilinear in the directions $[k]\setminus S$. To this end, fix some $d\in[k]\setminus S$ and take $x^1_{[k]\setminus S}, x^2_{[k]\setminus S}, x^3_{[k]\setminus S}\in B^i$ which differ in coordinate d and $x^1_d - x^2_d = x^3_d$. Write $D_2=\cap_{j\in[3]}\operatorname{dom}\theta_{x^j_{[k]\setminus S}}$ and $D_1 = D_2\cap\{y_S\in G_S\colon \rho_i(y_S)=0\}$. Observe that $\theta_{x^1_{[k]\setminus S}}-\theta_{x^2_{[k]\setminus S}}$ is a multilinear map that extends $\tau_{x^1_{[k]\setminus S}}-\tau_{x^2_{[k]\setminus S}}$ from D_1 to D_2 and maps z_S to 0. Also, $\theta_{x^3_{[k]\setminus S}}$ is a multilinear map that extends $\tau_{x^3_{[k]\setminus S}}$ from D_1 to D_2 and maps z_S to 0. But $\tau_{x^1_{[k]\setminus S}}-\tau_{x^2_{[k]\setminus S}}=\tau_{x^3_{[k]\setminus S}}$ on D_1, so by the uniqueness of extensions in Theorem 3.1, we have $\theta_{x^1_{[k]\setminus S}}-\theta_{x^2_{[k]\setminus S}}=\theta_{x^3_{[k]\setminus S}}$ on D_2, as desired. □

Apply the claim above with $i=s-n$. After that, it remains to remove the maps $\alpha_{[n]}$. From (4.2) and Theorem 2.7, we may find $m'\le mC_k^{\text{ranks}}(((2k^2+k+1)(m+1)2^{2k+3}+2k^2)^{D_k^{\text{ranks}}}+1)$ and further multilinear forms $\gamma_j\colon G_{J_j}\to\mathbb{F}$, $J_j\in\mathcal{F}\setminus\{S\}$, $j\in[m']$ such that

$$\Big\{x_{[k]}\in G_{[k]}\colon (\forall j\in[m'])\ \gamma_j(x_{J_j})=0\Big\}\subseteq\Big\{x_{[k]}\in G_{[k]}\colon (\forall j\in[n])\ \alpha_j(x_S)=0\Big\}.$$

Hence, the map Φ with domain

$$\begin{aligned}\operatorname{dom}\Phi = (B^{s-n}\times G_S)&\cap\{x_{[k]}\in G_{[k]}\colon(\forall j\in[m'])\gamma_j(x_{J_j})=0\}\\ &\cap\{x_{[k]}\in G_{[k]}\colon(\forall j\in[s+1,m])\beta_j(x_{I_j})=0\}\end{aligned}$$

and $\Phi=\psi^{s-n}$ on its domain is the desired map. Its domain $\operatorname{dom}\Phi$ has codimension at most $m^2+m+m'=O_k(m^{O_k(1)})$, which is the claimed bound. This completes the proof of Proposition 4.1 and with it the proof of Theorem 1.4.

5. The case of general finite fields

In this section, we use Theorem 1.4 to deduce Theorem 1.5. Hence, we fix a field $\mathbb{F}=\mathbb{F}_{p^r}$ for some prime p and integer r. We denote the field of order p by $\mathbb{F}_p$ and we stress that in this section $\mathbb{F}$ will always stand for the larger field. Since $\mathbb{F}$ is itself a vector space of dimension r over $\mathbb{F}_p$, there are $e_1,\dots,e_r\in\mathbb{F}$ which form a basis over $\mathbb{F}_p$. Using this basis, we may define the coordinates of $x\in\mathbb{F}$ as elements $x_1,\dots,x_r\in\mathbb{F}_p$ which satisfy $x=\sum_{i\in[r]}x_ie_i$. More generally, for a map $\alpha\colon X\to\mathbb{F}$, where X is an arbitrary set, we can write α_i for its ith coordinate, which is just a composition of α with taking the ith coordinate in $\mathbb{F}$.

We need the following lemma which essentially says that if an $\mathbb{F}_p$-multilinear map α with codomain $\mathbb{F}$ is nearly $\mathbb{F}$-linear in each coordinate, then it differs from an $\mathbb{F}$-multilinear form by a map of low partition rank.

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Lemma 5.1. *Let $\alpha\colon G_{[k]} \to \mathbb{F}$ be an $\mathbb{F}_p$-multilinear map that is $\mathbb{F}$-linear in each of the first $d-1$ coordinates. Suppose that for every $i \in [r]$ and $\lambda \in \mathbb{F}$,*

$$\operatorname{prank}_{\mathbb{F}_p}(x_{[k]} \mapsto \alpha_i(x_{[k]\setminus\{d\}}, \lambda x_d) - (\lambda\cdot\alpha)_i(x_{[k]})) \leq s.$$

Then there is an $\mathbb{F}_p$-multilinear map $\sigma\colon G_{[k]} \to \mathbb{F}$ that is $\mathbb{F}$-linear in each of the first d coordinates, such that for each $i \in [r]$

$$\operatorname{prank}_{\mathbb{F}_p}(\alpha_i - \sigma_i) \leq 2\,sr^2.$$

Proof. Let $\mathcal{M}$ be the $\mathbb{F}_p$-vector space of all $\mathbb{F}_p$-multilinear maps $\mu\colon G_{[k]} \to \mathbb{F}$ that are additionally $\mathbb{F}$-linear in coordinates $1, \dots, d-1$. Consider the $\mathbb{F}^\times$-action on $\mathcal{M}$ given by

$$\lambda \circ \mu \colon = \left(x_{[k]} \mapsto \lambda^{-1}\mu(x_{[k]\setminus\{d\}}, \lambda x_d)\right),$$

for every $\lambda \in \mathbb{F}^\times$ and $\mu \in \mathcal{M}$. This action can be viewed as a representation of the multiplicative group $\mathbb{F}^\times$. Let $V \leq \mathcal{M}$ be the $\mathbb{F}_p$-subspace $V = \langle \lambda\circ\alpha - \lambda'\circ\alpha \colon \lambda, \lambda' \in \mathbb{F}^\times\rangle_{\mathbb{F}_p}$. Then V is invariant under the above action: that is, V is a subrepresentation. Since $p = \operatorname{char}\mathbb{F}_p$ does not divide $|\mathbb{F}^\times|$, we may apply Maschke's theorem to find another subspace $S \leq \mathcal{M}$, also invariant under the action above, such that $\mathcal{M} = V \oplus S$. Write $\alpha = v + \sigma$ for $v \in V$ and $\sigma \in S$. Then for each $\lambda \in \mathbb{F}^\times$,

$$\alpha - \lambda\circ\alpha = (v - \lambda\circ v) + (\sigma - \lambda\circ\sigma).$$

Since V and S are invariant under the action, we have $v - \lambda\circ v \in V$ and $\sigma - \lambda\circ\sigma \in S$. However, $\alpha - \lambda\circ\alpha \in V$, so we in fact get $\sigma - \lambda\circ\sigma \in V \cap S = \{0\}$, and therefore $\sigma = \lambda\circ\sigma$ for each $\lambda \in \mathbb{F}^\times$. Hence, σ is actually $\mathbb{F}$-linear in coordinate d as well. In fact, σ satisfies the conditions in the conclusion of the lemma, as we shall now check.

Note that V is spanned by elements of the form $e_i\circ\alpha - e_j\circ\alpha$, so there are $\nu_{ij} \in \mathbb{F}_p$, $i, j \in [r]$, such that

$$\alpha - \sigma = v = \sum_{i,j\in[r]} \nu_{ij}(e_i\circ\alpha - e_j\circ\alpha).$$

For each $l \in [r]$, by assumptions, we have $\operatorname{prank}_{\mathbb{F}_p}((e_i\circ\alpha)_l - \alpha_l) \leq s$. Hence

$$\operatorname{prank}_{\mathbb{F}_p}(\alpha_l - \sigma_l) \leq 2\,sr^2,$$

as required. □

Proof of Theorem 1.5. Let B be an $\mathbb{F}$-multilinear variety of codimension at most d in $G_{[k]}$, where each G_i is a vector space over $\mathbb{F}$, and let $\phi\colon B \to H$ be an $\mathbb{F}$-multilinear map. We may view each G_i as a vector space over $\mathbb{F}_p$, B as an $\mathbb{F}_p$-multilinear variety of codimension at most rd and ϕ as an $\mathbb{F}_p$-multilinear map. Thus, we may apply Theorem 1.4, to obtain a global $\mathbb{F}_p$-multilinear map $\psi\colon G_{[k]} \to H$ such that $\psi = \phi$ holds on an $\mathbb{F}_p$-multilinear variety $V \subset B$ of codimension at most $C_k(rd)^{D_k}$.

Our goal now is to replace ψ by a global $\mathbb{F}$-multilinear map. We do this in k steps. At the i^{th} step, we obtain an $\mathbb{F}_p$-multilinear map $\psi^{(i)}\colon G_{[k]} \to H$, which is additionally $\mathbb{F}$-multilinear in the first i coordinates, and a non-empty $\mathbb{F}_p$-multilinear variety $V^{(i)}$ such

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that $\phi = \psi^{(i)}$ on $V^{(i)}$, of codimension at most $d_i = (krd)^{O_i(1)}$, where $O_i(1)$ stands for a parameter that depends only on i. Additionally, the varieties will be nested: that is, $V^{(i)} \subset V^{(i-1)}$. To start the inductive procedure, we set $\psi^{(0)} = \psi$ and $V^{(0)} = V$.

Assume that for some $i \geq 0$ we have obtained $\psi^{(i)}$ and $V^{(i)}$ with the properties described. We shall use this map and variety to define $\psi^{(i+1)}$ and $V^{(i+1)}$. Fix an $\mathbb{F}$-dot product $\cdot$ on H and define $\theta\colon G_{[k]} \times H \to \mathbb{F}$ by $\theta(x_{[k]}, h) = \psi^{(i)}(x_{[k]}) \cdot h$. (Below we also use the notation $\cdot$ for multiplication inside $\mathbb{F}$, but this should cause no confusion.) This is an $\mathbb{F}_p$-multilinear map, which is additionally $\mathbb{F}$-linear in each of the first i coordinates and in the last coordinate as well. In order to apply Lemma 5.1, we need to prove the following claim. □

Claim 5.2. *For each $j \in [r]$ and $\lambda \in \mathbb{F}$ we have*

$$\operatorname{prank}_{\mathbb{F}_p}\left((x_{[k]}, h) \mapsto \theta_j(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1}, h) - (\lambda \cdot \theta)_j(x_{[k]}, h)\right) \leq (2kd_i)^{O(1)}.$$

Proof of the claim. Observe that the analytic rank a of the given form over $\mathbb{F}_p$ satisfies

$$\begin{aligned}
p^{-a} &= \mathop{\mathbb{E}}_{x_{[k]},h} \omega^{\theta_j(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1}, h) - (\lambda\cdot\theta)_j(x_{[k]},h)} \\
&= \mathop{\mathbb{E}}_{x_{[k]},h} \omega^{(\psi^{(i)}(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1})\cdot h)_j - (\lambda\psi^{(i)}(x_{[k]})\cdot h)_j} \\
&= \mathop{\mathbb{E}}_{x_{[k]},h} \omega^{\left(\left(\psi^{(i)}(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1}) - \lambda\psi^{(i)}(x_{[k]})\right)\cdot h\right)_j} \\
&\geq \frac{1}{|G_{[k]}|}\left|\left\{x_{[k]} \in G_{[k]} \colon \psi^{(i)}(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1}) = \lambda\psi^{(i)}(x_{[k]})\right\}\right|.
\end{aligned}$$

Observe that if $x_{[k]}$ and $(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1})$ both belong to $V^{(i)}$, then $\psi^{(i)}(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1}) = \phi(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1})$ and $\psi^{(i)}(x_{[k]}) = \phi(x_{[k]})$. Additionally, $V^{(i)} \subset B$ so $\phi(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1}) = \lambda\phi(x_{[k]})$ and we deduce that

$$\psi^{(i)}(x_{[k]\setminus\{i+1\}}, \lambda x_{i+1}) = \lambda\psi^{(i)}(x_{[k]}).$$

Thus, the analytic rank a satisfies

$$p^{-a} \geq \frac{1}{|G_{[k]}|}\left|\left\{x_{[k]} \in G_{[k]} \colon (x_{[k]\setminus\{i+1\}}, \lambda x_{i+1}), (x_{[k]}) \in V^{(i)}\right\}\right|.$$

The set on the right-hand side of the expression is an intersection of two $\mathbb{F}_p$-multilinear varieties of codimension at most d_i. Since both varieties contain 0, we get that this set is also a non-empty $\mathbb{F}_p$-multilinear variety of codimension at most $2d_i$. By Lemma 2.3, we deduce that $p^{-a} \geq p^{-2kd_i}$. Thus, the analytic rank of the given $\mathbb{F}_p$-multilinear form is at most $2kd_i$. Theorem 2.7 then applies to bound its partition rank over $\mathbb{F}_p$ by $(2kd_i)^{O(1)}$, as required.

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Now we may apply Lemma 5.1 to obtain a further $\mathbb{F}_p$-multilinear map $\tilde{\theta} \colon G_{[k]} \times H \to \mathbb{F}$, which is $\mathbb{F}$-linear in the first $i+1$ coordinates and in the last one, such that for each $j \in [r]$

$$\operatorname{prank}_{\mathbb{F}_p}(\theta_j - \tilde{\theta}_j) \leq r^2(2kd_i)^{O(1)}.$$

We may find an $\mathbb{F}_p$-multilinear map $\psi^{(i+1)} \colon G_{[k]} \to H$, which is additionally $\mathbb{F}$-linear in the first $i+1$ coordinates, such that $\tilde{\theta}(x_{[k]}, h) = \psi^{(i+1)}(x_{[k]}) \cdot h$. This will be the desired map in this step of the inductive procedure. It remains to show that $\psi^{(i+1)}(x_{[k]})$ coincides with ϕ on an $\mathbb{F}_p$-multilinear variety of bounded codimension.

Using the bounds on the partition rank, for each $j \in [r]$, we may find $m_j \leq r^2(2kd_i)^{O(1)}$ and $\mathbb{F}_p$-multilinear forms $\alpha_l^{(j)} \colon G_{I_l^{(j)}} \to \mathbb{F}_p, \beta_l^{(j)} \colon G_{[k]\setminus I_l^{(j)}} \times H \to \mathbb{F}_p$, where $l \in [m_j]$ and $I_l^{(j)} \subset [k]$ are non-empty sets, such that for each $x_{[k]} \in G_{[k]}$ and $h \in H$,

$$\theta_j(x_{[k]}, h) - \tilde{\theta}_j(x_{[k]}, h) = \sum_{l \in [m_j]} \alpha_l^{(j)}(x_{I_l^{(j)}}) \beta_l^{(j)}(x_{[k]\setminus I_l^{(j)}}, h)$$

holds. Note that if $x_{[k]} \in G_{[k]}$ satisfies $\alpha_l^{(j)}(x_{I_l^{(j)}}) = 0$ for all $j \in [r]$ and $l \in [m_j]$, then in fact

$$\theta_j(x_{[k]}, h) - \tilde{\theta}_j(x_{[k]}, h) = 0$$

holds for all $h \in H$, so $\psi^{(i)}(x_{[k]}) \cdot h = \psi^{(i+1)}(x_{[k]}) \cdot h$ holds for all $h \in H$, and thus $\psi^{(i+1)}(x_{[k]}) = \psi^{(i)}(x_{[k]})$. Therefore, we set

$$V^{(i+1)} = \Big\{ x_{[k]} \in V^{(i)} \colon (\forall j \in [r])(\forall l \in [m_j])\, \alpha_l^{(j)}(x_{I_l^{(j)}}) = 0 \Big\}$$

which is a non-empty $\mathbb{F}_p$-multilinear variety of codimension at most $r^3(2kd_i)^{O(1)} \leq (krd)^{O_{i+1}(1)}$, and when $x_{[k]} \in V^{(i+1)}$ then $\psi^{(i+1)}(x_{[k]}) = \phi(x_{[k]})$, as desired.

Setting $i = k$ and taking $\psi^{(k)}$ and $V^{(k)}$ completes the proof. □

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