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NOTE ON OPERATIONS GENERATING THE GROUP OPERATIONS IN NILPOTENT GROUPS OF CLASS 3

ERNEST PŁONKA

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Let K be a class of groups and let $\omega(K)$ denote the set of all such words $w(x, y)$ that the group operations $1, x^{-1}, xy$ in every group $G \in K$ can be expressed as a superposition of w and the projections $e_1(x, y) = x$, $e_2(x, y) = y$. Clearly,

$$\omega(K) \supseteq \{xy^{-1}, x^{-1}, y, yx^{-1}, y^{-1}x\}$$

for arbitrary class K. The inverse does not hold in general (for example for a class of periodic nilpotent groups of class 2, see Hulanicki and Swierczkowski (1962), and one may ask about the class K for which

$$(*) \qquad \omega(K) = \{xy^{-1}, x^{-1}y, yx^{-1}, y^{-1}x).$$

Let N_k be the variety of all nilpotent groups of class k. In Padmanabhan (1969) it is shown that (*) holds for the Abelian variety N_1, and recently Fajtlowicz (1972) has proved the same for N_2. One might conjecture that (*) is also valid for N_3. Unexpectedly enough, it turns out that this is not the case and in this note we prove the following result.

THEOREM. *For all integers a we have*

$$xy^{-1}[y, x, x]^a \in \omega(N_3).$$

PROOF. Let us recall the identities

$$[xy, z] = [x, z][x, z, y][y, z]$$
$$[x, yz] = [x, z][x, y][x, y, z]$$

holding in any group. Using induction one can easily verify that the identities

$$[[x, y]^m, z] = [x, y, z]^m$$
$$[x^m, y, z] = [x, y^m, z] = [x, y, z^m] = [x, y, z]^m$$
$$[y^n, x^m] = [y, x]^{mn}[y, x, x]^{n\binom{m}{2}}[y, x, y]^{m\binom{n}{2}}$$

are fulfilled in every $G \in N_3$ and for all integers m, n. (For commutator identities see e.g. B. Huppert (1967)).

Let us observe that all operations of one variable can be expressed in terms of $w = xy^{-1}[y,x,x]^a$. Indeed, we have $w(x,x) = 1$, $w(1,y) = y^{-1}$ and for all $k \geqq 0$ $x^{k+1} = w(x^k, x^{-1}) \,.\, x^{-k-1} = w(x^{-k}, x)$. We verify that

$$xy = w\{w(x,y^{-1}), \{w(w(w(y,w(y,x)), w(w(y,w(y,x)),x)),x)\}\, x^{-a}\}.$$

Let

$$\begin{aligned} u &= w(y,w(y,x)) \\ &= w(y, yx^{-1}[x,y,y]^a) \\ &= y[x,y,y]^{-a}xy^{-1}[x^{-1},y,y]^a \\ &= yxy^{-1}[y,x,y]^{2a} \\ &= x[y,x][y,x,y]^{2a-1}. \end{aligned}$$

Hence

$$\begin{aligned} w(u,x) &= ux^{-1}[x,u,u]^a \\ &= x[y,x][y,x,y]^{2a-1}x^{-1} \\ &= [y,x][y,x,x]^{-1}[y,x,y]^{2a-1}. \end{aligned}$$

Then $v = w(u,w(u,x)) = x[y,x,x]$, so that $w(v,x) = [y,x,x]$. But then we have $w(w(x,y^{-1}),\ (w(v,x))^{-a}) = xy$, which completes the proof of the theorem.

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Instytut Matematyczny PAN
51–617 Wrocław
ul. Kopernika 18
Poland

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