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ON REVERSIBLE GROUP RINGS

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Let G be an arbitrary finite group, R be a finite associative ring with identity and RG be the group ring. We show that $\mathbb{Z}_2Q_8$ is the minimal reversible group ring which is not symmetric, and we also characterise the finite rings R for which RQ_8 is reversible. The first result extends a result of Gutan and Kisielewicz which shows that $\mathbb{Z}_2Q_8$ is the minimal reversible group algebra over a field which is not symmetric, and it answers a question raised by Marks for the group ring case.

1. INTRODUCTION AND PRELIMINARIES

Let R be an associative ring with identity. Define R to be reversible if $\alpha\beta = 0$ implies $\beta\alpha = 0$, and symmetric if $\alpha\beta\gamma = 0$ implies $\alpha\gamma\beta = 0$ for all $\alpha, \beta, \gamma \in R$.

Recently reversible rings have been discussed by several authors [**1, 5, 6, 7, 8**]. Symmetric rings are clearly reversible, but the converse is not true. In fact, Marks [**8**] showed that the group algebra $\mathbb{Z}_2Q_8$ of the quaternion group of order 8 over the two-element field is reversible, but not symmetric. In [**5**], Gutan and Kisielewicz asserted that it is the minimal group ring over a field with the above mentioned property. In this short note, we shall show that in fact, $\mathbb{Z}_2Q_8$ is the minimal group ring with this property, thereby providing a partial answer to a question raised by Marks [**8**]. We shall also characterise the finite rings R for which RQ_8 is reversible.

It was shown in [**7**] that if a group ring RG of a torsion group G over any associative ring with identity is reversible, then R is a reversible ring and G is either an Abelian group or a Hamiltonian group, that is, $G = Q_8 \times E_2 \times E_2'$, where Q_8 is the quaternion group of order 8, E_2 is an elementary Abelian 2-group, and E_2' is an Abelian group all of whose elements are of odd order. The following results shown in [**7**] are needed in the proof of Theorem 2.1.

LEMMA 1.1. ([**7**, Lemma 2.4].) *Let R be a ring with identity. If R contains a nonzero nilpotent element r such that $2r = 0$, then RQ_8 is not reversible.*

THEOREM 1.2. ([**7**, Theorem 2.5].) *$\mathbb{Z}_nQ_8$ is reversible if and only if $n = 2$.*

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The following remark follows immediately from Lemma 1.1 and Theorem 1.2.

REMARK 1.3. Let R be a finite ring with identity. If RQ_8 is reversible, then the characteristic of R must be 2 and R contains no non-zero nilpotent elements.

Throughout the paper, R always denotes a finite associative ring with identity and G denotes a finite group. The symbol $A(R)$ denotes the two-sided annihilator of R. As mentioned earlier, Q_8 is the quaternion group of order 8; and UT is the ring of upper-triangular 2×2 matrices over $GF(2)$. Our other notation follows that in [**9**].

2. Minimal reversible group rings

In this section, we investigate minimal reversible group rings. We first characterise when RQ_8 is reversible.

THEOREM 2.1. *Let R be a finite (not necessarily commutative) ring with identity. Then RQ_8 is reversible if and only if $R = \prod GF(2^{n_i})$ where each n_i is odd.*

PROOF: If RQ_8 is reversible, then it follows from Remark 1.3 that the characteristic of R must be 2 and R has no nonzero nilpotent elements. This implies that R is semisimple and so it is a direct product of full matrix rings over some division rings. Since R has no nonzero nilpotent elements and a finite division ring is always a field, we conclude that $R = \prod GF(2^{n_i})$ (a direct product of finite fields of characteristic 2). Since $RQ_8 \cong \prod GF(2^{n_i})Q_8$ is reversible, all $GF(2^{n_i})Q_8$ must be reversible. Therefore, all n_i must be odd by [**5**, Corollary 4.2].

The converse follows immediately from [**5**, Proposition 2.3 (i) and Corollary 4.2]. ∎

In [**8**], Marks asks whether $\mathbb{Z}_2Q_8$ is the smallest ring which is reversible but not symmetric. As mentioned earlier, Gutan and Kisielewicz [**5**] assert that this is the minimal group ring over a field with this property. Theorem 2.1 confirms that this is the case when finite group rings over commutative rings are considered (this result can also be obtained by using an argument on the orders of group rings). Next we prove that $\mathbb{Z}_2Q_8$ is indeed the smallest group ring with the property.

THEOREM 2.2. *$\mathbb{Z}_2Q_8$ is the minimal reversible group ring which is not symmetric.*

PROOF: We shall show that every reversible group ring RG having $|RG| \leqslant |\mathbb{Z}_2Q_8| = 256$ is symmetric except for $RG = \mathbb{Z}_2Q_8$.

If RG is reversible, then as mentioned in the introduction, R is reversible and G is either Abelian or Hamiltonian; and since Q_8 is the smallest Hamiltonian group, $\mathbb{Z}_2Q_8$ is the minimal reversible non-symmetric group ring with G Hamiltonian. Thus, we may suppose that G is Abelian, R is reversible but not commutative, and $|RG| \leqslant 256$. Since $|RG| = |R|^{|G|}$, we see that if $|G| \geqslant 3$, then $|R| < 7$; therefore, R is commutative, hence RG is both reversible and symmetric. Thus we may assume that $G = C_2$.

If $|RC_2| < 256$, then $|R| \leqslant 15$; and since UT is the only such non-commutative ring with 1, and it is clearly not reversible, we need only to consider the case RC_2 with R non-commutative and reversible, and $|R| = 16$. Hence Theorem 2.2 follows from the following theorem. $\square$

THEOREM 2.3. *There is a unique non-commutative reversible ring R_0 with 1 of order 16. Moreover, R_0C_2 is not reversible.*

The proof of this theorem employs the following lemma.

LEMMA 2.4. *If R is a non-commutative nil ring of order 8 with $2R = \{0\}$, then R is not reversible.*

PROOF: We regard R as a vector space over $GF(2)$. For $S \subseteq R$, denote by $\langle S \rangle$ the subspace generated by S.

Since a finite nil ring is nilpotent, the two-sided annihilator $A(R)$ contains a nonzero element w. Choose $a, b \in R$ such that $\{w, a, b\}$ is a basis. Since R is not commutative, $a^2 \in \langle w, a \rangle$ and $b^2 \in \langle w, b \rangle$; and it follows easily that each of a^2, b^2 is either 0 or w, so that $a^2, b^2 \in A(R)$. It is now easy to show that each of ab and ba must be either 0 or w; and since R is not commutative, one of ab and ba is 0 and the other is w. Thus, R is not reversible. $\square$

PROOF OF THEOREM 2.3: Let R be any ring with 1 of order 16 which is reversible but not commutative. Since $ab = 0$ implies $ba = 0 = bar = arb$ for any $r \in R$, it follows easily that the set N of nilpotent elements in R is an ideal.

We note that since R is reversible, all idempotents are central. For if $e^2 = e$ and $x \in R$, $(ex - exe)e = 0$, and hence $e(ex - exe) = ex - exe = 0$; and $e(xe - exe) = 0$, so $(xe - exe)e = xe - exe = 0$. Thus, if R has an idempotent $e \neq 0, 1$, $R = eR \bigoplus A(e)$ - a ring-theoretic direct sum. We cannot have both summands of order 4, since R would then be commutative; and we cannot have $R = \mathbb{Z}_2 \bigoplus UT$, since UT is not reversible. Thus 0 and 1 are the only idempotents in R; and since each element of a finite ring has an idempotent power, each element of R is nilpotent or invertible.

Next we show that $2R = \{0\}$. Note that in $(R, +)$, an element of maximal order may be taken as an element of a basis, hence if 1 has order 8, or if 1 has order 4 and $(R, +) = C_4 \bigoplus C_4$, then R is not non-commutative. Moreover, if 1 has order 4 and $(R, +) = C_4 \bigoplus C_2 \bigoplus C_2$, then $(R, +)$ has a basis $\{1, a, b\}$ where $a, b \in N$, so that $\{2, a, b\}$ is a basis for N and $|N| = 8$. Since $1 + N$ is the set of all invertible elements, N must be non-commutative. However, by Lemma 2.4 N is not reversible and thus R is not reversible. Hence this situation must be ruled out, and therefore, $2R = \{0\}$ as claimed.

Now it follows from [**3**, Theorem 1] that $|N| \geqslant 4$, so $|N| = 4$ or $|N| = 8$. But $|N| = 8$ is ruled out by Lemma 2.4, so $|N| = 4$. It now follows from [**2**, Theorem 1] that R is unique. Specifically, $R = GF(4) \times GF(4)$ with componentwise addition, and multiplication given by $(a, b)(c, d) = (ac, ad + bc^2)$. This ring, which we designate as R_0, has $N^2 = \{0\}$, hence is obviously reversible. (It is also easy to see that R_0 is symmetric.)

It remains only to show that R_0C_2 is not reversible. Let a be a generator of the multiplicative group of $GF(4)$, so that $a^3 = 1$ and $a^2 = a + 1$; and let $C_2 = \{e, g\}$, where e is the identity element. It is easily proved that

$$\big((a,a)e + (a,0)g\big)\big((a,a^2)e + (a,0)g\big) = 0,$$

but

$$\big((a,a^2)e + (a,0)g\big)\big((a,a)e + (a,0)g\big) = (0,1)e + (0,1)g \neq 0.$$

Hence R_0C_2 is not reversible. □

REMARK 2.5. We discovered Theorem 2.3 by a computer search using GAP [**4**], which identified 13 non-commutative rings with 1 of order 16 and showed that only one of them is reversible. However, the presentation of additive groups in GAP did not make it easy to see that R_0C_2 is not reversible.

REMARK 2.6. The ring R_0 is one of three minimal non-commutative duo rings identified by Xue in [**10**]. Specifically, R_0 is the ring R_3 in [**10**].

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