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CANONICAL BASES AND STANDARD MONOMIALS

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Let U be the quantized enveloping algebra associated to a simple Lie algebra $\mathbf{g}$ by Drinfel'd and Jimbo. Let λ be a classical fundamental weight for $\mathbf{g}$, and $V(\lambda)$ the irreducible, finite-dimensional type **1** highest weight U-module with highest weight λ. We show that the canonical basis for $V(\lambda)$ (see Kashiwara [**6**, §0] and Lusztig [**18**, 14.4.12]) and the standard monomial basis (see [**11**, §§2.4 and 2.5]) for $V(\lambda)$ coincide.

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1. Introduction

Let $\mathbf{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, and let $\mathcal{U}$ be its universal enveloping algebra. Then there is a well-developed theory of "standard monomials" for $\mathcal{U}$ (see, for example, [**13**]). For each integrable highest weight module M for $\mathcal{U}$ there is a subset of M (the standard monomials) which are conjectured to form a basis for M. In many cases this conjecture has been proved. More recently (see [**11**]), this theory has been extended to the quantized case, providing standard monomials for certain modules for quantized enveloping algebras. Also, Lakshmibai (see [**7**], [**8**] and [**9**]) and Littelmann (see [**14**] and [**15**]) have developed theories of monomial bases in modules for universal and quantized enveloping algebras via crystal bases and paths (in Littelmann's case, also for the algebras). Let U be the quantized enveloping algebra associated to $\mathbf{g}$ by Drinfel'd [**3**] and Jimbo [**5**]. For each dominant weight λ in the weight lattice of $\mathbf{g}$ there is an irreducible, finite-dimensional type **1** highest weight U-module $V(\lambda)$ with highest weight λ (see [**18**, 3.5.6, 6.2.3 and 6.3.4]; for the definition of type **1**, see [**2**, 10.1]). In particular, if $V = V(\lambda)$ is a fundamental module of classical type (see the start of Section 3) for U then the standard monomials in V are known to form a basis for V. There is a canonical basis for V defined independently by Kashiwara [**6**] and Lusztig [**18**, 14.4.12]. Using certain Kashiwara operators on V, and Theorem 19.3.5 in [**18**], we show the canonical basis and the standard monomial basis in V coincide.

2. Preliminaries

We use the treatment in [**18**, §§1–3]. Let $\mathbf{g}$ be a semisimple Lie algebra, with root system Φ, simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, and Killing form $(\ ,\ )$. Let $h_1, h_2, \ldots, h_n$ be a basis for a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$, satisfying $(h_i, h) = \alpha_i^*(h)$ for all h in $\mathbf{h}$ and all $i \in I = \{1, 2, \ldots, n\}$. Here $\alpha_i^* := 2\alpha_i/(\alpha_i, \alpha_i)$. Let Y be the $\mathbb{Z}$-lattice spanned by $h_1, h_2, \ldots, h_n$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the fundamental weights of $\mathbf{g}$, defined by $\omega_i(h_j) = \delta_{ij}$, and let X be the $\mathbb{Z}$-lattice spanned by them (the weight lattice). Let d be the minimal positive integer so that $d(\alpha_i, \alpha_i)$ is always even. (Note that then $d(\alpha_i, \alpha_j)$ is always an integer.) If the highest common factor of the $\frac{1}{2}d(\alpha_i, \alpha_i)$ is not 1, then replace d by d divided by this highest common factor. We then define $i \cdot j$ to be $d(\alpha_i, \alpha_j)$ for each $i, j \in I$, so $(I, \cdot)$ is a Cartan datum as in [**18**, 1.1.1]. For $\mu \in Y$ and $\lambda \in X$, define $\langle \mu, \lambda \rangle$ to be $\lambda(\mu)$. Define an imbedding of I into Y by $i \mapsto h_i$ and into X by $i \mapsto \alpha_i$ for all $i \in I$. We then have a root datum of type $(I, \cdot)$ as in [**18**, 2.2.1], with $\langle h_1, \alpha_j \rangle = \alpha_j(h_i) = A_{ij}$ the corresponding symmetrizable Cartan matrix. For each $i \in I$, we define d_i to be the integer $\frac{1}{2}d(\alpha_i, \alpha_i)$. Then $d_i A_{ij} = \frac{1}{2}d(\alpha_i, \alpha_i)\left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right) = d(\alpha_i, \alpha_j)$ for each $i, j \in I$, and is thus a symmetric matrix over $\mathbb{Z}$. We use the same numbering as [1, Planches I to IX].

Let $\mathbb{Q}(v)$ be the field of rational functions in an indeterminate v, and $\mathcal{A} \subseteq \mathbb{Q}(v)$ the ring $\mathbb{Z}[v, v^{-1}]$. For N, $M \in \mathbb{N}$ and $i \in I$ we put $v_i = v^{d_i}$ and define the following (all of which lie in $\mathcal{A}$):

$$[N]_i = \frac{v_i^N - v_i^{-N}}{v_i - v_i^{-1}}, \quad [N]_i^! = [N]_i[N-1]_i \cdots [1]_i, \quad \begin{bmatrix} M \\ N \end{bmatrix}_i = \frac{[M]_i^!}{[N]_i^![M-N]_i^!}.$$

These are referred to as quantized integers, quantized factorials and quantized binomial coefficients, respectively. If v is specialized to 1 they specialize to the usual integers, factorials and binomial coefficients.

We define the quantized enveloping algebra U corresponding to the above data (as in [**18**, 3.1.1 and 33.1.5]) to be the $\mathbb{Q}(v)$-algebra U with generators $1, E_1, E_2, \ldots, E_n$, $F_1, F_2, \ldots, F_n$, and K_μ for $\mu \in Y$, subject to the relations: (for each $i, j \in I$ and $\mu, \mu' \in Y$)

$$K_0 = 1,$$
$$K_\mu K_{\mu'} = K_{\mu+\mu'},$$
$$K_\mu E_i = v^{\alpha_i(\mu)} E_i K_\mu,$$
$$K_\mu F_i = v^{-\alpha_i(\mu)} F_i K_\mu,$$
$$E_i F_i - F_i E_i = \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v_i - v_i^{-1}},$$
$$E_i F_j - F_j E_i = 0, \quad i \neq j,$$
$$\sum_{p+p'=1-A_{ij}} (-1)^{p'} \begin{bmatrix} 1 - A_{ij} \\ p' \end{bmatrix}_i E_i^p E_j E_i^{p'} = 0, \quad i \neq j,$$
$$\sum_{p+p'=1-A_{ij}} (-1)^{p'} \begin{bmatrix} 1 - A_{ij} \\ p' \end{bmatrix}_i F_i^p F_j F_i^{p'} = 0, \quad i \neq j,$$

(where, for $i \in I$, we put $\tilde{K}_i = K_{d_i h_i}$ and $\tilde{K}_i^{-1} = K_{-d_i h_i}$). In the last two summations, p and p' are restricted to the non-negative integers.

We make the following definitions (see [**18**, 3.1.1 and 3.1.13]). For $M \in \mathbb{N}$, and $i \in I$, we put $E_i^{(M)} = E_i^M/[M]_i^!$, and $F_i^{(M)} = F_i^M/[M]_i^!$, which are called *divided powers*. We also put $K_i = K_{h_i}$ and $K_i^{-1} = K_{-h_i}$ for $i \in I$. Let $U_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of U generated by the elements $E_i^{(N)}, F_i^{(N)}, K_\mu$ for $i \in I$, $N \in \mathbb{N}$ and $\mu \in Y$. It is called the *integral form* of U. Let U^- be the subalgebra of U generated by the F_i, $i \in I$.

Let W be the Weyl group of $\mathbf{g}$. So W is the group:

$$W = \langle s_1 s_2, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \ (i \neq j) \rangle$$

where $m_{ij} = 2, 3, 4, 6$ if $A_{ij}A_{ji} = 0, 1, 2, 3$, respectively. For $r \in I$, let W^r be the set of distinguished left coset representatives of the parabolic subgroup W_r of W generated by $\{s_1, s_2, \ldots, s_n\} \backslash \{s_r\}$.

Let $X^+ \subseteq X$ be the set of dominant weights, i.e. those of the form $\lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n \in X$ where $\omega_1, \omega_2, \ldots, \omega_n$ are the fundamental weights of $\mathbf{g}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{N}$. Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n$ be a dominant weight. We follow the construction in [**18**, 3.4.5 and 3.5.6]. Let J be the left ideal of U generated by the elements E_i for $i \in I$ and the elements $K_\mu - v^{\langle\lambda(\mu)\rangle}$ for $\mu \in Y$. Then the map from U^- to U/J taking $x \in U^-$ to $x + J$ is a $\mathbb{Q}(v)$-vector space isomorphism, which can be used to transfer the left U-module structure of U/J to U^-. The resulting U-module we denote by $M(\lambda)$; it is then called a *Verma module*. Let $T(\lambda)$ be the left ideal of $M(\lambda)$ (as a $\mathbb{Q}(v)$-algebra) generated by the elements $F_i^{\lambda_i+1}$, for $i \in I$, and let $V(\lambda)$ be the quotient module $M(\lambda)/T(\lambda)$. Then, by [**18**, 6.2.3 and 6.3.4], $V(\lambda)$ is the unique irreducible, finite-dimensional type **1** highest weight U-module with highest weight λ. It is the quantized counterpart of the irreducible, finite-dimensional $\mathcal{U}$-module $V'(\lambda)$ with highest weight λ (where $\mathcal{U}$ is the universal enveloping algebra of $\mathbf{g}$). We fix x_1 as the image of $1 \in M(\lambda)$ under the natural map from $M(\lambda)$ to $V(\lambda)$. Then x_1 is a highest weight vector for $V(\lambda)$. We also write $V(\lambda)_{\mathcal{A}}$ for the integral form of $V(\lambda)$ (see [**18**. 19.3.1]). This is a $U_{\mathcal{A}}$-module, by [**18**, 19.3.2]. For each $r \in I$ we denote by V_r the type **1** module $V(\omega_r)$ with highest weight ω_r. This is called the r-th *fundamental module* for U.

We shall need the following definition of certain root operators (see [**6**, §2.2]):

Definition 2.1. Suppose that λ is a dominant weight, and $V(\lambda)$ is the corresponding U-module as above. Fix $i \in I$. Any element $m \in V(\lambda)$ can be written uniquely $m = \sum_{0 \le k \le k'} F_i^{(k)} x_{k,k'}$, where the $x_{k,k'}$ satisfy $E_i x_{k,k'} = 0$ and $K_i x_{k,k'} = v^{k'} x_{k,k'}$. Then define

$$\tilde{F}_i(m) = \sum_{0 \le k \le k'} F_i^{(k+1)} x_{k,k'}, \quad \text{and} \quad \tilde{E}_i(m) = \sum_{1 \le k \le k'} F_i^{(k-1)} x_{k,k'}.$$

Let $^-$ be the $\mathbb{Q}$-algebra automorphism from U to U taking E_i to E_i, F_i to F_i, and K_μ to $K_{-\mu}$, for each $i \in I$ and $\mu \in Y$, and v to v^{-1} (see [**18**, 3.1.12]). Let $\lambda \in X$. There is an induced $\mathbb{Q}$-linear automorphism (also denoted $^-$) of any module $V(\lambda)$ for U defined by $\overline{ux_1} = \bar{u}x_1$ for any $u \in U^-$ (see [**18**, 19.3.4]). Note that every element of $V(\lambda)$ is of the form ux_1 for some $u \in U^-$. For $\lambda \in X$ let $\mathbf{B}(\lambda)$ be the canonical basis for $V(\lambda)$ (see [**6**, §0] or [**18**, 14.4.12]).

3. Standard monomials – the unquantized case

Initially, we look at the unquantized case, i.e. standard monomials in modules for the universal enveloping algebra. We will use one of the descriptions in this case in the next section to describe the standard monomials in the quantized case in a way suitable for proving they coincide with the canonical basis in the classical fundamental modules. We consider the $\mathcal{U}$-module $V'(\lambda)$, which is the unquantized counterpart of the U-module $V(\lambda)$. Let x'_1 be a fixed highest weight vector in $V'(\lambda)$. We fix $\lambda = \omega_r$, a classical fundamental weight, i.e. a fundamental weight satisfying $|(\omega_r, \alpha^*)| \leq 2$ for all positive roots α, where $(\, , \,)$ is the Killing form.

Suppose $\tau, \phi \in W^r$. Then (τ, ϕ) is an *admissible pair* if there is a sequence $\{\tau_i\}$ of elements of W^r satisfying

$$\tau = \tau_0 > \tau_1 > \cdots > \tau_k = \phi, \tag{1}$$

where the relation involved is the Bruhat order on W. The sequence must also satisfy:

(1) For $i = 0, 1, \ldots, k-1$, there is a fundamental root β_i such that $\tau_i = s_{\beta_i}\tau_{i+1}$. So $\ell(\tau_i) = \ell(\tau_{i+1}) + 1$ (where $\ell(\cdot)$ is the usual length function on W).

(2) We have $(\tau_i(\omega_r), \beta_i^*) = 2$, for $i = 0, 1, \ldots, k-1$.

We call such a sequence a *defining sequence* for (τ, ϕ). For each admissible pair (τ, ϕ) in W^r there is a corresponding standard monomial $Q'(\tau, \phi) \in V'(\lambda)$. We make the definition inductively in the following way (following [**12**, §3.6]):

Definition 3.1. Firstly, we put $Q'(1, 1) = x'_1$, the highest weight vector. Next, we assume that we have already defined $Q'(\tau, \phi)$ for all admissible pairs (τ, ϕ) satisfying $\theta \geq \tau$ for some fixed $\theta \in W^r$. Let (τ_1, ϕ_1) be such a pair satisfying $(\tau_1(\omega_r) + \phi_1(\omega_r), \alpha_i^*) > 0$ and $\theta \not\geq s_{\alpha_i}\tau_1$ for some fundamental root α_i.

(a) If $(\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r)), \alpha_i^*) = 1$, we set:

$$Q'(s_{\alpha_i}\tau_1, s_{\alpha_i}\phi_1) = f_i Q'(\tau_1, \phi_1).$$

(b) If $(\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r)), \alpha_i^*) = 2$, we set:

(i) $Q'(s_{\alpha_i}\tau_1, \phi_1) = f_i Q'(\tau_1, \phi_1)$, and

(ii) $Q'(s_{\alpha_i}\tau_1, s_{\alpha_i}\phi_1) = \frac{f_i^2}{2} Q'(\tau_1, \phi_1)$.

Thus, when $(\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r)), \alpha_i^*) = 1$, we have defined one new $Q'(\tau, \phi)$ using $Q'(\tau_1, \phi_1)$ and when $(\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r)), \alpha_i^*) = 2$, we have defined two new elements of the form $Q'(\tau, \phi)$ using $Q'(\tau_1, \phi_1)$. By [**12**, §3.6], $Q'(\tau_1, \phi_1) \neq 0$, so $\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r))$ is a weight, and it follows that as ω_r is a classical weight, these are the only possibilities for the value of $(\frac{1}{2}(\tau_1(\omega_r)+\phi_1(\omega_r)), \alpha_i^*)$. By [**12**, §3.6], each admissible pair arises exactly once in the above process, so we get one standard monomial corresponding to each admissible pair. However, as there is a certain amount of choice in the way the $Q'(\tau, \phi)$ are defined it is not clear that if we make different choices we will get the same elements of the module. However, we shall see below (see Theorem 3.2) that whatever choices are made in the definition we shall always obtain the same elements $Q'(\tau, \phi)$ in the module (note that we have fixed a highest weight vector x_1').

For $i \in I$ and $m \in \mathbb{N}$, put $e_i^{(m)} = e_i^m/m!$ and $f_i^{(m)} = f_i^m/m!$ Standard monomials can also be described in the following way (as usual we are assuming that the highest weight λ is a classical fundamental weight):

Theorem 3.2. *Suppose (τ, ϕ) is an admissible pair.*

(*a*) *If (τ, ϕ) is trivial (that is, $\tau = \phi$), then let $s_{\gamma_t}s_{\gamma_{t-1}}\cdots s_{\gamma_1}$ be a reduced expression for τ, where each γ_i is a fundamental root and s_{γ_i} is the corresponding fundamental reflection. For $i = 1, 2, \ldots, t$, put $m_i = (s_{\gamma_{i-1}}\cdots s_{\gamma_1}(\omega_r), \gamma_i^*)$. We have:*

$$Q'(\tau, \tau) = f_{\gamma_t}^{(m_t)} f_{\gamma_{t-1}}^{(m_{t-1})} \cdots f_{\gamma_1}^{(m_1)} x_1'.$$

(*b*) *If (τ, ϕ) is not trivial, and $\{\tau_i\}$ is a defining sequence as above for (τ, ϕ), we have:*

$$Q'(\tau, \phi) = f_{\beta_0} f_{\beta_1} \cdots f_{\beta_{k-1}} Q'(\phi, \phi).$$

(*The product on the right hand side is independent of the defining sequence chosen*).

In particular, the elements $Q'(\tau, \phi)$ are independent of the choices made in Definition 3.1.

Proof. We first show that the result (a) is true for certain choices in Definition 3.1, by induction on the length t of τ. If $\ell(\tau) = 0$, then $\tau = 1$. By definition, $Q'(1, 1) = x_1'$ so the result is true for $t = 0$. Suppose the result is true for $t = u - 1$. Let $\tau = s_{\gamma_u}\cdots s_{\gamma_1} \in W$ be an element of length u, and for $i = 1, 2, \ldots, u$, put $m_i = (s_{\gamma_{i-1}}\cdots s_{\gamma_1}(\omega_r), \gamma_i^*)$. By the inductive hypothesis applied to $v = s_{\gamma_{u-1}}\cdots s_{\gamma_1}$, we have $Q'(v, v) = f_{\gamma_{u-1}}^{(m_{u-1})}\cdots f_{\gamma_1}^{(m_1)} x_1'$. Now apply Definition 3.1, with $\theta = \tau$, $\tau_1 = \phi_1 = v$, and $\alpha_i = \gamma_u$. Clearly the condition $\theta \not\geq \tau_1$ is satisfied. In the notation of the definition, we have $(\frac{1}{2}(\tau_1(\omega_r)+\phi_1(\omega_r)), \alpha_i^*) = (v(\omega_r), \gamma_u^*) = m_u$, whence $m_u = (\omega_r, v^{-1}(\gamma_u^*))$. But $\ell(v^{-1}s_{\gamma_u}) > \ell(v^{-1})$, so $v^{-1}(\gamma_u)$ is a positive root, and $(\omega_r, v^{-1}(\gamma_u)) > 0$, since ω_r is a fundamental weight. We have $m_u > 0$, and therefore, applying part (a) of the definition if $m_u = 1$ or (b)(i) if $m_u = 2$, $Q'(\tau, \tau) = f_{\gamma_u}^{(m_u)} f_{\gamma_{u-1}}^{(m_{u-1})}\cdots f_{\gamma_1}^{(m_1)} x_1'$, as required. So, the result is true for $t = u$ given it is true for $t = u - 1$. By induction it is true for all t.

We next show that whatever choices are made, we still get the same elements in the module, for admissible pairs of the form (τ, τ). Firstly, we note that to define $Q'(\tau, \tau)$ we must use either (a) or (b)(ii) in Definition 3.1, since if we used (b)(i), we would have that $(\tau_1, \phi_1) = (s_{\alpha_i}\tau, \tau)$ was an admissible pair, and thus that $s_{\alpha_i}\tau > \tau$ (note we cannot have equality). But in this situation, we would have $\theta \geq \tau_1 = s_{\alpha_i}\tau$ for some $\theta \in W^r$, and thus that $\theta \geq \tau = s_{\alpha_i}(s_{\alpha_i}\tau)$. But we must also have $\theta \not\geq s_{\alpha_i}\tau_1$, i.e. $\theta \not\geq \tau$, a contradiction. Thus, all of the elements $Q'(\tau, \tau)$ will be defined as in the above paragraph. So all that remains to be done is to check that $Q'(\tau, \tau)$ is independent of the reduced expression chosen. This follows from specializing the result in [**18**, 28.1.2] to the unquantized case. Thus part (a) is proved.

For part (b), see [**12**, §3.8]. Note that Theorem 5.1 later provides an alternative proof that the elements $Q'(\tau, \phi)$ are uniquely defined, since as the specialization of the elements $Q(\tau, \phi)$ (see Definition 4.1 and also Theorem 4.4) they must be equal to the specialization of the canonical basis of the module $V(\omega_r)$ for U, provided we specialize x_1 to x_1'. □

Theorem 3.3. *Suppose that ω_r is a classical fundamental weight. Then the set $\{Q'(\tau, \phi) : (\tau, \phi)$ is an admissible pair in $W^r\}$ forms a $\mathbb{C}$-basis for $V'(\omega_r)$. Each $Q'(\tau, \phi)$ has weight $\frac{1}{2}(\tau(\omega_r) + \phi(\omega_r))$.*

Proof. See [**12**, §3.6]. □

4. Standard monomials – the quantized case

The theory of standard monomials has been extended to the quantized case, also. We now use the description in the previous section to describe these standard monomials in a way suitable for our purposes. Again, assuming that ω_r is a classical weight, we look at standard monomials for the U-module $V(\omega_r)$. These monomials are defined in a similar way to Theorem 3.2, as follows below (see [**11**, §§2.4 and 2.5]). A standard monomial $Q(\tau, \phi)$ is again defined for each admissible pair, (τ, ϕ).

Definition 4.1. Suppose (τ, ϕ) is an admissible pair.

(a) If (τ, ϕ) is trivial (that is, $\tau = \phi$), then let $s_{\gamma_t} s_{\gamma_{t-1}} \cdots s_{\gamma_1}$ be a reduced expression for τ, where each γ_i is a fundamental root and s_{γ_i} is the corresponding fundamental reflection. For $i = 1, 2, \ldots, t$, put $m_i = (s_{\gamma_{i-1}} \cdots s_{\gamma_1}(\omega_r), \gamma_i^*)$. We put:
$$Q(\tau, \tau) = F_{\gamma_t}^{(m_t)} F_{\gamma_{t-1}}^{(m_{t-1})} \ldots F_{\gamma_1}^{(m_1)} x_1.$$

(b) If (τ, ϕ) is not trivial, and $\{\tau_i\}$ is a defining sequence as above for (τ, ϕ), we put:
$$Q(\tau, \phi) = F_{\beta_0} F_{\beta_1} \cdots F_{\beta_{k-1}} Q(\phi, \phi).$$

This element is independent of the defining sequence chosen.

We also have:

Theorem 4.2. *Suppose that ω_r is a classical fundamental weight. Then the set $\{Q(\tau, \phi) : (\tau, \phi)$ is an admissible pair in $W^r\}$ forms a $\mathbb{Q}(v)$-basis for $V(\omega_r)$. Each $Q(\tau, \phi)$ has weight $\frac{1}{2}(\tau(\omega_r) + \phi(\omega_r))$.*

Proof. See [**11**, §2.6]. □

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For each admissible pair (τ, ϕ) we shall define a monomial $Q^*(\tau, \phi)$ in $V(\lambda)$. These monomials will be seen to be the same as those defined above. We make the definition inductively in the following way (following the construction in [**12**, §3.6] for the unquantized case – see Definition 3.1):

Definition 4.3. Firstly, we put $Q^*(1,1) = x_1$, the highest weight vector. Next, we assume that we have already defined $Q^*(\tau, \phi)$ for all admissible pairs (τ, ϕ) satisfying $\theta \geq \tau$ for some fixed $\theta \in W^r$. Let (τ_1, ϕ_1) be such a pair satisfying $(\tau_1(\omega_r) + \phi_1(\omega_r), \alpha_i^*) > 0$ and $\theta \not\geq s_{\alpha_i}\tau_1$ for some simple root α_i.

(a) If $(\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r)), \alpha_i^*) = 1$, we set:

$$Q^*(s_{\alpha_i}\tau_1, s_{\alpha_i}\phi_1) = F_i Q^*(\tau_1, \phi_1).$$

(b) If $(\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r)), \alpha_i^*) = 2$, we set:

(i) $Q^*(s_{\alpha_i}\tau_1, \phi_1) = F_i Q^*(\tau_1, \phi_1)$, and

(ii) $Q^*(s_{\alpha_i}\tau_1, s_{\alpha_i}\phi_1) = F_i^{(2)} Q^*(\tau_1, \phi_1)$.

Thus, when $(\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r)), \alpha_i^*) = 1$, we have defined one new $Q^*(\tau, \phi)$ using $Q^*(\tau_1, \phi_1)$ and when $(\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r)), \alpha_i^*) = 2$, we have defined two new elements of the form $Q^*(\tau, \phi)$ using $Q^*(\tau_1, \phi_1)$. As in the unquantized case (see Definition 3.1), these are the only possibilities for the value of $(\frac{1}{2}(\tau_1(\omega_r) + \phi_1(\omega_r)), \alpha_i^*)$. By [**12**, §3.6], each admissible pair arises exactly once in the above process, so we get one standard monomial corresponding to each admissible pair. However, as there is a certain amount of choice in the way the $Q^*(\tau, \phi)$ are defined it is not clear that if we make different choices we will get the same elements of the module. However, we shall see below (see Theorem 4.4) that whatever choices are made in the definition we shall always obtain the same elements $Q^*(\tau, \phi)$ in the module.

Theorem 4.4. *We have, for all admissible pairs (τ, ϕ), that $Q^*(\tau, \phi) = Q(\tau, \phi)$ (where $Q(\tau, \phi)$ is as in Definition 4.1). In particular, the elements $Q^*(\tau, \phi)$ are independent of the choices made in Definition 4.3.*

Proof. The proof of this theorem goes through in exactly the same way as in the unquantized case (see Theorem 3.2). We use [**18**, 28.1.2]. Note that the definition of the monomials $Q^*(\tau, \phi)$ is the quantized version of the definition of standard monomials in [**12**, §3.6]. Note also that Theorem 5.1 below provides an alternative proof that the elements $Q^*(\tau, \phi)$ are independent of the choices made in their definition. □

5. Equality of bases

We now identify $Q(\tau, \phi)$ and $Q^*(\tau, \phi)$ for all admissible pairs (τ, ϕ). Since ω_r is classical, we have $|(\omega_r, \alpha^*)| \leq 2$ for any root α. Since (,) is W-invariant we have, for any root α and any $\tau \in W$, $(\omega_r, \alpha^*) = (\tau(\omega_r), \tau(\alpha^*)) = (\tau(\omega_r), (\tau(\alpha))^*)$, whence $|(\tau(\omega_r), \alpha^*)| \leq 2$

for any root α and any $\tau \in W$. Therefore

$$\left|\left(\frac{1}{2}(\tau(\omega_r)+\phi(\omega_r)), \alpha^*\right)\right| \leq 2 \tag{2}$$

for any admissible pair (τ, ϕ). Note also that if $\xi \in V_r$ has weight δ, then $K_i\xi = v^c\xi$, where $c = (\delta, \alpha_i^*)$, by the definition of weight.

We therefore have, in case (a) of Definition 4.3, $K_iQ(\tau_1, \phi_1) = vQ(\tau_1, \phi_1)$, and in case (b), $K_iQ(\tau_1, \phi_1) = v^2Q(\tau_1, \phi_1)$ (where i is as in the definition). In either case, suppose that $E_iQ(\tau_1, \phi_1) \neq 0$. Then $E_iQ(\tau_1, \phi_1)$ has weight $\frac{1}{2}(\tau_1(\omega_r)+\phi_1(\omega_r))+\alpha_i$. Because standard monomials form a basis of V_r, any weight of V_r must be of the form $\frac{1}{2}(\beta(\omega_r)+\nu(\omega_r))$, for some admissible pair (β, ν), by Theorem 4.2. Therefore,

$$\frac{1}{2}(\tau_1(\omega_r)+\phi_1(\omega_r))+\alpha_i = \frac{1}{2}(\beta(\omega_r)+\nu(\omega_r)),$$

for some such pair (β, ν). But also $(\frac{1}{2}(\tau_1(\omega_r)+\phi_1(\omega_r)), \alpha_i^*) \geq 1$. (In fact it is either 1 or 2 – see Definition 4.3.) Thus:

$$\left(\frac{1}{2}(\tau_1(\omega_r)+\phi_1(\omega_r))+\alpha_i, \alpha_i^*\right) \geq 1+2=3,$$

that is,

$$\left(\frac{1}{2}(\beta(\omega_r)+\nu(\omega_r)), \alpha_i^*\right) \geq 3,$$

which contradicts (2). We conclude that in case (a) or (b) of Definition 4.3, $E_iQ(\tau_1, \phi_1) = 0$. (Note that we have used here an argument similar to an argument used in the proof of Lemma 3.6 in [**12**].)

We use the root operator $\tilde{F}_i$ (see Definition 2.1). The results in the above paragraph shows that the decomposition (as in Definition 2.1) of the element $Q(\tau_1, \phi_1)$ in Definition 4.3 is $F_i^{(0)}x_{0,1}$ in case (a) and $F_i^{(0)}x_{0,2}$ in case (b), from which it follows easily that

in case (a): $Q(s_{\alpha_i}\tau_1, s_{\alpha_i}\phi_1) = \tilde{F}_iQ(\tau_1, \phi_1)$,

in case (b)(i): $Q(s_{\alpha_i}, \tau_1, \phi_1) = \tilde{F}_iQ(\tau_1, \phi_1)$, and

in case (b)(ii): $Q(s_{\alpha_i}\tau_1, s_{\alpha_i}\phi_1) = \tilde{F}_i^2Q(\tau_1, \phi_1)$.

Theorem 5.1. *The standard monomial basis for V_r is the canonical basis for V_r.*

Proof. By the above each standard monomial is of the form $\tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_t}x_1$ for some sequence $i_1, i_2, \ldots, i_t$ in I. It is clear that each standard monomial is not zero and lies in the $\mathcal{A}$-form $V_{r,\mathcal{A}}$. Since $\bar{}$ fixes each F_i and also $[N]_i$ for any $N \in \mathbb{N}$, $i \in I$, it is clear that each standard monomial basis element is fixed under $\bar{}$. Therefore by the characterization [**18**, 19.3.5] (note that, in Lusztig's notation, the assumption $b \neq 0$ is required to make this theorem correct), of the canonical basis, each standard monomial lies in the canonical basis. Since the standard monomials also form a basis, the standard monomial basis and the canonical basis for V_r coincide. □

Remark 5.2. The above theorem applies to fundamental modules of classical type. A natural question is whether it extends to the other fundamental modules for U (in the cases when there are any). For these modules as well there are standard monomials forming a basis (see [**11**, Appendix A]), which are defined in the following way. Let $\lambda = \omega_r$ be a fundamental weight of non-classical type, associated to a simple Lie algebra. First we define the vectors $Q(\tau, \tau)$ for $\tau \in W^r$ in the same way as in Definition 4.1. For the other weights, we need the indexing set $S = \{(\tau, \mu)_N : \tau, \mu \in W^r\}$, where τ, μ and N satisfy the following:

(a) There exists a sequence $\{\mu_i \in W^r, 0 \leq i \leq s+1\}$ such that

$$\tau = \mu_0 > \mu_1 > \cdots \mu_{s+1} = \mu, \qquad \ell(\mu_i) = \ell(\mu_{i+1}) + 1.$$

(b) Let $\mu_i = s_{\beta_i}\mu_{i+1}$ (where β_i is a positive root), and $m_i = (\mu_{i+1}(\lambda), \beta_i^*)$. There exist positive integers $n_i, 0 \leq i \leq s$, such that

$$1 > n_s/m_s \geq n_{s-1}/m_{s-1} \geq \cdots \geq n_0/m_0 > 0.$$

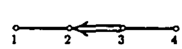
(In particular, note that each $m_i > 1$.)

(c) Let $p_t/q_t > p_{t-1}/q_{t-1} > \cdots > p_1/q_1$ be all the distinct numbers in the set $\{n_s/m_s, n_{s-1}/m_{s-1}, \ldots, n_0/m_0\}$; then $N = (p_t/q_t, p_{t-1}/q_{t-1}, \ldots, p_1/q_1)$.

To each $(\tau, \mu)_N$ we associate the vector $Q(\tau, \mu)_N = F_{\beta_0}^{(n_0)} F_{\beta_1}^{(n_1)} \cdots F_{\beta_s}^{(n_s)} Q(\mu, \mu)$. For β non-simple, F_β is to be understood as in [**16**], according to [**11**]. We take this to be [**17**], as such root vectors are not described in [**16**]. (See also the remark at the end of the second paragraph on page 123 in [**10**].)

It turns out that in case F_4, if we take $\lambda = \omega_2$, the fundamental weight corresponding to the adjoint representation, the standard monomial basis and the canonical basis for the corresponding finite-dimensional highest weight module differ. The Dynkin diagram and the numbering we are using for it are shown below:

F_4



Note that $d_1 = d_2 = 1$ and $d_3 = d_4 = 2$. In this case there exist sequences as described above where one of the β_i is not a simple root, and the corresponding root vector from [**17**] must be used. Put $\alpha = \alpha_1 + \alpha_2$; then the reflection s_α in the hyperplane orthogonal to α is $s_1 s_2 s_1 \in W$. We consider first the following:

$$\tau = \mu_0 = s_2 s_1 s_3 s_2 s_4 s_3 s_2 > \mu = \mu_1 = s_2 s_3 s_2 s_4 s_3 s_2,$$

with $\beta_0 = \alpha$, so $m_0 = 2$. Put $n_0 = 1$, so $N = (1/2)$. Then $Q(\tau, \mu)_N = F_\alpha F_2 F_3 F_2 F_4 F_3 F_2 x_1$. From [17, 6.6] we have $F_\alpha = -F_1 F_2 + vF_2 F_1$. Note that our numbering of the Dynkin diagram is *good* in the sense of [17, 4.3]. Now $F_1 F_2 . F_2 F_3 F_2 F_4 F_3 F_2 x_1 = 0$. To see this, start by applying the relation

$$F_2^{(3)} F_3 - F_2^{(2)} F_3 F_2 + F_2 F_3 F_2^{(2)} - F_3 F_2^{(3)} = 0. \tag{3}$$

Thus

$$F_1 F_2 . F_2 F_3 F_2 F_4 F_3 F_2 x_1 = [2] F_1 F_2^{(2)} F_3 F_2 F_4 F_3 F_2 x_1$$
$$= [2] F_1 F_2^{(3)} F_3 F_4 F_3 F_2 x_1 + [2] F_1 F_2 F_3 F_2^{(2)} F_4 F_3 F_2 x_1 - [2] F_1 F_3 F_2^{(3)} F_4 F_3 F_2 x_1.$$

Then, applying the relation $F_3 F_4 F_3 = F_3^{(2)} F_4 + F_4 F_3^{(2)}$ to the first term, and commutations to the other two, we have

$$\begin{aligned} F_1 F_2 . F_2 F_3 F_2 F_4 F_3 F_2 x_1 = {} & [2] F_1 F_2^{(3)} F_3^{(2)} F_4 F_2 x_1 + [2] F_1 F_2^{(3)} F_4 F_3^{(2)} F_2 x_1 \\ & + [2] F_1 F_2 F_3 F_4 F_2^{(2)} F_3 F_2 x_1 - [2] F_1 F_3 F_4 F_2^{(3)} F_3 F_2 x_1. \end{aligned}$$

Noting that $F_i x_1 = 0$ if $i \neq 2$, the first term can be seen to be zero by applying a commutation, the second term by using the relation $F_3^{(2)} F_2 - F_3 F_2 F_3 + F_2 F_3^{(2)} = 0$, and the last two by using again the relation (3).

So we should just consider $vF_2 F_1 F_2 F_3 F_2 F_4 F_3 F_2 x_1$. Let $\rho : U \to U^{\text{opp}}$ be the isomorphism of $\mathbf{Q}(v)$-algebras defined by Lusztig (see [**18**, 19.1.1]), given by:

$$\rho(E_i) = v_i \tilde{K}_i F_i, \quad \rho(F_i) = v_i \tilde{K}_i^{-1} E_i, \quad \rho(K_\mu) = K_\mu.$$

We will use (from [**18**, 19.1.2 and 19.3.5]):

Proposition 5.3. *Let λ be a dominant weight, and $V(\lambda)$ the corresponding finite dimensional type* 1 *highest weight module. There is a unique bilinear form $(\ ,\ )$ on $V(\lambda)$ such that*

(*a*) $(x_1, x_1) = 1$ *and*

(*b*) $(ux, y) = (x, \rho(u)y)$ *for all $x, y \in V(\lambda)$ and $u \in U$.*

Furthermore, if $b \in V(\lambda)$ then $b \in \pm\mathbf{B}(\lambda)$ if and only if $b \in V(\lambda)_A$, $\bar{b} = b$ and $(b, b) \equiv 1 \bmod v^{-1}\mathbf{Z}[v^{-1}]$. □

We have

$$(F_2 F_1 F_2 F_3 F_2 F_4 F_3 F_2 x_1,\ F_2 F_1 F_2 F_3 F_2 F_4 F_3 F_2 x_1)$$
$$= (\rho(F_2 F_1 F_2 F_3 F_2 F_4 F_3 F_2) F_2 F_1 F_2 F_3 F_2 F_4 F_3 F_2 x_1,\ x_1)$$

$$= (\rho(F_1F_2F_3F_2F_4F_3F_2)v_2\tilde{K}_2^{-1}E_2F_2F_1F_2F_3F_2F_4F_3F_2x_1, x_1)$$
$$= v(\rho(F_1F_2F_3F_2F_4F_3F_2)E_2F_2F_1F_2F_3F_2F_4F_3F_2x_1, x_1)$$
$$= v(\rho(F_1F_2F_3F_2F_4F_3F_2)F_2F_1F_3F_2F_4F_3F_2x_1, x_1).$$

In the last step we use the fact that:

$$E_i(F_{i_1}F_{i_2}\cdots F_{i_t}x_1) = \sum_{i_u=i}^{u} [\gamma(i_{u+1}, i_{u+2}, \ldots, i_t)]_i F_{i_1}\cdots \widehat{F_{i_u}}\cdots F_{i_t}x_1$$

where the hat indicates omission, and $\gamma(j_1, j_2, \ldots, j_s) \in \mathbb{Z}$ is defined by

$$\tilde{K}_i F_{j_1}F_{j_2}\cdots F_{j_s}x_1 = v_i^{\gamma(j_1, j_2, \ldots, j_s)}F_{j_1}F_{j_2}\cdots F_{j_s}x_1.$$

The square brackets indicate the quantum integer as usual. This fact follows easily from the relations of U. In this case all terms in the sum are zero except one.

We continue the calculation in this way, and get:

$$(F_2F_1F_2F_3F_2F_4F_3F_2x_1, F_2F_1F_2F_3F_2F_4F_3F_2x_1)$$
$$= v^{-1}[2](\rho(F_2F_3F_2F_4F_3F_2)F_2F_3F_2F_4F_3F_2x_1, x_1)$$
$$= (1+v^{-2})(\rho(F_3F_2F_4F_3F_2)F_3F_2F_4F_3F_2x_1.x_1)$$
$$= (1+v^{-2})(\rho(F_3F_2F_4F_3F_2)F_3F_2F_4F_3F_2x_1, x_1)$$
$$= (1+v^{-2})(\rho(F_2F_4F_3F_2)F_2F_4F_3F_2x_1, x_1)$$
$$= (1+v^{-2})(\rho(F_4F_3F_2)F_4F_3F_2x_1, x_1)$$
$$= (1+v^{-2})(\rho(F_3F_2)F_3F_2x_1, x_1)$$
$$= (1+v^{-2})(\rho)(F_2)F_2x_1, x_1$$
$$= (1+v^{-2})(x_1, x_1) = 1+v^{-2}$$

(in each case, the sum in the previous paragraph contains only one non-zero summand).

We thus have

$$(F_\alpha F_2F_3F_2F_4F_3F_2x_1, F_\alpha F_2F_3F_2F_4F_3F_2x_1) = v^2+1,$$

so $F_\alpha F_2F_3F_2F_4F_3F_2x_1$ does not lie in the canonical basis for $V(\lambda)$.

One may ask if an alternative root vector F_α would solve this problem. But consider also the following:

$$\tau = \mu_0 = s_1s_2s_3s_2s_1s_4s_3s_2 > \mu_1 = s_3s_1s_2s_1s_4s_3s_2 > \mu_2 = s_1s_2s_1s_4s_3s_2 = \mu,$$

with $\beta_0 = \alpha$ and $\beta_1 = \alpha_3$, so $m_0 = m_1 = 2$. Put $n_0 = n_1 = 1$, so $N = (1/2)$. Then $Q(\tau,\mu)_N = F_\alpha F_3 F_1 F_2^{(2)} F_1 F_4 F_3 F_2 x_1$. In this case $F_2 F_1.F_3 F_1 F_2^{(2)} F_1 F_4 F_3 F_2 x_1 = 0$. To see this, note that $K_1(F_2^{(2)} F_1 F_4 F_3 F_2 x_1) = v(F_2^{(2)} F_1 F_4 F_3 F_2 x_1)$, so if v is the weight of $F_2^{(2)} F_1 F_4 F_3 F_2 x_1$, then $(v, \alpha_1^*) = 1$. Note also that $v + \alpha_1$ is not a weight of the module. This is easy to see with bare hands: $v - \alpha_1 = \lambda - 3\alpha_2 - \alpha_3 - \alpha_4$, so if ux_1 were a non-zero element in such a weight space, with $u \in U^-$, then u would be a linear combination of monomials in the F_i, each containing F_2 three times and F_3 and F_4 once each. Suppose, for a contradiction, we had such a monomial which did not annihilate x_1. Such a monomial must end in $F_3 F_2$ as $F_i x_1 = 0$ if $i \neq 2$ and $F_2^2 x_1 = 0$. Since F_4 commutes with F_2, we are reduced to considering $F_4 F_2^{(2)} F_3 F_2 x_1$, but this is zero by the relation (3). Since the weights of $V(\lambda)$ are the same as those in the classical case, we have, by [**4**, Theorem 1, p. 112], that $v - 2\alpha_1$ is not a weight of $V(\lambda)$, from which it follows that $F_1^2 F_2^{(2)} F_1 F_4 F_3 F_2 x_1 = 0$, and so $F_2 F_1.F_3 F_1 F_2^{(2)} F_1 F_4 F_3 F_2 x_1 = 0$.

We therefore consider $F_1 F_2.F_3 F_1 F_2^{(2)} F_1 F_4 F_3 F_2 x_1$. We have

$$(F_1 F_2.F_3 F_1 F_2^{(2)} F_1 F_4 F_3 F_2 x_1, F_1 F_2.F_3 F_1 F_2^{(2)} F_1 F_4 F_3 F_2 x_1) = 1 + v^{-2},$$

in a similar way to the previous example. (Again the action of the E_i in each case produces a sum in which only one summand is non-zero.)

Thus, if we define F_α to be $\pm v^a F_1 F_2 \pm v^b F_2 F_1$ then a necessary condition for the standard monomial basis and the canonical basis to coincide for this module is that $a = b = 0$. For $i \in I$, let T_i be the unique algebra automorphism of U whose action on the generators is given by:

$$T_i(E_i) = -F_i K_i, \quad T_i E_j = \sum_{r+s=-A_{ij}} (-1)^r v^{-d_i s} E_i^{(r)} E_j E_i^{(s)}, \quad \text{if } i \neq j,$$

$$T_i(F_i) = -K_i^{-1} E_i, \quad T_i F_j = \sum_{r+s=-A_{ij}} (-1)^r v^{-d_i s} F_i^{(s)} F_j F_i^{(r)}, \quad \text{if } i \neq j,$$

$$T_i(K_j) = K_i K_j^{-A_{ij}}.$$

If $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ is a reduced expression for $w \in W$, then T_w is defined to be $T_{i_1} T_{i_2} \cdots T_{i_m}$; this product is independent of the reduced expression chosen (see [**17**]). In [**19**, 1.4], Xi defines a root vector in U of root $-\alpha$ to be an element of the form $T_w(F_i)$ where $w \in W$ satisfies $w^{-1}(\alpha) = \alpha_i$.

Using [**19**, 4.4] and interpreting the result in terms of U^- and negative roots, we see that all root vectors of root $-\alpha$ are of the form $T_1^{\pm 1}(F_2)$. Since $T_1^{-1}(F_2) = T_2(F_1)$ it is easy to see that these are indeed root vectors of root $-\alpha$. Thus the root vectors of root $-\alpha$ are $T_1(F_2) = vF_1F_2 - F_2F_1$ and $T_2(F_1) = -F_1F_2 + vF_2F_1$, and we see that for all root vectors of root $-\alpha$ in this sense, the standard monomial basis and the canonical basis for $V(\omega_2)$ in type F_2 do not coincide.

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