Nagoya Math. J., **239** (2020), 173–204
DOI 10.1017/nmj.2018.35

FOUR IDENTITIES FOR THIRD ORDER MOCK THETA FUNCTIONS

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Abstract. In 2005, using a famous lemma of Atkin and Swinnerton-Dyer (Some properties of partitions, Proc. Lond. Math. Soc. (3) **4** (1954), 84–106), Yesilyurt (Four identities related to third order mock theta functions in Ramanujan's lost notebook, Adv. Math. **190** (2005), 278–299) proved four identities for third order mock theta functions found on pages 2 and 17 in Ramanujan's lost notebook. The primary purpose of this paper is to offer new proofs in the spirit of what Ramanujan might have given in the hope that a better understanding of the identities might be gained. Third order mock theta functions are intimately connected with ranks of partitions. We prove new dissections for two rank generating functions, which are keys to our proof of the fourth, and the most difficult, of Ramanujan's identities. In the last section of this paper, we establish new relations for ranks arising from our dissections of rank generating functions.

§1. Introduction

On pages 2 and 17 in his *Lost Notebook* [26], Ramanujan recorded four identities involving the rank generating function. Of course, Ramanujan would not have used this terminology, because the rank of a partition was not defined until 1944 by Dyson [13]. He defined the *rank of a partition* to be the largest part minus the number of parts. For example, the rank of the partition $4+1$ is $4-2=2$. Let $N(m,n)$ denote the number of partitions of the positive integer n with rank m. Dyson showed that the generating function for $N(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}N(m,n)q^nz^m=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(q/z)_n}=:G(z,q),\qquad |q|<1. \tag{1.1}$$

Received September 12, 2017. Revised August 21, 2018. Accepted August 21, 2018.
2010 Mathematics subject classification. Primary 33D15, 11P83.
The third author was partially supported by the Singapore Ministry of Education Academic Research Fund, Tier 2, project number MOE2014-T2-1-051, ARC40/14

Ramanujan's four identities involve special cases of $G(z, q)$. Here, we use the standard notation

$$(a)_n := (a; q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$n \geqslant 1, (a)_0 := (a; q)_0 := 1.$$

Ramanujan's four identities were first proved in a wonderful paper by Yesilyurt [29]. Although Yesilyurt's proofs using the Atkin–Swinnerton-Dyer lemma [7] are ingenious, it is doubtful that Ramanujan would have used such an approach. First, in his voluminous work in q-series, there is no evidence that he would have used such a means via complex analysis. Second, the Atkin–Swinnerton-Dyer lemma was established many years after Ramanujan died in 1920, although, of course, it is conceivable that he could have discovered an equivalent theorem himself. Third, to apply the Atkin–Swinnerton-Dyer lemma, it would seem that prior knowledge of the identities' existence would be necessary. For these reasons, the authors were compelled for several years to find more natural proofs using methods from the theory of q-series. The purpose of this paper is indeed to provide such proofs, which also lead us to a better understanding of these identities. Moreover, as emphasized above, these functions are intimately connected with ranks of partitions, and our work has also led to the discovery of some new facts about ranks.

In the sequel, we also use the notation

$$\begin{aligned}
(a_1, a_2, \ldots, a_m; q)_n &:= (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \quad n \geqslant 0,\\
(a)_\infty &:= (a; q)_\infty := \lim_{n\to\infty} (a; q)_n,\\
(a_1, a_2, \ldots, a_m; q)_\infty &:= (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty,\\
[a; q]_n &:= (a; q)_n (q/a; q)_n, \quad n \geqslant 0, a \neq 0,\\
[a_1, a_2, \ldots, a_m; q]_n &:= [a_1; q]_n [a_2; q]_n \cdots [a_m; q]_n, \quad n \geqslant 0,\\
[a]_\infty &:= [a; q]_\infty := \lim_{n\to\infty} [a; q]_n,\\
[a_1, a_2, \ldots, a_m; q]_\infty &:= [a_1; q]_\infty [a_2; q]_\infty \cdots [a_m; q]_\infty.
\end{aligned}$$

Throughout this paper, $|q| < 1$.

To state the aforementioned four identities of Ramanujan, we need to define Ramanujan's theta function $\psi(q)$,

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = (-q; q)_\infty^2 (q; q)_\infty = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.2}$$

by the Jacobi triple product identity (given in its general form in (3.1) below) and Euler's theorem. Appearing in each of the four identities are instances of

$$f_a(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+aq+q^2)(1+aq^2+q^4)\cdots(1+aq^n+q^{2n})}, \tag{1.3}$$

where a is any real number. Observe that

$$f_a(q) = G\left(\frac{-a \pm \sqrt{a^2-4}}{2}, q\right),$$

where $G(z,q)$ is defined in (1.1). A focus in this paper is the special case when $a=\sqrt{2}$, for which we can write

$$f_{\sqrt{2}}(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(e^{3\pi i/4}q)_n(e^{5\pi i/4}q)_n} = G(e^{3\pi i/4}, q). \tag{1.4}$$

Furthermore, define

$$\tilde{\phi}(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(iq)_n(q/i)_n} = G(i,q), \tag{1.5}$$

which is featured in Ramanujan's fourth identity.

We are now ready to state the four identities of Ramanujan, which were first proved by Yesilyurt [29].

ENTRY 1.1. [26, p. 2] *Suppose that a and b are real numbers such that $a^2+b^2=4$. Recall that $f_a(q)$ is defined by (1.3). Then*

$$\begin{aligned} &\frac{b-a+2}{4} f_a(-q) + \frac{b+a+2}{4} f_{-a}(-q) - \frac{b}{2} f_b(q) \\ &= \frac{(q^4;q^4)_\infty}{(-q;q^2)_\infty} \prod_{n=1}^{\infty} \frac{1-bq^n+q^{2n}}{1+(a^2b^2-2)q^{4n}+q^{8n}}. \end{aligned} \tag{1.6}$$

ENTRY 1.2. [26, p. 2] *Let a and b be real numbers with $a^2+ab+b^2=3$. Then, with $f_a(q)$ defined by (1.3),*

$$\begin{aligned} &(a+1)f_{-a}(q) + (b+1)f_{-b}(q) - (a+b-1)f_{a+b}(q) \\ &= 3\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty} \prod_{n=1}^{\infty} \frac{1}{1+ab(a+b)q^{3n}+q^{6n}}. \end{aligned} \tag{1.7}$$

ENTRY 1.3. [26, p. 17] *Let $f_a(q)$ and $\psi(q)$ be defined by (1.3) and (1.2), respectively. Then*

$$\begin{aligned}
&\frac{1+\sqrt{3}}{2} f_{-1}(-q) + \frac{3+\sqrt{3}}{6} f_1(-q) \\
&\quad = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+\sqrt{3}q+q^2)\cdots(1+\sqrt{3}q^n+q^{2n})} \\
&\qquad + \frac{2}{\sqrt{3}}\psi(-q)\frac{(q^4;q^4)_\infty}{(q^6;q^6)_\infty}\prod_{n=1}^{\infty}\frac{1}{1+\sqrt{3}q^n+q^{2n}}. \qquad (1.8)
\end{aligned}$$

ENTRY 1.4. [26, p. 17] *Let $\tilde{\phi}(q)$ be defined by (1.5) and $\psi(q)$ be defined by (1.2). Then*

$$\begin{aligned}
&\frac{1}{2}(1+e^{\pi i/4})\tilde{\phi}(iq) + \frac{1}{2}(1+e^{-\pi i/4})\tilde{\phi}(-iq) \\
&\quad = f_{\sqrt{2}}(q) + \frac{1}{\sqrt{2}}\psi(-q)(-q^2;q^4)_\infty \prod_{n=1}^{\infty}\frac{1}{1+\sqrt{2}q^n+q^{2n}}. \qquad (1.9)
\end{aligned}$$

Yesilyurt's proofs [29] of Entries 1.1–1.4 depend upon the following famous lemma of Atkin and Swinnerton-Dyer [7].

LEMMA 1.5. *Let q, $|q|<1$, be fixed. Suppose that $\vartheta(z)$ is an analytic function of z, except for possibly a finite number of poles, in every annulus $0 < z_1 \leqslant |z| \leqslant z_2$. If*

$$\vartheta(zq) = Az^k\vartheta(z)$$

for some integer k (positive, negative, or 0) and some constant A, then either $\vartheta(z)$ has k more poles than zeros in the region $|q| < |z| \leqslant 1$, or $\vartheta(z)$ vanishes identically.

Since it is very unlikely that Ramanujan would have given proofs of Entries 1.1–1.4 using complex analysis, in particular, using Lemma 1.5, as stated earlier, the primary purpose of this paper is to give completely different proofs using q-series, perhaps more in line with what Ramanujan might have devised. However, although our proofs of Entries 1.1–1.3 are not difficult, our proof of Entry 1.4 is considerably more difficult. Our proof of Entry 1.4 relies on 2-dissections for two special cases of the rank generating function $G(z,q)$, when $z=i$ and when z is a primitive eighth root of unity. These two 2-dissections of the rank, with their immediate consequences, comprise a second major focus of this paper. Their proofs will be given in Sections 4–6.

THEOREM 1.6. *The 2-dissection of the rank function $G(i, q)$ is given by*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(iq)_n(q/i)_n} = \frac{2}{(q^{16};q^{16})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+8n}}{1+q^{16n+2}} - \frac{[q^4;q^{16}]_\infty^2 (q^{16};q^{16})_\infty}{[-q^2,-q^2,-q^6;q^{16}]_\infty}$$

$$+\frac{2}{(q^{16};q^{16})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+24n+5}}{1+q^{16n+6}} + q\frac{[q^4;q^{16}]_\infty^2 (q^{16};q^{16})_\infty}{[-q^2,-q^6,-q^6;q^{16}]_\infty}. \tag{1.10}$$

THEOREM 1.7. *Let a be a primitive eighth root of unity. Then*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(aq)_n(q/a)_n} = \frac{2-a-1/a}{(q^{16};q^{16})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+8n}}{1-q^{16n+2}}$$

$$+\frac{(a+1/a-1)[q^4;q^{16}]_\infty^2 (q^{16};q^{16})_\infty}{[q^2,q^2,q^6;q^{16}]_\infty}$$

$$+q\frac{[q^4;q^{16}]_\infty^2 (q^{16};q^{16})_\infty}{[q^2,q^6,q^6;q^{16}]_\infty} + \frac{a+1/a}{(q^{16};q^{16})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+24n+5}}{1-q^{16n+6}}. \tag{1.11}$$

Atkin and Swinnerton-Dyer [7] gave the generating functions for the rank differences modulo 5 and 7. In a sequel, Atkin and Hussain [6] established the rank differences modulo 11. From Garvan's paper [14], we know that these are equivalent to the 5, 7, and 11-dissections of the rank modulo 5, 7, and 11, respectively. Except for a 3-dissection of the rank modulo 3 by the third author and Mao [12], there have been no further results on dissections for the rank besides those obtainable from [7] and [6].

Subsequent to our proofs of Theorems 1.6 and 1.7, Mortenson [25] found shorter proofs based on 2-dissections for deviations of ranks, which we now define. Let

$$N(a,m;n) := \text{the number of partitions of } n \text{ with rank } \equiv a \pmod{m}.$$

Then the *deviation* of ranks from the expected value is defined by

$$D(a,m;q) := \sum_{n=0}^{\infty} \left(N(a,m;n) - \frac{p(n)}{m} \right) q^n, \quad |q|<1,$$

https://doi.org/10.1017/nmj.2018.35 Published online by Cambridge University Press

where $p(n)$ denotes the number of ordinary partitions of n. (It is understood that $p(0) = 0$.)

§2. Proofs of Entries 1.1–1.3

Our starting point is a corollary of Lemma 2.3.2 from [3, p. 19]. (It is to be assumed in the sequel that parameters, such as z and ζ below, are chosen so that all relevant expressions are well defined.)

THEOREM 2.1. *For any complex numbers z, ζ,*

$$\begin{aligned} R(z,\zeta,q) &:= \zeta^2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{3n} q^{n(3n+1)/2}}{1 - z\zeta q^n} + \zeta \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{-3n} q^{n(3n+1)/2}}{1 - zq^n/\zeta} \\ &\quad - \zeta \frac{(\zeta^2, q/\zeta^2; q)_\infty}{(\zeta, q/\zeta; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1 - zq^n} \\ &= \frac{z(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty}. \end{aligned} \tag{2.1}$$

Proof. Define

$$\begin{aligned} S(z,\zeta,q) &:= \zeta^3 \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{3n} q^{3n(n+1)/2}}{1 - z\zeta q^n} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{-3n} q^{3n(n+1)/2}}{1 - zq^n/\zeta} \\ &\quad - \zeta \frac{(\zeta^2, q/\zeta^2; q)_\infty}{(\zeta, q/\zeta; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 - zq^n}. \end{aligned}$$

Then from [3, p. 19, Lemma 2.3.2],

$$S(z,\zeta,q) = \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty}.$$

Hence, to conclude the proof of Theorem 2.1, we are required to show that

$$S(z,\zeta,q) = \frac{1}{z} R(z,\zeta,q).$$

Using the pentagonal number theorem in the second equality below, we find that

$$S(z,\zeta,q) = \zeta^3 \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{3n} q^{n(3n+1)/2}}{1 - z\zeta q^n} \left(\frac{1}{z\zeta} - \frac{1}{z\zeta}(1 - z\zeta q^n) \right)$$

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$$
\begin{aligned}
&+\sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{-3n} q^{n(3n+1)/2}}{1-zq^n/\zeta}\left(\frac{\zeta}{z}-\frac{\zeta}{z}\left(1-\frac{z}{\zeta}q^n\right)\right)\\
&-\frac{\zeta(\zeta^2, q/\zeta^2;q)_\infty}{(\zeta, q/\zeta;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n}\left(\frac{1}{z}-\frac{1}{z}(1-zq^n)\right)\\
=&\ \frac{1}{z}R(z,\zeta,q)-\frac{\zeta}{z}\Bigg(\zeta\sum_{n=-\infty}^{\infty}(-1)^n\zeta^{3n}q^{n(3n+1)/2}\\
&+\sum_{n=-\infty}^{\infty}(-1)^n\zeta^{-3n}q^{n(3n+1)/2}-\frac{(\zeta^2, q/\zeta^2;q)_\infty}{(\zeta, q/\zeta;q)_\infty}(q;q)_\infty\Bigg)\\
=&\ \frac{1}{z}R(z,\zeta,q),
\end{aligned}
$$

because the expression inside the large parentheses equals 0, as it is a formulation of the quintuple product identity [3, p. 221, equation (8.2.18)], [9, p. 18]. In particular, if we take the formulation from [9, p. 18]

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty} q^{3n^2+n}(z^{3n}q^{-3n}-z^{-3n-1}q^{3n+1})\\
&=(q^2;q^2)_\infty(qz;q^2)_\infty(q/z;q^2)_\infty(z^2;q^4)_\infty(q^4/z^2;q^4)_\infty,
\end{aligned}
\tag{2.2}
$$

replace q by $\sqrt{q}$, and then set $z=-\zeta\sqrt{q}$, we find that the sum of the expressions within large parentheses on the far right side above equals 0.

We frequently use the observation that if $a=-t-1/t$, then

$$
\sum_{n=0}^{\infty}\frac{q^{n^2}}{(tq)_n(q/t)_n}=\frac{1-t}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1-tq^n},
\tag{2.3}
$$

by [2, p. 263, equation (12.2.3)].

Proof of Entry 1.1. First we note that we may parameterize the circle $a^2+b^2=4$ by

$$
b=-2\cos\theta, \qquad a=-2\sin\theta,
$$

and with $z=e^{i\theta}$, we find that

$$
b=-z-z^{-1} \qquad \text{and} \qquad a=i(z-z^{-1}).
\tag{2.4}
$$

Hence,

$$
a^2b^2-2=-z^4-z^{-4}.
\tag{2.5}
$$

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Let us now set $\zeta = i\ (=\sqrt{-1})$ in Theorem 2.1. Thus, by (2.3), (2.4), and (2.5),

$$-\sum_{n=-\infty}^{\infty}\frac{(-1)^n i^{3n} q^{n(3n+1)/2}}{1-ziq^n}+i\sum_{n=-\infty}^{\infty}\frac{(-1)^n i^{-3n} q^{n(3n+1)/2}}{1+ziq^n}$$
$$-\frac{i(-1,-q;q)_\infty}{(i,-iq;q)_\infty}\frac{(q;q)_\infty}{(1-z)}f_b(q)$$
$$=\frac{z(i,-iq,-1,-q,q,q;q)_\infty}{(-iz,qi/z,z,q/z,iz,-iq/z;q)_\infty}$$
$$=\frac{2z(1-i)(-q;-q)_\infty(q^4;q^4)_\infty}{(1+z^2)(1-z)(-q;q^2)_\infty}\prod_{n=1}^{\infty}\frac{(1-bq^n+q^{2n})}{(1+(a^2b^2-2)q^{4n}+q^{8n})}. \tag{2.6}$$

Multiply both sides of (2.6) by

$$\frac{(1+z^2)(1-z)}{2z(1-i)(-q;-q)_\infty}.$$

Upon doing so, we then see that the right-hand side of (2.6) becomes the right-hand side of (1.6). The third expression on the left-hand side of (2.6) then becomes

$$\frac{(1+z^2)(1-z)(-i)(-1,-q;q)_\infty(q;q)_\infty}{2z(1-i)(-q;-q)_\infty(i,-iq;q)_\infty(1-z)}f_b(q)=-\frac{b}{2}f_b(q).$$

Therefore, we will complete the proof if we can show that

$$\frac{(1+z^2)(1-z)}{2z(1-i)(-q;-q)_\infty}\left(-\sum_{n=-\infty}^{\infty}\frac{(-1)^n i^{3n} q^{n(3n+1)/2}}{1-ziq^n}\right.$$
$$\left.+i\sum_{n=-\infty}^{\infty}\frac{(-1)^n i^{-3n} q^{n(3n+1)/2}}{1+ziq^n}\right)$$
$$=\frac{b-a+2}{4}f_a(-q)+\frac{b+a+2}{4}f_{-a}(-q)$$
$$=\frac{b-a+2}{4}\frac{(1+iz)}{(-q;-q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(-q)^{n(3n+1)/2}}{1+iz(-q)^n}$$
$$+\frac{b+a+2}{4}\frac{(1-iz)}{(-q;-q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(-q)^{n(3n+1)/2}}{1-iz(-q)^n}, \tag{2.7}$$

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where we have twice used (2.3). Proving (2.7) is equivalent to proving that

$$
\begin{aligned}
&-\sum_{n=-\infty}^{\infty}\frac{(-1)^n i^{3n} q^{n(3n+1)/2}}{1-ziq^n}+i\sum_{n=-\infty}^{\infty}\frac{(-1)^n i^{-3n} q^{n(3n+1)/2}}{1+ziq^n}\\
&=i\sum_{n=-\infty}^{\infty}\frac{(-1)^n(-q)^{n(3n+1)/2}}{1+iz(-q)^n}-\sum_{n=-\infty}^{\infty}\frac{(-1)^n(-q)^{n(3n+1)/2}}{1-iz(-q)^n}. \qquad (2.8)
\end{aligned}
$$

Combining sums on each side of (2.8), we see that our task has been reduced to proving that

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+z^2q^{2n}}\left(-i^{3n}(1+ziq^n)+i^{1-3n}(1-ziq^n)\right)\\
&=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+z^2q^{2n}}\Big(i(-1)^{n(3n+1)/2}(1-iz(-q)^n)\\
&\quad -(-1)^{n(3n+1)/2}(1+iz(-q)^n)\Big),
\end{aligned}
$$

and this follows immediately because

$$-i^{3n}+i^{1-3n}=(-1)^{n(3n+1)/2}(i-1)$$

and

$$-i^{1+3n}-i^{2-3n}=-(-1)^{n(3n+1)/2+n}(-1+i).$$

The last two assertions are most easily proved by noting that each expression is periodic with period 4, and that the assertions hold for $n=0,1,2,3$.

Proof of Entry 1.2. First we note that we may parameterize the ellipse $a^2+ab+b^2=3$ by $a=2\cos(\theta+\frac{2}{3}\pi)$, $b=2\cos\theta$. So with $z=e^{i\theta}$, we find that

$$b=z+z^{-1} \qquad \text{and} \qquad a=z\omega+(z\omega)^{-1},$$

where $\omega=e^{2\pi i/3}$. Hence,

$$a+b=-z\omega^2-(z\omega^2)^{-1} \qquad (2.9)$$

and

$$ab(a+b)=-z^3-z^{-3}.$$

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Therefore, we now set $\zeta=\omega$ in (2.1). Thus, the resulting right-hand side, by (2.9), equals

$$\frac{z(1-\omega)(1-\omega^2)(q^3;q^3)_\infty^2}{(1-z^3)(z^3q^3,z^{-3}q^3;q^3)_\infty} \tag{2.10}$$
$$=\frac{z(q;q)_\infty}{(1-z^3)}\frac{3(q^3;q^3)_\infty^2}{(q;q)_\infty\prod_{n=1}^\infty(1+ab(a+b)q^{3n}+q^{6n})}.$$

We now observe that the latter quotient on the right-hand side of (2.10) is the same as the right-hand side of (1.7). We are thus led to multiply the left-hand side of (2.1) with $\zeta=\omega$ by

$$\frac{(1-z^3)}{z(q;q)_\infty}$$

to deduce, with the help of three applications of (2.3), that

$$\frac{(1-z^3)}{z(q;q)_\infty}\left(\omega^2\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}}{1-z\omega q^n}+\omega\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}}{1-z\omega^2q^n}\right.$$
$$\left.-\frac{\omega(1-\omega^2)}{(1-\omega)}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}}{1-zq^n}\right)$$
$$=\frac{(1-z^3)\omega^2}{z(1-\omega z)}f_{-a}(q)+\frac{(1-z^3)\omega}{z(1-\omega^2z)}f_{a+b}(q)-\frac{(1-z^3)\omega(1-\omega^2)}{z(1-\omega)(1-z)}f_{-b}(q)$$
$$=(a+1)f_{-a}(q)-(a+b-1)f_{a+b}(q)+(b+1)f_{-b}(q),$$

which is the left-hand side of (1.7). This completes the proof.

Proof of Entry 1.3. Let $a=1$ and $b=\sqrt{3}$ in Entry 1.1 to deduce that

$$\frac{1+\sqrt{3}}{4}f_1(-q)+\frac{3+\sqrt{3}}{4}f_{-1}(-q)-\frac{\sqrt{3}}{2}f_{\sqrt{3}}(q) \tag{2.11}$$
$$=\frac{(q^4;q^4)_\infty}{(-q;q^2)_\infty}\prod_{n=1}^\infty\frac{1-\sqrt{3}q^n+q^{2n}}{1+q^{4n}+q^{8n}}.$$

Now multiply both sides of (2.11) by $2/\sqrt{3}$ to arrive at

$$\frac{3+\sqrt{3}}{6}f_1(-q)+\frac{1+\sqrt{3}}{2}f_{-1}(-q)-f_{\sqrt{3}}(q) \tag{2.12}$$
$$=\frac{2}{\sqrt{3}}\frac{(q^4;q^4)_\infty}{(-q;q^2)_\infty}\prod_{n=1}^\infty\frac{1-\sqrt{3}q^n+q^{2n}}{1+q^{4n}+q^{8n}}.$$

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Examining (1.8) and (2.12), we see that we are required to show that

$$\psi(-q) = \frac{(q^6;q^6)_\infty}{(-q;q^2)_\infty} \prod_{n=1}^{\infty} \frac{(1-\sqrt{3}q^n+q^{2n})(1+\sqrt{3}q^n+q^{2n})}{1+q^{4n}+q^{8n}}. \tag{2.13}$$

To that end,

$$\begin{aligned}
&\frac{(q^6;q^6)_\infty}{(-q;q^2)_\infty} \prod_{n=1}^{\infty} \frac{(1-\sqrt{3}q^n+q^{2n})(1+\sqrt{3}q^n+q^{2n})}{1+q^{4n}+q^{8n}} \\
&= \frac{(q^6;q^6)_\infty}{(-q;q^2)_\infty} \prod_{n=1}^{\infty} \frac{1-q^{2n}+q^{4n}}{1+q^{4n}+q^{8n}} \\
&= \frac{(q^6;q^6)_\infty}{(-q;q^2)_\infty} \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1+q^{2n}+q^{4n})(1-q^{2n})} \\
&= \frac{(q^6;q^6)_\infty}{(-q;q^2)_\infty} \frac{(q^2;q^2)_\infty}{(q^6;q^6)_\infty} = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \psi(-q),
\end{aligned}$$

by (1.2). Thus, we have shown that (2.13) holds, and so the proof of Entry 1.3 is finished.

§3. Proof of Entry 1.4; part 1

We show in this section that Entry 1.4 follows from the two 2-dissections for two special cases of the rank generating function $G(z, q)$ given in Theorems 1.6 and 1.7.

Proof of Entry 1.4. We need knowledge of theta functions. After Ramanujan, set, for $|ab| < 1$,

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{3.1}$$

where the latter equality is the Jacobi triple product identity [8, p. 35, Entry 19]. To simplify the product on the right side of (1.9), use (1.2) and (3.1) to deduce that

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\psi(-q)(-q^2;q^4)_\infty \prod_{n=1}^{\infty} \frac{1}{1+\sqrt{2}q^n+q^{2n}} \\
&= \frac{1}{\sqrt{2}} \frac{(q^4;q^8)_\infty^2}{(q^2;q^4)_\infty^2} (qe^{\pi i/4}, qe^{-\pi i/4}, q;q)_\infty \\
&= \frac{1}{\sqrt{2}} \frac{(q^4;q^8)_\infty^2}{(q^2;q^4)_\infty^2} \frac{f(-e^{\pi i/4}, -qe^{-\pi i/4})}{(1-e^{\pi i/4})}. \qquad (3.2)
\end{aligned}$$

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We need a special case of an identity of Ramanujan for theta functions [8, p. 48, Entry 31]. To that end, if $U_n := a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n := a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer n, then

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{3.3}$$

We apply (3.3) with $a = U_1 = -e^{\pi i/4}$, $b = V_1 = -qe^{-\pi i/4}$, and $n = 4$ to the theta function in the last equality of (3.2). Thus,

$$\begin{aligned}
&f(-e^{\pi i/4}, -qe^{-\pi i/4})\\
&\quad = f(-q^6, -q^{10}) - e^{\pi i/4} f(-q^{10}, -q^6) + iqf(-q^{14}, -q^2)\\
&\qquad + e^{-\pi i/4} q^3 f(-q^{18}, -q^{-2})\\
&\quad = (1 - e^{\pi i/4}) f(-q^6, -q^{10}) + (i - e^{-\pi i/4}) qf(-q^2, -q^{14}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{f(-e^{\pi i/4}, -qe^{-\pi i/4})}{(1 - e^{\pi i/4})}\\
&\quad = f(-q^6, -q^{10}) - (\sqrt{2} + 1) f(-q^{14}, -q^2)\\
&\quad = (q^6; q^{16})_\infty (q^{10}; q^{16})_\infty (q^{16}; q^{16})_\infty\\
&\qquad - (\sqrt{2} + 1) q(q^2; q^{16})_\infty (q^{14}; q^{16})_\infty (q^{16}; q^{16})_\infty\\
&\quad = [q^6; q^{16}]_\infty (q^{16}; q^{16})_\infty - (\sqrt{2} + 1) q[q^2; q^{16}]_\infty (q^{16}; q^{16})_\infty, \qquad (3.4)
\end{aligned}$$

where we made two applications of (3.1). Hence, inserting (3.4) into (3.2), we deduce that

$$\begin{aligned}
&\frac{1}{\sqrt{2}} \psi(-q)(-q^2; q^4)_\infty \prod_{n=1}^{\infty} \frac{1}{1 + \sqrt{2}q^n + q^{2n}}\\
&\quad = \frac{1}{\sqrt{2}} \frac{(q^4; q^8)_\infty^2}{(q^2; q^4)_\infty^2} \left([q^6; q^{16}]_\infty (q^{16}; q^{16})_\infty - (1 + \sqrt{2}) q[q^2; q^{16}]_\infty (q^{16}; q^{16})_\infty\right)\\
&\quad = \frac{1}{\sqrt{2}} \frac{[q^4; q^{16}]_\infty^2 (q^{16}; q^{16})_\infty}{[q^2, q^2, q^6; q^{16}]_\infty} - \left(1 + \frac{1}{\sqrt{2}}\right) q \frac{[q^4; q^{16}]_\infty^2 (q^{16}; q^{16})_\infty}{[q^2, q^6, q^6; q^{16}]_\infty}.
\end{aligned}$$

Therefore, identity (1.9) is equivalent to

$$f_{\sqrt{2}}(q) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)(\tilde{\phi}(iq) + \tilde{\phi}(-iq)) - \frac{1}{\sqrt{2}} \frac{[q^4; q^{16}]_\infty^2 (q^{16}; q^{16})_\infty}{[q^2, q^2, q^6; q^{16}]_\infty}$$

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$$+\frac{1}{2\sqrt{2}}(i\tilde{\phi}(iq)-i\tilde{\phi}(-iq))+\left(1+\frac{1}{\sqrt{2}}\right)q\frac{[q^4;q^{16}]_\infty^2(q^{16};q^{16})_\infty}{[q^2,q^6,q^6;q^{16}]_\infty}. \tag{3.5}$$

We now apply Theorem 1.6 twice, with q replaced by iq and $-iq$, obtaining, by (1.5), $\tilde{\phi}(iq)$ and $\tilde{\phi}(-iq)$, respectively. We next apply Theorem 1.7 with $a=e^{3\pi i/4}$ thereby obtaining, by (1.4), $f_{\sqrt{2}}(q)$. If we substitute these three representations into (3.5), we see indeed that (3.5) is valid. It therefore remains to prove Theorems 1.6 and 1.7, which we do in the following sections.

§4. Proof of Entry 1.4; part 2, identities for theta functions and Lambert series

We offer the following lemmas that are needed to simplify the dissections in the proofs of Theorems 1.6 and 1.7.

LEMMA 4.1. *We have*

$$\frac{1}{(q)_\infty}=\frac{(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2}\left([-q^6;q^{16}]_\infty+q[-q^2;q^{16}]_\infty\right). \tag{4.1}$$

Proof. By (1.2) and (3.3) with $a=q$, $b=q^3$, and $n=2$,

$$\begin{aligned}\frac{1}{(q)_\infty}&=\frac{1}{(q^2;q^2)_\infty^2}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=\frac{1}{(q^2;q^2)_\infty^2}\sum_{n=-\infty}^{\infty}q^{n(2n+1)}\\&=\frac{(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2}\left([-q^6;q^{16}]_\infty+q[-q^2;q^{16}]_\infty\right),\end{aligned}$$

and so the proof is complete.

Next, we state Halphen's identity [16, p. 187] in the form discovered and presented in [5],

$$\begin{aligned}H(a,b,c,q):&=\frac{[ab,bc,ca]_\infty(q)_\infty^2}{[a,b,c,abc]_\infty}\\&=1+F(a,q)+F(b,q)+F(c,q)-F(abc,q),\end{aligned} \tag{4.2}$$

where

$$F(x):=F(x,q):=\sum_{k=0}^{\infty}\frac{xq^k}{1-xq^k}-\sum_{k=1}^{\infty}\frac{q^k/x}{1-q^k/x},\quad |q|<1. \tag{4.3}$$

Below are three identities that are consequences of Halphen's identity [11, Corollary 4.4].

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COROLLARY 4.2. *We have*

$$H(a,b,c,q^2) - H(a,b,d,q^2) = H(c,1/d,abd,q^2), \tag{4.4}$$

$$H(a,a,q/a,q^2) + H(b,b,q/b,q^2) = 2H(a,q/a,b,q^2), \tag{4.5}$$

$$H(a,a,q/a,q^2) - H(b,b,q/b,q^2) = 2H(a,q/a,b/q,q^2). \tag{4.6}$$

Setting $r=0$ and $s=3$ in [10, Theorem 2.1] and then replacing q by q^{16}, we derive the generalized Lambert series identity

$$\begin{aligned}\frac{(q^{16};q^{16})_\infty^2}{[b,c,d;q^{16}]_\infty} &= \frac{1}{[c/b,d/b;q^{16}]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{24n(n+1)}}{1-bq^{16n}}\left(\frac{b^2}{cd}\right)^n \\ &\quad + \frac{1}{[b/c,d/c;q^{16}]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{24n(n+1)}}{1-cq^{16n}}\left(\frac{c^2}{bd}\right)^n \\ &\quad + \frac{1}{[b/d,c/d;q^{16}]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{24n(n+1)}}{1-dq^{16n}}\left(\frac{d^2}{bc}\right)^n.\end{aligned}$$

Multiplying both sides by $[c/b,d/b;q^{16}]_\infty$ and rearranging, we deduce the following lemma.

LEMMA 4.3. *We have*

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{24n(n+1)}}{1-bq^{16n}}\left(\frac{b^2}{cd}\right)^n - \frac{c}{b}\frac{[d/b;q^{16}]_\infty}{[d/c;q^{16}]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{24n(n+1)}}{1-cq^{16n}}\left(\frac{c^2}{bd}\right)^n \\ &\quad = \frac{(q^{16};q^{16})_\infty^2[c/b,d/b;q^{16}]_\infty}{[b,c,d;q^{16}]_\infty} \\ &\qquad + \frac{d}{b}\frac{[c/b;q^{16}]_\infty}{[c/d;q^{16}]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{24n(n+1)}}{1-dq^{16n}}\left(\frac{d^2}{bc}\right)^n.\end{aligned} \tag{4.7}$$

Let

$$\begin{aligned}S(z,\zeta) &:= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{24n(n+1)}\zeta^n}{1-zq^{16n}}, \\ S^*(\zeta) &:= \sum_{n=-\infty}^{\infty}{}'\frac{(-1)^n q^{24n(n+1)}\zeta^n}{1-q^{16n}},\end{aligned} \tag{4.8}$$

where the prime in the second sum denotes the omission of the term $n=0$. Then by substituting $(b,c,d)=(q^8,q^{12},-q^2)$ and $(q^8,q^{12},-q^{-6})$ in

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Lemma 4.3, we find that, respectively,

$$q^6 S(q^8,-q^2) - q^{14} S(q^{12},-q^{14})$$
$$= \frac{(q^{16};q^{16})_\infty^2[-q^6;q^{16}]_\infty}{[-q^2,q^8;q^{16}]_\infty} - \frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty} S(-q^2,q^{-16}), \tag{4.9}$$
$$q^{12} S(q^8,-q^{10}) - q^{22} S(q^{12},-q^{22})$$
$$= q^4 \frac{(q^{16};q^{16})_\infty^2[-q^2;q^{16}]_\infty}{[-q^6,q^8;q^{16}]_\infty} - \frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2-8n}}{1+q^{16n-6}}$$
$$= q^4 \frac{(q^{16};q^{16})_\infty^2[-q^2;q^{16}]_\infty}{[-q^6,q^8;q^{16}]_\infty} + q^{16} \frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} S(-q^{10},q^{16}), \tag{4.10}$$

where we replaced n by $n+1$ in the previous line. Similarly, substituting $(b,c,d) = (-1,-q^4,q^{18}), (-q^8,-q^{12},q^2), (-1,-q^{12},q^6)$, and $(-q^4,-q^8,q^6)$ in Lemma 4.3, we obtain, respectively,

$$S(-1,-q^{-22}) - q^2 S(-q^4,-q^{-10})$$
$$= -\frac{(q^{16};q^{16})_\infty^2[-q^2,q^4;q^{16}]_\infty}{[-1,q^2,-q^4;q^{16}]_\infty} - q^{32} \frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} S(q^{18},q^{32})$$
$$= -\frac{(q^{16};q^{16})_\infty^2[-q^2,q^4;q^{16}]_\infty}{[-1,q^2,-q^4;q^{16}]_\infty} + \frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} S(q^2,q^{-16}), \tag{4.11}$$
$$q^6 S(-q^8,-q^2) - q^{14} S(-q^{12},-q^{14})$$
$$= \frac{(q^{16};q^{16})_\infty^2[q^4,-q^6;q^{16}]_\infty}{[q^2,-q^8,-q^{12};q^{16}]_\infty} - \frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty} S(q^2,q^{-16}), \tag{4.12}$$
$$S(-1,-q^{-18}) - q^{18} S(-q^{12},-q^{18})$$
$$= \frac{(q^{16};q^{16})_\infty^2[q^4,-q^6;q^{16}]_\infty}{[-1,-q^4,q^6;q^{16}]_\infty} - q^6 \frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty} S(q^6,1), \tag{4.13}$$
$$S(-q^4,-q^{-6}) - q^6 S(-q^8,-q^6)$$
$$= \frac{(q^{16};q^{16})_\infty^2[-q^2,q^4;q^{16}]_\infty}{[-q^4,q^6,-q^8;q^{16}]_\infty} - q^2 \frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} S(q^6,1). \tag{4.14}$$

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LEMMA 4.4. *We have*

$$\sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{24n(n+1)}}{1-q^{16n}} \left(\frac{1}{cd}\right)^n - c\frac{[d;q^{16}]_\infty}{[d/c;q^{16}]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n(n+1)}}{1-cq^{16n}} \left(\frac{c^2}{d}\right)^n$$
$$= \sum_{n=0}^{\infty}\left(-\frac{cq^{16n}}{1-cq^{16n}} + \frac{q^{16n+16}/c}{1-q^{16n+16}/c} - \frac{dq^{16n}}{1-dq^{16n}} + \frac{q^{16n+16}/d}{1-q^{16n+16}/d}\right)$$
$$+ d\frac{[c;q^{16}]_\infty}{[c/d;q^{16}]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n(n+1)}}{1-dq^{16n}} \left(\frac{d^2}{c}\right)^n,$$

where the prime on the sum on the left-hand side denotes the omission of the term $n=0$.

Proof. Multiply both sides of (4.7) by $(b-1)$, differentiate with respect to b, and let $b \to 1$. We find that the only terms that remain are those in Lemma 4.4.

LEMMA 4.5. *Recall that $S(z,\zeta)$ and $S^*(\zeta)$ are defined in (4.8). Then*

$$\frac{1}{2} + S^*(-q^{-22}) - S^*(-q^{-14}) - q^2 S(q^4,-q^{-10}) + q^4 S(q^4,-q^{-2})$$
$$= \frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} S(-q^2,q^{-16}) + q^{16}\frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty} S(-q^{10},q^{16})$$
$$- \frac{1}{2}\frac{[q^4,q^8,q^8;q^{16}]_\infty (q^{16};q^{16})_\infty^2}{[-q^2,-q^2,-q^6,-q^6;q^{16}]_\infty}. \tag{4.15}$$

Proof. Substituting $c=q^4$ and $d=-q^{18}$ in Lemma 4.4, we find that

$$S^*(-q^{-22}) - q^2 S(q^4,-q^{-10})$$
$$= \sum_{n=0}^{\infty}\left(-\frac{q^{16n+4}}{1-q^{16n+4}} + \frac{q^{16n+12}}{1-q^{16n+12}} + \frac{q^{16n+18}}{1+q^{16n+18}} - \frac{q^{16n-2}}{1+q^{16n-2}}\right)$$
$$- q^{32}\frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+56n}}{1+q^{16n+18}}$$
$$= -1 + \sum_{n=0}^{\infty}\left(-\frac{q^{16n+4}}{1-q^{16n+4}} + \frac{q^{16n+12}}{1-q^{16n+12}} + \frac{q^{16n+2}}{1+q^{16n+2}} - \frac{q^{16n+14}}{1+q^{16n+14}}\right)$$
$$+ \frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} S(-q^2,q^{-16}). \tag{4.16}$$

Similarly, substituting $c=q^4$ and $d=-q^{10}$ in Lemma 4.4 gives

$$\begin{aligned} &S^*(-q^{-14}) - q^4 S(q^4, -q^{-2}) \\ &\quad = \sum_{n=0}^{\infty}\left(-\frac{q^{16n+4}}{1-q^{16n+4}} + \frac{q^{16n+12}}{1-q^{16n+12}} + \frac{q^{16n+10}}{1+q^{16n+10}} - \frac{q^{16n+6}}{1+q^{16n+6}}\right) \\ &\qquad - q^{16}\frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty} S(-q^{10}, q^{16}). \end{aligned} \tag{4.17}$$

Taking the difference between (4.16) and (4.17), we obtain

$$\begin{aligned} &S^*(-q^{-22}) - S^*(-q^{-14}) - q^2 S(q^4, -q^{-10}) + q^4 S(q^4, -q^{-2}) \\ &\quad = -1 + \sum_{n=0}^{\infty}\left(\frac{q^{16n+2}}{1+q^{16n+2}} + \frac{q^{16n+6}}{1+q^{16n+6}} - \frac{q^{16n+10}}{1+q^{16n+10}} - \frac{q^{16n+14}}{1+q^{16n+14}}\right) \\ &\qquad + \frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} S(-q^2, q^{-16}) + q^{16}\frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty} S(-q^{10}, q^{16}). \end{aligned} \tag{4.18}$$

Note that by Halphen's identity (4.2) with q replaced by q^{16} and $(a,b,c)=(-q^2,-q^2,-q^6)$,

$$\begin{aligned} &-\frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{q^{16n+2}}{1+q^{16n+2}} + \frac{q^{16n+6}}{1+q^{16n+6}} - \frac{q^{16n+10}}{1+q^{16n+10}} - \frac{q^{16n+14}}{1+q^{16n+14}}\right) \\ &\quad = -\frac{1}{2}\frac{[q^4,q^8,q^8;q^{16}]_\infty (q^{16};q^{16})_\infty^2}{[-q^2,-q^2,-q^6,-q^6;q^{16}]_\infty}. \end{aligned}$$

Therefore, (4.18) is equivalent to

$$\begin{aligned} &\frac{1}{2} + S^*(-q^{-22}) - S^*(-q^{-14}) - q^2 S(q^4, -q^{-10}) + q^4 S(q^4, -q^{-2}) \\ &\quad = \frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty} S(-q^2, q^{-16}) + q^{16}\frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty} S(-q^{10}, q^{16}) \\ &\qquad - \frac{1}{2}\frac{[q^4,q^8,q^8;q^{16}]_\infty (q^{16};q^{16})_\infty^2}{[-q^2,-q^2,-q^6,-q^6;q^{16}]_\infty}, \end{aligned}$$

which is identical with (4.15).

§5. Proof of Entry 1.4; part 3, proof of Theorem 1.6

Proof of Theorem 1.6. Invoking the partial fraction identity (2.3) with $t=i$, we find that the left side of (1.10) is

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$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(iq)_n(q/i)_n} = \frac{1-i}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-iq^n}$$
$$= \frac{1-i}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+iq^n)}{1+q^{2n}}$$
$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}[(1+q^n)-i(1-q^n)]}{1+q^{2n}}. \tag{5.1}$$

Replacing n by $-n$ and multiplying both the numerator and denominator by q^{2n}, we see that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+3)/2}}{1+q^{2n}}.$$

Therefore, the identity (5.1) simplifies to

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(iq)_n(q/i)_n} = \frac{2}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}}. \tag{5.2}$$

We focus our attention on the series on the right side above. By subdividing the index of summation into residue classes modulo 4 and using the definitions (4.8), we find that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}} = \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+2n}}{1+q^{8n}} - \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+14n+2}}{1+q^{8n+2}}$$
$$+ \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+26n+7}}{1+q^{8n+4}} - \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+38n+15}}{1+q^{8n+6}}$$
$$= \frac{1}{2} + \sum_{n=-\infty}^{\infty}{}' \frac{q^{24n^2+2n}(1-q^{8n})}{1-q^{16n}} - \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+14n+2}(1-q^{8n+2})}{1-q^{16n+4}}$$
$$+ \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+26n+7}(1-q^{8n+4})}{1-q^{16n+8}} - \sum_{n=-\infty}^{\infty} \frac{q^{24n^2+38n+15}(1-q^{8n+6})}{1-q^{16n+12}}$$
$$= \frac{1}{2} + S^*(-q^{-22}) - S^*(-q^{-14}) - q^2 S(q^4, -q^{-10}) + q^4 S(q^4, -q^{-2})$$
$$+ q^7 S(q^8, -q^2) - q^{11} S(q^8, -q^{10}) - q^{15} S(q^{12}, -q^{14})$$
$$+ q^{21} S(q^{12}, -q^{22}). \tag{5.3}$$

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Therefore, by (5.2), (4.1), (5.3), (4.9), (4.10), and (4.15),

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(iq)_n(q/i)_n}&=\left\{\frac{(-q^6,-q^{10},q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2}+q\frac{(-q^2,-q^{14},q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2}\right\}\\&\times\left\{2\frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty}S(-q^2,q^{-16})+2q^{16}\frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty}S(-q^{10},q^{16})\right.\\&-\frac{[q^4,q^8,q^8;q^{16}]_\infty(q^{16};q^{16})_\infty^2}{[-q^2,-q^2,-q^6,-q^6;q^{16}]_\infty}+2q\frac{(q^{16};q^{16})_\infty^2[-q^6;q^{16}]_\infty}{[-q^2,q^8;q^{16}]_\infty}\\&-2q\frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty}S(-q^2,q^{-16})-2q^3\frac{(q^{16};q^{16})_\infty^2[-q^2;q^{16}]_\infty}{[-q^6,q^8;q^{16}]_\infty}\\&\left.-2q^{15}\frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty}S(-q^{10},q^{16})\right\}.\end{aligned}\tag{5.4}$$

We now prove two identities that we use in our collection of the even powers from (5.4). First, we prove the identity

$$[-q^6;q^{16}]_\infty^2-q^2[-q^2;q^{16}]_\infty^2=[q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty.\tag{5.5}$$

Replacing q^2 by $-q$ and noting that $[-q,-q^3;q^8]_\infty=[q^2;q^8]_\infty/[q,q^3;q^8]_\infty$, we see that (5.5) is equivalent to

$$[q^3;q^8]_\infty^2+q[q;q^8]_\infty^2=\frac{[q^2,q^2,q^2,q^4;q^8]_\infty}{[q,q^3;q^8]_\infty}.\tag{5.6}$$

Dividing (5.6) throughout by $[q,q,q^4;q^8]_\infty$ and rearranging, we see that (5.5) is in turn equivalent to

$$\frac{[q^3,q^3,q^2;q^8]_\infty}{[q,q,q^2,q^4;q^8]_\infty}-\frac{[q^2,q^2,q^2;q^8]_\infty}{[q,q,q,q^3;q^8]_\infty}=-q\frac{1}{[q^4;q^8]_\infty}=\frac{[q,q^2,q^5;q^8]_\infty}{[q^2,1/q,q^3,q^4;q^8]_\infty},$$

and this follows from (4.4) with q replaced by q^4 and $(a,b,c,d)=(q,q,q^2,q)$.

Second, we establish the identity

$$\begin{aligned}&[q^8,q^8;q^{16}]_\infty-2q^2[-q^2,-q^2,q^2,q^4,q^6;q^{16}]_\infty\\&=[q^2,q^2,q^4,q^4,q^4,q^6,q^6,q^8;q^{16}]_\infty.\end{aligned}\tag{5.7}$$

Replacing q^2 by $-q$, we find that (5.7) is equivalent to

$$\begin{aligned}&[q^4,q^4;q^8]_\infty+2q[q,q,-q,q^2,-q^3;q^8]_\infty\\&=[-q,-q,q^2,q^2,q^2,-q^3,-q^3,q^4;q^8]_\infty,\end{aligned}$$

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which in turn is equivalent to

$$[q^4, q^4; q^8]_\infty + 2q\frac{[q, q^2, q^2; q^8]_\infty}{[q^3; q^8]_\infty} = \frac{[q^2, q^2, q^2, q^2, q^2, q^4; q^8]_\infty}{[q, q, q^3, q^3; q^8]}. \tag{5.8}$$

We thus want to prove (5.8). We apply (4.6) with q replaced by q^4 and with $(a, b) = (q, q^2)$ to obtain the identity

$$\begin{aligned}
&\frac{[q^2, q^4, q^4; q^8]_\infty (q^8; q^8)_\infty^2}{[q, q, q^3, q^3; q^8]_\infty} - \frac{[q^4, q^4, q^4; q^8]_\infty (q^8; q^8)_\infty^2}{[q^2, q^2, q^2, q^2; q^8]_\infty}\\
&\quad = 2\frac{[q^{-1}, q, q^4; q^8]_\infty (q^8; q^8)_\infty^2}{[q, q^3, q^{-2}, q^2; q^8]_\infty}\\
&\quad = 2q\frac{[q, q^4; q^8]_\infty (q^8; q^8)_\infty^2}{[q^2, q^2, q^3; q^8]_\infty}. \qquad (5.9)
\end{aligned}$$

Multiplying both sides by $[q^2; q^8]_\infty^4$ and then dividing both sides by $[q^4; q^8]_\infty$ $(q^8; q^8)_\infty^2$, we see that (5.9) is (5.8) upon rearrangement.

Invoking (5.5) in the second equality below and (5.7) in the last equality below, we find that the even powers from (5.4) are equal to

$$\begin{aligned}
&\frac{[-q^6; q^{16}]_\infty (q^{16}; q^{16})_\infty}{(q^2; q^2)_\infty^2} \times \Bigg\{ 2\frac{[q^4; q^{16}]_\infty}{[-q^2; q^{16}]_\infty} S(-q^2, q^{-16})\\
&\qquad + 2q^{16}\frac{[q^4; q^{16}]_\infty}{[-q^6; q^{16}]_\infty} S(-q^{10}, q^{16}) - \frac{[q^4, q^8, q^8; q^{16}]_\infty (q^{16}; q^{16})_\infty^2}{[-q^2, -q^2, -q^6, -q^6; q^{16}]_\infty} \Bigg\}\\
&\qquad + q\frac{[-q^2; q^{16}]_\infty (q^{16}; q^{16})_\infty}{(q^2; q^2)_\infty^2} \times \Bigg\{ 2q\frac{(q^{16}; q^{16})_\infty^2 [-q^6; q^{16}]_\infty}{[-q^2, q^8; q^{16}]_\infty}\\
&\qquad - 2q\frac{[q^4; q^{16}]_\infty}{[-q^6; q^{16}]_\infty} S(-q^2, q^{-16}) - 2q^3\frac{(q^{16}; q^{16})_\infty^2 [-q^2; q^{16}]_\infty}{[-q^6, q^8; q^{16}]_\infty}\\
&\qquad - 2q^{15}\frac{[q^4; q^{16}]_\infty}{[-q^2; q^{16}]_\infty} S(-q^{10}, q^{16}) \Bigg\}\\
&\quad = \frac{2(q^{16}; q^{16})_\infty}{(q^2; q^2)_\infty (q^4; q^4)_\infty} S(-q^2, q^{-16}) \left([-q^6; q^{16}]_\infty^2 - q^2[-q^2; q^{16}]_\infty^2\right)\\
&\qquad - \frac{[q^4, q^8, q^8; q^{16}]_\infty (q^{16}; q^{16})_\infty^3}{(q^2; q^2)_\infty^2 [-q^2, -q^2, -q^6; q^{16}]_\infty}\\
&\qquad + 2q^2\frac{(q^{16}; q^{16})_\infty^3}{(q^2; q^2)_\infty^2 [-q^6, q^8; q^{16}]_\infty} \left([-q^6; q^{16}]_\infty^2 - q^2[-q^2; q^{16}]_\infty^2\right)
\end{aligned}$$

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$$\begin{aligned}
&= \frac{2(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^4;q^4)_\infty} S(-q^2,q^{-16})[q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty \\
&\quad - \frac{[q^4,q^8,q^8;q^{16}]_\infty(q^{16};q^{16})_\infty^3}{(q^2;q^2)_\infty^2[-q^2,-q^2,-q^6;q^{16}]_\infty} \\
&\quad + 2q^2\frac{(q^{16};q^{16})_\infty^3}{(q^2;q^2)_\infty^2[-q^6,q^8;q^{16}]_\infty}[q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty \\
&= \frac{2(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^4;q^4)_\infty} S(-q^2,q^{-16})[q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty \\
&\quad - \frac{[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty^3}{(q^2;q^2)_\infty^2[-q^2,-q^2,-q^6;q^{16}]_\infty} \\
&\quad \times \left([q^8,q^8;q^{16}]_\infty - 2q^2[-q^2,-q^2,q^2,q^4,q^6;q^{16}]_\infty\right) \\
&= \frac{2}{(q^{16};q^{16})_\infty}S(-q^2,q^{-16}) - \frac{[q^4;q^{16}]_\infty^2(q^{16};q^{16})_\infty}{[-q^2,-q^2,-q^6;q^{16}]_\infty}. \qquad (5.10)
\end{aligned}$$

We now collect the odd powers of q from (5.4). In the analysis below, we need to use the identity

$$\begin{aligned}
&2[q^2,q^4,q^6,-q^6,-q^6;q^{16}]_\infty - [q^8,q^8;q^{16}]_\infty \\
&\quad = [q^2,q^2,q^4,q^4,q^4,q^6,q^6,q^8;q^{16}]_\infty, \qquad (5.11)
\end{aligned}$$

which we now prove. Replacing q^2 by $-q$ and rearranging, we see that (5.11) is equivalent to

$$[q^4;q^8]_\infty^2 + \frac{[q^2;q^8]_\infty^5[q^4;q^8]_\infty}{[q,q^3;q^8]_\infty^2} = 2\frac{[q^2,q^2,q^3;q^8]_\infty}{[q;q^8]_\infty}. \qquad (5.12)$$

Multiplying both sides of (5.12) by $[q^4;q^8]_\infty(q^8;q^8)_\infty^2$ and dividing both sides by $[q^2;q^8]_\infty^4$, we obtain

$$\frac{[q^4;q^8]_\infty^3(q^8;q^8)_\infty^2}{[q^2;q^8]_\infty^4} + \frac{[q^2,q^4,q^4;q^8]_\infty(q^8;q^8)_\infty^2}{[q,q^3;q^8]_\infty^2} = 2\frac{[q^3,q^4;q^8]_\infty(q^8;q^8)_\infty^2}{[q,q^2,q^2;q^8]_\infty}. \qquad (5.13)$$

If in (4.5), we replace q by q^4 and set $(a,b)=(q^2,q)$, we arrive at (5.13). Hence, (5.11) has been established.

In the first equality below, we employ (5.5) twice, and in the penultimate equality below, we utilize (5.11). Thus, collecting the odd powers of q from (5.4), we find that

$$
\begin{aligned}
&-2q^{15}\frac{[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2[-q^2,-q^6;q^{16}]_\infty}S(-q^{10},q^{16})([-q^6;q^{16}]_\infty^2-q^2[-q^2;q^{16}]_\infty^2)\\
&\quad+2q\frac{(q^{16};q^{16})_\infty^3}{(q^2;q^2)_\infty^2[-q^2,q^8;q^{16}]_\infty}([-q^6;q^{16}]_\infty^2-q^2[-q^2;q^{16}]_\infty^2)\\
&\quad-q\frac{[q^4,q^8,q^8;q^{16}]_\infty(q^{16};q^{16})_\infty^3}{(q^2;q^2)_\infty^2[-q^2,-q^6,-q^6;q^{16}]_\infty}\\
&=-q^{15}\frac{2}{(q^{16};q^{16})_\infty}S(-q^{10},q^{16})\\
&\quad+2q\frac{(q^{16};q^{16})_\infty^3}{(q^2;q^2)_\infty^2[-q^2,q^8;q^{16}]_\infty}[q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty\\
&\quad-q\frac{[q^4,q^8,q^8;q^{16}]_\infty(q^{16};q^{16})_\infty^3}{(q^2;q^2)_\infty^2[-q^2,-q^6,-q^6;q^{16}]_\infty}\\
&=-q^{15}\frac{2}{(q^{16};q^{16})_\infty}S(-q^{10},q^{16})+q\frac{[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty^3}{(q^2;q^2)_\infty^2[-q^2,-q^6,-q^6;q^{16}]_\infty}\\
&\quad\times(2[q^2,q^4,q^6,-q^6,-q^6;q^{16}]_\infty-[q^8,q^8;q^{16}]_\infty)\\
&=-q^{15}\frac{2}{(q^{16};q^{16})_\infty}S(-q^{10},q^{16})\\
&\quad+q\frac{[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty^3}{(q^2;q^2)_\infty^2[-q^2,-q^6,-q^6;q^{16}]_\infty}[q^2,q^2,q^4,q^4,q^4,q^6,q^6,q^8;q^{16}]_\infty\\
&=-q^{15}\frac{2}{(q^{16};q^{16})_\infty}S(-q^{10},q^{16})+q\frac{[q^4;q^{16}]_\infty^2(q^{16};q^{16})_\infty}{[-q^2,-q^6,-q^6;q^{16}]_\infty}\\
&=q^5\frac{2}{(q^{16};q^{16})_\infty}S(-q^6,1)+q\frac{[q^4;q^{16}]_\infty^2(q^{16};q^{16})_\infty}{[-q^2,-q^6,-q^6;q^{16}]_\infty}.
\end{aligned}
\tag{5.14}
$$

We now return to (5.4). On the right side of (5.4), we substitute the expressions that we found for the even powers in (5.10) and the representation for the odd powers from (5.14). We immediately obtain the proposed identity (1.10), thus completing the proof of Theorem 1.6.

§6. Proof of Entry 1.4; part 4, proof of Theorem 1.7

Proof of Theorem 1.7. Let a be a primitive eighth root of unity. Then by the partial fraction identity (2.3),

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$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(aq)_n(q/a)_n} = \frac{1-a}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-aq^n}$$
$$= \frac{1-a}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+aq^n+a^2q^{2n}+a^3q^{3n})}{1+q^{4n}}$$
$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{4n}} \big[(1-a^4q^{3n}) - a(1-a^2q^{3n})$$
$$+ aq^n(1-a^2q^n) - a^2q^n(1-q^n)\big]. \tag{6.1}$$

Replacing n by $-n$ and multiplying both the numerator and denominator of the summands by q^{4n}, we see that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2} q^{kn}}{1+q^{4n}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2} q^{(3-k)n}}{1+q^{4n}}.$$

Therefore, we deduce that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1-a^4q^{3n})}{1+q^{4n}} = 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{4n}}, \tag{6.2}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1-a^2q^{3n})}{1+q^{4n}} = (1-a^2)\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{4n}}, \tag{6.3}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2} q^n(1-a^2q^n)}{1+q^{4n}} = (1-a^2)\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+3)/2}}{1+q^{4n}}, \tag{6.4}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2} q^n(1-q^n)}{1+q^{4n}} = 0. \tag{6.5}$$

Thus, noting that $-a^3 = 1/a$ and putting (6.2)–(6.5) in (6.1), we find that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(aq)_n(q/a)_n} = \frac{2-a-1/a}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{4n}}$$
$$+ \frac{a+1/a}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+3)/2}}{1+q^{4n}}. \tag{6.6}$$

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We now proceed as we did in the proof of Theorem 1.6. We first work out the dissections of the relevant Lambert series. For each of the two Lambert series below, we divide the index of summation n into residue classes modulo 4 and express each sum in terms of $S(z,\zeta)$, defined in (4.8). By applying (4.11) and (4.12) in the first case and (4.13) and (4.14) in the second, we find that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{4n}} &= S(-1,-q^{-22})-q^2S(-q^4,-q^{-10})\\ &\quad+q^7S(-q^8,-q^2)-q^{15}S(-q^{12},-q^{14})\\ &= -\frac{(q^{16};q^{16})_\infty^2[-q^2,q^4;q^{16}]_\infty}{[-1,q^2,-q^4;q^{16}]_\infty}+\frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty}S(q^2,q^{-16})\\ &\quad+q\frac{(q^{16};q^{16})_\infty^2[q^4,-q^6;q^{16}]_\infty}{[q^2,-q^4,-q^8;q^{16}]_\infty}-q\frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty}S(q^2,q^{-16})\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(3n^2+3n)/2}}{1+q^{4n}} &= S(-1,-q^{-18})-q^3S(-q^4,-q^{-6})\\ &\quad+q^9S(-q^8,-q^6)-q^{18}S(-q^{12},-q^{18})\\ &= \frac{(q^{16};q^{16})_\infty^2[q^4,-q^6;q^{16}]_\infty}{[-1,-q^4,q^6;q^{16}]_\infty}-q^6\frac{[q^4;q^{16}]_\infty}{[-q^6;q^{16}]_\infty}S(q^6,1)\\ &\quad-q^3\frac{(q^{16};q^{16})_\infty^2[-q^2,q^4;q^{16}]_\infty}{[-q^4,q^6,-q^8;q^{16}]_\infty}+q^5\frac{[q^4;q^{16}]_\infty}{[-q^2;q^{16}]_\infty}S(q^6,1).\end{aligned} \tag{6.8}$$

To study (6.7), we first verify two theta function identities, both used in the second equality of (6.13) below. The first equality that we shall employ is

$$[-q^8;q^{16}]_\infty-q^2[-1;q^{16}]_\infty=[q^2,q^2,q^4,q^6,q^6;q^{16}]_\infty(q^8;q^{16})_\infty, \tag{6.9}$$

which, after multiplying both sides by $(q^{16};q^{16})_\infty$ and utilizing the Jacobi triple product identity (3.1), is equivalent to the elementary identity

$$\begin{aligned}\sum_{n=-\infty}^{\infty}q^{8n^2}-2q^2\sum_{n=0}^{\infty}q^{8n(n+1)} &= \sum_{n=-\infty}^{\infty}q^{2(2n)^2}-\sum_{n=-\infty}^{\infty}q^{2(2n+1)^2}\\ &= \sum_{n=-\infty}^{\infty}(-1)^nq^{2n^2}.\end{aligned}$$

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The second equality that we shall use is given by

$$\begin{aligned}&[-1,-q^6,-q^6;q^{16}]_\infty-[-q^2,-q^2,-q^8;q^{16}]_\infty\\ &\quad=[q^2,q^2,q^4,q^4;q^{16}]_\infty(q^8;q^{16})_\infty,\end{aligned}\tag{6.10}$$

which we now prove. Multiplying both sides of (6.10) by $[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty^2$ and then dividing both sides by $[-1,q^2,q^2,-q^4,-q^8;q^{16}]_\infty$, we see that (6.10) is equivalent to

$$\begin{aligned}&\frac{[q^4,-q^6,-q^6;q^{16}]_\infty(q^{16};q^{16})_\infty^2}{[q^2,q^2,-q^4,-q^8;q^{16}]_\infty}-\frac{[-q^2,-q^2,q^4;q^{16}]_\infty(q^{16};q^{16})_\infty^2}{[-1,q^2,q^2,-q^4;q^{16}]_\infty}\\ &\quad=\frac{[q^4,q^4,q^4;q^{16}]_\infty(q^8;q^{16})_\infty(q^{16};q^{16})_\infty^2}{[-1,-q^4,-q^8;q^{16}]_\infty}=\frac{[q^4,q^4,q^8;q^{16}]_\infty(q^{16};q^{16})_\infty^2}{[-1,-q^4,-q^4,-q^8;q^{16}]_\infty}.\end{aligned}\tag{6.11}$$

Note that

$$[q^4;q^{16}]_\infty(q^8;q^{16})_\infty=\frac{(q^8;q^{16})_\infty}{(-q^4;q^4)_\infty}=\frac{(q^8;q^{16})_\infty}{(-q^4;q^8)_\infty(-q^8;q^8)_\infty}=\frac{(q^8;q^{16})_\infty^2}{[-q^4;q^{16}]_\infty},\tag{6.12}$$

which we use on the right side of (6.11). Then (6.11) follows from (4.4) upon replacing q by q^8 and setting $(a,b,c,d)=(q^2,q^2,-q^4,-1)$.

We now focus on (6.7). By invoking (4.1) and using (5.5) in the second equality below and, as we mentioned above, (6.9) and (6.10) as well, we find that

$$\begin{aligned}&\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}}{1+q^{4n}}\\ &=\frac{[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2[-q^2,-q^6;q^{16}]_\infty}S(q^2,q^{-16})([-q^6;q^{16}]_\infty^2-q^2[-q^2;q^{16}]_\infty^2)\\ &\quad-\frac{(q^{16};q^{16})_\infty^3[-q^2,q^4,-q^6;q^{16}]_\infty([-q^8;q^{16}]_\infty-q^2[-1;q^{16}]_\infty)}{(q^2;q^2)_\infty^2[-1,q^2,-q^4,-q^8;q^{16}]_\infty}\\ &\quad+q\frac{(q^{16};q^{16})_\infty^3[q^4;q^{16}]_\infty([-1,-q^6,-q^6;q^{16}]_\infty-[-q^2,-q^2,-q^8;q^{16}]_\infty)}{(q^2;q^2)_\infty^2[-1,q^2,-q^4,-q^8;q^{16}]_\infty}\\ &=\frac{[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2[-q^2,-q^6;q^{16}]_\infty}S(q^2,q^{-16})[q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty\end{aligned}$$

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$$- \frac{(q^{16};q^{16})_\infty^3[-q^2,q^4,-q^6;q^{16}]_\infty}{(q^2;q^2)_\infty^2[-1,q^2,-q^4,-q^8;q^{16}]_\infty}[q^2,q^2,q^4,q^6,q^6;q^{16}]_\infty(q^8;q^{16})_\infty$$

$$+ q\frac{(q^{16};q^{16})_\infty^3[q^4;q^{16}]_\infty}{(q^2;q^2)_\infty^2[-1,q^2,-q^4,-q^8;q^{16}]_\infty}[q^2,q^2,q^4,q^4;q^{16}]_\infty(q^8;q^{16})_\infty$$

$$= \frac{1}{(q^{16};q^{16})_\infty}S(q^2,q^{-16}) - \frac{(q^{16};q^{16})_\infty[q^4;q^{16}]_\infty^2}{2[q^2,q^2,q^6;q^{16}]_\infty} + q\frac{(q^{16};q^{16})_\infty[q^4;q^{16}]_\infty^2}{2[q^2,q^6,q^6;q^{16}]_\infty}. \tag{6.13}$$

To study (6.8), we again need to first establish a certain theta function identity, namely,

$$[-q^6,-q^6,-q^8;q^{16}]_\infty - q^4[-1,-q^2,-q^2;q^{16}]_\infty = [q^4,q^4,q^6,q^6;q^{16}]_\infty(q^8;q^{16})_\infty. \tag{6.14}$$

Multiplying both sides of (6.14) by $[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty^2$ and then dividing both sides by $[-1,-q^4,q^6,q^6,-q^8;q^{16}]_\infty$, we see that (6.14) is equivalent to

$$\frac{[q^4,-q^6,-q^6;q^{16}]_\infty(q^{16};q^{16})_\infty^2}{[-1,q^6,q^6,-q^4;q^{16}]_\infty} - \frac{[-q^2,-q^2,q^4;q^{16}]_\infty(q^{16};q^{16})_\infty^2}{[q^6,q^6,-q^{-4},-q^8;q^{16}]_\infty} = \frac{[q^4,q^4,q^8;q^{16}]_\infty(q^{16};q^{16})_\infty^2}{[-1,-q^4,-q^4,-q^8;q^{16}]_\infty}, \tag{6.15}$$

where we applied (6.11) and the elementary identity

$$[-q^4;q^{16}]_\infty = q^4[-q^{-4};q^{16}]_\infty.$$

In (4.4), replace q by q^8 and set $(a,b,c,d) = (q^6,q^6,-1,-q^{-4})$. Then (6.15) follows immediately.

We now give our attention to (6.8). As before, we utilize (4.1), and then applying (5.5), (6.14), and (6.9) in the second equality below, we find that

$$\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+3)/2}}{1+q^{4n}}$$

$$= q^5\frac{[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2[-q^2,-q^6;q^{16}]_\infty}S(q^6,1)([-q^6;q^{16}]_\infty^2 - q^2[-q^2;q^{16}]_\infty^2)$$

$$+ \frac{(q^{16};q^{16})_\infty^3[q^4;q^{16}]_\infty([-q^6,-q^6,-q^8;q^{16}]_\infty - q^4[-1,-q^2,-q^2;q^{16}]_\infty)}{(q^2;q^2)_\infty^2[-1,-q^4,q^6,-q^8;q^{16}]_\infty}$$

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$$
\begin{aligned}
&+ q\frac{(q^{16};q^{16})_\infty^3[-q^2,q^4,-q^6;q^{16}]_\infty([-q^8;q^{16}]_\infty - q^2[-1;q^{16}]_\infty)}{(q^2;q^2)_\infty^2[-1,-q^4,q^6,-q^8;q^{16}]_\infty}\\
&= q^5\frac{[q^4;q^{16}]_\infty(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty^2[-q^2,-q^6;q^{16}]_\infty}S(q^6,1)[q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty\\
&\quad+\frac{(q^{16};q^{16})_\infty^3[q^4;q^{16}]_\infty}{(q^2;q^2)_\infty^2[-1,-q^4,q^6,-q^8;q^{16}]_\infty}[q^4,q^4,q^6,q^6;q^{16}]_\infty(q^8;q^{16})_\infty\\
&\quad+q\frac{(q^{16};q^{16})_\infty[q^4;q^{16}]_\infty^2}{2[q^2,q^6,q^6;q^{16}]_\infty}\\
&= q^5\frac{1}{(q^{16};q^{16})_\infty}S(q^6,1)+\frac{(q^{16};q^{16})_\infty[q^4;q^{16}]_\infty^2}{2[q^2,q^2,q^6;q^{16}]_\infty}+q\frac{(q^{16};q^{16})_\infty[q^4;q^{16}]_\infty^2}{2[q^2,q^6,q^6;q^{16}]_\infty}.
\end{aligned}
\tag{6.16}
$$

We can now complete the proof of Theorem 1.7 by substituting (6.13) and (6.16) into (6.6).

§7. Some consequences of Theorems 1.6 and 1.7

In this final section, we give some immediate consequences of Theorems 1.6 and 1.7 related to ranks and to mock theta functions. First, let $N(k,t,n)$ denote the number of partitions of n with rank congruent to k modulo t. Denote

$$R_{b,c}(t,l,d)=\sum_{n=0}^{\infty}[N(b,t,ln+d)-N(c,t,ln+d)]q^n.$$

Then the following results follow from Theorems 1.6 and 1.7.

Corollary 7.1. *We have*

$$R_{0,2}(4,4,0)=R_{0,4}(8,4,0), \tag{7.1}$$

$$R_{0,2}(4,4,2)=-R_{0,4}(8,4,2), \tag{7.2}$$

$$R_{0,2}(4,4,1)=R_{0,4}(8,4,1)+2R_{1,3}(8,4,1), \tag{7.3}$$

$$R_{0,2}(4,4,3)=-R_{0,4}(8,4,3)-2R_{1,3}(8,4,3), \tag{7.4}$$

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that is, for $n \geqslant 0$,

$$N(0,4,2n) - N(2,4,2n) = (-1)^n[N(0,8,2n) - N(4,8,2n)], \tag{7.5}$$

$$\begin{aligned} &N(0,4,2n+1) - N(2,4,2n+1) \\ &\quad = (-1)^n \left[N(0,8,2n+1) + 2N(1,8,2n+1)\right. \\ &\qquad \left. - 2N(3,8,2n+1) - N(4,8,2n+1)\right]. \end{aligned} \tag{7.6}$$

Proof. On page 72 of [14], Garvan explains how one can obtain results on rank differences from the dissections of the rank generating functions. We follow his argument here. First, note that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(iq)_n(q/i)_n} = \sum_{n=0}^{\infty}\sum_{k=0}^{3} N(k,4,n) i^k q^n = \sum_{n=0}^{\infty}(N(0,4,n) - N(2,4,n))q^n,$$

since $N(1,4,n) = N(3,4,n)$ [14, equations (1.09), (1.10)] and $i^3 = -i$. By extracting the terms with even powers of q on both sides of (1.10), we deduce that

$$\begin{aligned} &\sum_{n=0}^{\infty}(N(0,4,2n) - N(2,4,2n))q^{2n} \\ &\quad = \frac{2}{(q^{16};q^{16})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+8n}}{1+q^{16n+2}} - \frac{[q^4;q^{16}]_\infty^2 (q^{16};q^{16})_\infty}{[-q^2,-q^2,-q^6;q^{16}]_\infty}. \end{aligned} \tag{7.7}$$

As above, let a denote a primitive eighth root of unity. Using the relations $a^2 = -a^6$, $N(2,8,n) = N(6,8,n)$, and $a^3 + a^5 = -(a + a^7)$, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(aq)_n(q/a)_n} &= \sum_{n=0}^{\infty}\sum_{k=0}^{7} N(k,8,n) a^k q^n \\ &= \sum_{n=0}^{\infty} \left\{[N(0,8,n) - N(4,8,n)] + (a+a^7)[N(1,8,n) - N(3,8,n)]\right\} q^n. \end{aligned}$$

Since 1 and $a + a^7$ are linearly independent over the set of integers, extracting the terms with even powers of q on both sides of (1.11), we obtain the two identities

$$\begin{aligned} &\sum_{n=0}^{\infty}(N(0,8,2n) - N(4,8,2n))q^{2n} \\ &\quad = \frac{2}{(q^{16};q^{16})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{24n^2+8n}}{1-q^{16n+2}} - \frac{[q^4;q^{16}]_\infty^2 (q^{16};q^{16})_\infty}{[q^2,q^2,q^6;q^{16}]_\infty}, \end{aligned} \tag{7.8}$$

$$\sum_{n=0}^{\infty}(N(1,8,2n)-N(3,8,2n))q^{2n}$$
$$=-\frac{1}{(q^{16};q^{16})_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{24n^2+8n}}{1-q^{16n+2}}+\frac{[q^4;q^{16}]_\infty^2(q^{16};q^{16})_\infty}{[q^2,q^2,q^6;q^{16}]_\infty}.$$

We see that the right side of (7.7) is exactly (7.8) but with q^2 replaced by $-q^2$. This implies (7.1), (7.2), and (7.5). Equations (7.3), (7.4), and (7.6) are proved similarly by selecting the terms with odd powers of q on both sides of (1.10) and (1.11).

REMARK 7.2. Santa-Gadea and Lewis proved many results on ranks and cranks modulo 4 and 8. See, for example, [18–24, 27, 28]. Equation (4.5) of Santa-Gadea's Thesis [27] gives the generating function of the relation

$$N(0,4,n)-N(2,4,n)=N(0,8,n)-2N(2,8,n)+N(4,8,n). \tag{7.9}$$

Through (7.9), the relations given in [27, equations (4.1)–(4.4)] are immediately seen to be equivalent to (7.5) and (7.6). The relations (4.1)–(4.4) in [27] were originally conjectured by Lewis in two papers [18], [20]. They were proved again in a later paper by Santa-Gadea and Lewis [24].

Each of Ramanujan's mock theta functions satisfies a transformation formula involving the rank function $G(z,q)$, defined in (1.1). For example, the famous mock theta conjectures, first proved by Hickerson [17, equations (0.9), (0.10)], are given by

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q)_n}=2-\frac{2}{1-q^2}G(q^2,q^{10})+\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{[q;q^5]_\infty},$$
$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q)_n}=\frac{2}{q}-\frac{2}{q(1-q^4)}G(q^4,q^{10})+\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{[q^2;q^5]_\infty}.$$

By summing the odd and even indices n's separately, we can write down the 2-dissection of the rank function modulo 4 directly as

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(iq)_n(q/i)_n}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{4n^2}}{(-q^2;q^2)_{2n}}+q\sum_{n=0}^{\infty}\frac{q^{4(n^2+n)}}{(-q^2;q^2)_{2n+1}}.$$

By applying (2.3) and the identity [14, equation (7.10)]

$$\frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1-zq^n}=-1+\frac{1}{1-z}G(z,q),$$

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in Theorem 1.6, we can then derive analogous identities.

COROLLARY 7.3. *We have*

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_{2n}} = \frac{2}{1+q} G(-q,q^8) - \frac{[q^2,q^2;q^8]_\infty (q^8;q^8)_\infty}{[-q,-q,-q^3;q^8]_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^{2(n^2+n)}}{(-q;q)_{2n+1}} = \frac{2}{q} - \frac{2}{q(1+q^3)} G(-q^3,q^8) + \frac{[q^2,q^2;q^8]_\infty (q^8;q^8)_\infty}{[-q,-q^3,-q^3;q^8]_\infty}.$$

This is not the first time these two functions have been studied. For example, in [1, equations (1.14), (1.15)]), the first author gave Hecke-type series representations for the two functions

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q)_{2n}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2+n}(1-q^{6n+3}) \sum_{j=-n}^{n} (-1)^j q^{-j^2},$$

$$\sum_{n=0}^{\infty} \frac{q^{2(n^2+n)}}{(-q;q)_{2n+1}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-j^2}.$$

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https://doi.org/10.1017/nmj.2018.35 Published online by Cambridge University Press

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