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# CELLULAR AUTOMATA OVER ALGEBRAIC STRUCTURES

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**Abstract.** Let $G$ be a group and $A$ a set equipped with a collection of finitary operations. We study cellular automata $\tau : A^G \to A^G$ that preserve the operations of $A^G$ induced componentwise from the operations of $A$. We show that $\tau$ is an endomorphism of $A^G$ if and only if its local function is a homomorphism. When $A$ is entropic (i.e. all finitary operations are homomorphisms), we establish that the set $\mathrm{EndCA}(G; A)$, consisting of all such endomorphic cellular automata, is isomorphic to the direct limit of $\mathrm{Hom}(A^S, A)$, where $S$ runs among all finite subsets of $G$. In particular, when $A$ is an $R$-module, we show that $\mathrm{EndCA}(G; A)$ is isomorphic to the group algebra $\mathrm{End}(A)[G]$. Moreover, when $A$ is a finite Boolean algebra, we establish that the number of endomorphic cellular automata over $A^G$ admitting a memory set $S$ is precisely $(k|S|)^k$, where $k$ is the number of atoms of $A$.

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**1. Introduction.** The theory of cellular automata (CA) has important connections with many areas of mathematics, such as group theory, topology, symbolic dynamics, theoretical computer science, coding theory and cryptography. In its classical setting, CA are studied over grids $\mathbb{Z}^d$ and finite alphabets $A$. However, in recent years, various generalisations have gained considerable interest; notably, CA over arbitrary groups instead of grids, *linear CA* over vector spaces as alphabets and *additive CA* over commutative monoids as alphabets have been thoroughly investigated (e.g. see [**2**, **3**, **4**] and [**16**, pp. 952–953]). One-dimensional CA over arbitrary algebraic structures have been studied in [**15**]. In this paper, we propose a general framework that encompasses all these settings by considering CA over arbitrary groups and alphabets with an arbitrary algebraic structure.

For any group $G$ and any set $A$, let $A^G$ be the set of all functions of the form $x : G \to A$. A *cellular automaton* over $A^G$ is a function $\tau : A^G \to A^G$ defined via a local function $\mu : A^S \to A$, where $S$ is a finite subset of $G$ called a *memory set* for $\tau$.

We shall assume that the set $A$ has an *algebraic structure*, which means that $A$ is equipped with a collection of finitary operations. In this situation, $A^G$ inherits an algebraic structure via operations defined componentwise from the operations of $A$. Naturally, we restrict our attention to CA that are *endomorphisms* of $A^G$, i.e. CA $\tau : A^G \to A^G$ that preserve the operations of $A^G$. Denote by $\mathrm{CA}(G; A)$ and $\mathrm{EndCA}(G; A)$ the sets of all CA over $A^G$ and all endomorphic CA over $A^G$, respectively.

The paper is structured as follows. In Section 2, we review the definition and basic facts of CA over arbitrary groups. In Section 3, we give a brief introduction to algebraic structures in the context of universal algebra, including their direct products and homomorphisms. In Section 4, we show that the operations of $A^G$ are continuous in its prodiscrete topology and that the shift action of $G$ on $A^G$ preserves the operations. We establish that $\mathrm{CA}(G; A)$ is always an algebra of the same type as $A$, and when $A$ is entropic (i.e. all its operations are homomorphisms), the set $\mathrm{EndCA}(G; A)$ is a subalgebra $\mathrm{CA}(G; A)$. Our main result of the section (Theorem 3) establishes that a cellular automaton is an endomorphism of $A^G$ if and only if its local function is a homomorphism. This implies, when $A$ is entropic, that $\mathrm{EndCA}(G; A)$ is isomorphic to the direct limit of the directed family $\{\mathrm{Hom}(A^S, A) : S \subseteq G, |S| < \infty\}$. Finally, in Section 5, we focus on two particular situations: when $A$ is an $R$-module, we show that $\mathrm{EndCA}(G; A)$ is isomorphic to the group algebra $\mathrm{End}(A)[G]$ (Theorem 5), and when $A$ is a finite Boolean algebra, we establish that there are precisely $(k|S|)^k$ endomorphic CA admitting a memory set $S \subseteq G$, where $k$ is the number of atoms of $A$ (Theorem 6).

Our results, particularly Theorems 3 and 5, generalise important facts known for linear CA. Moreover, we consider that the argument given in the proof of Theorem 5 is more concise and elegant than the corresponding proof found in the literature for linear CA (c.f. [**4**, Theorem 8.5.2.]); hence, we believe that this illustrates the strength and beauty of introducing universal algebras in the context of CA.

**2. Cellular automata.** Let $G$ be a group and $A$ a set. The set $A^G$ of all functions $x : G \to A$ is usually called the *configuration space* in this context. The *shift action* of $G$ on $A^G$ is defined by $g \cdot x(h) := x(g^{-1}h)$, for all $x \in A^G$, $g, h \in G$.

The following definition is taken from [**4**, Section 1.4].

DEFINITION 1. A *cellular automaton* over $A^G$ is a function $\tau : A^G \to A^G$ such that there is a finite subset $S \subseteq G$, called a *memory set* of $\tau$, and a *local function* $\mu : A^S \to A$ satisfying

$$\tau(x)(g) = \mu((g^{-1} \cdot x)|_S), \quad \forall x \in A^G, g \in G,$$

where $|_S$ denotes the restriction to $S$ of a configuration in $A^G$.

REMARK 1. Let $\tau : A^G \to A^G$ be a cellular automaton with memory set $S$. The local defining function $\mu : A^S \to A$ may be recovered from $\tau : A^G \to A^G$ via

$$\mu(y) = \tau(\overline{y})(e), \quad \forall y \in A^S,$$

where $\overline{y} \in A^G$ is any extension of the function $y : S \to A$ and $e \in G$ is the identity element of the group.

REMARK 2. A memory set for a cellular automaton $\tau : A^G \to A^G$ is normally not unique. Indeed, if $S \subseteq G$ is a memory set for $\tau$, with local defining function $\mu : A^S \to A$, then any superset $S' \supseteq S$ is also a memory set for $\tau$: the local defining function $\mu' : A^{S'} \to A$ associated with $S'$ is given by $\mu'(x) = \mu(x|_S)$, for any $x \in A^{S'}$.

The most famous example of a cellular automaton is John Conway's *Game of Life*, which is defined over $\{0, 1\}^{\mathbb{Z}^2}$ and has memory set $S = \{-1, 0, 1\}^2$.

A notorious family of examples are the so-called *elementary CA*, which are defined over $A^{\mathbb{Z}}$, with $A = \{0, 1\}$, and have memory set $S = \{-1, 0, 1\}$; they are labelled as 'Rule $M$', where $M$ is a number from 0 to 255. In each case, the local function $\mu_M : A^S \to A$ of Rule $M$ is determined as follows: let $M_1 \ldots M_8$ be the binary representation of $M$ and

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write the elements of $A^S$ in lexicographical descending order, i.e. 111, 110, ..., 000, then the image of the $i$-th element of $A^S$ under $\mu_M$ is $M_i$.

EXAMPLE 1. Let $G = \mathbb{Z}$ and $A = \{0, 1\}$. We may identify the elements of $A^{\mathbb{Z}}$ with bi-infinite sequences, i.e. for any $x \in A^{\mathbb{Z}}$, we may write

$$x = (\dots, x_{-2}, x_{-1}, \cdot x_0, x_1, x_2, \dots),$$

where $x_i = x(i) \in A$, for all $i \in \mathbb{Z}$, and the dot $\cdot$ is used to distinguish the image of zero in the sequence. Note that the action of $k \in \mathbb{Z}$ on $x \in A^{\mathbb{Z}}$ is given by

$$k \cdot x = (\dots, x_{-k-2}, x_{-k-1}, \cdot x_{-k}, x_{-k+1}, x_{-k+2}, \dots).$$

Let $S = \{-1, 0, 1\} \subseteq G$ and define $\mu : A^S \to A$ by the following table

| $x \in A^S$ | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 |
|---|---|---|---|---|---|---|---|---|
| $\mu(x)$ | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |

The cellular automaton $\tau : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ with memory set $S$ and local function $\mu$ as above is the elementary cellular automaton Rule 110. Remarkably, this cellular automaton is known to be Turing complete [5].

EXAMPLE 2. For any group $G$ and set $A$, fix a function $\phi : A \to A$. The map $\tau : A^G \to A^G$ defined by

$$\tau(x) = \phi \circ x, \quad \forall x \in A^G$$

is cellular automaton over $A^G$ with memory set $S = \{e\}$ and local function $\mu : A^S \to A$ given by $\mu(y) = \phi(y(e))$, $\forall y \in A^S$. Indeed, we verify that, for any $x \in A^G$, $g \in G$,

$$\tau(x)(g) = \phi(x(g)) = \phi\left(g^{-1} \cdot x(e)\right) = \mu\left((g^{-1} \cdot x)|_{\{e\}}\right),$$

which satisfies Definition 1.

We endow $A^G$ with the *prodiscrete topology*, which is the product topology of the discrete topology on $A$ (see [13] for a comprehensive introduction to product topologies). For every $g \in G$, the projection maps $\pi_g : A^G \to A$, defined by $\pi_g(x) := x(g)$, $\forall x \in A^G$, are continuous and the preimage sets

$$\pi_g^{-1}(a) = \left\{x \in A^G : x(g) = a\right\}, \quad \text{for all } g \in G, a \in A,$$

form a sub-basis of the prodiscrete topology of $A^G$ (i.e. every open set in $A^G$ may be written as a union of finite intersections of these preimage sets). A function $\tau : A^G \to A^G$ is continuous if and only if, for all $g \in G$, the functions $\pi_g \circ \tau : A^G \to A$ are continuous.

It is known that every cellular automaton over $A^G$ commutes with the shift action (i.e. $\tau(g \cdot x) = g \cdot \tau(x)$, for all $x \in A^G$, $g \in G$) and is continuous in the prodiscrete topology of $A^G$. Moreover, when $A$ is finite, Curtis–Hedlund theorem [4, Theorem 1.8.1] establishes that every shift commuting continuous function $\tau : A^G \to A^G$ is a cellular automaton.

For any group $G$ and set $A$, define

$$\mathrm{CA}(G; A) := \{\tau : A^G \to A^G \mid \tau \text{ is a cellular automaton}\}.$$

As the composition of any two CA is a cellular automaton, the set $\mathrm{CA}(G; A)$ equipped with composition is a monoid (see [4, Corollary 1.4.11]).

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**3. Algebraic structures.** In this section, we introduce some concepts and notation coming from universal algebra. For more details, see [1].

Let $A$ be a non-empty set and $n$ a non-negative integer. An *n-ary operation* on $A$ is a function $f : A^n \to A$. A *finitary operation* is an $n$-ary operation, for some $n$. When $n = 0$, then $A^0 = \{\emptyset\}$, so 0-ary operation $f$ simply corresponds to a *distinguished element* $e_f \in A$.

An *algebra type* is a set $\mathcal{F}$ of function symbols with a non-negative integer $n$ (the *arity*) associated with each $f \in \mathcal{F}$; in such case, we say that $f \in \mathcal{F}$ is an $n$-ary function symbol. An *algebraic structure of type* $\mathcal{F}$, or simply an *algebra of type* $\mathcal{F}$, is a non-empty set $A$ together with a family $F$ of finitary operations on $A$ such that each $n$-ary operation $f^A \in F$ is indexed by an $n$-ary function symbol $f \in \mathcal{F}$.

Let $A$ and $B$ algebras of the same type $\mathcal{F}$. We say that $A$ is a subalgebra of $B$ if $A$ is a subset of $B$, and for every $f \in \mathcal{F}$, we have $f^A = f^B|_A$.

Given two algebras $A$ and $B$ of the same type $\mathcal{F}$, the *direct product* $A \times B$ is an algebra of type $\mathcal{F}$ with componentwise operations: for every $n$-ary $f \in \mathcal{F}$, $a_1, \dots, a_n \in A$, $b_1, \dots, b_n \in B$,

$$f^{A\times B}((a_1, b_1), \dots, (a_n, b_n)) = (f^A (a_1, \dots, a_n), f^B(b_1, \dots, b_n)).$$

Hence, for each algebra $A$ and $m \geq 0$, we may define an algebra $A^m$. This definition of direct product may be generalised to arbitrary direct products.

A *homomorphism* from $A$ to $B$ is a function $\phi : A \to B$ such that, for all $n$-ary $f \in \mathcal{F}$, $a_1, \dots, a_n \in A$, we have

$$\left(\phi \circ f^A\right)(a_1, \dots a_n) = f^B(\phi(a_1), \dots, \phi(a_n)).$$

If $e_A$ and $e_B$ are distinguished elements of $A$ and $B$, respectively, corresponding to the same 0-ary operation symbol, a homomorphism $\phi : A \to B$ satisfies $\phi(e_A) = e_B$. An *endomorphism* of $A$ is simply a homomorphism from $A$ to $A$. Define the sets

$$\begin{aligned} \mathrm{Hom}(A, B) &:= \{\phi : A \to B \mid \phi \text{ is a homomorphism}\}, \\ \mathrm{End}(A) &:= \mathrm{Hom}(A, A). \end{aligned}$$

Two algebras $A$ and $B$ of type $\mathcal{F}$ are isomorphic if there exists a bijective homomorphism from $A$ to $B$; in such case, we write $A \cong B$

An algebra $A$ is called *entropic* if for all $n$-ary $f \in \mathcal{F}$, we have $f^A \in \mathrm{Hom}(A^n, A)$ [6]. Entropic algebras are also known in the literature as *medial*, *commutative*, *bi-commutative*, *abelian*, among other names [10].

LEMMA 1 ([9]). *Let $A$ be an algebra of type $\mathcal{F}$. The following statements are equivalent:*

1. *$A$ is entropic.*
2. *For every $n$-ary $f \in \mathcal{F}$ and every $m$-ary $g \in \mathcal{F}$,*

$$\begin{aligned} &f^A\left(g^A(a_{11}, \dots, a_{m1}), \dots, g^A(a_{1n}, \dots, a_{mn})\right) \\ &\quad = g^A\left(f^A(a_{11}, \dots, a_{1n}), \dots f^A(a_{m1}, \dots, a_{mn})\right), \end{aligned}$$

*for every $a_{ij} \in A$, $i = 1, \dots, m$, $j = 1, \dots, n$.*

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3. *For every algebra $X$ of type $\mathcal{F}$,* $\mathrm{Hom}(X, A)$ *is also an algebra of type $\mathcal{F}$ by defining*

$$f^{\mathrm{Hom}(X,A)}(\phi_1, \ldots, \phi_n)(x) := f^A(\phi_1(x), \ldots, \phi_n(x)),$$

*for every $f \in \mathcal{F}$, $\phi_1, \ldots, \phi_n \in \mathrm{Hom}(X, A)$, $x \in X$.*

In particular, if $A$ is an entropic algebra, then $\mathrm{End}(A)$ is an algebra of the same type as $A$. Examples of entropic algebras are commutative semigroups (which include abelian groups) and modules over commutative rings (which include vector spaces). Note that a magma $A$ (i.e. an algebra with a single binary operation) is entropic if and only if $(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d)$, for all $a, b, c, d \in A$.

**4. Cellular automata over algebras.** Throughout this section, let $A$ be an algebra of type $\mathcal{F}$. Then, the configuration space $A^G$ is also an algebra of type $\mathcal{F}$; for any $n$-ary operation $f^A : A^n \to A$, we have an $n$-ary operation $f^{A^G} : (A^G)^n \to A^G$ induced componentwise as follows: for any $x_1, x_2, \ldots, x_n \in A^G$ and $g \in G$,

$$f^{A^G}(x_1, x_2, \ldots, x_n)(g) := f^A(x_1(g), x_2(g), \ldots, x_n(g)).$$

A *topological algebra of type $\mathcal{F}$* is an algebra $A$ of type $\mathcal{F}$ that is also a topological space in which all $n$-ary operations $f^A : A^n \to A$ are continuous (considering $A^n$ with the product topology).

THEOREM 1. *With respect to the prodiscrete topology, $A^G$ is a topological algebra of type $\mathcal{F}$.*

*Proof.* Consider any $n$-ary $f \in \mathcal{F}$. First, observe that the operation $f^A : A^n \to A$ is continuous as both spaces $A^n$ and $A$ have the discrete topology (as the finite product of discrete spaces is discrete).

Fix $g \in G$ and consider the function $\pi'_g : (A^G)^n \to A^n$ defined as $\pi'_g(x_1, x_2, \ldots, x_n) = (x_1(g), x_2(g), \ldots, x_n(g))$, for every $x_1, x_2, \ldots, x_n \in A^G$. The preimage of any $(a_1, a_2, \ldots, a_n) \in A^n$ under $\pi'_g$ is

$$\begin{aligned}(\pi'_g)^{-1}(a_1, a_2, \ldots, a_n) &= \left\{(x_1, x_2, \ldots, x_n) \in \left(A^G\right)^n : x_i(g) = a_i, \forall i\right\} \\ &= \left\{x_1 \in A^G : x_1(g) = a_1\right\} \times \cdots \times \left\{x_n \in A^G : x_n(g) = a_n\right\} \\ &= \pi_g^{-1}(a_1) \times \cdots \times \pi_g^{-1}(a_n).\end{aligned}$$

This is an open set in $(A^G)^n$ as it is a Cartesian product of the open sets $\pi_g^{-1}(a_i)$ of $A^G$. Thus, $\pi'_g$ is a continuous function for any $g \in G$.

The operation $f^{A^G} : (A^G)^n \to A^G$ is continuous if and only if $\pi_g \circ f^{A^G} : (A^G)^n \to A$ is continuous for all $g \in G$. Note that

$$\pi_g \circ f^{A^G} = f^A \circ \pi'_g.$$

As both $f^A$ and $\pi'_g$ are continuous, it follows that $\pi_g \circ f^{A^G}$ is continuous. □

Let $G$ be a group acting on an algebra $X$. We say that $X$ is a *$G$-algebra* if $G$ acts by homomorphisms, i.e. for all $g \in G$, the function $\varphi_g : X \to X$ defined by $\varphi_g(x) = g \cdot x$, $\forall x \in X$, is an endomorphism of the algebra.

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THEOREM 2. *With respect to the shift action, $A^G$ is a G-algebra.*

*Proof.* Let $n \geq 1$ and observe that for all $n$-ary $f \in \mathcal{F}$, $x_1, \ldots, x_n \in A^G$, $g, h \in G$,

$$\begin{aligned} f^{A^G}(\varphi_g(x_1), \ldots, \varphi_g(x_n))(h) &= f^A(g \cdot x_1(h), \ldots, g \cdot x_n(h)) \\ &= f^A(x_1(g^{-1}h), \ldots, x_n(g^{-1}h)) \\ &= f^{A^G}(x_1, \ldots, x_n)(g^{-1}h) \\ &= g \cdot f^{A^G}(x_1, \ldots, x_n)(h). \end{aligned}$$

Thus,

$$f^{A^G}(\varphi_g(x_1), \ldots, \varphi_g(x_n)) = (\varphi_g \circ f^{A^G})(x_1, \ldots, x_n),$$

which proves that $\varphi_g$ is an endomorphism of $A^G$.

When $n = 0$, a distinguished element $e_{A^G} \in A^G$ induced componentwise from a distinguished element $e_A \in A$ is defined by $e_{A^G}(g) = e_A \in A$, for all $g \in G$. Hence, for all $g, h \in G$, we have

$$g \cdot e_{A^G}(h) = e_{A^G}(g^{-1}h) = e_A = e_{A^G}(h).$$

Thus, $\varphi_g(e_{A^G}) = e_{A^G}$ for all $g \in G$. □

LEMMA 2. *The set* $\mathrm{CA}(G; A)$ *is a subalgebra of* $(A^G)^{A^G}$.

*Proof.* Let $X := (A^G)^{A^G}$. We must show that for any $n$-ary $f \in \mathcal{F}$ and $\tau_1, \ldots, \tau_n \in \mathrm{CA}(G; A)$ we have $f^X(\tau_1, \ldots, \tau_n) \in \mathrm{CA}(G; A)$. Let $S_i$ and $\mu_i : A^{S_i} \to A$ be the memory set and local function of $\tau_i$, respectively, for $i = 1, \ldots, n$. Define $S := \bigcup_{i=1}^n S_i$ and $\mu : A^S \to A$ by

$$\mu(y) := f^A(\mu_1(y|_{S_1}), \ldots, \mu_n(y|_{S_n})), \quad \forall y \in A^S.$$

Then, for all $x \in A^G$, $g \in G$,

$$\begin{aligned} f^X(\tau_1, \ldots, \tau_n)(x)(g) &= f^{A^G}(\tau_1(x), \ldots, \tau_n(x))(g) \\ &= f^A\left(\tau_1(x)(g), \ldots, \tau_n(x)(g)\right) \\ &= f^A\left(\mu_1\left((g^{-1} \cdot x)\,|_{S_1}\right), \ldots, \mu_n\left((g^{-1} \cdot x)\,|_{S_n}\right)\right) \\ &= \mu\left((g^{-1} \cdot x)|_S\right). \end{aligned}$$

This shows that $f^X(\tau_1, \ldots, \tau_n)$ is a cellular automaton with memory set $S$ and local function $\mu : A^S \to A$. □

For any $S \subset G$, define

$$\mathrm{CA}(G, S; A) := \{\tau : A^G \to A^G \mid \tau \text{ is a cellular automaton with memory set contained in } S\}.$$

This is not a monoid under composition, as the memory set of $\tau \circ \sigma$, for $\tau, \sigma \in \mathrm{CA}(G, S; A)$, may not be contained in $S$. However, a similar argument as in Lemma 2 shows that $\mathrm{CA}(G, S; A)$ is a subalgebra of $\mathrm{CA}(G; A)$.

Naturally, we are interested on CA that preserve the algebraic structure of $A^G$. Define

$$\mathrm{EndCA}(G, S; A) := \mathrm{End}(A^G) \cap \mathrm{CA}(G, S; A),$$

$$\mathrm{EndCA}(G; A) := \mathrm{End}(A^G) \cap \mathrm{CA}(G : A).$$

We call the elements of $\mathrm{EndCA}(G; A)$ *endomorphic CA* over $A^G$.

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LEMMA 3. *If $A$ is an entropic algebra, then both* $\mathrm{EndCA}(G, S; A)$ *and* $\mathrm{EndCA}(G; A)$ *are subalgebras of* $\mathrm{CA}(G : A)$

*Proof.* By Lemma 1, $\mathrm{End}(A^G)$ is an algebra with the induced componentwise operations. Hence, being intersections of algebras, both $\mathrm{EndCA}(G, S; A)$ and $\mathrm{EndCA}(G; A)$ are algebras. □

EXAMPLE 3. Suppose that $A$ is an abelian group with operation $+$, and consider a finite subset $S \subseteq G$. Define $\mu : A^S \to A$ by

$$\mu(z) = \sum_{s \in S} z(s), \quad \forall z \in A^S.$$

The cellular automaton $\tau : A^G \to A^G$ defined by the local function $\mu : A^S \to A$ preserves the addition of $A^G$. Indeed, for all $x, y \in A^G$, $g \in G$,

$$\begin{aligned} \tau(x+y)(g) &= \mu((g^{-1} \cdot (x+y)|_S) \\ &= \sum_{s \in S} g^{-1} \cdot (x+y)(s) \\ &= \sum_{s \in S} g^{-1} \cdot x(s) + \sum_{s \in S} g^{-1} \cdot y(s) \\ &= \tau(x)(g) + \tau(y)(g). \end{aligned}$$

Therefore, $\tau(x+y) = \tau(x) + \tau(y)$, for all $x, y \in A^G$, so $\tau \in \mathrm{EndCA}(G; A)$.

EXAMPLE 4. If we consider $A = \{0, 1\}$ as an abelian group with addition modulo 2, then the elementary cellular automaton $\tau : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ known as Rule 110 (defined in Example 1) does not preserve the addition of $A^{\mathbb{Z}}$. This may be verified, for example, by defining $x, y \in A^{\mathbb{Z}}$ as

$$x = (\dots, 0, 0, \cdot 1, 0, 0, \dots) \quad \text{and} \quad y = (\dots, 0, 0, \cdot 0, 1, 0, \dots),$$

and noting that $\tau(x) + \tau(y) \neq \tau(x+y)$.

EXAMPLE 5. Let $A$ be any algebra and fix $\phi \in \mathrm{End}(A)$. We claim that the map $\tau : A^G \to A^G$ defined by

$$\tau(x) = \phi \circ x, \quad \forall x \in A^G$$

is an endomorphic cellular automaton over $A^G$. By Example 2, this is a cellular automaton with memory set $S = \{e\}$ and local function $\mu : A^S \to A$ given by $\mu(y) = \phi(y(e))$, $\forall y \in A^S$. Now we check that $\tau$ is an endomorphism of $A^G$: for every $n$-ary $f \in \mathcal{F}$, $x_1, x_2, \dots, x_n \in A^G$, and $g \in G$, we have

$$\begin{aligned} f^{A^G}(\tau(x_1), \tau(x_2), \dots, \tau(x_n))(g) &= f^{A^G}(\phi \circ x_1, \phi \circ x_2, \dots, \phi \circ x_n)(g) \\ &= f^A(\phi \circ x_1(g), \phi \circ x_2(g), \dots, \phi \circ x_n(g)) \\ &= \phi \circ f^A(x_1(g), x_2(g), \dots, x_n(g)) \\ &= \left(\tau \circ f^{A^G}\right)(x_1, x_2, \dots, x_n)(g). \end{aligned}$$

This generalises Example 8.1.3. (b) in [4] which was given for linear CA.

EXAMPLE 6. Let $A$ be any algebra and fix $g_0 \in G$. Consider the bijection $R_{g_0}$ of $G$ defined by the right multiplication by $g_0$, i.e. $R_{g_0} : G \to G$ is the map defined by $R_{g_0}(g) = gg_0$, for all $g \in G$. We claim that the map $\tau : A^G \to A^G$ defined by

$$\tau(x) = x \circ R_{g_0}, \quad \forall x \in A^G$$

is an endomorphic cellular automaton over $A^G$. Note that $\tau$ is a cellular automaton with memory set $S = \{g_0\}$ and local function $\mu : A^S \to A$ given by $\mu(y) = y(g_0)$, $\forall y \in A^S$. We check that $\tau$ is an endomorphism of $A^G$: for every $n$-ary $f \in \mathcal{F}$, $x_1, x_2, \ldots, x_n \in A^G$, and $g \in G$,

$$\begin{aligned} f^{A^G}(\tau(x_1), \tau(x_2), \ldots, \tau(x_n))(g) &= f^A\left(x_1 \circ R_{g_0}(g), x_2 \circ R_{g_0}(g), \ldots, x_n \circ R_{g_0}(g)\right) \\ &= f^A(x_1(gg_0), x_2(gg_0), \ldots, x_n(gg_0)) \\ &= f^{A^G}(x_1, x_2, \ldots, x_n)(gg_0) \\ &= f^{A^G}(x_1, x_2, \ldots, x_n) \circ R_{g_0}(g) \\ &= \left(\tau \circ f^{A^G}\right)(x_1, x_2, \ldots, x_n)\,(g) \end{aligned}$$

This generalises Example 8.1.3. (c) in [4] which was given for linear CA.

The following two lemmas are technical, but they are required to show the main result of this section.

LEMMA 4. *Consider an n-ary operation $f^A : A^n \to A$ and $S \subseteq G$. For any $x_1, \ldots, x_n \in A^G$, we have*

$$f^{A^G}(x_1, \ldots, x_n)|_S = f^{A^S}(x_1|_S, \ldots, x_n|_S).$$

*Proof.* The above equality compares two elements of $A^S$. Note that, for every $s \in S$, the left-hand side gives us

$$f^{A^G}(x_1, \ldots, x_n)|_S(s) = f^A(x_1(s), \ldots, x_n(s)).$$

On the other hand, for every $s \in S$, the right-hand side gives us

$$f^{A^S}(x_1|_S, \ldots, x_n|_S)(s) = f^A(x_1(s), \ldots, x_n(s)).$$

The result follows. □

For any $S \subseteq G$ and $y \in A^S$, denote by $\overline{y}$ an element of $A^G$ such that $\overline{y}|_S = y$.

LEMMA 5. *Let $S \subseteq G$. For any n-ary $f \in \mathcal{F}$ and $y_1, \ldots, y_n \in A^S$, we have*

$$\overline{f^{A^S}(y_1, \ldots, y_n)}|_S = f^{A^G}\left(\overline{y_1}, \ldots, \overline{y_n}\right)|_S.$$

*Proof.* For every $s \in S$, we have

$$\begin{aligned} \overline{f^{A^S}(y_1, \ldots, y_n)}(s) &= f^{A^S}(y_1, \ldots, y_n)(s) \\ & f^A(y_1(s), \ldots, y_n(s)) \\ & f^A(\overline{y_1}(s), \ldots, \overline{y_n}(s)) \\ & f^{A^G}(\overline{y_1}, \ldots, \overline{y_n})(s). \end{aligned}$$

□

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The following result shows that a cellular automaton is an endomorphism of $A^G$ if and only if its local function is a homomorphism. This is a significant generalisation of [4, Proposition 8.1.1.], which was proved for linear CA.

THEOREM 3. *Let $\tau : A^G \to A^G$ be a cellular automaton with memory set $S \subseteq G$ and local function $\mu : A^S \to A$. Then, $\tau \in \mathrm{End}(A^G)$ if and only if $\mu \in \mathrm{Hom}(A^S, A)$.*

*Proof.* Suppose that $\tau \in \mathrm{End}(A^G)$. Recall (see Remark 1) that $\mu(y) = \tau(\overline{y})(e)$ for every $y \in A^S$. Then, for any $n$-ary $f \in \mathcal{F}$ and $y_1, \ldots y_n \in A^S$, we have

$$\mu\left(f^{A^S}(y_1, \ldots, y_n)\right) = \tau\left(\overline{f^{A^S}(y_1, \ldots, y_n)}\right)(e). \tag{4.1}$$

It follows from the definition of CA that if $x_1, x_2 \in A^G$ are two configurations such that $x_1|_S = x_2|_S$, then $\tau(x_1)(e) = \tau(x_2)(e)$. Hence, by Lemma 5, we have

$$\tau\left(\overline{f^{A^S}(y_1, \ldots y_n)}\right)(e) = \tau\left(f^{A^G}(\overline{y_1}, \ldots, \overline{y_n})\right)(e). \tag{4.2}$$

Combining (4.1) and (4.2), and using the fact that $\tau \in \mathrm{End}(A^G)$, we obtain

$$\begin{aligned}\mu\left(f^{A^S}(y_1, \ldots y_n)\right) &= \tau\left(f^{A^G}(\overline{y_1}, \ldots, \overline{y_n})\right)(e)\\ &= f^{A^G}(\tau(\overline{y_1}), \ldots, \tau(\overline{y_n}))(e)\\ &= f^{A}(\tau(\overline{y_1})(e), \ldots, \tau(\overline{y_n})(e))\\ &= f^{A}(\mu(y_1), \ldots, \mu(y_n)).\end{aligned}$$

Therefore, $\mu \in \mathrm{Hom}(A^S, A)$.

Suppose now that $\mu \in \mathrm{Hom}(A^S, A)$. For any $n$-ary $f \in \mathcal{F}$, $x_1, \ldots, x_n \in A^G$, and $g \in G$, we apply the definition of cellular automaton

$$\tau\left(f^{A^G}(x_1, \ldots, x_n)\right)(g) = \mu\left(\left(g^{-1} \cdot f^{A^G}(x_1, \ldots, x_n)\right)|_S\right).$$

By Theorem 2 and Lemma 4,

$$\begin{aligned}\mu\left(\left(g^{-1} \cdot f^{A^G}(x_1, \ldots, x_n)\right)|_S\right) &= \mu\left(\left(f^{A^G}(g^{-1} \cdot x_1, \ldots, g^{-1} \cdot x_n)\right)|_S\right)\\ &= \mu\left(f^{A^S}\left((g^{-1} \cdot x_1)|_S, \ldots, (g^{-1} \cdot x_n)|_S\right)\right)\\ &= f^{A}\left(\mu\left((g^{-1} \cdot x_1)|_S\right), \ldots, \mu\left((g^{-1} \cdot x_n)|_S\right)\right)\\ &= f^{A}(\tau(x_1)(g), \ldots, \tau(x_n)(g))\\ &= f^{A^G}(\tau(x_1), \ldots, \tau(x_n))(g).\end{aligned}$$

Hence, $\tau \in \mathrm{End}(A^G)$. □

COROLLARY 1. *The number of endomorphic cellular automata admitting a memory set $S \subseteq G$ is $|\mathrm{Hom}(A^S, A)|$.*

THEOREM 4. *Let $A$ be an entropic algebra. Then,*

$$\mathrm{EndCA}(G, S; A) \cong \mathrm{Hom}(A^S, A).$$

*Proof.* Consider the function $\Phi : \mathrm{EndCA}(G, S; A) \to \mathrm{Hom}(A^S, A)$ defined by $\Phi(\tau) = \mu$, where $\tau \in \mathrm{EndCA}(G, S; A)$ and $\mu : A^S \to A$ is the local function of $\tau$. By Theorem 3,

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$\Phi$ is well-defined and it is easy to see that $\Phi$ is bijective (using Remark 1). In order to show that $\Phi$ is a homomorphism, let $f \in \mathcal{F}$ be an $n$-ary function symbol. Let $\tau_1, \dots, \tau_n \in \mathrm{EndCA}(G, S; A)$ have local functions $\mu_1, \dots, \mu_n \in \mathrm{Hom}(A^S, A)$, respectively. We claim that the local function of $f^{\mathrm{EndCA}(G,S;A)}(\tau_1, \dots, \tau_n)$ is $f^{\mathrm{Hom}(A^S,A)}(\mu_1, \dots, \mu_n)$. Indeed, for all $x \in A^G$, we have

$$\begin{aligned} f^{\mathrm{EndCA}(G,S;A)}(\tau_1, \dots, \tau_n)(x)(e) &= f^A(\tau_1(x)(e), \dots, \tau_n(x)(e)) \\ &= f^A(\mu_1(x|_S), \dots, \mu_n(x|_S)) \\ &= f^{\mathrm{Hom}(A^S,A)}(\mu_1, \dots, \mu_n)(x|_S). \end{aligned}$$

This shows that

$$\Phi\left(f^{\mathrm{EndCA}(G,S;A)}(\tau_1, \dots, \tau_n)\right) = f^{\mathrm{Hom}(A^S,A)}\left(\Phi(\tau_1), \dots, \Phi(\tau_n)\right),$$

proving that $\Phi$ is an isomorphism. □

A partially ordered set $(I, \leq)$ is a *directed set* if for every pair $i, j \in I$, there exists $z \in I$ such that $i \leq z$ and $j \leq z$. A *directed family of algebras* is collection $\mathcal{A} := \{A_i : i \in I\}$ of algebras of the same type $\mathcal{F}$ indexed by a directed set $I$ together with a collection of maps $\{\phi_{ij} : A_i \to A_j : i, j \in I, i \leq j\}$, such that $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ if $i \leq j \leq k$ and $\phi_{ii}$ is the identity map for all $i \in I$. In this situation, as shown in [7, Section 21], one may construct an algebra of type $\mathcal{F}$ called the *direct limit* of $\mathcal{A}$ and denoted by $\varinjlim(\mathcal{A})$.

Let $\mathcal{P}_{<\infty}(G)$ be the set of all finite subsets of $G$. Ordered by inclusion, $\mathcal{P}_{<\infty}(G)$ is a directed set as an upper bound for a pair $S_1, S_2 \in \mathcal{P}_{<\infty}(G)$ is $S_1 \cup S_2 \in \mathcal{P}_{<\infty}(G)$. Together with the inclusion maps, the collection $\mathcal{A} := \{\mathrm{EndCA}(G, S; A) : S \in \mathcal{P}_{<\infty}(G)\}$ is a directed family and its direct limit is given by its union:

$$\varinjlim(\mathcal{A}) = \bigcup_{S \in \mathcal{P}_{<\infty}(G)} \mathrm{EndCA}(G, S; A).$$

As every cellular automaton has a finite memory set by definition, it is easy to see that

$$\mathrm{EndCA}(G; A) = \bigcup_{S \in \mathcal{P}_{<\infty}(G)} \mathrm{EndCA}(G, S; A).$$

Hence, Theorem 4 implies the following result.

COROLLARY 2. *Let $A$ be an entropic algebra. The algebra* $\mathrm{EndCA}(G; A)$ *is isomorphic to the direct limit of the directed family* $\{\mathrm{Hom}(A^S, A) : S \in \mathcal{P}_{<\infty}(G)\}$.

## 5. Cellular automata over particular algebras.

**5.1. *R*-modules.** In this section, we assume that $R$ is a ring (associative with 1) and $M$ is an $R$-module. In the language of universal algebra, this means that $M$ is an algebra of type $\mathcal{F} = \{+, 0, r\cdot \mid r \in R\}$, where $+$ is binary operation, 0 is a distinguished element and $\{r\cdot \mid r \in R\}$ is a family of unary operations. Important classes of $R$-modules are vector spaces (in which $R$ is a field) and abelian groups (in which $R = \mathbb{Z}$).

For any $x \in M^G$, define the support of $x$ as the set $\mathrm{supp}(x) := \{g \in G : x(g) \neq 0\}$. Let $M[G]$ be the subset of $M^G$ consisting of all configurations with finite support; this is an

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$R$-module with operations induced componentwise. Furthermore, we define a new binary operation on $M[G]$, called the *convolution product*, as follows

$$(x \cdot y)(g) := \sum_{h \in G} x(h) y(h^{-1} g), \quad \forall x, y \in M[G].$$

Notice that the sum above is finite because both $x$ and $y$ have finite support. In the case when $M = R$, $R[G]$ is known as the *group ring* of $G$ over $R$ and has significant importance in the representation theory of $G$ (see [12]).

Equivalently, $M[G]$ may be seen as the set of formal sums $\sum_{g \in G} m_g g$, such that $m_g \in M$, for all $g \in G$, and the set $\{g \in G : m_g \neq 0\}$ is finite.

REMARK 3. As $M[G]$ consists of functions with finite support, we have

$$M[G] = \bigcup_{S \in \mathcal{P}_{<\infty}(G)} M^S.$$

If $A$ is an $R$-module, then both $\mathrm{End}(A)$ and $\mathrm{End}(A)[G]$ are $Z(R)$-modules, where $Z(R)$ denotes the centre of $R$. The following result is a generalisation of [4, Theorem 8.5.2.].

THEOREM 5. *Let $A$ be an $R$-module and $G$ a group. Then,* $\mathrm{EndCA}(G; A) \cong \mathrm{End}(A)[G]$ *as $Z(R)$-modules.*

*Proof.* If $A$ is an $R$-module, then $\mathrm{Hom}(A^2, A) \cong \mathrm{End}(A) \times \mathrm{End}(A)$ (see [14, Corollary 2.32]). Using this and Theorem 4, we obtain that for every finite $S \subseteq G$,

$$\mathrm{EndCA}(G, S; A) \cong \mathrm{Hom}(A^S, A) \cong \prod_{s \in S} \mathrm{End}(A) = \mathrm{End}(A)^S.$$

The result follows because

$$\mathrm{EndCA}(G; A) \cong \bigcup_{S \in \mathcal{P}_{<\infty}(G)} \mathrm{Hom}(A^S, A) \cong \bigcup_{S \in \mathcal{P}_{<\infty}(G)} \mathrm{End}(A)^S = \mathrm{End}(A)[G]. \qquad \square$$

An explicit isomorphism from $\mathrm{Hom}(A^S, A)$ to $\prod_{s \in S} \mathrm{End}(A)$ is defined as follows: the image of $\mu : A^S \to A$ is $\prod_{s \in S} (\mu \circ j_s)$, where $j_s : A \to A^S$ is the natural embedding given by

$$j_s(a)(r) := \begin{cases} a & \text{if } r = s \\ 0 & \text{otherwise,} \end{cases} \quad \forall a \in A, r \in S.$$

Hence, an explicit isomorphism $\Psi : \mathrm{EndCA}(G; A) \to \mathrm{End}(A)[G]$ may be defined as follows: the image of a cellular automaton $\tau : A^G \to A^G$ with memory set $S$ and local function $\mu : A^S \to A$ corresponds to the element $\sum_{s \in S} (\mu \circ j_s) s$ in $\mathrm{End}(A)[G]$, given in the formal sums notation. From here, we see that the minimal memory set of $\tau$ corresponds to the support of $\Psi(\tau)$. It follows, by the same proof as in [4, Theorem 8.5.2.] that $\Psi$ satisfies

$$\Psi(\tau \circ \sigma) = \Psi(\tau) \Psi(\sigma), \quad \forall \tau, \sigma \in \mathrm{EndCA}(G; A),$$

where the operation on the right-hand side is the convolution product.

EXAMPLE 7. Let $A$ be an abelian group. Using the formal sums notation, the cellular automaton $\tau : A^G \to A^G$ with memory set $S \subseteq G$ and local function $\mu : A^S \to A$ defined in Example 3 corresponds to the element $\sum_{s \in S} \mathrm{id}\, s$ of $\mathrm{End}(A)[G]$, where $\mathrm{id} : A \to A$ is the identity endomorphism.

EXAMPLE 8. Let $A$ be an $R$-module. Using the formal sums notation, the cellular automaton $\tau : A^G \to A^G$ defined via $\phi \in \mathrm{End}(A)$ as in Example 2 corresponds to the element $\phi e$ of $\mathrm{End}(A)[G]$.

REMARK 4. By Theorem 5, the number of CA in $\mathrm{EndCA}(G; A)$ admitting a memory set $S \subseteq G$ is $|\mathrm{End}(A)|^{|S|}$. This formula is specially useful when $\mathrm{End}(A)$ is finite.

COROLLARY 3. *Let $G$ be a group and $S \subseteq G$ a finite subset of size $s$.*

1. *Suppose $A$ is a vector space of dimension $n < \infty$ over a finite field $F$. The number of linear CA $\tau : A^G \to A^G$ admitting a memory set $S$ is $|F|^{n^2 s}$.*
2. *Let $A$ be a group isomorphic to $\mathbb{Z}_n$. The number of CA $\tau : A^G \to A^G$ that are group homomorphisms, or $\mathbb{Z}$-module homomorphisms, admitting a memory set $S$ is $n^s$.*

*Proof.* When $A$ is a vector space, it is well-known that $\mathrm{End}(A)$ is isomorphic to the algebra of matrices $M_{n\times n}(F)$. Hence, $|\mathrm{End}(A)| = |M_{n\times n}(F)| = |F|^{n^2}$. Part (1.) follows by the previous remark.

For part (2.), it is easy to see that $|\mathrm{End}(\mathbb{Z}_n)| = n$, so the result again follows by the previous remark. □

EXAMPLE 9. Consider $A = \{0, 1\}$ as a group isomorphic to $\mathbb{Z}_2$. Then, the number of *linear* (or *additive*) elementary CA over $A^{\mathbb{Z}}$ is $2^3 = 8$, as elementary CA admit a memory set $S = \{-1, 0, 1\}$; explicitly, these are Rules 0, 60, 90, 102, 150, 170, 204 and 240.

**5.2. Boolean algebras.** A *Boolean algebra* is a set $B$ equipped with two binary operations $\wedge : B \times B \to B$ and $\vee : B \times B \to B$ called *meet* and *join*, respectively, an unary operation $\neg : B \to B$ called *complement* and two distinguished elements $0 \in B$ and $1 \in B$, called bottom and top, respectively. The meet and join are commutative and associative, and distributive between each other, 0 is an identity for $\vee$, 1 is an identity of $\wedge$ and complements satisfy $x \vee \neg x = 1$ and $x \wedge \neg x = 0$, for all $x \in B$ (see [8] for details).

The smallest example of a Boolean algebra is $\mathbf{2} := \{0, 1\}$, which only contains the two distinguished elements. If $X$ is a set, the power set $\mathcal{P}(X)$ with meet, join and complement given by the intersection, union and complement of subsets, respectively, is a Boolean algebra. In fact, $\mathcal{P}(X) \cong \mathbf{2}^X$ and every finite Boolean algebra is isomorphic to $\mathcal{P}(S)$, for some finite non-empty set $S$ ([8, Chapter 15]).

We may define a partial order relation $\leq$ on a Boolean algebra $B$ as follows: $x \leq y$ if and only if $x \vee y = y$ (or, equivalently, if and only if $x \wedge y = x$). The non-zero minimal elements of $B$ with respect to this order are called the *atoms* of $B$. For example, when $B$ is the power set $\mathcal{P}(S)$ of a set $S$, the atoms are precisely the singleton sets $\{x\}$ with $x \in S$. Thus, $\mathcal{P}(S)$ has exactly $|S|$ atoms.

An *ideal* of a Boolean algebra $B$ is a subset $I \subseteq B$ such that $0 \in I$, for all $a, b \in I$, we have $a \vee b \in I$, and for all $a \in I$, $x \in B$, we have $a \wedge x \in I$. An ideal $I$ of $B$ is *maximal* if $I$ is properly contained in $B$, and there is no ideal $J$ of $B$ such that $I \subset J \subset B$. The *ideal generated* by a subset $E$ of $B$, denoted by $\langle E \rangle$, is the smallest ideal of $B$ that contains $E$. An ideal generated by a singleton is called a *principal ideal*. In fact, for any $y \in B$, a theorem of Stone (see [8, Chapter 18]) implies that

$$\langle y \rangle = \{x \in B : x \leq y\}.$$

The kernel of a Boolean homomorphism $\phi : B \to B'$ between Boolean algebras is the set

$$\ker(\phi) := \{b \in B : \phi(b) = 0\},$$

Kernels of Boolean homomorphisms are always ideals.

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LEMMA 6. *Let $G$ be a group and consider the Boolean algebra $\mathbf{2} = \{0, 1\}$. Then, the number of Boolean cellular automata over $\mathbf{2}^G$ with memory set $S \subseteq G$ is $|S|$.*

*Proof.* By Corollary 1, the number of Boolean CA over $\mathbf{2}^G$ admitting memory set $S \subseteq G$ is equal to $|\mathrm{Hom}(\mathbf{2}^S, \mathbf{2})|$. By [8, Chapter 22], the set $\mathrm{Hom}(\mathbf{2}^S, \mathbf{2})$ is in bijection with the set of maximal ideals of $\mathbf{2}^S$ via $\mu \mapsto \ker(\mu)$, for every $\mu \in \mathrm{Hom}(\mathbf{2}^S, \mathbf{2})$. As $\mathbf{2}^S$ is finite, every ideal is principal [8, Corollary 18.2], so the maximal ideals of $\mathbf{2}^S$ are given by $\langle \neg a \rangle$, where $a \in \mathbf{2}^S$ is an atom. As $\mathbf{2}^S \cong \mathcal{P}(S)$ has precisely $|S|$ atoms (corresponding to the singletons $\{s\}$, $s \in S$), then $\mathbf{2}^S$ has precisely $|S|$ maximal ideals, and the result follows. □

COROLLARY 4. *Let $G$ be a group and consider the Boolean algebra $\mathbf{2} = \{0, 1\}$. Let $\tau : \mathbf{2}^G \to \mathbf{2}^G$ be a cellular automaton with memory set $S$. Then, $\tau$ is a Boolean homomorphism if and only its local function is a projection $\pi_s : \mathbf{2}^S \to \mathbf{2}$, for some $s \in S$.*

*Proof.* Observe that $\pi_s \in \mathrm{Hom}(\mathbf{2}^S, \mathbf{2})$, for each $s \in S$, and use Lemma 6 and Theorem 3. □

EXAMPLE 10. There are precisely 3 elementary CA over $\mathbf{2}^{\mathbb{Z}}$ that are Boolean homomorphisms; explicitly, these are Rules 170, 204 and 240.

THEOREM 6. *Let $G$ be a group and $A$ be a finite Boolean algebra. The number of Boolean cellular automata over $A^G$ with memory set $S \subseteq G$ is $(k|S|)^k$, where $k$ is the number of atoms of $A$.*

*Proof.* By Corollary 1, we must show that $|\mathrm{Hom}(A^S, A)| = (k|S|)^k$. By [8, Corollary 15.1], $A$ is isomorphic to $\mathbf{2}^k$, where $k$ is the number of atoms of $A$, and $A^S$ is isomorphic to $\mathbf{2}^{ks}$, where $s = |S|$, so

$$\mathrm{Hom}(A^S, A) \cong \mathrm{Hom}(\mathbf{2}^{ks}, \mathbf{2}^k).$$

In general, for algebras $B$, $B_i$, $i \in I$, the set $\mathrm{Hom}(B, \prod_{i\in I} B_i)$ is in bijection with $\prod_{i\in I} \mathrm{Hom}(B, B_i)$ via $\phi \mapsto (\pi_i \circ \phi)_{i\in I}$, for any $\phi \in \mathrm{Hom}(B, \prod_{i\in I} B_i)$ (c.f. [11, p. 70]). Hence, the result follows by Lemma 6:

$$|\mathrm{Hom}(\mathbf{2}^{ks}, \mathbf{2}^k)| = \left| \prod_{i=1}^{k} \mathrm{Hom}(\mathbf{2}^{ks}, \mathbf{2}) \right| = \prod_{i=1}^{k} |\mathrm{Hom}(\mathbf{2}^{ks}, \mathbf{2})| = \prod_{i=1}^{k} ks = (ks)^k. \qquad \square$$

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