Proceedings of the Royal Society of Edinburgh, **152**, 402–427, *2022*
DOI:10.1017/prm.2021.14

Existence and multiplicity of periodic solutions to differential equations with attractive singularities

José Godoy and Robert Hakl
Institute of Mathematics, Czech Academy of Sciences, Žižkova 22, 616 62 Brno, Czech Republic (godoy@math.cas.cz; hakl@ipm.cz)

Xingchen Yu
School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, P.R. China (yuxingchen0@yeah.net)

(Received 8 December 2020; accepted 15 March 2021)

The existence and multiplicity of T-periodic solutions to a class of differential equations with attractive singularities at the origin are investigated in the paper. The approach is based on a new method of construction of strict upper and lower functions. The multiplicity results of Ambrosetti–Prodi type are established using *a priori* estimates and certain properties of topological degree.

Keywords: periodic solutions; upper and lower functions; multiplicity results

2010 *Mathematics Subject Classification:* 34C25; 34B18

1. Introduction

The study of differential equations with some type of singularity has been attracting the attention of many researchers during the recent decades. The main reason is the applicability of theoretical results to important problems arising in natural sciences (see, e.g., [**1**, **8**, **16**, **24**, **28**, **29**]). From the mathematical point of view, differential equations with singularities can be divided into three major classes according to the type of singularity involved in the equation—attractive, repulsive, and mixed (attractive–repulsive) type. Each one of the above-mentioned classes possesses its own property, that means a different approach in their study. As for the problems with attractive-type singularity we recommend the papers [**3**, **4**, **9–14**, **17–22**, **25**, **27**] (see also the references therein).

One of the first important works studying the solvability of a periodic problem for such equations is the well-known paper of Lazer and Solimini published in 1987 (see [**17**]) where the authors dealt, in particular, with the family of equations

$$u'' + \frac{\nu}{u^\lambda} = h(t), \tag{1.1}$$

supposing $\lambda > 0$, $\nu \in \mathbb{R} \setminus \{0\}$, and h is a T-periodic function. They proved that (1.1) in the attractive case, i.e. in the case when $\nu > 0$, admits a T-periodic solution if and only if $\overline{h} > 0$ provided h is a continuous function (condition on the regularity of the external force). The boundedness of the function h in the result of Lazer and Solimini is essential and it cannot be extended to the general case $h \in L^p(\mathbb{R}/T\mathbb{Z})$ for all $\lambda > 0$. Recently, Hakl and Zamora established a relation between the existence of a T-periodic solution and the orders of regularity of h and singularity λ (see [**12**]). More precisely, the main results of [**12**] can be formulated as follows:

- If $h \in L^p(\mathbb{R}/T\mathbb{Z})$, $\lambda \geqslant 1/(2p-1)$ then (1.1) with $\nu > 0$ has a positive T-periodic solution iff $\overline{h} > 0$. Moreover, this solution is unique.
- If $\lambda \in (0, 1/(2p-1))$ then there exists $h \in L^p(\mathbb{R}/T\mathbb{Z})$ with $\overline{h} > 0$ such that (1.1) with $\nu > 0$ has no positive T-periodic solution.

Another step in this direction was done in [**13**] where the ideas of the paper [**12**] were generalized for the equation of the form

$$u'' + \frac{g(t)}{u^\lambda} = h(t)u^\delta,$$

with $g, h \in L(\mathbb{R}/T\mathbb{Z})$, $g(t) \geqslant 0$ for a. e. $t \in [0, T]$, $\delta \in [0, 1)$, $\lambda > 0$. The main importance of the results established in [**13**] lies in the fact that the function g is not necessarily bounded from below by some positive constant. The results obtained there play an important role also in the present work where the existence and multiplicity of the positive T-periodic solutions to the differential equation of the form

$$u'' + \frac{g(t)}{u^\lambda} = h(t, u) + \mu f(t), \tag{1.2}$$

is studied. Here, in addition, h is a Carathéodory function, $f \in L(\mathbb{R}/T\mathbb{Z})$ with $\overline{f} > 0$, and $\mu \in \mathbb{R}$ is a parameter. The aim of this paper is to find conditions guaranteeing the existence of a critical parameter $\mu_\dagger$ such that the equation (1.2) has at least two, at least one, or no solution provided $\mu > \mu_\dagger$, $\mu = \mu_\dagger$, or $\mu < \mu_\dagger$, respectively.

The basic tool used to handle with the equation (1.2) relies on *a priori* estimates of all possible T-periodic solutions to (1.2), the construction of well-ordered strict lower and upper functions, and the direct application of the degree theory. The idea how to obtain multiplicity results is similar to the previous works, usually we use homotopy invariance and additivity property of Leray-Schauder degree (see, e.g. [**2**, **6**, **7**, **15**, **19**, **23**, **30**]). However, the nature of this kind of problems request quite new approach in how to construct the strict lower and upper functions.

The paper is organized as follows: the statement of the problem and some basic notation are introduced in § 2. Section 3 is devoted to the main results; theorems 3.2–3.6 deal with the general case, the effective conditions for a particular case are established in theorems 3.8–3.10. Auxiliary propositions and the construction of strict lower and upper functions are included in §§ 4 and 5, respectively. The proofs of the main results can be found in § 6.

2. Statement of problem

Our contribution to the above-described topics deals with the existence and multiplicity of T-periodic solutions to the following differential equation with an attractive singularity

$$u'' + \frac{g(t)}{u^\lambda} = h(t,u) + \mu f(t). \tag{2.1}$$

Here $\lambda > 0$, $\mu \in \mathbb{R}$ is a parameter, f, g are T-periodic Lebesgue integrable functions with positive mean values, i.e., $f, g \in L\big(\mathbb{R}/T\mathbb{Z}\big)$ and

$$\int_0^T f(s)\,\mathrm{d}s > 0, \qquad \int_0^T g(s)\,\mathrm{d}s > 0, \tag{2.2}$$

$g(t) \geqslant 0$ for a. e. $t \in [0,T]$, and $h : \mathbb{R}/T\mathbb{Z} \times \mathbb{R}_+ \to \mathbb{R}$ is a Carathéodory function, i.e.

(i) $h(\cdot, x) : [0,T] \to \mathbb{R}$ is measurable for every $x \in \mathbb{R}_+$;

(ii) $h(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is continuous for a. e. $t \in [0,T]$;

(iii) for every $r > 0$ there exist non-negative functions ψ_r^+, $\psi_r^- \in L\big(\mathbb{R}/T\mathbb{Z}\big)$ such that

$$-\psi_r^-(t) \leqslant h(t,x) \leqslant \psi_r^+(t) \qquad \text{for a. e. } t \in [0,T], \quad x \in [0,r]. \tag{2.3}$$

Remark 2.1. Without loss of generality we can assume that the functions ψ_r^+ are nondecreasing with respect to r, i.e., $\psi_r^+(t) \leqslant \psi_s^+(t)$ for a. e. $t \in [0,T]$ whenever $r \leqslant s$. Moreover, we assume that the functions ψ_r^+ are nontrivial, i.e.,

$$\int_0^T \psi_r^+(s)\,\mathrm{d}s > 0 \qquad \text{for } r > 0. \tag{2.4}$$

These relations will be used later in the paper.

By a T-periodic solution to the equation (2.1) we understand a positive function u that is absolutely continuous together with its first derivative and satisfies the equality (2.1) almost everywhere on $[0,T]$. We write $u(t;\mu)$ in order to emphasize that u is a solution to (2.1) with a parameter μ.

For convenience, we introduce the following notation that is used throughout the paper.

$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ are the sets of natural, integer, and real numbers, respectively, $\mathbb{R}_+ = [0,+\infty)$, $\mathbb{R}_- = (-\infty, 0]$.

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

$L^p(\mathbb{R}/T\mathbb{Z})$, where $p \geqslant 1$, is the Banach space of T-periodic functions $y : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ that are Lebesgue integrable on $[0, T]$ in the p-th power, endowed with the norm

$$\|y\|_p = \left(\int_0^T |y(s)|^p \, \mathrm{d}s \right)^{1/p}.$$

We write $L(\mathbb{R}/T\mathbb{Z})$ instead of $L^1(\mathbb{R}/T\mathbb{Z})$. If $y \in L(\mathbb{R}/T\mathbb{Z})$ then

$$\overline{y} = \frac{1}{T} \int_0^T y(s) \, \mathrm{d}s, \qquad [y]_+(t) = \frac{|y(t)| + y(t)}{2}, \qquad [y]_-(t) = \frac{|y(t)| - y(t)}{2}.$$

$L^\infty(\mathbb{R}/T\mathbb{Z})$ is the Banach space of T-periodic functions $y : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ that are essentially bounded with the norm

$$\|y\|_\infty = \text{ess sup} \{ |y(t)| : t \in [0, T] \}.$$

$AC^1(\mathbb{R}/T\mathbb{Z})$ is a set of T-periodic functions $y : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ such that y and y' are absolutely continuous on $[0, T]$.

$C(\mathbb{R}/T\mathbb{Z})$ is the Banach space of T-periodic continuous functions $y : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ with the norm

$$\|y\|_C = \max \{ |y(t)| : t \in [0, T] \}.$$

For every Carathéodory function $y : \mathbb{R}/T\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ we define a continuous function $\overline{y} : \mathbb{R} \to \mathbb{R}$ by setting

$$\overline{y}(x) = \frac{1}{T} \int_0^T y(s, x) \, \mathrm{d}s \qquad \text{for } x \in \mathbb{R}.$$

If $E \subseteq \mathbb{R}$ then $\operatorname{meas} E$ is the Lebesgue measure of the set E.

Now we introduce (strict) lower and upper functions in a form suitable for us. For more details about the topics see, e.g., [**5**, **25**, **26**] and references therein.

Definition 2.2. A function $\alpha \in AC^1(\mathbb{R}/T\mathbb{Z})$ is said to be a *lower function* to (2.1) if $\alpha(t) > 0$ for $t \in [0, T]$ and

$$\alpha''(t) + \frac{g(t)}{\alpha^\lambda(t)} \geqslant h(t, \alpha(t)) + \mu f(t) \quad \textit{for a.e. } t \in [0, T]. \tag{2.5}$$

A lower function α to (2.1) is said to be *strict* if every T-periodic solution u to (2.1) satisfying $u(t) \geqslant \alpha(t)$ for $t \in [0, T]$ admits the inequality $u(t) > \alpha(t)$ for $t \in [0, T]$.

Definition 2.3. A function $\beta \in AC^1(\mathbb{R}/T\mathbb{Z})$ is said to be an *upper function* to (2.1) if $\beta(t) > 0$ for $t \in [0, T]$ and

$$\beta''(t) + \frac{g(t)}{\beta^\lambda(t)} \leqslant h(t, \beta(t)) + \mu f(t) \quad \textit{for a.e. } t \in [0, T]. \tag{2.6}$$

An upper function β to (2.1) is said to be *strict* if every T-periodic solution u to (2.1) satisfying $u(t) \leqslant \beta(t)$ for $t \in [0, T]$ admits the inequality $u(t) < \beta(t)$ for $t \in [0, T]$.

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

The following assertion can be found in [5].

PROPOSITION 2.4. *Let $\beta \in AC^1(\mathbb{R}/T\mathbb{Z})$ be such that $\beta(t) > 0$ for $t \in [0,T]$. Assume that β is not a T-periodic solution to (2.1) and let there exist $\varepsilon > 0$ such that*

$$\beta''(t) + \frac{g(t)}{x^\lambda} \leqslant h(t,x) + \mu f(t) \quad \text{for a. e. } t \in [0,T], \quad x \in [\beta(t)-\varepsilon, \beta(t)]. \tag{2.7}$$

Then β is a strict upper function to (2.1).

3. Main results and applications

In order to obtain *a priori estimates* for all possible T-periodic solutions to the equation (2.1) we need some assumptions. The first one is a technical condition that helps us to get an upper bound.

(H_1) Suppose that there exists $r > 0$ such that

$$-\eta(t,x) \leqslant h(t,x) \leqslant h^+(t,x) - h^-(t,x) \quad \text{for a. e. } t \in [0,T], \quad x \geqslant r, \tag{3.1}$$

where $\eta, h^+, h^- : \mathbb{R}/T\mathbb{Z} \times \mathbb{R}_+ \to \mathbb{R}_+$ are Carathéodory functions non-decreasing with respect to the second variable satisfying

$$\lim_{x\to+\infty} \frac{\overline{\eta}(x)}{x} = 0, \qquad \lim_{x\to+\infty} \overline{h^-}(x) = +\infty. \tag{3.2}$$

Furthermore, we assume that there exists $\zeta \in (0,1)$ such that

$$L \overset{\text{def}}{=} \limsup_{x\to+\infty} \frac{\overline{h^+}(x)}{\overline{h^-}((1-\zeta)x)} < 1. \tag{3.3}$$

REMARK 3.1. Note that according to (2.3) we can assume without loss of generality that the first inequality in (3.1) holds for all $x \geqslant 0$, i.e., we will assume

$$-\eta(t,x) \leqslant h(t,x) \qquad \text{for a. e. } t \in [0,T], \quad x \geqslant 0. \tag{3.4}$$

The following assumptions help us to obtain a lower bound.

(H_2) Let $\psi_r^+, f \in L^p(\mathbb{R}/T\mathbb{Z})$ $(r > 0, p \geqslant 1)$ and let for every $r > 0$ there exists $\varphi_r \in L^q(\mathbb{R}/T\mathbb{Z})$ $(q \geqslant 1)$ such that[1]

$$\psi_r^+(t) + |f(t)| \leqslant \varphi_r(t) g^{(q-1)/q}(t) \qquad \text{for a. e. } t \in [0,T].$$

(H_3) Assume that

$$\lim_{x\to t^+} \int_x^{t+T} \frac{g(s)}{(s-t)^\sigma}\,\mathrm{d}s + \lim_{x\to t^-} \int_{t-T}^x \frac{g(s)}{(t-s)^\sigma}\,\mathrm{d}s = +\infty \quad \text{for } t \in [0,T],$$

where $\sigma = \frac{\lambda(2p-1)q}{p}$.

[1] if $q = 1$ then we put $g^{(q-1)/q}(t) = 1$ for $t \in [0,T]$.

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

3.1. Main results

THEOREM 3.2. *Let* (H_1)–(H_3) *be fulfilled. Then there exist* $\mu_*, \mu^* \in \mathbb{R}$ *such that* $\mu_* \leqslant \mu^*$ *and*

(i) *the equation* (2.1) *has no* T*-periodic solution provided* $\mu < \mu_*$*;*

(ii) *the equation* (2.1) *has at least one* T*-periodic solution provided* $\mu = \mu^*$*;*

(iii) *the equation* (2.1) *has at least two* T*-periodic solutions provided* $\mu > \mu^*$*.*

REMARK 3.3. Note that, in general, $\mu_* \neq \mu^*$ as shown in example 3.12. In order to guarantee the equality $\mu_* = \mu^*$ we have to strengthen conditions imposed on the functions h and f.

THEOREM 3.4. *Let* (H_1)–(H_3) *be fulfilled. Let, moreover, the function* $h(t,x)/x$ *is nondecreasing with respect to* x*, i.e.,*

$$\frac{h(t,x)}{x} \leqslant \frac{h(t,y)}{y} \quad \textit{for a. e. } t \in [0,T], \quad 0 < x \leqslant y. \tag{3.5}$$

Then there exists a critical value $\mu_\dagger > 0$ *such that*

(i) *the equation* (2.1) *has no* T*-periodic solution provided* $\mu < \mu_\dagger$*;*

(ii) *the equation* (2.1) *has at least one* T*-periodic solution provided* $\mu = \mu_\dagger$*;*

(iii) *the equation* (2.1) *has at least two* T*-periodic solutions provided* $\mu > \mu_\dagger$*.*

REMARK 3.5. An example of h satisfying all the assumptions of theorem 3.4 is a function

$$h(t,x) = \sum_{i=1}^{n} \left[h_i(t) x^{\delta_i} + \frac{k_i(t)}{(1+x)^{\lambda_i}} \right] + q(t) \quad \text{for a. e. } t \in [0,T], \quad x \in \mathbb{R}_+,$$

where $h_i(t) \leqslant 0$, $k_i(t) \leqslant 0$, $q(t) \leqslant 0$ for a. e. $t \in [0,T]$, $\delta_i \in (0,1)$, and $\lambda_i > 0$ $(i = 1, \ldots, n)$.

THEOREM 3.6. *Let* (H_1)–(H_3) *be fulfilled, and let there exist* $h_0 \in L\big(\mathbb{R}/T\mathbb{Z}\big)$ *and a continuous function* $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *such that*

$$h_0(t) \geqslant 0 \quad \textit{for a. e. } t \in [0,T], \qquad \varphi(x,x) = 0 \quad \textit{for } x \in \mathbb{R}_+, \tag{3.6}$$

$$h(t,x) - h(t,y) \leqslant h_0(t)\varphi(x,y) \quad \textit{for a. e. } t \in [0,T], \quad 0 \leqslant x \leqslant y. \tag{3.7}$$

Let, moreover, there exist $c > 0$ *such that*

$$h_0(t) \leqslant c f(t) \qquad \textit{for a. e. } t \in [0,T]. \tag{3.8}$$

Then there exists a critical value $\mu_\dagger \in \mathbb{R}$ *such that the items* (i)–(iii) *of theorem* 3.4 *hold.*

In particular, from theorem 3.6 we obtain the following assertion.

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Corollary 3.7. *Let* (H_1)–(H_3) *be fulfilled, and let there exist* $h_0 \in L^\infty(\mathbb{R}/T\mathbb{Z})$ *and a continuous function* $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *such that* (3.6) *and* (3.7) *hold. Let, moreover,*

$$\text{ess inf}\,\big\{f(t) : t \in [0, T]\big\} > 0. \tag{3.9}$$

Then the conclusion of theorem 3.6 *is valid.*

3.2. A particular case

We apply the results established above to a particular case of the equation (2.1), namely to the equation of the form

$$u'' + \frac{g(t)}{u^\lambda} = \sum_{i=1}^{n} \left[h_i(t) u^{\delta_i} + \frac{k_i(t)}{(c_i + u)^{\lambda_i}} \right] + q(t) + \mu f(t). \tag{3.10}$$

Here, g, f, μ, and λ are as in (2.1), $h_i, k_i, q \in L(\mathbb{R}/T\mathbb{Z})$, $\delta_i \in (0, 1)$, $\lambda_i > 0$, $c_i > 0$ $(i = 1, \ldots, n)$, and $\delta_1 > \cdots > \delta_n$.

Theorem 3.8. *Let* $[h_i]_+, [k_i]_+, [q]_+, f \in L^p(\mathbb{R}/T\mathbb{Z})$ $(p \geqslant 1; i = 1, \ldots, n)$, $\overline{h_1} < 0$, *and let there exist* $c_0 > 0$, $\alpha_i, \beta_i \geqslant 0$, $t_i \in \mathbb{R}$ $(i = 1, \ldots, m)$ *such that* $t_1 < t_2 < \cdots < t_m < t_1 + T$ *and*

$$g(t) \geqslant c_0 (t_{i+1} - t)^{\alpha_{i+1}} (t - t_i)^{\beta_i} \quad \text{for a. e. } t \in (t_i, t_{i+1}), \quad i = 1, \ldots, m-1, \tag{3.11}$$

$$g(t) \geqslant c_0 (t_1 + T - t)^{\alpha_1} (t - t_m)^{\beta_m} \quad \text{for a. e. } t \in (t_m, t_1 + T). \tag{3.12}$$

Let, moreover,

$$\lambda \geqslant \frac{1 + \gamma_0}{2p - 1} \quad \text{if } p = 1 \text{ or } \gamma = 0, \quad \lambda > \frac{(1 + \gamma_0)(1 + \gamma p)}{(1 + \gamma)(2p - 1)} \quad \text{otherwise} \tag{3.13}$$

where

$$\gamma_0 = \max\big\{ \min\{\alpha_i, \beta_i\} : i = 1, \ldots, m \big\}, \quad \gamma = \max\big\{ \alpha_i, \beta_i : i = 1, \ldots, m \big\}. \tag{3.14}$$

Then there exist $\mu_*, \mu^* \in \mathbb{R}$ *such that* $\mu_* \leqslant \mu^*$ *and*

(i) *the equation* (3.10) *has no* T*-periodic solution provided* $\mu < \mu_*$*;*

(ii) *the equation* (3.10) *has at least one* T*-periodic solution provided* $\mu = \mu^*$*;*

(iii) *the equation* (3.10) *has at least two* T*-periodic solutions provided* $\mu > \mu^*$*.*

Theorem 3.9. *Let* $f \in L^p(\mathbb{R}/T\mathbb{Z})$ $(p \geqslant 1)$, $\overline{h_1} < 0$,

$$h_i(t) \leqslant 0, \quad k_i(t) \leqslant 0, \quad q(t) \leqslant 0 \qquad \text{for a. e. } t \in [0, T] \quad (i = 1, \ldots, n),$$

and let there exist $c_0 > 0$, $\alpha_i, \beta_i \geqslant 0$, $t_i \in \mathbb{R}$ $(i = 1, \ldots, m)$ *such that* $t_1 < t_2 < \cdots < t_m < t_1 + T$ *and* (3.11)–(3.13) *is fulfilled with* γ_0 *and* γ *given by* (3.14). *Then there exists a critical value* $\mu_\dagger > 0$ *such that*

(i) *the equation* (3.10) *has no* T*-periodic solution provided* $\mu < \mu_\dagger$*;*

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

(ii) *the equation* (3.10) *has at least one T-periodic solution provided $\mu = \mu_\dagger$;*

(iii) *the equation* (3.10) *has at least two T-periodic solutions provided $\mu > \mu_\dagger$.*

THEOREM 3.10. *Let $[h_i]_+, [k_i]_+, [q]_+, f \in L^p(\mathbb{R}/T\mathbb{Z})$ $(p \geqslant 1; i = 1, \dots, n)$, $\overline{h_1} < 0$, and let there exist $c_0 > 0$, $\alpha_i, \beta_i \geqslant 0$, $t_i \in \mathbb{R}$ $(i = 1, \dots, m)$ such that $t_1 < t_2 < \dots < t_m < t_1 + T$ and* (3.11)–(3.13) *is fulfilled with γ_0 and γ given by* (3.14)*. Let, moreover, there exists $c > 0$ such that*

$$\sum_{i=1}^{n}([h_i]_-(t) + [k_i]_+(t)) \leqslant cf(t) \qquad \text{for a. e. } t \in [0, T].$$

Then there exists a critical value $\mu_\dagger \in \mathbb{R}$ such that the items (i)–(iii) *of theorem* 3.9 *hold.*

In particular, from theorem 3.10 we obtain the following assertion.

COROLLARY 3.11. *Let $[h_i]_+, [q]_+, f \in L^p(\mathbb{R}/T\mathbb{Z})$, $[h_i]_-, [k_i]_+ \in L^\infty(\mathbb{R}/T\mathbb{Z})$ $(p \geqslant 1; i = 1, \dots, n)$, $\overline{h_1} < 0$, and let there exist $c_0 > 0$, $\alpha_i, \beta_i \geqslant 0$, $t_i \in \mathbb{R}$ $(i = 1, \dots, m)$ such that $t_1 < t_2 < \dots < t_m < t_1 + T$ and* (3.11)–(3.13) *is fulfilled with γ_0 and γ given by* (3.14)*. Let, moreover,* (3.9) *hold. Then the conclusion of theorem* 3.10 *is valid.*

3.3. An example

EXAMPLE 3.12. Let us consider a particular case of the equation (3.10), namely the equation

$$u'' + \frac{g(t)}{u^\lambda} = h(t)u^\delta + \mu f(t), \tag{3.15}$$

where f, g, h are 2π-periodic functions given by

$$h(t) = \begin{cases} \dfrac{\pi}{\varepsilon} & \text{for } t \in [0, \varepsilon), \\ 0, & \text{for } t \in [\varepsilon, \pi - \varepsilon], \\ -\dfrac{\pi + \eta}{\varepsilon} & \text{for } t \in (\pi - \varepsilon, \pi], \end{cases} \qquad h(t) = h(2\pi - t) \quad \text{for } t \in (\pi, 2\pi],$$

$$g(t) = \eta(2 + \cos t)^\lambda, \qquad f(t) = h(t)(2 + \cos t)^\delta - \eta + \cos t \quad \text{for } t \in [0, 2\pi],$$

numbers $\lambda > 0$, $\delta \in (0, 1)$ are arbitrary but fixed, and the constants $\varepsilon \in (0, \pi/2)$ and $\eta > 0$ are such that

$$\left[(2 + \cos\varepsilon)^\delta - (2 - \cos\varepsilon)^\delta\right]\pi > \left[(2 - \cos\varepsilon)^\delta + \pi\right]\eta.$$

Then it can be easily verified that $g(t) \geqslant \eta$ for $t \in [0, 2\pi]$, $\overline{h} = -2\eta$, and

$$\begin{aligned} \int_0^{2\pi} f(s)\,\mathrm{d}s &= 2\left[\int_0^{\varepsilon} \frac{\pi}{\varepsilon}(2 + \cos s)^\delta\,\mathrm{d}s - \int_{\pi-\varepsilon}^{\pi} \frac{\pi + \eta}{\varepsilon}(2 + \cos s)^\delta\,\mathrm{d}s\right] - 2\pi\eta \\ &\geqslant 2[\pi(2 + \cos\varepsilon)^\delta - (\pi + \eta)(2 - \cos\varepsilon)^\delta] - 2\pi\eta > 0. \end{aligned}$$

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Furthermore, one can easily verify that the assumptions of theorem 3.8 are fulfilled with $\alpha_i = 0$, $\beta_i = 0$, $c_0 = \eta$, and $p \geqslant (1+\lambda)/2\lambda$. Consequently, there exist $\mu_*, \mu^* \in \mathbb{R}$ such that the conclusion of theorem 3.8 is fulfilled with $T = 2\pi$. Moreover, the equation (3.15) with $\mu = 0$ has no 2π-periodic solution. Indeed, assume on the contrary that there is a 2π-periodic solution to (3.15) with $\mu = 0$. Then dividing both sides of (3.15) by u^δ and integrating it over $[0, 2\pi]$ we find

$$\int_0^{2\pi} \frac{g(s)}{u^{\lambda+\delta}}\,\mathrm{d}s \leqslant \int_0^{2\pi} h(s)\,\mathrm{d}s,$$

that contradicts $\overline{h} < 0$. Therefore, necessarily, $\mu^* > 0$.

On the other hand, it can be easily verified that $u(t) = 2 + \cos t$ for $t \in [0, 2\pi]$ is a 2π-periodic solution to (3.15) with $\mu = -1$. That means, $\mu_* \leqslant -1$.

4. Auxiliary propositions

First we introduce two lemmas established in [**11**] and [**10**], respectively, which will be useful to obtain a priori estimates.

Lemma 4.1 (see [**11**, lemma 2.4]). *Let $u \in AC^1(\mathbb{R}/T\mathbb{Z})$. Then*

$$\big(M - m\big)^2 \leqslant \frac{T}{4}\int_0^T u'^2(s)\,ds,$$

where

$$M = \max\big\{u(t) : t \in [0,T]\big\}, \qquad m = \min\big\{u(t) : t \in [0,T]\big\}. \tag{4.1}$$

Lemma 4.2 (see [**10**, lemma 2.4]). *Let $u \in AC^1(\mathbb{R}/T\mathbb{Z})$. Then*

$$M - m \leqslant \frac{T}{4}\int_0^T [u'']_+(s)\,ds,$$

where M and m are given by (4.1).

Lemma 4.3. *Let (H_1) be fulfilled. Then*

$$\lim_{x\to+\infty} \frac{\overline{h^-}(x)}{x} = 0 \tag{4.2}$$

and

$$\lim_{x\to+\infty} \frac{\overline{h^+}(x)}{x} = 0. \tag{4.3}$$

Proof. The integration of (3.1) over $[0, T]$ yields

$$\overline{\eta}(x) \geqslant \overline{h^-}(x) - \overline{h^+}(x) \qquad \text{for } x \geqslant r. \tag{4.4}$$

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

According to (3.2) there exists $r_0 \geqslant r$ such that $\overline{h^-}((1-\zeta)x) > 0$ for $x \geqslant r_0$. Therefore, from (4.4), since $\overline{h^-}$ is nondecreasing, we obtain

$$\frac{\overline{\eta}(x)}{\overline{h^-}(x)} \geqslant 1 - \frac{\overline{h^+}(x)}{\overline{h^-}(x)} \geqslant 1 - \frac{\overline{h^+}(x)}{\overline{h^-}((1-\zeta)x)} \qquad \text{for } x \geqslant r_0. \tag{4.5}$$

Consequently, on account of (3.3), from (4.5) it follows that

$$\liminf_{x\to+\infty} \frac{\overline{\eta}(x)}{\overline{h^-}(x)} \geqslant 1 - L > 0.$$

Therefore, there exists $r_1 \geqslant r_0$ such that

$$\overline{\eta}(x) \geqslant \frac{(1-L)}{2}\overline{h^-}(x) > 0 \qquad \text{for } x \geqslant r_1,$$

and thus (4.2) follows from the first relation in (3.2).

Further, since $\overline{h^-}$ is nondecreasing, we have

$$\frac{\overline{h^+}(x)}{\overline{h^-}(x)} \leqslant \frac{\overline{h^+}(x)}{\overline{h^-}((1-\zeta)x)} \qquad \text{for } x \geqslant r_0,$$

and so, with respect to (3.3), there exists $r_2 \geqslant r_0$ such that

$$0 \leqslant \overline{h^+}(x) \leqslant (L+1)\overline{h^-}(x) \qquad \text{for } x \geqslant r_2.$$

Consequently, the latter inequality together with (4.2) implies (4.3). □

Lemma 4.4. *Let* (H_1) *and* (3.5) *be fulfilled. Then*

$$h(t,x) \leqslant 0 \qquad \textit{for a. e. } t \in [0,T], \quad x > 0, \tag{4.6}$$

and for every $c > 1$ *and every* $x_0 > 0$ *there exists* $\varepsilon > 0$ *such that*

$$ch(t,x) \leqslant h(t,y) \qquad \textit{for a. e. } t \in [0,T], \quad y \in [cx-\varepsilon, cx], \quad x \geqslant x_0. \tag{4.7}$$

Proof. First we will show that (4.6) holds. Assume on the contrary that there exists $x_1 > 0$ and $E \subseteq [0,T]$ such that

$$\operatorname{meas} E > 0, \qquad h(t,x_1) > 0 \qquad \text{for } t \in E.$$

Then, in view of (H_1) and (3.5) we have

$$\begin{aligned} 0 < \int_E \frac{h(t,x_1)}{x_1}\mathrm{d}t &\leqslant \int_E \frac{h(t,x)}{x}\mathrm{d}t \\ &\leqslant \int_E \frac{h^+(t,x)}{x}\mathrm{d}t \leqslant \frac{T\overline{h^+}(x)}{x} \qquad \text{for } x > \max\{x_1, r\}. \end{aligned} \tag{4.8}$$

However, according to lemma 4.3 the relation (4.3) holds and this contradicts (4.8).

Now let $c > 1$ and $x_0 > 0$ be arbitrary but fixed. Choose $\varepsilon \in (0, (c-1)x_0)$, and let $y \in [cx-\varepsilon, cx]$ for some $x \geqslant x_0$. Then, obviously, $x < y$ and $y/x \leqslant c$. Consequently, from (3.5) in view of (4.6) we obtain (4.7). □

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Notation 4.5. *Let $u(\cdot;\mu_n)$ $(n \in \mathbb{N})$ be a sequence of T-periodic solutions to (2.1) with $\mu = \mu_n$. Then, in what follows, for the sake of brevity we put*

$$M_n = \max\big\{u(t;\mu_n) : t \in [0,T]\big\}, \qquad m_n = \min\big\{u(t;\mu_n) : t \in [0,T]\big\}.$$

Lemma 4.6. *Let (H_1) hold and let $u_n = u(\cdot;\mu_n)$ $(n \in \mathbb{N})$ be a sequence of T-periodic solutions to (2.1) with $\mu = \mu_n$ such that*

$$\lim_{n\to+\infty} M_n = +\infty, \qquad \lim_{n\to+\infty} \frac{|\mu_n|}{M_n} = 0. \tag{4.9}$$

Then

$$\lim_{n\to+\infty} \frac{m_n}{M_n} = 1. \tag{4.10}$$

Proof. Multiplying both sides of (2.1) by u_n^λ and integrating it on $[0,T]$, for every $n \in \mathbb{N}$, we get

$$\int_0^T u_n''(s)u_n^\lambda(s)\,\mathrm{d}s = -\int_0^T g(s)ds + \int_0^T h(s,u_n(s))u_n^\lambda(s)\,\mathrm{d}s + \int_0^T \mu_n f(s)u_n^\lambda(s)\,\mathrm{d}s,$$

which together with (3.4) leads to

$$-\lambda\int_0^T u_n^{\lambda-1}(s)u_n'^2(s)\,\mathrm{d}s \geqslant -T\overline{g} - \int_0^T \eta(s,u_n(s))u_n^\lambda(s)\,\mathrm{d}s - |\mu_n|\int_0^T |f(s)|u_n^\lambda(s)\,\mathrm{d}s.$$

The latter inequality implies

$$\lambda\int_0^T u_n^{\lambda-1}(s)u_n'^2(s)\,\mathrm{d}s \leqslant T(\overline{g} + \overline{\eta}(M_n)M_n^\lambda + |\mu_n|\overline{|f|}M_n^\lambda). \tag{4.11}$$

On the other hand, according to lemma 4.1, we have

$$\begin{aligned}\left(M_n^{\frac{\lambda+1}{2}} - m_n^{\frac{\lambda+1}{2}}\right)^2 &\leqslant \frac{T}{4}\int_0^T \left(\left(u_n^{\frac{\lambda+1}{2}}(s)\right)'\right)^2 \mathrm{d}s \\ &= \frac{T(\lambda+1)^2}{16}\int_0^T u_n^{\lambda-1}(s)u_n'^2(s)\,\mathrm{d}s. \end{aligned}\tag{4.12}$$

Therefore, using (4.11) in (4.12) we obtain

$$\left(M_n^{\frac{\lambda+1}{2}} - m_n^{\frac{\lambda+1}{2}}\right)^2 \leqslant \frac{T^2(\lambda+1)^2}{16\lambda}\left[\overline{g} + \overline{\eta}(M_n)M_n^\lambda + |\mu_n|\overline{|f|}M_n^\lambda\right],$$

and, consequently,

$$\left(1 - \left(\frac{m_n}{M_n}\right)^{\frac{\lambda+1}{2}}\right)^2 \leqslant \frac{T^2(\lambda+1)^2}{16\lambda}\left[\frac{\overline{g}}{M_n^{\lambda+1}} + \frac{\overline{\eta}(M_n)}{M_n} + \frac{|\mu_n|\overline{|f|}}{M_n}\right].$$

Now, passing to the limit as $n \to +\infty$ together with (3.2) and (4.9) we obtain (4.10). □

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

4.1. Upper bounds

LEMMA 4.7. *Let* (H_1) *be fulfilled. Then there exists a nondecreasing function* $\gamma : \mathbb{R}_+ \to (0,+\infty)$ *such that every* T*-periodic solution* u *to* (2.1) *admits the inequality*

$$u(t;\mu) \leqslant \gamma(\mu) \qquad \textit{for } t \in [0,T], \quad \mu \in \mathbb{R}_+. \tag{4.13}$$

Proof. First we show that for every fixed $\mu_0 \in \mathbb{R}_+$ there exists $\gamma_0(\mu_0) > 0$ such that

$$u(t;\mu) \leqslant \gamma_0(\mu_0) \qquad \text{for } t \in [0,T], \quad \mu \in [0,\mu_0]. \tag{4.14}$$

Assume on the contrary that (4.14) does not hold. Then there exists a sequence of T-periodic solutions $u_n = u(\cdot;\mu_n)$ $(n \in \mathbb{N})$ to (2.1) with $\mu = \mu_n \in [0,\mu_0]$ such that $M_n > n$ $(n \in \mathbb{N})$. Obviously, (4.9) holds, and so, according to lemma 4.6 we have that (4.10) is fulfilled. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$m_n \geqslant (1-\zeta)M_n \geqslant r \qquad \text{for } n \geqslant n_0. \tag{4.15}$$

Now the integration of (2.1) over $[0,T]$ results in

$$0 < \int_0^T \frac{g(t)}{u_n^{\lambda}(t)}\,\mathrm{d}t = \int_0^T h(t,u_n(t))\,\mathrm{d}t + \mu_n \int_0^T f(t)\,\mathrm{d}t, \tag{4.16}$$

and in view of (2.2), (3.1), and (4.15), from (4.16) it follows that

$$0 < \overline{h^+}(M_n) - \overline{h^-}(m_n) + \mu_n \overline{f} \leqslant \overline{h^+}(M_n) - \overline{h^-}((1-\zeta)M_n) + \mu_0 \overline{f} \quad \text{for } n \geqslant n_0. \tag{4.17}$$

However, from (4.17) we get

$$0 < \frac{\overline{h^+}(M_n)}{\overline{h^-}((1-\zeta)M_n)} - 1 + \frac{\mu_0 \overline{f}}{\overline{h^-}((1-\zeta)M_n)} \qquad \text{for } n \geqslant n_1, \tag{4.18}$$

where $n_1 \geqslant n_0$ is such that $\overline{h^-}((1-\zeta)M_n) > 0$ for $n \geqslant n_1$. Passing to the limit as $n \to +\infty$, in view of (3.2) and (4.9), from (4.18) we get $0 \leqslant L - 1$ that contradicts (3.3).

Now we put

$$\gamma(\mu) \overset{\text{def}}{=} \inf\left\{\gamma_0(\mu_0) : \mu_0 \geqslant \mu\right\} \qquad \text{for } \mu \in \mathbb{R}_+.$$

Obviously, γ is the required nondecreasing function satisfying (4.13). □

LEMMA 4.8. *Let* (H_1) *be fulfilled. Then there exists a positive constant* A *such that every* T*-periodic solution* u *to* (2.1) *admits the inequality*

$$u(t;\mu) \leqslant A(1+|\mu|) \qquad \textit{for } t \in [0,T], \quad \mu \in \mathbb{R}_-. \tag{4.19}$$

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Proof. Assume on the contrary that (4.19) does not hold. Then there exists a sequence of T-periodic solutions $u_n = u(\cdot;\mu_n)$ $(n \in \mathbb{N})$ to (2.1) with $\mu = \mu_n \in \mathbb{R}_-$ such that

$$M_n > n(1 + |\mu_n|) \qquad \text{for } n \in \mathbb{N}. \tag{4.20}$$

Obviously, from (4.20) it follows that

$$\frac{1}{n} > \frac{1}{M_n} + \frac{|\mu_n|}{M_n} \qquad \text{for } n \in \mathbb{N},$$

and so (4.9) holds. Thus, according to lemma 4.6 we have that (4.10) is fulfilled. Therefore, there exists $n_0 \in \mathbb{N}$ such that (4.15) is valid. Now the integration of (2.1) over $[0, T]$ results in (4.16), and in view of (2.2), (3.1), and (4.15), from (4.16) it follows that

$$0 < \overline{h^+}(M_n) - \overline{h^-}(m_n) + \mu_n \overline{f} \leqslant \overline{h^+}(M_n) - \overline{h^-}((1-\zeta)M_n) \quad \text{for } n \geqslant n_0. \tag{4.21}$$

However, from (4.21) we get

$$0 < \frac{\overline{h^+}(M_n)}{\overline{h^-}((1-\zeta)M_n)} - 1 \qquad \text{for } n \geqslant n_1, \tag{4.22}$$

where $n_1 \geqslant n_0$ is such that $\overline{h^-}((1-\zeta)M_n) > 0$ for $n \geqslant n_1$. Passing to the limit as $n \to +\infty$, in view of (3.2) and (4.9), from (4.22) we get $0 \leqslant L - 1$ which contradicts (3.3). □

Now we put

$$\rho(\mu) \overset{\text{def}}{=} \begin{cases} \gamma(\mu) & \text{for } \mu \in \mathbb{R}_+, \\ A(1+|\mu|) & \text{for } \mu \in (-\infty, 0), \end{cases} \tag{4.23}$$

where γ is a function appearing in lemma 4.7 and A is a number from lemma 4.8. Obviously, the following lemma is valid.

Lemma 4.9. *Let* (H_1) *be fulfilled. Then every* T*-periodic solution* u *to* (2.1) *admits the inequality*

$$u(t;\mu) \leqslant \rho(\mu) \qquad \textit{for } t \in [0, T], \quad \mu \in \mathbb{R}, \tag{4.24}$$

where ρ *is given by* (4.23).

4.2. Lower bound

The approach how to get a lower bound for possible T-periodic solutions to (2.1) is based on results obtained in [**13**]. Therefore, together with the assumption (H_1) we assume that (H_2) and (H_3) hold, as well.

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Lemma 4.10. *Let* (H_1)–(H_3) *be fulfilled. Then, for every* $\mu \in \mathbb{R}$, *there exists a positive function* $\alpha(\cdot;\mu) \in AC^1\big(\mathbb{R}/T\mathbb{Z}\big)$ *such that*

$$\alpha(t;\mu) \geqslant \alpha(t;\nu) \quad \textit{for } t \in [0,T] \quad \textit{whenever } |\mu| \leqslant |\nu|, \tag{4.25}$$

and every T*-periodic solution* u *to* (2.1) *admits the estimate*

$$\alpha(t;\mu) < u(t;\mu) \qquad \textit{for } t \in [0,T]. \tag{4.26}$$

Proof. Let $\mu \in \mathbb{R}$ be arbitrary but fixed, and let u be an arbitrary T-periodic solution to (2.1). Then, according to lemma 4.9 the estimate (4.24) is fulfilled, and consequently, in view of (2.3), we have

$$h(t,u(t;\mu)) \leqslant \psi^+_{\rho(\mu)}(t) \qquad \text{for a. e. } t \in [0,T]. \tag{4.27}$$

According to [**13**, theorem 1], the equation

$$\alpha'' + \frac{g(t)}{2^{1+\lambda}\alpha^\lambda} = \psi^+_{\rho(\mu)}(t) + \psi^+_{\rho(-\mu)}(t) + |\mu f(t)| \tag{4.28}$$

has a unique positive T-periodic solution $\alpha(\cdot;\mu) \in AC^1\big(\mathbb{R}/T\mathbb{Z}\big)$. Note also that

$$\alpha(t;\mu) = \alpha(t;-\mu) \qquad \text{for } t \in [0,T]. \tag{4.29}$$

We will show that the inequality $2\alpha(t;\mu) \leqslant u(t;\mu)$ for $t \in [0,T]$ holds. Then, obviously, (4.26) holds, as well. Put

$$z(t;\mu) \overset{\text{def}}{=} u(t;\mu) - 2\alpha(t;\mu) \qquad \text{for } t \in \mathbb{R} \tag{4.30}$$

and assume on the contrary that there exists an interval $I \subset \mathbb{R}$ such that

$$z(t;\mu) < 0 \qquad \text{for } t \in I. \tag{4.31}$$

Obviously, $z \in AC^1\big(\mathbb{R}/T\mathbb{Z}\big)$ and in view of (2.1), (4.27), (4.28), and (4.30) we have

$$\begin{aligned} z''(t;\mu) &= -\frac{g(t)}{u^\lambda(t;\mu)} + \frac{g(t)}{\big(2\alpha(t;\mu)\big)^\lambda} + h(t,u(t;\mu)) \\ &\quad - 2\psi^+_{\rho(\mu)}(t) - 2\psi^+_{\rho(-\mu)}(t) \ + \mu f(t) - 2|\mu f(t)| \\ &\leqslant -\frac{g(t)}{u^\lambda(t;\mu)} + \frac{g(t)}{\big(2\alpha(t;\mu)\big)^\lambda} - \psi^+_{\rho(\mu)}(t) \quad \text{for a. e. } t \in I. \end{aligned} \tag{4.32}$$

Consequently, on account of (4.30) and (4.31), from (4.32) it follows that

$$z''(t;\mu) \leqslant -\psi^+_{\rho(\mu)}(t) \leqslant 0 \qquad \text{for a. e. } t \in I. \tag{4.33}$$

Thus (4.31) implies (4.33). We will discuss two cases.

Case 1. There exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$,

$$z(t_1;\mu) = 0, \qquad z(t_2;\mu) = 0, \tag{4.34}$$

and (4.31) is fulfilled with $I = (t_1,t_2)$. Then, according to the above-proven, the inequality (4.33) holds, a contradiction.

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Case 2. The inequality (4.31) is fulfilled with $I = [0, T]$. Consequently, (4.33) holds again. However, the integration of (4.33) over $[0, T]$ with respect to the inclusion $z \in AC^1(\mathbb{R}/T\mathbb{Z})$ and (2.4) yields

$$0 = \int_0^T z''(t;\mu)\,\mathrm{d}t \leqslant -\int_0^T \psi^+_{\rho(\mu)}(t)\,\mathrm{d}t < 0,$$

a contradiction.

Consequently, $z(t;\mu) \geqslant 0$ for $t \in [0, T]$, which together with (4.30) results in (4.26). It remains to show that (4.25) is true. Assume that $|\mu| \leqslant |\nu|$ and put

$$w(t) \stackrel{\text{def}}{=} \alpha(t;\mu) - \alpha(t;\nu) \qquad \text{for } t \in \mathbb{R}. \tag{4.35}$$

If $|\mu| = |\nu|$ then (4.25) holds trivially because of the uniqueness of α and (4.29). Suppose therefore that $|\mu| < |\nu|$ and assume on the contrary that there exists an interval $I \subset \mathbb{R}$ such that

$$w(t) < 0 \qquad \text{for } t \in I. \tag{4.36}$$

Then, in view of (4.28), remark 2.1, and (4.23), we have

$$\begin{aligned} w''(t) = &-\frac{g(t)}{2^{1+\lambda}}\left(\frac{1}{\alpha^\lambda(t;\mu)} - \frac{1}{\alpha^\lambda(t;\nu)}\right) + \psi^+_{\rho(\mu)}(t) + \psi^+_{\rho(-\mu)}(t) \\ &- \psi^+_{\rho(\nu)}(t) - \psi^+_{\rho(-\nu)}(t) + (|\mu| - |\nu|)|f(t)| \\ \leqslant &-\frac{g(t)}{2^{1+\lambda}}\left(\frac{1}{\alpha^\lambda(t;\mu)} - \frac{1}{\alpha^\lambda(t;\nu)}\right) + (|\mu| - |\nu|)|f(t)| \quad \text{for a. e. } t \in I. \end{aligned} \tag{4.37}$$

Consequently, in view of (4.35) and (4.36), from (4.37) we obtain

$$w''(t) \leqslant (|\mu| - |\nu|)|f(t)| \leqslant 0 \qquad \text{for a. e. } t \in I.$$

Moreover, in view of (4.35) we have $w \in AC^1(\mathbb{R}/T\mathbb{Z})$. Therefore, in the same manner as above, with respect to (2.2), one can prove that the assumption (4.36) leads to a contradiction. That implies $w(t) \geqslant 0$ for $t \in [0, T]$, i.e. (4.25) holds. □

5. Construction of upper and lower functions

LEMMA 5.1. *Let* (H$_1$)–(H$_3$) *be fulfilled. Then, for every* $\nu \in \mathbb{R}$, *the function* $\alpha(\cdot;\nu)$ *constructed in Lemma* 4.10 *is a strict lower function to the equation* (2.1) *with* $\mu \in [-|\nu|, |\nu|]$.

Proof. Let $\nu \in \mathbb{R}$ be arbitrary but fixed and let $\mu \in [-|\nu|, |\nu|]$. Let, moreover, u be an arbitrary T-periodic solution to (2.1). Then according to lemmas 4.9 and 4.10

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

we have

$$\alpha(t;\nu) \leqslant \alpha(t;\mu) < u(t;\mu) \leqslant \rho(\mu) \leqslant \max\big\{\rho(\nu), \rho(-\nu)\big\} \quad \text{for } t \in [0,T]. \tag{5.1}$$

Consequently, in view of definition 2.2 it is sufficient to show that

$$\alpha''(t;\nu) + \frac{g(t)}{\alpha^{\lambda}(t;\nu)} \geqslant h(t, \alpha(t;\nu)) + \mu f(t) \qquad \text{for a. e. } t \in [0,T]. \tag{5.2}$$

Remind that α is a T-periodic solution to (4.28), i.e.,

$$\alpha''(t;\nu) + \frac{g(t)}{2^{1+\lambda}\alpha^{\lambda}(t;\nu)} = \psi^{+}_{\rho(\nu)}(t) + \psi^{+}_{\rho(-\nu)}(t) + |\nu f(t)| \quad \text{for a. e. } t \in [0,T], \tag{5.3}$$

and note that, on account of (2.3) and (5.1), we have

$$h(t, \alpha(t;\nu)) \leqslant \psi^{+}_{\rho(\nu)}(t) + \psi^{+}_{\rho(-\nu)}(t) \qquad \text{for a. e. } t \in [0,T].$$

Therefore, using the latter inequality in (5.3), on account of $|\mu| \leqslant |\nu|$, we get (5.2). □

LEMMA 5.2. *Let (H_1) hold. Then, for every μ sufficiently large, there exists a strict upper function $\beta(\cdot;\mu)$ to the equation (2.1). Moreover,*

$$\lim_{\mu\to+\infty} \beta(t;\mu) = +\infty \qquad \textit{uniformly on } [0,T]. \tag{5.4}$$

Proof. Let $\mu > 0$ and let w be a solution to the Dirichlet boundary value problem

$$w''(t;\mu) = -(g(t) + \eta(t, B\mu))\frac{\overline{f}}{\overline{g} + \overline{\eta}(B\mu)} + f(t) \qquad \text{for a. e. } t \in [0,T], \tag{5.5}$$

$$w(0;\mu) = 0, \qquad w(T;\mu) = 0, \tag{5.6}$$

where $B = 1 + (T^2\overline{f_+}/4)$. Note that

$$w'(0;\mu) = w'(T;\mu) \qquad \text{for } \mu > 0, \tag{5.7}$$

and therefore we can consider the T-periodic extension of w, i.e., $w \in AC^1\big(\mathbb{R}/T\mathbb{Z}\big)$. According to lemma 4.2 we have

$$M_w - m_w \leqslant \frac{T^2\overline{f_+}}{4}, \tag{5.8}$$

where

$$M_w = \max\big\{w(t;\mu) : t \in [0,T]\big\}, \qquad m_w = \min\big\{w(t;\mu) : t \in [0,T]\big\}.$$

Put

$$\beta(t;\mu) \overset{\text{def}}{=} \left(\frac{\overline{g} + \overline{\eta}(B\mu)}{\mu\overline{f}}\right)^{1/\lambda} + \mu\left(w(t;\mu) - m_w + \frac{1}{2}\right) \quad \text{for } t \in \mathbb{R}, \quad \mu > 0. \tag{5.9}$$

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Then, on account of (5.6) and (5.7) it can be easily verified that $\beta(\cdot;\mu) \in AC^1(\mathbb{R}/T\mathbb{Z})$ for $\mu > 0$, and from (5.9) it follows that

$$\beta(t;\mu) \geqslant \left(\frac{\overline{g}+\overline{\eta}(B\mu)}{\mu\overline{f}}\right)^{1/\lambda} + \frac{\mu}{2} \qquad \text{for } t \in [0,T], \quad \mu > 0. \tag{5.10}$$

Further, in view of (3.2) we have

$$\lim_{\mu\to+\infty} \frac{\overline{g}+\overline{\eta}(B\mu)}{\mu\overline{f}} = 0. \tag{5.11}$$

Finally, from (5.9) in view of (5.5) it follows that

$$\beta''(t;\mu) = -(g(t)+\eta(t,B\mu))\frac{\mu\overline{f}}{\overline{g}+\overline{\eta}(B\mu)} + \mu f(t) \qquad \text{for a. e. } t \in [0,T]. \tag{5.12}$$

Now, according to (5.11) there exists $\mu_0 > 0$ such that

$$\left(\frac{\overline{g}+\overline{\eta}(B\mu)}{\mu\overline{f}}\right)^{1/\lambda} \leqslant \min\left\{\frac{\mu}{2}, 1\right\} \qquad \text{for } \mu \geqslant \mu_0. \tag{5.13}$$

Consequently, from (5.9) with respect to (5.8) and (5.13) we obtain

$$\beta(t;\mu) \leqslant \frac{\mu}{2} + \mu\left(\frac{T^2\overline{f_+}}{4} + \frac{1}{2}\right) = B\mu \qquad \text{for } t \in [0,T], \quad \mu \geqslant \mu_0. \tag{5.14}$$

Therefore, since η is nondecreasing with respect to the second variable, on account of (5.13) and (5.14) we get

$$\eta(t,B\mu)\frac{\mu\overline{f}}{\overline{g}+\overline{\eta}(B\mu)} \geqslant \eta(t,x) \quad \text{for a. e. } t \in [0,T],$$
$$x \in \left[\beta(t;\mu) - \frac{\mu_0}{2}, \beta(t;\mu)\right], \ \mu \geqslant \mu_0. \tag{5.15}$$

On the other hand, from (5.10) it follows that

$$\beta(t;\mu) - \frac{\mu_0}{2} \geqslant \left(\frac{\overline{g}+\overline{\eta}(B\mu)}{\mu\overline{f}}\right)^{1/\lambda} \qquad \text{for } t \in [0,T], \quad \mu \geqslant \mu_0, \tag{5.16}$$

and so we have

$$\frac{\mu\overline{f}}{\overline{g}+\overline{\eta}(B\mu)} \geqslant \frac{1}{x^\lambda} \quad \text{for } x \in \left[\beta(t;\mu) - \frac{\mu_0}{2}, \beta(t;\mu)\right], \quad t \in [0,T], \quad \mu \geqslant \mu_0. \tag{5.17}$$

Now using (5.15) and (5.17) in (5.12) with respect to (3.4) we obtain

$$\beta''(t;\mu) \leqslant -\frac{g(t)}{x^\lambda} - \eta(t,x) + \mu f(t) \leqslant -\frac{g(t)}{x^\lambda} + h(t,x) + \mu f(t) \quad \text{for a. e. } t \in [0,T],$$
$$x \in \left[\beta(t;\mu) - \frac{\mu_0}{2}, \beta(t;\mu)\right], \quad \mu \geqslant \mu_0.$$

Moreover, (5.16) guarantees that $\beta(\cdot;\mu)$ ($\mu \geqslant \mu_0$) is not a T-periodic solution to (2.1). Consequently, according to Proposition 2.4, $\beta(\cdot;\mu)$ ($\mu \geqslant \mu_0$) is a strict upper function to the equation (2.1). Finally, (5.10) implies (5.4). □

6. Proof of main results

6.1. Nonexistence of solutions

LEMMA 6.1. *Let* (H_1)–(H_3) *be fulfilled. Then there exists* $\mu_* \leqslant 0$ *such that the equation* (2.1) *with* $\mu < \mu_*$ *has no* T*-periodic solution.*

Proof. Assume on the contrary that there exist a sequence of parameters $\{\mu_n\}_{n=1}^{+\infty} \subset \mathbb{R}_-$ such that

$$\lim_{n\to+\infty} \mu_n = -\infty \tag{6.1}$$

and the corresponding sequence of T-periodic solutions $u_n = u(\cdot;\mu_n)$ $(n \in \mathbb{N})$ to (2.1) with $\mu = \mu_n$. Then the integration of (2.1) from 0 to T results in

$$\int_0^T \frac{g(t)}{u_n^\lambda(t)}\,\mathrm{d}t = \int_0^T h(t, u_n(t))\,\mathrm{d}t + \mu_n \int_0^T f(t)\,\mathrm{d}t \qquad \text{for } n \in \mathbb{N},$$

whence in view of (2.2), (2.3) and (3.1) we get

$$0 < \int_0^T \frac{g(t)}{u_n^\lambda(t)}\,\mathrm{d}t \leqslant \int_0^T \left[\psi_r^+(t) + h^+(t, u_n(t))\right]\mathrm{d}t - |\mu_n| \int_0^T f(t)\,\mathrm{d}t. \tag{6.2}$$

According to lemma 4.8, from (6.2) it follows that

$$\overline{f} < \frac{\overline{\psi_r^+}}{|\mu_n|} + 2A\frac{\overline{h^+}(A(1+|\mu_n|))}{A(1+|\mu_n|)} \qquad \text{for } n \geqslant n_0 \tag{6.3}$$

where $n_0 \in \mathbb{N}$ is such that $|\mu_n| \geqslant 1$ for $n \geqslant n_0$. Now passing to the limit as n tends to $+\infty$, with respect to lemma 4.3 and (6.1), the inequality (6.3) yields $\overline{f} \leqslant 0$, a contradiction to (2.2). □

6.2. Functional setting and multiplicity results

In order to prove our main results we rewrite the periodic problem for (2.1) as an operator equation. For this purpose, let

$$\Lambda \stackrel{\text{def}}{=} \left\{u \in C\big(\mathbb{R}/T\mathbb{Z}\big) : u(t) > 0 \ \text{ for } \ t \in [0, T]\right\}$$

and, for every $\mu \in \mathbb{R}$, define an operator $\Psi_\mu : \Lambda \to C\big(\mathbb{R}/T\mathbb{Z}\big)$ by

$$\Psi_\mu[u](t) \stackrel{\text{def}}{=} \int_0^T G(t,s)\left[-\frac{g(s)}{u^\lambda(s)} + h(s, u(s)) + \mu f(s) - u(s)\right]\mathrm{d}s \quad \text{for } t \in \mathbb{R}/T\mathbb{Z},$$

where G is the Green's function to the periodic boundary value problem

$$u'' - u = 0, \qquad u(0) = u(T), \qquad u'(0) = u'(T).$$

Obviously, Ψ_μ is a completely continuous operator and the problem of finding a T-periodic solution to (2.1) is equivalent to the problem of finding a fixed point to the operator equation

$$u = \Psi_\mu[u], \qquad u \in \Lambda. \tag{6.4}$$

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

LEMMA 6.2. *Let* (H_1)–(H_3) *be fulfilled. Let, moreover, there exist* $\nu, \mu_1 \in \mathbb{R}$ *and a strict upper function* β *to* (2.1) *with* $\mu = \mu_1$ *such that*

$$\alpha(t;\nu) < \beta(t;\mu_1) \qquad \text{for } t \in [0,T],$$

where $|\nu| \geqslant |\mu_1|$ *and* α *is the function appearing in lemma* 4.10*. Then the equation* (2.1) *with* $\mu = \mu_1$ *has at least two* T*-periodic solutions.*

Proof. According to lemma 5.1, the function $\alpha(\cdot;\nu)$ is a strict lower function to (2.1) with $\mu = \mu_1$. We put

$$\Omega_1 \overset{\text{def}}{=} \big\{u \in C\big(\mathbb{R}/T\mathbb{Z}\big) : \alpha(t;\nu) < u(t) < \beta(t;\mu_1) \ \text{ for } \ t \in [0,T]\big\}.$$

Obviously, $\Omega_1 \subset \Lambda$. According to [**5**, Theorem III-1.8] (see also [**26**, Theorem 2.4]) we have

$$\deg(I - \Psi_{\mu_1}, \Omega_1) = 1, \tag{6.5}$$

and, in particular, there exists a T-periodic solution to (2.1) with $\mu = \mu_1$ in Ω_1. According to lemma 6.1 there exists $\mu_2 < -|\nu|$ such that (2.1) with $\mu = \mu_2$ has no T-periodic solution. Put

$$\Omega \overset{\text{def}}{=} \left\{u \in C\big(\mathbb{R}/T\mathbb{Z}\big) : \frac{\alpha(t;\mu_2)}{2} < u(t) < \rho(\mu_2) + \rho(-\mu_2) + \beta(t;\mu_1) \text{ for } t \in [0,T]\right\},$$

where the function ρ is given by (4.23) and $\alpha(\cdot;\mu_2)$ is the function appearing in lemma 4.10. Then $\overline{\Omega_1} \subset \Omega \subset \Lambda$, and

$$\deg(I - \Psi_{\mu_2}, \Omega) = 0.$$

Moreover, according to lemmas 4.9 and 4.10, there is no fixed point to (6.4) on $\partial\Omega$ for $\mu \in [\mu_2, \mu_1]$. Therefore,

$$\deg(I - \Psi_{\mu_1}, \Omega) = 0. \tag{6.6}$$

Now, let $\Omega_2 \overset{\text{def}}{=} \Omega \setminus \overline{\Omega_1}$. Then, in view of (6.5) and (6.6) we find

$$\deg(I - \Psi_{\mu_1}, \Omega_2) = \deg(I - \Psi_{\mu_1}, \Omega) - \deg(I - \Psi_{\mu_1}, \Omega_1) = -1,$$

and, consequently, there is another T-periodic solution to (2.1) with $\mu = \mu_1$ in $\overline{\Omega_2}$. □

6.3. Proof of theorem 3.2

According to lemmas 5.1 and 5.2, for every μ sufficently large there exist well-ordered strict lower and upper functions α and β to (2.1). More precisely, there

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

exists $\mu_0 > 0$ such that

$$\alpha(t;\mu) < \beta(t;\mu) \qquad \text{for } t \in [0,T], \quad \mu > \mu_0.$$

Therefore, according to lemma 6.2 the equation (2.1) has at least two T-periodic solutions for every $\mu > \mu_0$. Define a set of parameters S by

$$S \stackrel{\text{def}}{=} \big\{\tau \in \mathbb{R} : \text{equation (2.1) has at least two} \\ T\text{-periodic solutions for every } \mu > \tau\big\}. \tag{6.7}$$

In view of the above-proven we have $\mu_0 \in S$, i.e., the set S is nonempty. Moreover, according to lemma 6.1 there exists $\mu_* \in \mathbb{R}$ such that $\mu_* \leqslant \mu_0$ and the equation (2.1) has no T-periodic solution provided $\mu < \mu_*$. Therefore, the set S is bounded from below. Put

$$\mu^* \stackrel{\text{def}}{=} \inf S. \tag{6.8}$$

Obvioulsy, $\mu_* \leqslant \mu^*$ and the assertions (i) and (iii) hold. It remains to show that (ii) is valid. Let, therefore, $\{\mu_n\}_{n=1}^{+\infty}$ be a sequence of parameters such that

$$\mu_n > \mu^* \qquad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to+\infty} \mu_n = \mu^*. \tag{6.9}$$

According to (6.7)–(6.9) there exists a sequence of T-periodic solutions $u_n = u(\cdot;\mu_n)$ $(n \in \mathbb{N})$ to (2.1). Moreover, on account of (6.9) and lemmas 4.9 and 4.10 there exist positive constants K_1 and K_2 such that

$$K_1 \leqslant u_n(t) \leqslant K_2 \qquad \text{for } t \in [0,T], \quad n \in \mathbb{N}. \tag{6.10}$$

Consequently, from (2.1) with respect to (6.9) and (6.10) we obtain

$$|u_n'(t)| \leqslant K_3 \qquad \text{for } t \in [0,T], \quad n \in \mathbb{N} \tag{6.11}$$

for a suitable constant K_3. Therefore, the sequence $\{u_n\}_{n=1}^{+\infty}$ is uniformly bounded and equicontinuous. Thus, according to Arzelà-Ascoli theorem without loss of generality we can assume that there exists $u_0 \in C\big(\mathbb{R}/T\mathbb{Z}\big)$ such that

$$\lim_{n\to+\infty} u_n(t) = u_0(t) \qquad \text{uniformly on } [0,T]. \tag{6.12}$$

On the other hand, the solutions u_n satisfy

$$u_n = \Psi_{\mu_n}[u_n] \qquad \text{for } n \in \mathbb{N}, \tag{6.13}$$

and so, passing to the limit as n tends to $+\infty$ in (6.13), in view of (6.9) and (6.12) we obtain

$$u_0 = \Psi_{\mu^*}[u_0]. \tag{6.14}$$

Therefore, $u_0 \in AC^1\big(\mathbb{R}/T\mathbb{Z}\big)$ and it is a T-periodic solution to (2.1) with $\mu = \mu^*$. □

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

6.4. Proof of theorem 3.4

Lemma 6.3. *Let all the assumptions of theorem 3.4 be fulfilled. Let, moreover, there exist a T-periodic solution u to (2.1) with $\mu = \mu_0$. Then $\mu_0 > 0$ and for every $\mu_1 > \mu_0$ the function β given by*

$$\beta(t; \mu_1) \overset{\text{def}}{=} \frac{\mu_1}{\mu_0} u(t; \mu_0) \qquad \textit{for } t \in [0, T] \tag{6.15}$$

is a strict upper function to (2.1) with $\mu = \mu_1$.

Proof. First we show that $\mu_0 > 0$. Assume on the contrary that $\mu_0 \leqslant 0$. According to lemma 4.4, the inequality (4.6) holds. Therefore, the integration of (2.1) with $\mu = \mu_0$ over $[0, T]$, in view of (2.2) and (4.6) results in

$$0 < \int_0^T \frac{g(t)}{u^\lambda(t; \mu_0)} \, \mathrm{d}t = \int_0^T h(t, u(t; \mu_0)) \, \mathrm{d}t + \mu_0 \int_0^T f(t) \, \mathrm{d}t \leqslant 0,$$

a contradiction.

Now let $\mu_1 > \mu_0$ be arbitrary but fixed and define β by (6.15). Then there exists $\varepsilon > 0$ such that

$$\left(\frac{\mu_1}{\mu_0}\right)^{1+\lambda} \geqslant \left(\frac{\beta(t; \mu_1)}{\beta(t; \mu_1) - \varepsilon}\right)^\lambda \qquad \text{for } t \in [0, T] \tag{6.16}$$

and, according to lemma 4.4,

$$\frac{\mu_1}{\mu_0} h(t, u(t; \mu_0)) \leqslant h(t, y) \quad \text{for a. e. } t \in [0, T], \quad y \in [\beta(t; \mu_1) - \varepsilon, \beta(t; \mu_1)]. \tag{6.17}$$

In view of (6.16) we have

$$\frac{\mu_1 g(t)}{\mu_0 u^\lambda(t; \mu_0)} = \left(\frac{\mu_1}{\mu_0}\right)^{1+\lambda} \frac{g(t)}{\beta^\lambda(t; \mu_1)} \geqslant \frac{g(t)}{(\beta(t; \mu_1) - \varepsilon)^\lambda} \geqslant \frac{g(t)}{y^\lambda} \quad \text{for a. e. } t \in [0, T], \\ y \in [\beta(t; \mu_1) - \varepsilon, \beta(t; \mu_1)]. \tag{6.18}$$

On the other hand,

$$\beta''(t; \mu_1) + \frac{\mu_1 g(t)}{\mu_0 u^\lambda(t; \mu_0)} = \frac{\mu_1}{\mu_0} h(t, u(t; \mu_0)) + \mu_1 f(t),$$

whence, in view of (6.17) and (6.18) we obtain that β is not a T-periodic solution to (2.1) with $\mu = \mu_1$ and

$$\beta''(t; \mu_1) + \frac{g(t)}{y^\lambda} \leqslant h(t, y) + \mu_1 f(t) \quad \text{for a. e. } t \in [0, T], \quad y \in [\beta(t; \mu_1) - \varepsilon, \beta(t; \mu_1)].$$

Thus, according to Proposition 2.4, β is a strict upper function to (2.1) with $\mu = \mu_1$. □

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Proof of theorem 3.4. Let S be defined by (6.7) and put

$$\mu_\dagger \stackrel{\text{def}}{=} \inf S. \tag{6.19}$$

Then, according to theorem 3.2, the items (ii) and (iii) are fulfilled, and consequently, in view of lemma 6.3 we have $\mu_\dagger > 0$.

It remains to show that (i) holds. Let $\mu_0 \in \mathbb{R}$ be such that the equation (2.1) with $\mu = \mu_0$ has a T-periodic solution u. Let, moreover, $\mu_1 > \mu_0$ be arbitrary but fixed. Then, according to lemma 6.3, $\mu_0 > 0$ and the function β given by (6.15) is a strict upper function to (2.1) with $\mu = \mu_1$. Moreover, in view of lemma 4.10 we have

$$\alpha(t;\mu_1) \leqslant \alpha(t;\mu_0) < u(t;\mu_0) < \frac{\mu_1}{\mu_0}u(t;\mu_0) = \beta(t;\mu_1) \qquad \text{for } t \in [0,T].$$

Therefore, on account of lemma 5.1, the functions $\alpha(\cdot;\mu_1)$ and $\beta(\cdot;\mu_1)$ are respectively strict lower and upper functions to (2.1) with $\mu = \mu_1$ that are well-ordered. According to lemma 6.2 there exist at least two T-periodic solutions to (2.1) with $\mu = \mu_1$. However, since μ_1 was chosen arbitrarily, we obtain $\mu_0 \in S$. Consequently, in view of (6.19) we have $\mu_\dagger \leqslant \mu_0$ and so the item (i) holds. □

6.5. Proof of theorem 3.6

LEMMA 6.4. *Let all the assumptions of theorem 3.6 be fulfilled. Let, moreover, there exist a T-periodic solution u to (2.1) with $\mu = \mu_0$. Then for every $\mu_1 > \mu_0$ there exists $\varepsilon > 0$ such that the function β given by*

$$\beta(t;\mu_1) \stackrel{\text{def}}{=} u(t;\mu_0) + \varepsilon \qquad \textit{for } t \in [0,T] \tag{6.20}$$

is a strict upper function to (2.1) with $\mu = \mu_1$.

Proof. Let $\mu_1 > \mu_0$ be arbitrary but fixed. Since φ is continuous and satisfies (3.6), there exists $\varepsilon > 0$ such that

$$\varphi(x,y) \leqslant \frac{\mu_1 - \mu_0}{c} \qquad \text{for } m \leqslant x \leqslant M, \quad x \leqslant y \leqslant x + \varepsilon, \tag{6.21}$$

where m and M are given by (4.1). Thus, from (3.7) with respect to (3.6), (3.8), and (6.21) we have

$$h(t,x) - h(t,y) \leqslant (\mu_1 - \mu_0)f(t) \quad \text{for a. e. } t \in [0,T],\ m \leqslant x \leqslant M,\ x \leqslant y \leqslant x + \varepsilon,$$

whence, on account of (6.20), we get

$$\begin{aligned} h(t,u(t;\mu_0)) - h(t,y) \leqslant & (\mu_1 - \mu_0)f(t) \quad \text{for a. e. } t \in [0,T], \\ y \in & [\beta(t;\mu_1) - \varepsilon, \beta(t;\mu_1)]. \end{aligned} \tag{6.22}$$

On the other hand, on account of (6.20) we have

$$\beta''(t;\mu_1) + \frac{g(t)}{u^\lambda(t;\mu_0)} = h(t,u(t;\mu_0)) + \mu_0 f(t) \qquad \text{for a. e. } t \in [0,T]. \tag{6.23}$$

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Using (6.22) in (6.23) we obtain

$$\beta''(t;\mu_1) + \frac{g(t)}{y^\lambda} \leqslant h(t,y) + \mu_1 f(t) \quad \text{for a. e. } t \in [0,T],$$
$$y \in [\beta(t;\mu_1) - \varepsilon, \beta(t;\mu_1)].$$

Obviously, β is not a T-periodic solution to (2.1) with $\mu = \mu_1$. Thus, according to proposition 2.4, β is a strict upper function to (2.1) with $\mu = \mu_1$. □

Proof of theorem 3.6. Let S be defined by (6.7) and define $\mu_\dagger$ by (6.19). Then, according to theorem 3.2, the items (ii) and (iii) are fulfilled.

It remains to show that (i) holds. Let $\mu_0 \in \mathbb{R}$ be such that the equation (2.1) with $\mu = \mu_0$ has a T-periodic solution u. Let, moreover, $\mu_1 > \mu_0$ be arbitrary but fixed. Then, according to lemma 6.4, there exists $\varepsilon > 0$ such that the function β given by (6.20) is a strict upper function to (2.1) with $\mu = \mu_1$. Let $\nu \overset{\text{def}}{=} \max\{|\mu_0|, |\mu_1|\}$. Then, in view of lemma 4.10 we have

$$\alpha(t;\nu) \leqslant \alpha(t;\mu_0) < u(t;\mu_0) < u(t;\mu_0) + \varepsilon = \beta(t;\mu_1) \qquad \text{for } t \in [0,T].$$

Therefore, on account of lemma 5.1, the functions $\alpha(\cdot;\nu)$ and $\beta(\cdot;\mu_1)$ are respectively strict lower and upper functions to (2.1) with $\mu = \mu_1$ that are well-ordered. According to lemma 6.2 there exist at least two T-periodic solutions to (2.1) with $\mu = \mu_1$. However, since μ_1 was chosen arbitrarily, we obtain $\mu_0 \in S$. Consequently, in view of (6.19) we have $\mu_\dagger \leqslant \mu_0$ and so the item (i) holds. □

6.6. Proofs of theorems 3.8–3.10

LEMMA 6.5. *Let the assumptions of theorem 3.8 be fulfilled. Then* (H_1)–(H_3) *hold with*

$$h(t,x) \overset{\text{def}}{=} \sum_{i=1}^{n} \left[h_i(t)x^{\delta_i} + \frac{k_i(t)}{(c_i + x)^{\lambda_i}} \right] + q(t) \quad \textit{for a. e. } t \in [0,T], \quad x \in \mathbb{R}_+. \tag{6.24}$$

Proof. Put

$$\eta(t,x) \overset{\text{def}}{=} \sum_{i=1}^{n} \left([h_i]_-(t)x^{\delta_i} + \frac{[k_i]_-(t)}{c_i^{\lambda_i}} \right) + [q]_-(t) \quad \text{for a. e. } t \in [0,T], \quad x \in \mathbb{R}_+,$$

$$h^+(t,x) \overset{\text{def}}{=} \sum_{i=1}^{n} \left([h_i]_+(t)x^{\delta_i} + \frac{[k_i]_+(t)}{c_i^{\lambda_i}} \right) + [q]_+(t) \quad \text{for a. e. } t \in [0,T], \quad x \in \mathbb{R}_+,$$

$$h^-(t,x) \overset{\text{def}}{=} \sum_{i=1}^{n} [h_i]_-(t)x^{\delta_i} + [q]_-(t) \qquad \text{for a. e. } t \in [0,T], \quad x \in \mathbb{R}_+,$$

and, on account of $\overline{h_1} < 0$, choose $\zeta \in (0,1)$ such that

$$\int_0^T [h_1]_+(t)dt < (1-\zeta)^{\delta_1} \int_0^T [h_1]_-(t)\,\mathrm{d}t.$$

Then it can be easily verified that (H_1) is fulfilled.

https://doi.org/10.1017/prm.2021.14 Published online by Cambridge University Press

Now we will show that (H_2) and (H_3) hold. First assume that $p > 1$ and $\gamma > 0$. Put

$$g_0(t) \overset{\text{def}}{=} \begin{cases} c_0(t_{i+1}-t)^{\alpha_{i+1}}(t-t_i)^{\beta_i} & \text{for } t \in [t_i, t_{i+1}), \quad i = 1, \dots, m-1, \\ c_0(t_1+T-t)^{\alpha_1}(t-t_m)^{\beta_m} & \text{for } t \in [t_m, t_1+T), \end{cases}$$

$$g_0(t) \overset{\text{def}}{=} g_0(t-kT) \qquad \text{for } t \in [t_1+kT, t_1+(k+1)T), \quad k \in \mathbb{Z} \setminus \{0\}$$

and

$$\varphi_r(t) \overset{\text{def}}{=} \big(h^+(t,r) + 2|f(t)|\big) g_0^{(1-q)/q}(t) \qquad \text{for a. e. } t \in [0,T]$$

where $1 \leqslant q < (1+\gamma)p/(1+\gamma p)$ is such that

$$\lambda \geqslant \frac{(1+\gamma_0)p}{(2p-1)q}. \tag{6.25}$$

Using the Hölder's inequality we obtain

$$\begin{aligned} \int_0^T \varphi_r^q(t)\,\mathrm{d}t &= \int_0^T \big(h^+(t,r) + 2|f(t)|\big)^q g_0^{1-q}(t)\,\mathrm{d}t \\ &\leqslant \|h^+(\cdot,r) + |f|\|_p^q \left(\int_0^T g_0^{\frac{p(1-q)}{p-q}}(t)\,\mathrm{d}t\right)^{\frac{p-q}{p}} < +\infty, \end{aligned}$$

i.e., $\varphi_r \in L^q\big(\mathbb{R}/T\mathbb{Z}\big)$. Moreover,

$$h^+(t,r) + 2|f(t)| \leqslant \varphi_r(t) g^{\frac{q-1}{q}}(t) \qquad \text{for a. e. } t \in [0,T],$$

and thus (H_2) is fulfilled with $\psi_r^+ = h^+(\cdot,r) + |f|$ (the condition (2.2) implies (2.4) also in the case when $\overline{h^+}(r) = 0$). The validity of the assumption (H_3) follows immediately from (3.11), (3.12), (3.14), and (6.25).

If $p = 1$ or $\gamma = 0$ then we put $q \overset{\text{def}}{=} p$,

$$\varphi_r(t) \overset{\text{def}}{=} \big(h^+(t,r) + 2|f(t)|\big) c_0^{(1-p)/p} \qquad \text{for a. e. } t \in [0,T],$$

and it can be easily verified that (H_2) and (H_3) are fulfilled with $\psi_r^+ = h^+(\cdot,r) + |f|$. □

Proof of theorem 3.8. The assertion immediately follows from lemma 6.5 and theorem 3.2. □

Proof of theorem 3.9. From lemma 6.5 it follows that the assumptions (H_1)–(H_3) are fulfilled. Moreover, it can be easily verified that (3.5) holds with h defined by (6.24). Thus the assertion follows from theorem 3.4. □

Proof of theorem 3.10. From lemma 6.5 it follows that the assumptions (H_1)–(H_3) are fulfilled. Moreover, it can be easily verified that (3.6)–(3.8) hold with

$$h_0(t) \overset{\text{def}}{=} \sum_{i=1}^{n}([h_i]_-(t) + [k_i]_+(t)) \qquad \text{for a. e. } t \in [0, T],$$

$$\varphi(x, y) \overset{\text{def}}{=} \sum_{i=1}^{n}\left[y^{\delta_i} - x^{\delta_i} + \frac{1}{(c_i + x)^{\lambda_i}} - \frac{1}{(c_i + y)^{\lambda_i}}\right] \qquad \text{for } 0 \leqslant x \leqslant y,$$

and h defined by (6.24). Thus the assertion follows from theorem 3.6. □

7. Open problems

Example 3.12 shows that the numbers μ_* and μ^* guaranteed by theorem 3.2 do not coincide, in general, and that there may exist other T-periodic solutions to (2.1) for $\mu \in]\mu_*, \mu^*[$ that are not connected to the set of T-periodic solutions to (2.1) for $\mu \geqslant \mu^*$.

Open Problem 7.1. Let the assumptions of theorem 3.2 be fulfilled. Find conditions guaranteeing $\mu_* < \mu^*$. Describe sets of T-periodic solutions to (2.1) for $\mu \in [\mu_*, \mu^*[$.

The assumptions of both theorems 3.4 and 3.6 guarantee that $\mu_* = \mu^*$. Moreover, it can be easily verified that the hypotheses of theorem 3.4 do not imply the ones of theorem 3.6 and vice versa. Maybe there exists a weaker assumption in order to get the validity of both theorems 3.4 and 3.6 at the same time.

Open Problem 7.2. Let the assumptions of theorem 3.2 be fulfilled. Find other conditions guaranteeing $\mu_* = \mu^*$.

Acknowledgments

Research of J. Godoy and R. Hakl was supported by RVO: 67985840, research of X. Yu was supported by China Scholarship Council (CSC No. 201908320531) and the project of Innovation in Scientific Research for Graduate Students of Jiangsu Province (No. SJKY_0957).

References

1 J. D. Benson, C. C. Chicone and J. K. Critser. A general model for the dynamics of cell volume, global stability and optimal control. *J. Math. Biol.* **63** (2011), 339–359.

2 C. Bereanu and J. Mawhin. Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian. *J. Diff. Equ.* **243** (2007), 536–557.

3 V. Bevc, J. L. Palmer and C. Süsskind. On the design of the transition region of axisymmetric, magnetically focused beam valves. *J. British Inst. Radio Eng.* **18** (1958), 696–708.

4 J. Chu, P. J. Torres and F. Wang. Twist periodic solutions for differential equations with acombined attractive-repulsive singularity. *J. Math. Anal. Appl.* **437** (2016), 1070–1083.

5 C. De Coster and P. Habets. *Two-point boundary value problems: lower and upper solutions* (Amsterdam: Elsevier, 2006).

6 A. Fonda and A. Sfecci. On a singular periodic Ambrosetti-Prodi problem. *Nonlinear Anal.* **149** (2017), 146–155.

7 J. Godoy, N. Morales and M. Zamora. Existence and multiplicity of periodic solutions to an indefinite singular equation with two singularities. The degenerate case.. *Discrete Contin. Dyn. Syst.* **39** (2019), 4137–4156.

8 W. Greiner. *Classical mechanics: point particles and relativity* (New York: Springer, 2004).

9 P. Habets and L. Sanchez. Periodic solutions of some Liénard equations with singularities. *Proc. Amer. Math. Soc.* **109** (1990), 1135–1144.

10 R. Hakl and P. J. Torres. On periodic solutions of second-order differential equations with attractive-repulsive singularities. *J. Diff. Equ.* **248** (2010), 111–126.

11 R. Hakl, P. J. Torres and M. Zamora. Periodic solutions of singular second order differential equations: upper and lower functions. *Nonlinear Anal.* **74** (2011), 7078–7093.

12 R. Hakl and M. Zamora. On the open problems connected to the results of Lazer and Solimini. *Proc. R. Soc. Edinburgh Sect. A Math.* **144** (2014), 109–118.

13 R. Hakl and M. Zamora. Existence and uniqueness of a perodic solution to an indefinite attractive singular equation. *Ann. Mat. Pura Appl.* **195** (2016), 995–1009.

14 R. Hakl and M. Zamora. Periodic solutions to the Liénard type equations with phase attractive singularities. *Bound. Value Probl.* **47** (2013), doi: 10.1186/1687-2770-2013-47, 1–20.

15 R. Hakl and M. Zamora. Existence and multiplicity of periodic solutions to indefinite singular equations having a non-monotone term with two singularities. *Adv. Nonlinear Stud.* **19** (2019), 317–332.

16 J. A. Hernández. A general model for the dynamics of the cell volume. *Bull. Math. Biol.* **69** (2007), 1631–1648.

17 A. C. Lazer and S. Solimini. On periodic solutions of nonlinear differential equations with singularities. *Proc. Amer. Math. Soc.* **99** (1987), 109–114.

18 A. Lomtatidze. Theorems on differential inequalities and periodic boundary value problem for second-order ordinary differential equations. *Mem. Diff. Equ. Math. Phys.* **67** (2016), 1–129.

19 S. Lu and X. Yu. Periodic solutions for second order differential equations with indefinite singularities. *Adv. Nonlinear Anal.* **9** (2020), 994–1007.

20 S. Lu and X. Yu. Existence of positive periodic solutions for Liénard equations with an indefinite singularity of attractive type. *Bound. Value Probl.* **101** (2018). doi: 10.1186/s13661-018-1020-0.

21 P. Martínez-Amores and P. J. Torres. Dynamics of a periodic differential equation with a singular nonlinearity of attractive type. *J. Math. Anal. Appl.* **202** (1996), 1027–1039.

22 J. Mawhin. *Topological degree and boundary value problems for nonlinear differential equations.* In Topological methods for ordinary differential equations (Montecatini Terme, 1991). *Lecture Notes in Mathematics* (ed. M. Furi and P. Zecca), vol. 1537, pp. 74–142 (Berlin, Germany: Springer, 1993).

23 J. Mawhin. The periodic Ambrosetti-Prodi problem for nonlinear perturbations of the ϕ-Laplacian. *J. Eur. Math. Soc.* **8** (2006), 375–388.

24 G. D. Montesinos, V. M. Perez-García and P. J. Torres. Stabilization of solitons of the multidimensional nonlinear Schrödinger equation: matter-wave breathers. *Phys. D* **191** (2004), 193–210.

25 I. Rachůnková, S. Staněk and M. Tvrdý. Solvability of nonlinear singular problems for ordinary differential equations. *Contemporary Mathematics and its Applications,* vol. 5 (New York: Hindawi Publishing Corporation, 2008).

26 I. Rachůnková and M Tvrdý. Nonlinear systems of differential inequalities and solvability of certain boundary value problems. *J. Inequal. Appl.* **6** (2001), 199–226.

27 I. Rachůnková, M. Tvrdý and I. Vrkoč. Existence of nonnegative and nonpositive solutions for second–order periodic boundary–value problems. *J. Diff. Equ.* **176** (2001), 445–469.

28 P. J. Torres. Periodic oscillations of a model for membrane permeability with fluctuating environmental conditions. *J. Math. Biol.* **71** (2015), 57–68.

29 P. J. Torres. *Mathematical models with singularities — a zoo of singular creatures* (Paris: Atlantis Press, 2015)

30 X. Yu and S. Lu. A multiplicity result for periodic solutions of Liénard equations with an attractive singularity. *Appl. Math. Comput.* **346** (2019), 183–192.